A numerical model for growing sandpiles on partially open tables

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MATHEMATICS AND ITS APPLICATIONS
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Qualitative methods for HJ equations and applications

Outline

1 Sandpile growth on a plane support

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2 The open table problem
- Two recent differential models
- Asymptotic behavior and equilibria characterization
- A numerical scheme for the two-layer system

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- Asymptotic behavior and equilibria (1D)
- Asymptotic behavior and equilibria (2D)
- Numerical experiments

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1. Sandpile growth on a plane support
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   - Asymptotic behavior and equilibria characterization
   - A numerical scheme for the two-layer system
3. The partially open table problem
   - Asymptotic behavior and equilibria (1D)
   - Asymptotic behavior and equilibria (2D)
   - Numerical experiments
4. Conclusion and developments

Connection with other mathematical research fields

- *Optimal mass transport, Monge-Kantorovich problem*
- *River networks, semiconductor magnetization, elastoplastic deformation*
- *Infinity Laplacian, absolute minimizers and optimal Lipschitz extensions of given boundary data*
- *Hamilton-Jacobi equations*
- *Nonlocal geometric curvature motion*

Sandpile growth on a plane support

Differential approach: main references

1. G. Aronsson, *A mathematical model in sand mechanics*, SIAM J.Appl.Math., **22** ('72)
2. J. -P. Bouchaud, M. E. Cates, J. Ravi Prakash and S. F. Edwards, *A model for the dynamics of sandpile surfaces*, J. Phys.I Fr., **4** ('94)
3. G. Aronsson, L.C. Evans and Y. Wu, *Fast/slow diffusion and growing sandpiles*, J.Diff.Equat., **131** ('96)
4. L. Prigozhin, *Variational model of sandpile growth*, Euro.J.Appl.Math., **7** ('96)
5. K.P. Hadeler and C. Kuttler, *Dynamical models for granular matter*, Granular Matter, **2** ('99)
6. P. Cannarsa and P.Cardaliaguet, *Representation of equilibrium solutions to the table problem for growing sandpiles*, JEMS, **6** ('04)

The table problem: notations

- $\Omega \subseteq \mathbb{R}^2$: bounded *table* , $\Omega_T = \Omega \times (0, T)$
- $\partial\Omega = \Gamma_0 \cup \Gamma_w$, Γ_0 : *open boundary*, Γ_w : *walls*
- $f(x) \geq 0$: vertical *source* , $D_f = \overline{\{x : f > 0\}}$
- $u(x, t)$, *pile height* in $x \in \Omega$ at time t
- $u_0(x)$: *initial profile* (here $\boxed{u_0 \equiv 0}$)
- $|\nabla u| \leq a$: *critical slope* (here $\boxed{a = 1}$)
- $d(x) = \mathrm{dist}(x, \Gamma_0)$: *distance* from Γ_0
- S : *cut locus* of Ω (singular set of d)

Boundary conditions for the table problem

Boundary conditions

1. Dirichlet ($\partial\Omega = \Gamma_0$) $\Rightarrow$ open table problem
2. Mixed ($\Gamma_0, \Gamma_w \neq \emptyset$) $\Rightarrow$ partially open table problem
3. Neumann ($\partial\Omega = \Gamma_w$) $\Rightarrow$ closed table problem (silo)

The problem changes very much according to boundary conditions. Here we only recall case (1) and discuss (2), where, as time grows,

$$u(x,t) \to \overline{u}(x) \qquad (\textit{equilibria}).$$

For the silo problem (3) (see e.g. [Hadeler-Kuttler, '99 and '01]) instead:

$$u(x,t) \to \overline{u}(x) + ct \qquad (\textit{similarity solutions}).$$

A variational model [Aronsson-Evans-Wu, Prigozhin, '96]

$$(P)\begin{cases} \partial_t u - \nabla\cdot(v\nabla u) = f & \text{in } \Omega_T \\ |\nabla u| \le 1\ , \quad |\nabla u| < 1 \Rightarrow v = 0 & \text{in } \Omega_T \\ u = 0 \quad \text{on } \partial\Omega\ , \qquad u(\cdot,0) = 0 & \text{in } \Omega \end{cases}$$

- $v(x,t) \ge 0$ is an auxiliary unknown which controls the surface flow (a dynamic Lagrange multiplier for the constraint on ∇u)
- (P) is equivalent to the variational inequality

$$\begin{cases} u(t) \in K = \{v \in W_0^{1,\infty}(\Omega) : |\nabla v| \le 1\}, \quad u(0) = 0, \\ (\partial_t u(t) - f, \phi - u(t)) \ge 0, \quad \forall \phi \in K,\ \forall t > 0 \end{cases}$$

- Existence and uniqueness of the solution (by penalty method)

A two-layer system [Hadeler-Kuttler, '99]

$$(HK)\begin{cases} \partial_t v = \nabla\cdot(v\nabla u) - (1-|\nabla u|)v + f & \text{in } \Omega_T \\ \partial_t u = (1-|\nabla u|)v & \text{in } \Omega_T \\ u = 0 \quad \text{on } \partial\Omega\,, \quad u(\cdot,0)=0 & \text{in } \Omega \end{cases}$$

where u : *standing layer*, v : *rolling layer*, $u+v$: *pile height*.

