Maximal solutions of viscous Hamilton–Jacobi equations with degeneracy

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Qualitative Methods for Hamilton–Jacobi Equations and Applications

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Work *in progress* jointly with

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Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set and let us consider the following second order degenerate elliptic equation

$$-\mathrm{tr}\left(A(x)D^2u\right) + |Du|^p + u = f(x)\,, \quad x \in \Omega\,, \qquad \textbf{(E)}$$

where $A : \overline{\Omega} \mapsto \mathcal{S}_N$ is a continuous map from $\overline{\Omega}$ into the space of symmetric $N \times N$ matrices satisfying

$$O \le A(x) \le \Lambda\, Id_N \qquad \forall\, x \in \overline{\Omega}\,,$$

$$\sqrt{A} \text{ is Lipschitz continuous in } \overline{\Omega}\,,$$

and with

$$p > 1\,, \qquad f \in C(\overline{\Omega})\,.$$

We are going to focus on *viscosity solutions* of equation (**E**) satisfying *special* boundary conditions.

Where does equation (**E**) come from?

Equations like (**E**) arise in degenerate *stochastic control* problems. Indeed, let us consider the following stochastic differential equation

$$dX_t = a(X_t)\,dt + \sqrt{A}(X_t)\,dW_t\,, \quad X_0 = x\,,$$

where $a(X_t)$ is interpreted as a *feedback control* and W_t is a standard Brownian motion.

By using the control a, we want to force the solution X_t to stay in Ω with probability 1 for all $t \geq 0$ and for all initial points $x \in \Omega$; in other words, we impose a *state constraint* on the controlled system.

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Note that if A is non degenerate and if a is bounded, then the probability that X_t hits the boundary $\partial\Omega$ is positive for all time $t > 0$. Thus, in this case, the only way we have to keep the solution X_t constrained in the domain is to use an *unbounded* control a which pushes back the state process with an infinite intensity.

We then define in such a way the class $\mathcal{A}_x$ of admisssible controls for the initial point $x \in \Omega$, and we consider for $a \in \mathcal{A}_x$ (provided $\mathcal{A}_x$ is non empty) the following *cost functional* associated with the problem

$$J(x, a) = E \int_0^\infty \left[f(X_t) + \frac{1}{q} |a(X_t)|^q \right] \mathrm{e}^{-t} \, dt \, .$$

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If $q = \frac{p}{p-1}$, then equation (**E**) (up to a multiplicative constant in front of $|Du|^p$) is expected for the value function

$$u(x) = \inf_{a \in \mathcal{A}_x} J(x,a)\,.$$

Moreover, the state contraint on the process X_t yields the boundary condition for u

$$-\mathrm{tr}\left(A(x)D^2u\right) + |Du|^p + u \geq f(x)\,, \quad x \in \partial\Omega\,.$$

In other words, the value function u turns out to be a *solution* in Ω and a *supersolution* in $\overline{\Omega}$.

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Some basic references

For the deterministic case (i.e. $A(x) \equiv O$)

- M.H. Soner, *Optimal control with state-space constraint I & II*, SIAM J. Control Optim. **24** (1986)
- I. Capuzzo Dolcetta & P.L. Lions, *Hamilton–Jacobi equations with state constraints*, Trans. AMS **318** (1990)
- H. Ishii & S. Koike, *A new formulation of state constraint problems for first–order pdes*, SIAM J. Control Optim. **34** (1996)

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Related papers for the stochastic case (with A depending on the control and degenerating on the boundary)

- M.A. Katsoulakis, *Viscosity solutions of second order fully nonlinear elliptic equations with state constraints*, Indiana Univ. Math. J. **43** (1994)
- H. Ishii & P. Loreti, *A class of stochastic optimal control problems with state constraint*, Indiana Univ. Math. J. **51** (2002)

Maximal Solutions

Motivated by the above discussion, we give the following

Definition

A maximal solution *of equation* (**E**) *is a function* $u \in C(\Omega)$ *which is a viscosity solution in* Ω *and such that*

$$u_*(x) = \begin{cases} u(x) & \text{if } x \in \Omega\,, \\ \liminf\limits_{\substack{y\in\Omega \\ y\to x}} u(y) & \text{if } x \in \partial\Omega\,, \end{cases}$$

is a viscosity supersolution in $\overline{\Omega}$.

In the language of *generalized viscosity solutions*, a maximal solution is nothing but a generalized viscosity solution of equation (**E**) equipped with the boundary condition

$$u = +\infty \qquad \text{on} \;\; \partial\Omega \, .$$

Then, the following result is very natural.

Proposition

Let u be a maximal solution. If $v \in C(\overline{\Omega})$ is a viscosity subsolution in Ω, then $v(x) \leq u_*(x)$ for every $x \in \overline{\Omega}$.

