

Recent advances on large time–step schemes for Mean Curvature Motion

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- "Level set" techniques and MCM (in codimension one)
- Analytical setting and Soner-Touzi representation formula
- Discretization and convergence analysis
- Numerical results (part 1a)
- Adaptive time-stepping
- Numerical results (part 1b)
- MCM in codimension two - The Ambrosio-Soner approach
- Post-processing of the solution
- Numerical results (part 2)

Interfaces evolution and "level set" techniques

- *Geometric problem*: to follow the evolution of a manifold (interface) which moves in the direction of the normal vector with a speed dependent (for example) on position and curvature.
- *Motivations*: Combustion, porous media, semiconductors,...

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- *Classical approach*: Let the manifold evolve using a parametrization
- *"Level set" approach*: Consider the manifold as a level surface of the solution of a suitable evolutive PDE

Model problem: Motion by Mean Curvature

The most typical case of application of *level set* techniques to interface propagation is the degenerate parabolic problem:

$$\begin{cases} v_t(x, t) = |Dv| \operatorname{div} \left(\frac{Dv(x, t)}{|Dv(x, t)|} \right) \\ v(x, 0) = v_0(x) \end{cases} \quad (1)$$

in the unknown $v = v(x, t)$, with $x \in \mathbb{R}^N$, $t \geq 0$ and with v_0 a continuous function such that

$$\Gamma_0 = \{x \in \mathbb{R}^N \mid v_0(x) = 0\}$$

(initial interface position)

Analytical questions (1)

Equation (1) is:

- **nonlinear** (it is in the form $v_t + H(u, Dv, D^2v) = 0$)

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Such problems have been solved from an analytical point of view by introducing the notion of viscosity solution (Evans-Spruck, Chen-Giga-Goto, 1991).

Analytical questions (2)

On the basis of the theory of viscosity solutions, it is possible to define a **generalized evolution** of Γ_0 by Mean Curvature as

$$\Gamma_t = \{x \in \mathbb{R}^N \mid v(x, t) = 0\} \quad \forall t \geq 0.$$

- It is possible to prove that such definition **matches the classical one** as far as the latter is well-defined, but can also handle **generation of singularities** and **topology changes** of the interface (in this situations, MCM cannot be defined in a classical sense).

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- Note that Γ_t **does not depend on v_0** (i.e. on how the initial position of the interface is represented)

Soner–Touzi formula (1)

Recently Soner and Touzi (2002) have proposed a *stochastic representation formula* for solutions of a large class of geometric second–order Hamilton–Jacobi equations, including (1); in our case (and provided Dv does not vanish - it can be generalized to situations in which $Dv = 0$ at some point) it reads

$$v(x, t) = \mathbb{E}\{v_0(y(x, t, t))\} \quad (2)$$

where the curves $y(x, t, s)$ play the role of *generalized characteristics*.

Soner–Touzi formula (2)

The function $y(x, t, s)$ appearing in (2) solves the stochastic initial value problem (written in \mathbb{R}^2 for simplicity):

$$\begin{cases} dy(x, t, s) = \sigma(y, t, s)dW(s) \\ y(x, t, 0) = x, \end{cases} \quad (3)$$

where dW is the differential of a standard Wiener process, and

$$\sigma(y, t, s) = \frac{\sqrt{2}}{|Dv(y, t - s)|} \begin{pmatrix} v_{x_2}(y, t - s) \\ -v_{x_1}(y, t - s) \end{pmatrix}$$

(such matrix projects the diffusion on the space *orthogonal to the gradient* of the solution v)

Discretization (1)

Writing Soner–Touzi representation formula on a single time step $t_k \rightarrow t_{k+1}$ one obtains:

$$v(x, t_{k+1}) = \mathbb{E}\{v(y(x, t_{k+1}, \Delta t), t_k)\} \quad (4)$$

where $y(x, t_{k+1}, \Delta t)$ is a single step (backwards from the point (x, t_{k+1})) along the generalized characteristic:

$$\begin{cases} dy(x, t_{k+1}, s) &= \frac{\sqrt{2}}{|Dv(y, t_{k+1}-s)|} \begin{pmatrix} v_{x_2}(y, t_{k+1}-s) \\ -v_{x_1}(y, t_{k+1}-s) \end{pmatrix} dW(s) \\ y(x, t_{k+1}, 0) &= x. \end{cases} \quad (5)$$

Discretization (2)

In order to set up (4), (5) in a fully discrete form:

- The computation of $v(\cdot, t_k)$ is replaced by a numerical reconstruction $I[v^k](\cdot)$ (Lagrange, ENO, WENO,...)