- Extension of the BCRE + de Gennes model (transport velocity proportional to the pile slope)
- No existence and uniqueness results known

Model comparison

Both the models describe granular surface flow and pile surface dynamics, neglecting avalanche phenomena (realistic for small source intensities). However, their characteristic are very different:

- (P) Surface flow allowed only at critical slopes. Well-suited for describing large piles or long distance phenomena (small details are negligible) [large spatiotemporal scale]

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- (HK) Surface flow allowed upon subcritical slopes. Well-suited for describing fast processes and small details (*sand ripples formation*, *contact angle near pile bottom*) [short spatiotemporal scale]
- Rescaled (HK) converges (in the long-scale limit) to (P) [Prigozhin-Zaltzman, '01]

Asymptotic behavior and equilibria

- Existence of equilibria: for both the (P) and the (HK) model

$$u_t \geq 0 \ , \quad u(.,t) \leq d(.) \quad \Rightarrow \quad u(x,t) \rightarrow \ \overline{u}(x) \ .$$

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- The two models have different dynamics, but formally the same admissible equilibrium configurations, solutions of

$$(E) \begin{cases} -\nabla \cdot (v \nabla u) = f & \text{in } \Omega \\ |\nabla u| = 1 & \text{in } \{v > 0\} \\ |\nabla u| \leq 1 \ , \quad u, v \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Remark

System (E) is not able to determine u in regions where $v = 0$!

Special equilibria

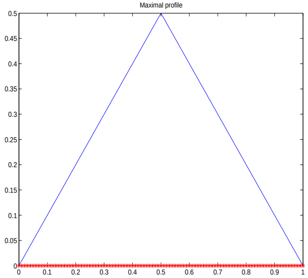
Maximal:

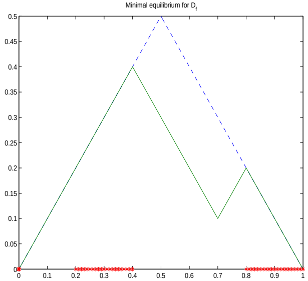
$$u(x) = d(x)$$

Minimal w.r. to D_f:

$$u_*(x) = \max_{y \in D_f} \{d(y) - |x - y|\}^+$$

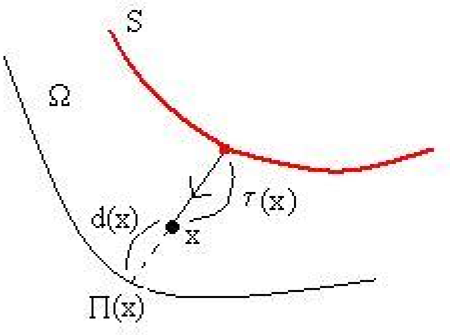
(*physical* solution)





Characterization of equilibria

Let $k(x)$ be the curvature of $\partial\Omega$ at the boundary projection $\Pi(x)$ of $x \in \Omega$, and $\tau(x) = \min\{t \geq 0 : x + t\nabla d(x) \in \overline{S}\}$



Characterization of equilibria [Cannarsa-Cardaliaguet '04]

Theorem

Let $\partial\Omega \in C^2$, $f \in C^0(\Omega)$; *then :*

- *Existence:* (u, v) *solves* (E), *with*

$$u = d \quad in\ \Omega, \qquad v = 0 \quad on\ \overline{S}$$

$$(*) \quad v(x) = \int_0^{\tau(x)} f(x + t\nabla d(x)) \frac{1 - (d(x) + t)k(x)}{1 - d(x)k(x)}\, dt, \ \ in\ \Omega \setminus \overline{S}$$

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- *Almost uniqueness*: *if* (u', v') *is another solution of (E), then*

$$v' = v \quad in\ \Omega\ , \qquad u' = d \quad in\ \{x \in \Omega : v' > 0\}.$$

Asymptotic behavior of the two models

1. $\boxed{S \subset \overline{D}_f}$: (E) **has one and only one solution** $(\overline{u}, \overline{v})$,
with $\boxed{\overline{u} \equiv d}$ **and** $\overline{v}$ **given by the integral formula** (*)
$\Rightarrow$ same equilibrium for the two models

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$\Rightarrow$ same equilibrium for the two models
2. $\boxed{S \not\subset \overline{D}_f}$: **no uniqueness for** $\overline{u}$ in (E)
   - (P) $\boxed{\overline{u} = u_*}$
     The active region $\Omega_P^+ = \{x \in \Omega : \overline{u} > 0\}$ is completely determined by D_f (source intensity only affects $\overline{v}$!)
   - (HK) $\boxed{\overline{u} > u_*}$
     $\overline{u}$ and the active region $\Omega_{HK}^+ = \{x \in \Omega : \overline{u} > 0\} \supseteq \Omega_P^+$ are not mathematically characterized: both depend on D_f and on the source intensity f too !