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For the proof, use the comparison principle by

G. Barles, E. Rouy & P.E. Souganidis, *Remarks on the Dirichlet problem for quasilinear elliptic and parabolic equations*, in Stochastic Analysis, Control, Optimization and Applications, Birkhäuser, Boston, 1999

jointly with a technicality to deal with the power–like nonlinearity of equation (**E**) as in

G. Barles & F. Da Lio, *On the generalized Dirichlet problem for viscous Hamilton–Jacobi equations*, J. Math. Pures Appl. **83** (2004). ∎

Conversely, we have the following

Proposition

Let $u \in C(\Omega)$ be a viscosity subsolution such that $u \geq v$ for every subsolution $v \in USC(\Omega)$. Then, u is a maximal solution.

The proof can be obtained by arguing as in the first order case (see e.g. Capuzzo Dolcetta & Lions) and by using the "bump lemma" of the *User's guide*. ∎

Goal: existence, uniqueness and regularity properties for maximal solutions.

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Known results for the deterministic case

If $A \equiv O$, then equation (**E**) reduces to the Hamilton–Jacobi equation

$$|Du|^p + u = f \qquad \text{in} \ \ \Omega\,.$$

It has been proved that there exists a unique maximal solution $u \in C(\overline{\Omega})$, which can be characterized also as the unique generalized supersolution of the associated homogeneus Neumann problem.
In this case, the maximal solution u is Lipschitz continuous in $\overline{\Omega}$ and the optimal control is $a(x) = -|Du(x)|^{p-2}Du(x)$.

Remark

Note that the Lipschitz continuity holds for *any* bounded from below *subsolution*.

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Known results for the uniformly stochastic case

In this case, equation (**E**) has the form

$$-\Delta u + |Du|^p + u = f \qquad \text{in} \ \ \Omega\,,$$

and it has been shown to have a unique maximal solution $u \in C(\Omega)$ which is *locally* Lipschitz continuous.

Moreover:

- if $p \leq 2$, then u uniformly blows up at the boundary, with a rate of order $\mathrm{dist}(x, \partial\Omega)^{\frac{p-2}{p-1}}$ if $p < 2$, and like $|\log \mathrm{dist}(x, \partial\Omega)|$ if $p = 2$. Then, u is a so called *large solution*.

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- if $p > 2$, then u is bounded in Ω and it can be extended to a globally Hölder continuos function with exponent $\alpha = \frac{p-2}{p-1}$.

In any case (with an additional assumption if $p > 2$), the optimal feedback control is $a(x) = -|Du|^{p-2}Du$, which is unbounded on $\partial\Omega$.

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The case of a general A, $p > 2$

Proposition

If $u \in C(\Omega)$ is a maximal solution, then there exist constants $m < M$ depending only on Ω, $p > 2$ and f such that

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Our main result is the following regularity theorem.

Theorem

Every viscosity subsolution $u \in BUSC(\Omega)$ of equation $(\mathbf{E})$ can be extended up to the boundary to a function satisfying

$$u \in C^{0,\alpha}(\overline{\Omega})\,, \qquad \alpha = \frac{p-2}{p-1}\,.$$

The idea of the proof is to use strongly the *coercitivity* of the first order term, as to partially absorbe the second order perturbation.
The α–hölderianity is the sharp regularity for subsolutions, as it is exhibited by the viscosity subsolution $u(x) = |x|^\alpha$ in any ball centered at the origin (if the dimension N is at least 2).

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As a consequence of the above regularity result, one easily gets the *existence* of a maximal solution by using any approximation argument (on the matrix $A(x)$, or adding to f a forcing datum defined on $\mathbb{R}^N$ and blowing up on the complement of Ω, or.....) Note that any approximating sequence of solutions will be bounded and equicontinuous, and thus uniformly converging to a solution.

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As for the regularity of *solution* is concerned, by adapting the Bernstein technique developped in Lasry & Lions, one can obtain the *local* Lipschitz continuity of any bounded solution by assuming f to be Lipschitz.

More precisely, for any solution $u \in C(\overline{\Omega})$ one gets the bound

$$|Du(x)| \leq \frac{C}{d(x)^{1-\alpha}}\,, \qquad x \in \Omega\,, \quad d(x) = \mathrm{dist}(x, \partial\Omega)\,.$$

As in the uniformly stochastic case, then one can show that the maximal solution u is the value function of the initial stochastic control problem, and $a(x) = -|Du|^{p-2}Du(x)$ is the optimal control.

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The case of a general A, $p \leq 2$

If $p \leq 2$, in general a maximal solution u satisfies Dirichlet boundary conditions of mixed type (bounded and unbounded).
In fact, one can easily show in this case that if u is any bounded from below supersolution in $\overline{\Omega}$, then

$$u_*(x) = +\infty \qquad \forall\, x \in \partial\Omega \text{ such that } \; A(x)\nu(x) \cdot \nu(x) > 0\,,$$

where $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at the point x.
On the other hand, in the boundary region where A degenerates along the normal direction, u is expected to be bounded (as in the deterministic case)
This makes the uniqueness still open, unless one specifies the rate of blowing up at the boundary.

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