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- Partial derivatives $v_{x_i}(y, t_{k+1})$ are replaced by finite differences computed at time t_k (a further error term is introduced to avoid an implicit scheme)
- An approximation of the expectation $\mathbb{E}\{v(y(x, t_{k+1}, \Delta t), t_k)\}$ is computed by weak convergence scheme for SDEs

Discretization (3)

Given the stochastic Cauchy problem (which we assume as scalar for simplicity)

$$\begin{cases} dy(x, t) = a(t, y(x, t)) + \sigma(t, y(x, t))dW(t) \\ y(x, t_0) = x, \end{cases}$$

the simplest choice in order to approximate $\mathbb{E}\{h(y(x, \Delta t))\}$ is the **weak stochastic Euler method**:

$$\begin{cases} y_1(\Delta W) = x + a(t_0, x)\Delta t + \sigma(t_0, x)\Delta W \\ \mathbb{E}\{h(y(x, \Delta t))\} = \frac{1}{2} \left(h(y_1(\sqrt{\Delta t})) + h(y_1(-\sqrt{\Delta t})) \right) + O(\Delta t^2) \end{cases}$$

In this case, the expectation is approximated with the **same order of consistency** as in the deterministic case (provided $a(\cdot, \cdot), \sigma(\cdot, \cdot), h(\cdot)$ are smooth enough)

Discretization (4)

We finally obtain the fully discrete scheme (in \mathbb{R}^2):

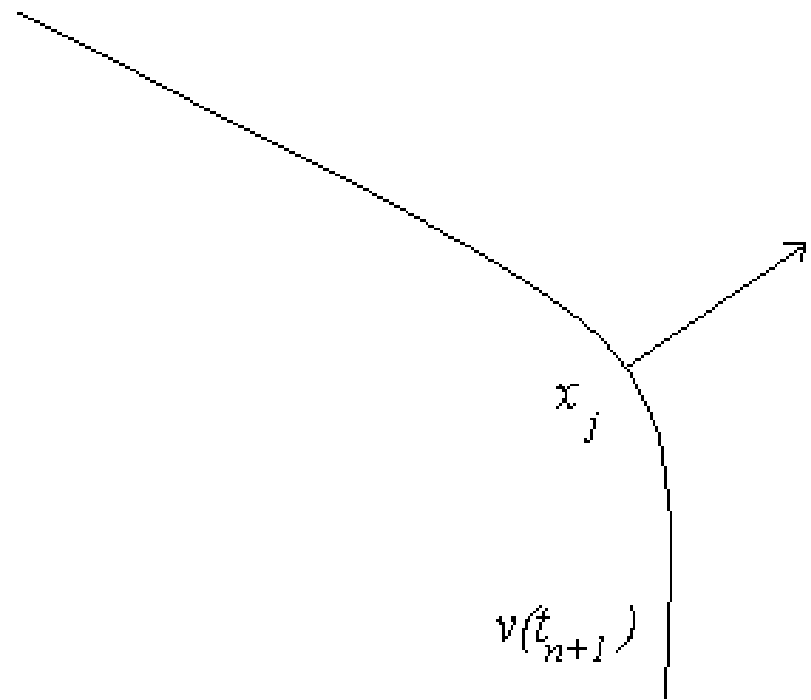
$$v_{ij}^{n+1} = \frac{1}{2} \left(I[v^n](x_{ij} + \sigma_{ij}^n \sqrt{\Delta t}) + I[v^n](x_{ij} - \sigma_{ij}^n \sqrt{\Delta t}) \right)$$