An explicit f.d. scheme for the two-layer system (HK)

[Falcone - F.V., SIAM J.Sci.Comput., '06]

$$(HK_{1D})\begin{cases} v_t + [-u_x v]_x = v_t + [F_u(v)]_x = f - (1-|u_x|)v \\ u_t = (1-|u_x|)v \\ u(0,t) = u(1,t) = 0\ , \qquad u(x,0)=0 \quad x \in \Omega = (0,1)\ . \end{cases}$$

$$(S_{HK})\begin{cases} v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(H^n_{i+\frac{1}{2}} - H^n_{i-\frac{1}{2}}) + \Delta t\,[\, f_i - (1-|Du_i^n|)\, v_i^n] \\ u_i^{n+1} = u_i^n + \Delta t\ (1-|Du_i^n|)\, v_i^{n+1} \\ u_i^0 = v_i^0 = 0 \quad (i=1,\dots,N)\ , \quad u_1^n = u_N^n = 0 \quad \forall n. \end{cases}$$

Finite difference formulas

At any node x_i : $D^+u_i = \frac{u_{i+1}-u_i}{h}$, $D^-u_i = \frac{u_i-u_{i-1}}{h}$

maxmod difference

$$|Du_i| \equiv \max\left(\left|D^+u_i\right|, \left|D^-u_i\right|\right)$$

upwind numeric flow in (x_i, x_{i+1})

$$H_{i+\frac{1}{2}} \equiv -\frac{u_{i+1}-u_i}{h} \ \mathrm{upw}(v_i, v_{i+1})$$

$$\mathrm{upw}(v_i, v_{i+1}) \equiv \begin{cases} v_i & \text{if } u_i > u_{i+1} \\ v_{i+1} & \text{if } u_i \le u_{i+1} \end{cases}$$

Properties of the scheme (S_{HK})

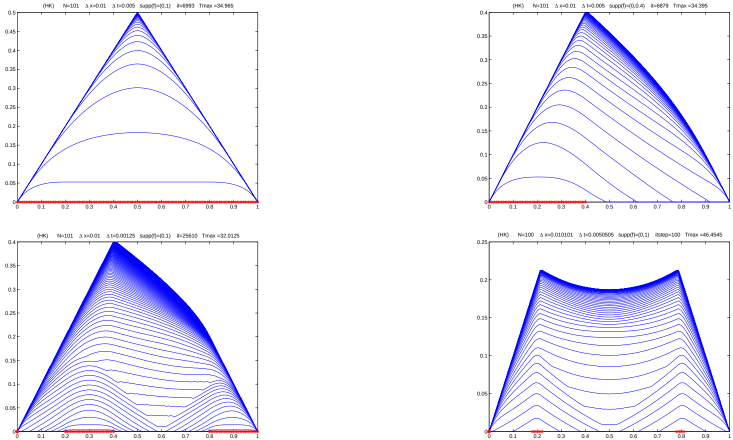
Theorem

Let $f \geq 0$ *in* Ω, *and* $\frac{\Delta t}{h} \leq \min\left(\frac{1}{2}, \frac{c}{\|f\|_\infty}\right)$ *; then for any* n *:*

- *(Positivity and monotonicity in* u*)* $0 \leq u^n \leq u^{n+1}$
- *(Positivity in* v*)* $v^n \geq 0$
- *(Gradient constraint in* u*)* $|Du^n| \leq 1$

$\Rightarrow$ *Under the previous stability conditions:* $(u^n, v^n) \to (\bar{u}, \bar{v})$, *equilibrium of the discrete system such that* $(1 - |D\bar{u}|)\bar{v} = 0$.

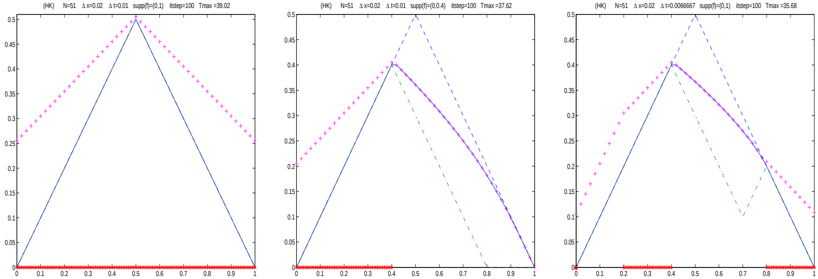
Examples of growing sandpiles



$f \equiv 0.5$, $u_h(\cdot, 100n\Delta t)$; $D_f = [0,1]$, $[0,0.4]$, $[0.2,0.4] \cup [0.8,1]$, $[0.18,0.22] \cup [0.78,0.82]$.

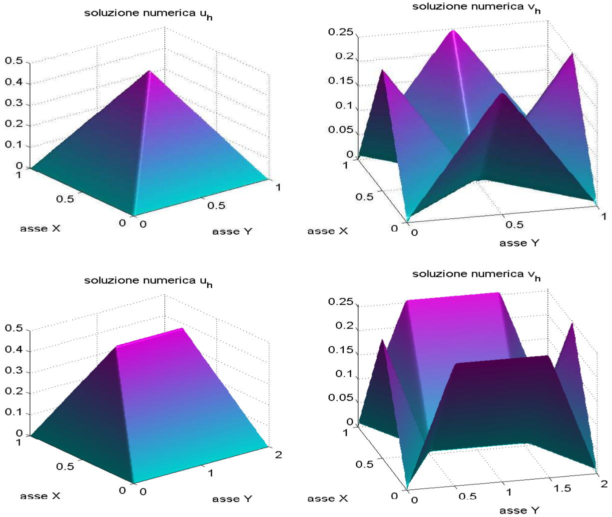
Examples of equilibria for different source supports

$\overline{u}$ versus u_*



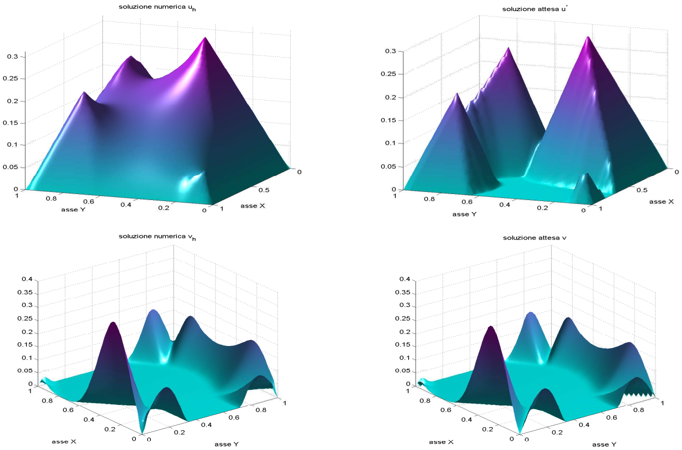
$f \equiv 0.5$, stationary u_h [—] and $(u_h + v_h)$ [++], the distance function d [- -] and the minimal solution u_* [-·-] when D_f is $[0, 1]$, $[0, 0.4]$ and $[.2, 0.4] \cup [0.8, 1]$.

Maximal equilibria (square and rectangular tables)



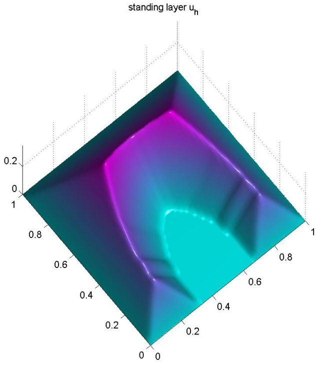
HK-2D : $D_f = \Omega$, $\Omega = (0,1)^2$ (top), $\Omega = (0,1) \times (0,2)$ (bottom).

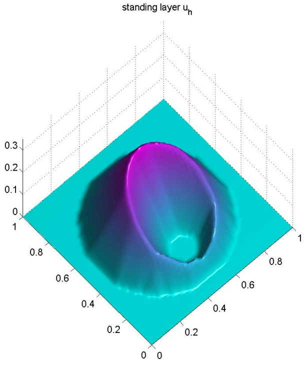
Different standing (same rolling) layers



HK-2D : $\Omega = (0,1)^2$, $D_f \subset \Omega$, $N = 51$, u_h versus u_*, v_h versus v_*.

More general tables: $|\nabla u| \leq 1_{\Omega'}(x),\ \Omega' \subset \Omega$





HK-2D : examples of non convex or non simply connected open tables

The partially open table problem

Let $\boxed{\partial\Omega = \Gamma_0 \cup \Gamma_w}$, $\Gamma_0, \Gamma_w \neq \emptyset$

Γ_0 : open boundary ; Γ_w : (infinite) vertical walls.