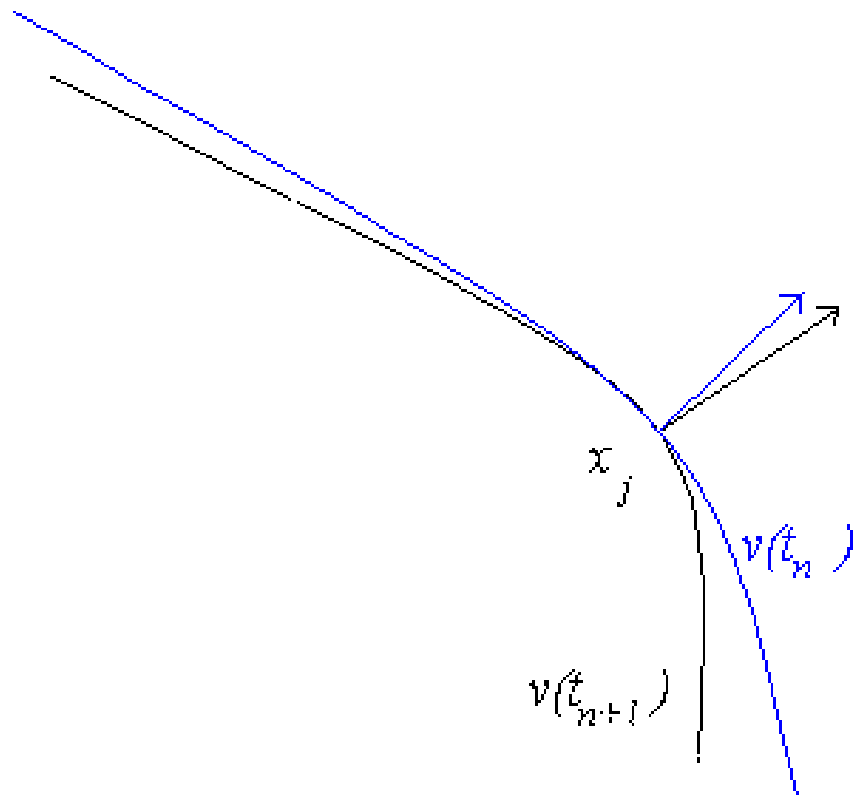
in which the derivatives in σ_{ij}^n have been replaced with finite differences:

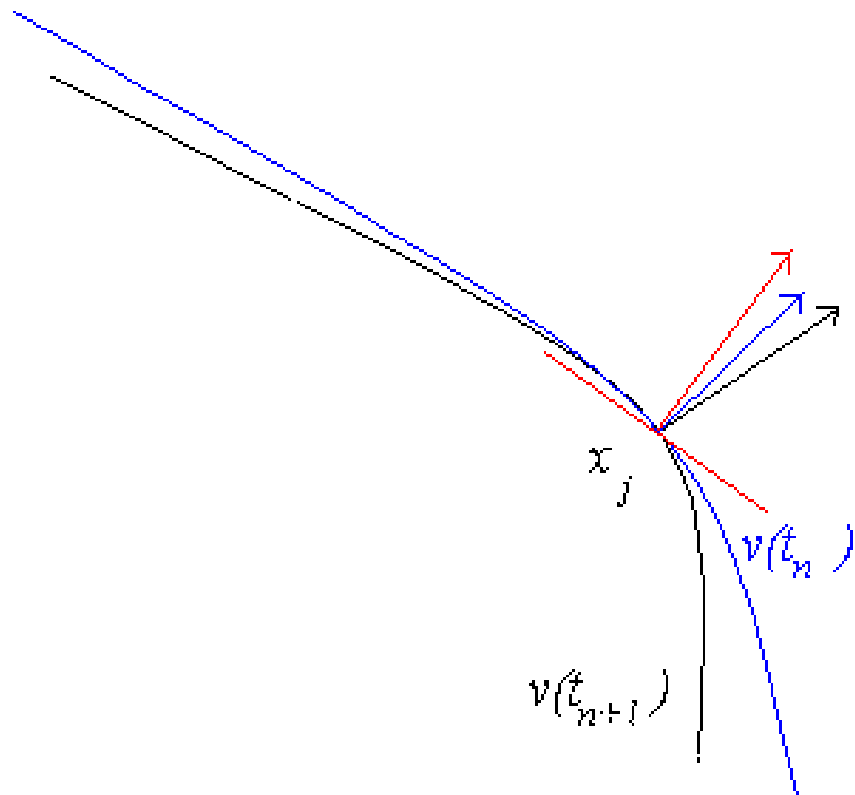
$$\sigma_{ij}^n = \frac{\sqrt{2}}{\sqrt{D_{1,ij}^n{}^2 + D_{2,ij}^n{}^2}} \begin{pmatrix} D_{2,ij}^n \\ -D_{1,ij}^n \end{pmatrix}$$