The two-layer system for the growing sandpiles becomes:

$$(HK_w) \begin{cases} \partial_t v = \nabla \cdot (v \nabla u) - (1 - |\nabla u|)v + f & \text{in } \Omega_T \\ \partial_t u = (1 - |\nabla u|)v & \text{in } \Omega_T \\ u(\cdot, 0) = 0 & \text{in } \Omega \\ u = 0 \text{ on } \Gamma_0 , \quad v \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_w & \end{cases}$$

The wall boundary condition

Summing up the two equations at the equilibrium:

$$0 = \frac{d}{dt}\int_\Omega (u+v)\ dx = \int_\Omega \nabla\cdot(v\nabla u)\ dx + \int_\Omega f\ dx =$$

$$= \int_{\Gamma_w} v\frac{\partial u}{\partial n}\ d\sigma + \int_{\Gamma_0} v\frac{\partial u}{\partial n}\ d\sigma + \int_\Omega f\ dx\ ;$$

since the last two terms (the sand leaving the table through Γ_0 and the incoming sand from the source) have to cancel at the equilibrium, the natural boundary condition at the wall becomes

$$\boxed{v\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_w.}$$

Equilibria

Special equilibria in this case:

- (maximal) $d_0(x) = dist(x, \Gamma_0)$
- (minimal w.r.t. to D_f) $u_*(x) = \max_{y \in D_f} \{d_0(y) - |x - y|\}^+$

A system for the equilibria

$$(E_w) \begin{cases} u, v \geq 0, & u \in Lip_1(\Omega), \;\; v \in BV(\Omega) \\ -\nabla \cdot (v \nabla u) = f & \text{in } \Omega \\ u_* \leq u \leq d_0 & \text{in } \overline{\Omega} \\ v \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_w \end{cases}$$

Asymptotic behavior and equilibria (1D)

Assume $\Omega = (0,1)$ and that sand can leave the table only from the left-hand side($\Gamma_0 = \{0\}, \ \Gamma_w = \{1\}$). Let $D_f = (x_1, x_2) \subseteq \Omega$; then

- Case (1) $\boxed{x_2 = 1}$ **There is only one possible equilibrium** :

$$\overline{u}(x) = d_0(x) = x \qquad \overline{v}(x) = \int_x^1 f(s)ds \ ;$$

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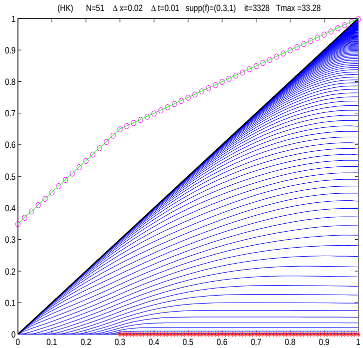
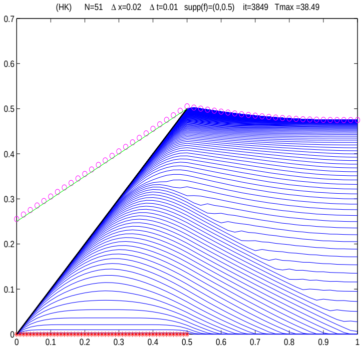
- Case (2) $\boxed{x_2 < 1}$ $\overline{u}$ **is not uniquely determined** :

$$\overline{u}(x) = \begin{cases} d_0(x) & \text{if } x \leq x_2 \\ ? & \text{if } x > x_2 \end{cases} \qquad \overline{v}(x) = \begin{cases} \int_x^{x_2} f(s)ds & \text{if } x < x_2 \\ 0 & \text{if } x \geq x_2 \end{cases}$$

The wall-problem (1D)





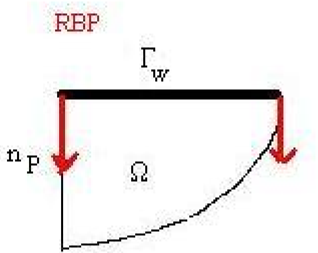
$f \equiv 0.5$: growing u_h [blue], final $u_h + v_h$ [red balls], $\overline{u}$ [black], $\overline{u} + \overline{v}$ [green].

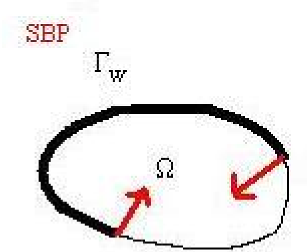
Asymptotic behavior and equilibria (2D)

- Ω convex $\Rightarrow$ all the transport rays towards Γ_0 are segments;
- $D_f = \Omega \Rightarrow$ the maximal equilibrium is expected.

Definition

Let $P \in \partial\Gamma_0$, n_P the normal direction to $\partial\Omega$ in P and $t \in \mathbb{R}$:
if $P + tn_P \notin \Omega$ for any $t \Rightarrow P$ is a regular boundary point (RBP),
if $P + tn_P \in \Omega$ for some $t \Rightarrow P$ is a singular boundary point (SBP).





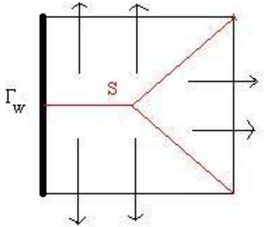
An example of regular boundary points

Example 1

- $\Omega = Q = (0,1)^2$ (a square table);
- $\Gamma_w =$ one side of Q (the transport rays are parallel).

There exists a unique (continuous) equilibrium:

$$\begin{cases} \overline{u} = d_0 \ , & \overline{v} = 0 \text{ on } \overline{S} \\ \overline{v}(x) = \int_0^{\tau(x)} f(x + t\nabla d_0(x))\, dt, & \forall x \in \Omega \backslash \overline{S} \end{cases}$$



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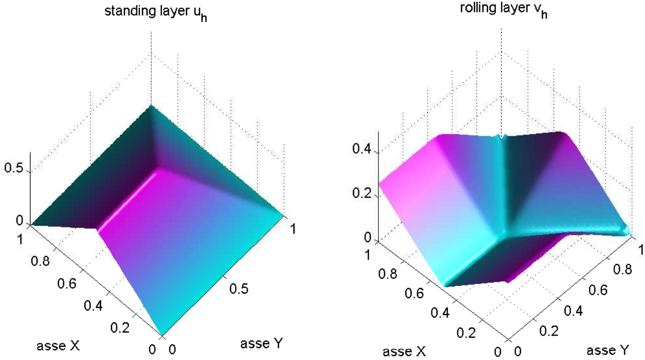


$$f \equiv 0.5, \; N = 41, \; \Gamma_w = \{0 < x < 1, \; y = 0\}.$$

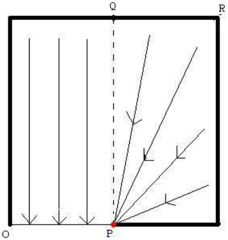
An example of singular boundary points

The presence of a SBP radically changes the situation:

Example 2

- $\Omega = Q$, $f \equiv 1$;
- $\Gamma_0 = \{0 \le x \le 0.5,\ y = 0\}$;
- O is a RBP, P is a SBP

[*there exist infinitely many transport rays through* P]



An example of singular boundary points

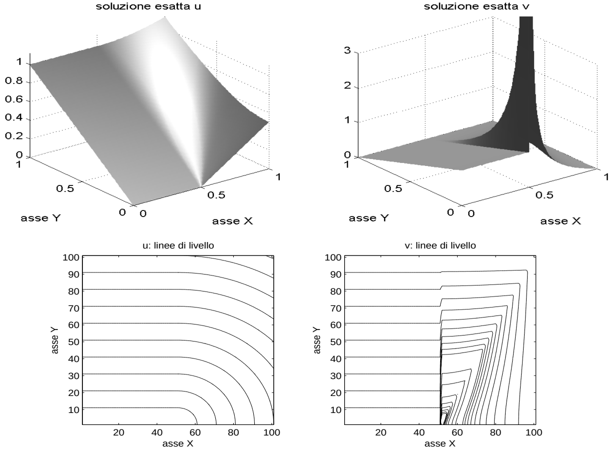
Explicit computation for the equilibrium is possible by decomposition along $\overline{PQ}$ and polar coordinates in P ($\theta \in [0, \frac{\pi}{2}]$, $0 < \rho < \tau(\theta) = dist_\theta(P, \partial\Omega)$, $r = \sqrt{x^2 + y^2}$):

$$\overline{u} = d_0 \ , \quad \overline{v}(x,y) = \begin{cases} 1 - y & \text{if } x \leq 0.5 \\ \int_r^{\tau(\theta)} \frac{\rho}{r} \, d\rho & \text{if } x > 0.5 \end{cases}$$

Then

- $\overline{v}$ is discontinuous along the segment $\overline{PQ}$
- $\overline{v}$ is unbounded in P but $\overline{v} \in L^1(\Omega)$. From the system: $\int_\Omega \overline{v} = \int_\Omega \overline{v} |\nabla d_0|^2 = -\int_\Omega d_0 \nabla \cdot (\overline{v} \nabla d_0) = \int_\Omega f d_0 < \infty$
- $\nabla \overline{v}$ is discontinuous along $\overline{PR}$

An example of a SBP: exact stationary solutions



Equilibria in the general 2D case

Existence results for the stationary solutions and their characterization are not easy in the general case (it is not clear in which sense a discontinuous function like $\overline{v}$ in Example 2 globally solves the differential system (E_w)).

We are able to extend the [CC] existence result to this case under the following assumptions:

- (H1) $\Omega \subset \mathbb{R}^2$ convex Lipschitz domain;
- (H2) $\Gamma_0 = \cup_{i=1}^{N} \Gamma_i$, Γ_i pairwise disjoint C^2 connected arcs of $\partial\Omega$ (with endpoints A_i and B_i).