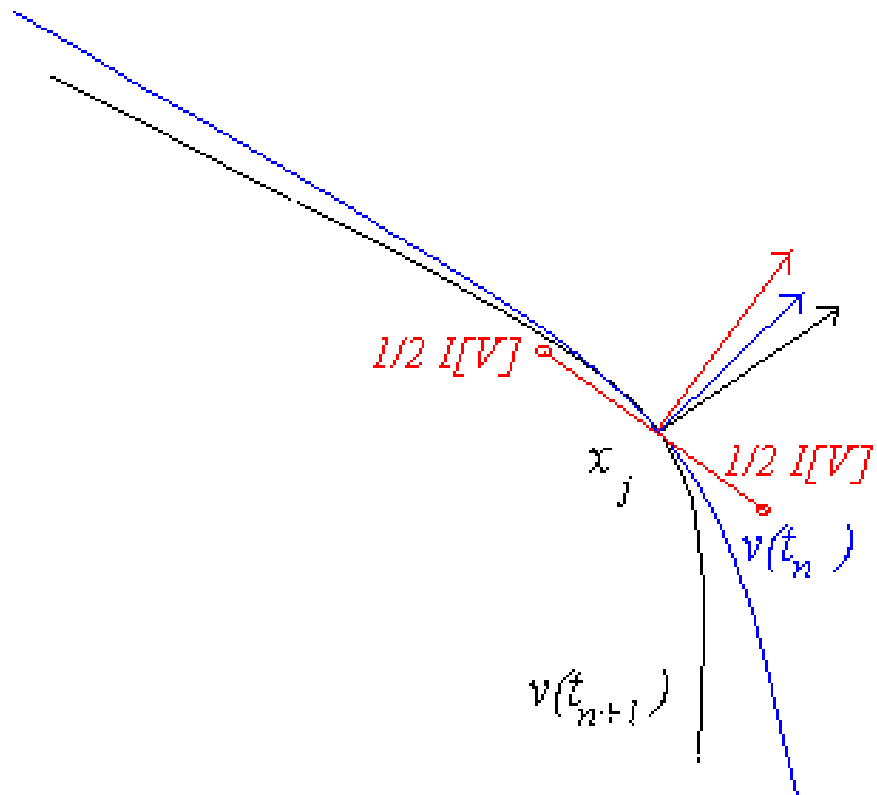
and, for example:

$$D_{1,ij}^n = \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta x}, \quad D_{2,ij}^n = \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta x}$$









Consistency

Assume the scheme approximating the SDE to converge with order 1, the interpolation $I[v](\cdot)$ to be of degree r and the approximation of derivatives to converge with order q . Then, the consistency error has the form:

$$L_{\Delta x, \Delta t}(x, t) = O\left(\Delta t^{1/2} + \frac{\Delta x^{r+1}}{\Delta t} + \frac{\Delta x^q}{\Delta t^{1/2}}\right).$$

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- It is possible to determine a relationship of the form $\Delta x = \Delta t^\alpha$ which makes the scheme consistent.
- Suitable conditions ensure consistency of the scheme where $Dv = 0$.

Monotonicity

Consistency along with monotonicity would imply convergence by a general convergence result (Barles-Souganidis, 1991).

Unfortunately, even perturbing the scheme with an artificial viscosity term, the relationship $\Delta t/\Delta x$ giving monotonicity is opposite to the relationship giving consistency.

Numerical tests (1)

Evolution of a circle

We take as initial condition the function

$$v_0(x) = \max\{(1 - (x_1^2 + x_2^2))^{10}, 0\}.$$

on the domain $[-2, 2]^2$. In this case the exact solution can be explicitly computed as

$$v(x, t) = \max\{(1 - (x_1^2 + x_2^2) - 2t)^{10}, 0\}.$$

The following table reports numerical errors computed at $t = 0.16$, with a scheme based on a first-order reconstruction.

Numerical tests (2)

Evolution of a circle - error table

Δx	Δt	$\ \cdot \ _{\infty}$	$\ \cdot \ _1$	L^{∞} - order	L^1 - order
0.04	0.08	$6.44 \cdot 10^{-2}$	$6.37 \cdot 10^{-4}$		
0.02	0.053	$2.40 \cdot 10^{-2}$	$4.54 \cdot 10^{-4}$	1.4	0.5
0.01	0.032	$1.05 \cdot 10^{-2}$	$1.49 \cdot 10^{-4}$	1.2	1.6
0.005	0.02	$8.36 \cdot 10^{-3}$	$4.55 \cdot 10^{-5}$	0.3	1.7

Numerical tests (3)