Decomposable domains

Let $\Omega^* = \Omega \backslash S$ and $\Pi_0 : \Omega^* \to \Gamma_0$ given by

$$\Pi_0(x) = \{y \in \Gamma_0 : d_0(x) = |x - y|\};$$

then, if (H1)-(H2) hold, Ω^* can be uniquely decomposed as

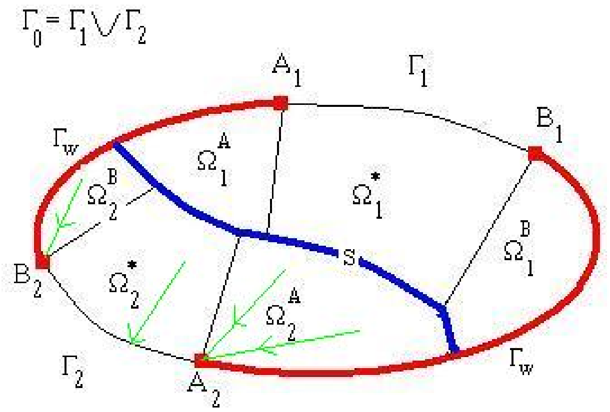
$$\Omega^* = \cup_{i=1}^{N} (\Omega_i^* \cup \Omega_i^A \cup \Omega_i^B),$$

where

$$\Omega_i^* = \{x \in \Omega^* : \Pi_0(x) \in int(\Gamma_i)\},$$

$$\Omega_i^A = \{x \in \Omega^* : \Pi_0(x) = A_i\}, \quad \Omega_i^B = \{x \in \Omega^* : \Pi_0(x) = B_i\}.$$

Example of decomposition



Characterization of 2D equilibria [Crasta-F.V.]

Theorem

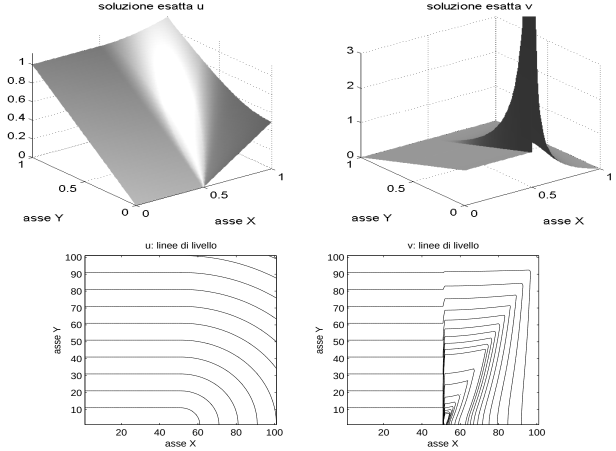
Assume (H1)-(H2); then (d_0, v) is a solution of (E_w), with $v = 0$ on $\overline{S}$ and

$$v(x) = \int_0^{\tau(x)} f(x + t\nabla d_o(x)) M_x(t)\, dt, \quad in\ \Omega \setminus S.$$

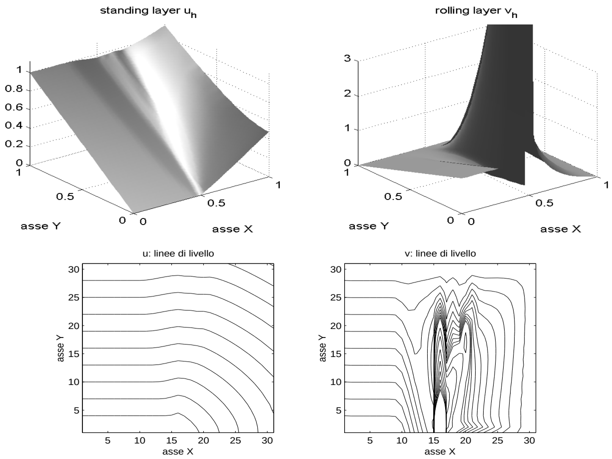
where

$$M_x(t) = \begin{cases} \frac{d_0(x)+t}{d_0(x)} & \text{if } x \in \Omega_i^A \cup \Omega_i^B, \\ \frac{1-(d_0(x)+t)k(y)}{1-d_0(x)k(y)} & \text{if } x \in \Omega_i^*,\ y = \Pi_0(x). \end{cases}$$