Evolution of a square

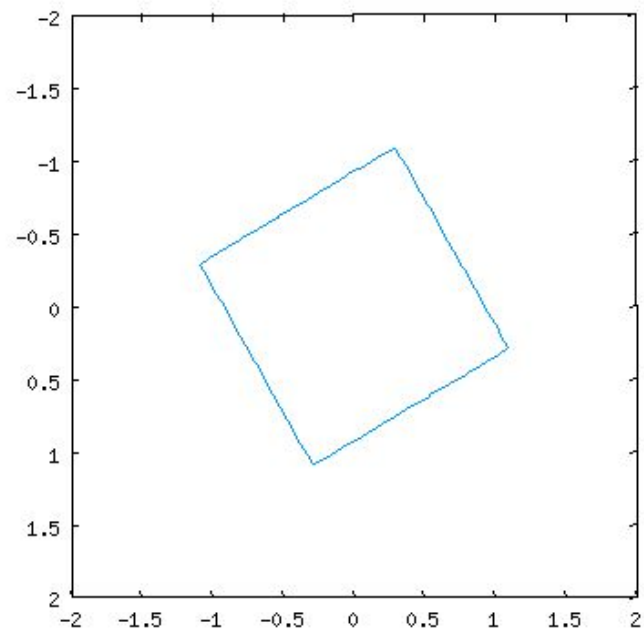
We take as an initial condition the function

$$v_0(x) = 1 - \|Qx\|_1$$

with Q a rotation matrix (in order to avoid alignment between the level curve and the grid).