An example of a SBP: exact stationary solutions



Numerical stationary solutions by standard algorithm



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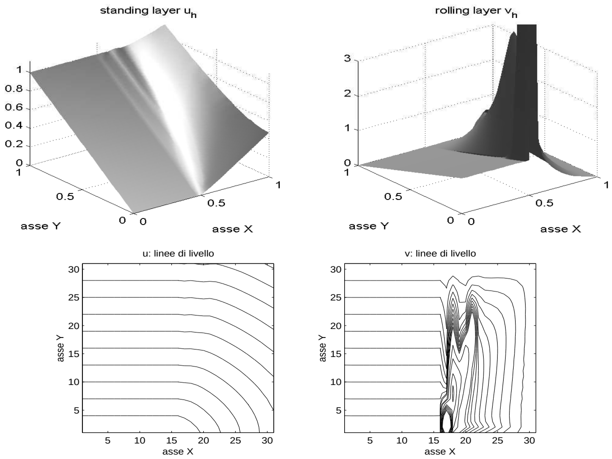
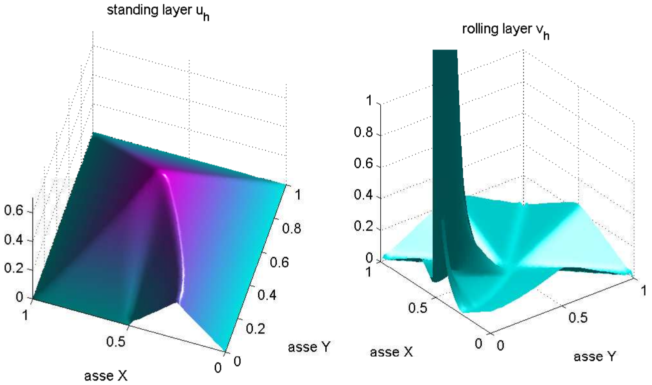


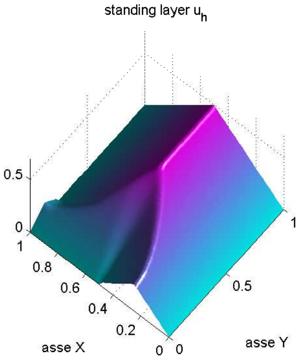
Growing sandpile by standard algorithm

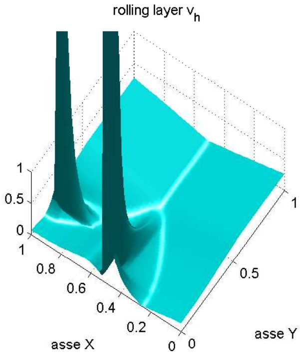
Other examples: one wall



$N = 41$, $\Gamma_w = \{0 < x < 0.5,\ y = 0\}$.

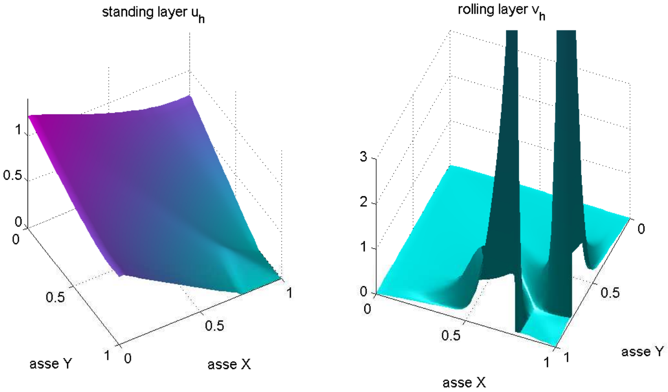
Other examples: three walls





$N = 41$, $\Gamma_w = \{0 < x < 0.5,\ y = 0\} \cup \{x = 0,\ 0 < y < 0.25\} \cup \{0 < x < 1,\ y = 1\}$.

Other examples: exit at the corner



$N = 41$, $\Gamma_0 = \{0.75 < x < 1,\ y = 1\} \cup \{x = 1,\ 0.75 < y < 1\}$.

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- Such extension requires a more careful formulation for the equilibrium system, giving sense to unbounded and discontinuous solutions. A partial result can be given on a priori decomposable tables.
- From a numerical point of view, the finite-difference scheme used for the description of the growing process and the equilibrium detection in the o.t. problem, can be adapted to the p.o.t. problem. But more efforts are necessary to take care efficiently of the wall boundary conditions and of the internal developing singularities (first tests)

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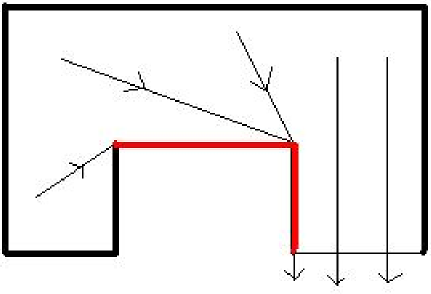

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- Numerical study of other boundary conditions (*silos*) and different model problems (*collapsing sandpiles, obstacles, optimal mass transport*)