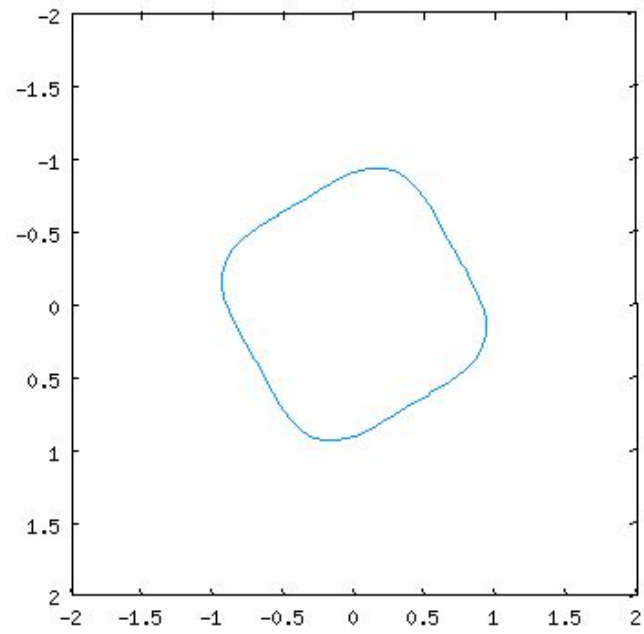
n°iter.= 0

0.2



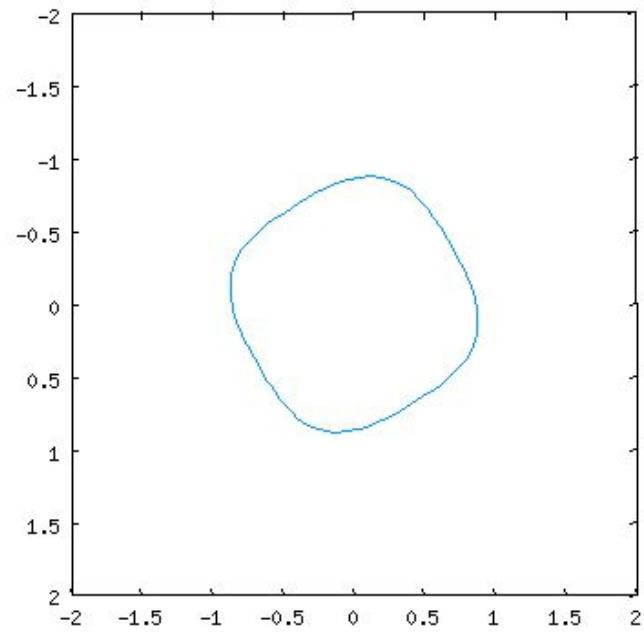
n°iter.= 5

0.2



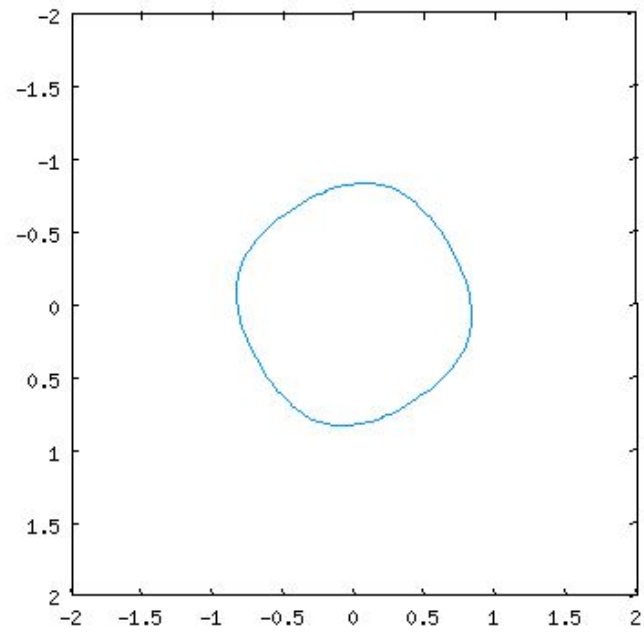
n°iter.=10

0.2



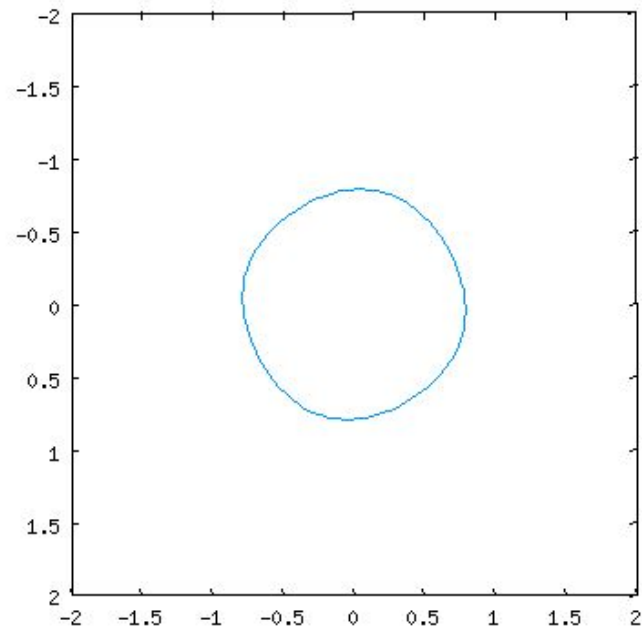
n°iter.=15

0.2



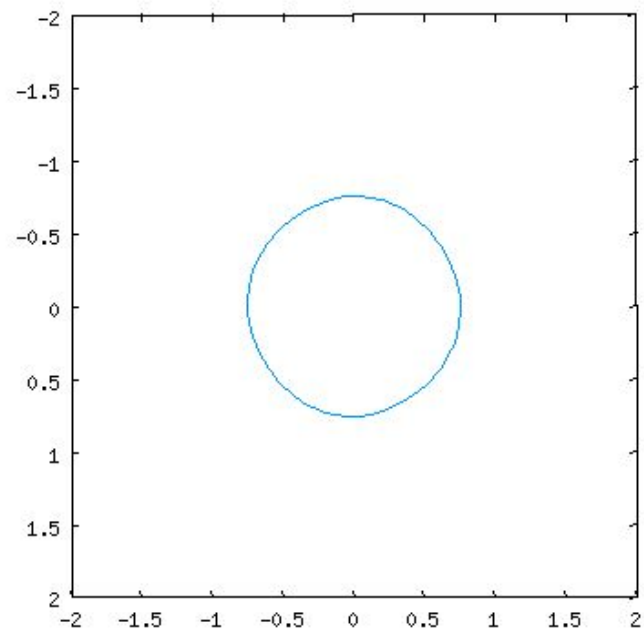
n°iter.=20

0.2



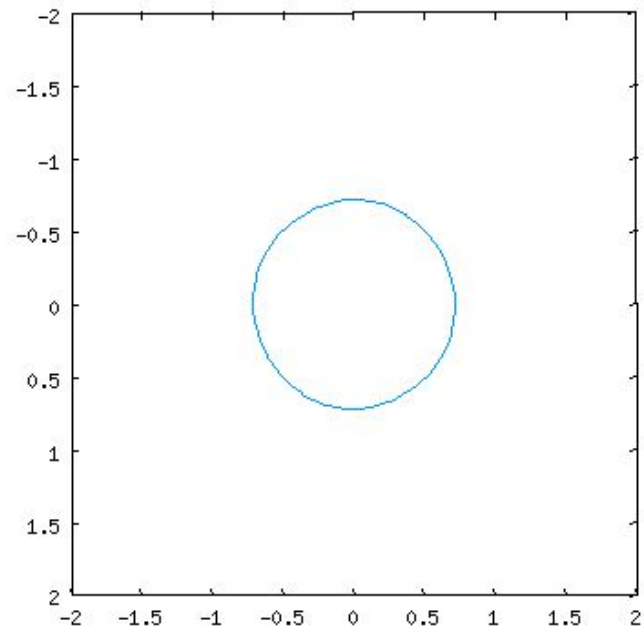
n°iter.=25

0.2



n°iter.=30

0.2



Numerical tests (4)

Fattening

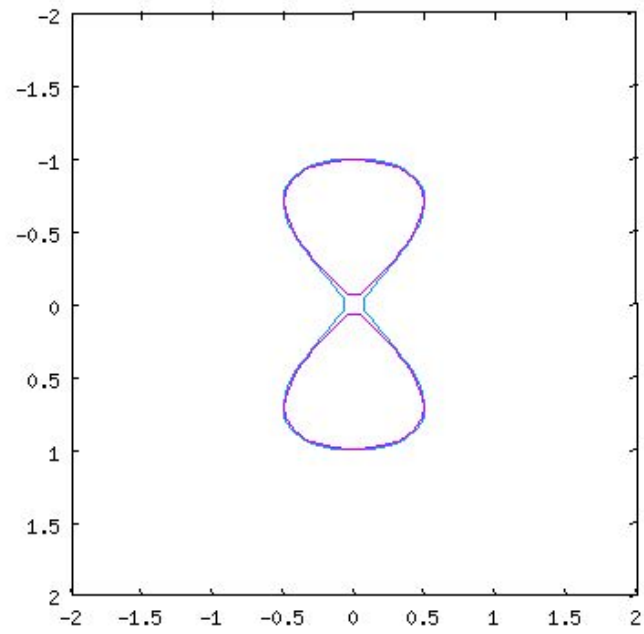
We use an initial condition given by

$$v_0(x) = x_1^4 - x_1^2 + x_2^2 + 1$$

which has a **saddle point** at critical level $v = 1$. In this situation, it is well-known that the level curve at the critical level may develop an interior. This phenomenon is known as *fattening* and in the following figures it has been illustrated by plotting two level curves, one slightly above and one slightly below the critical level.

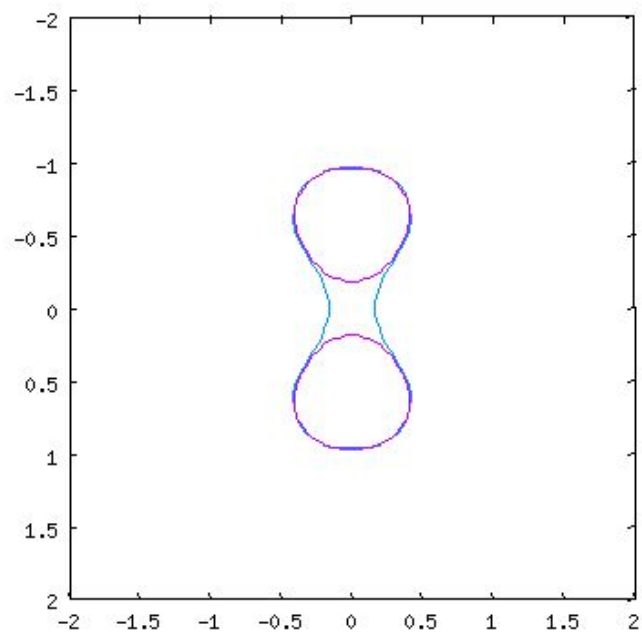
n°iter.= 0

1 —
0.995 —



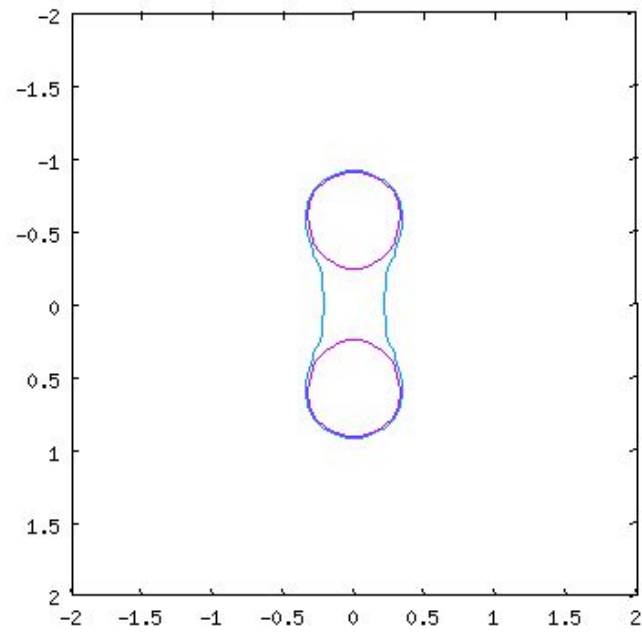
n°iter.= 5

1 —
0.995 —



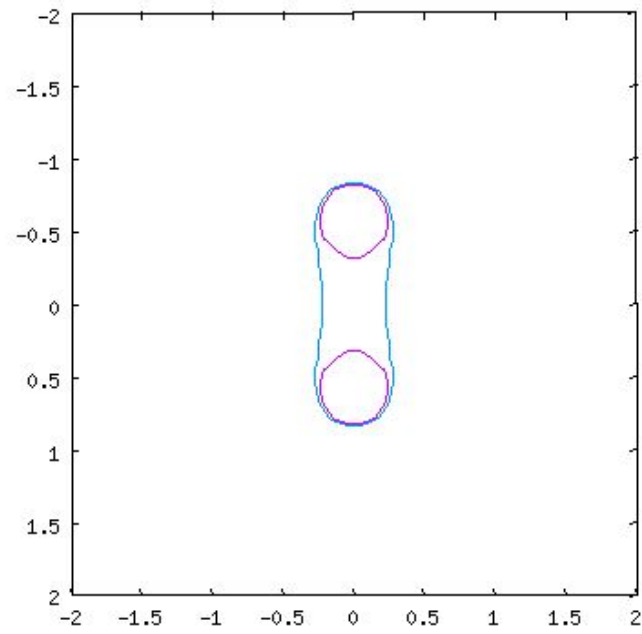
n°iter.=10

1 —
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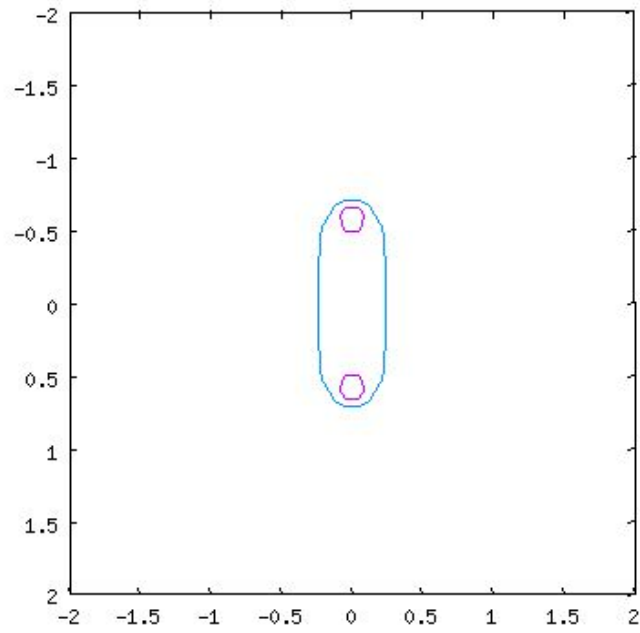
n°iter.=15

1 —
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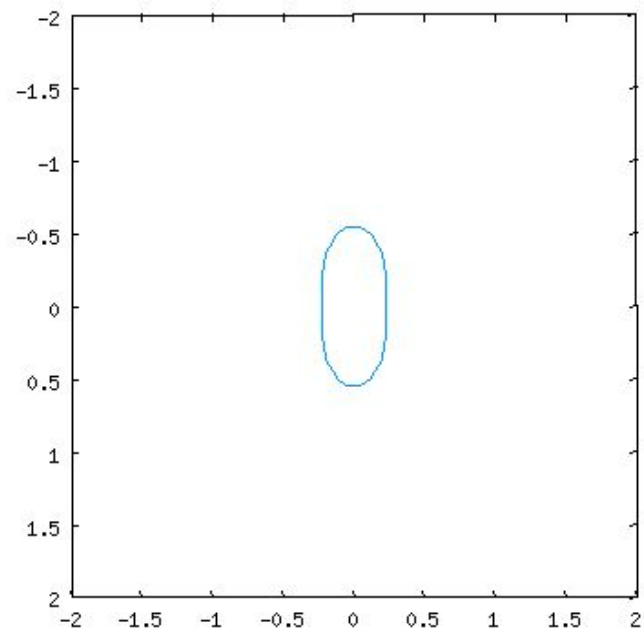
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1 —
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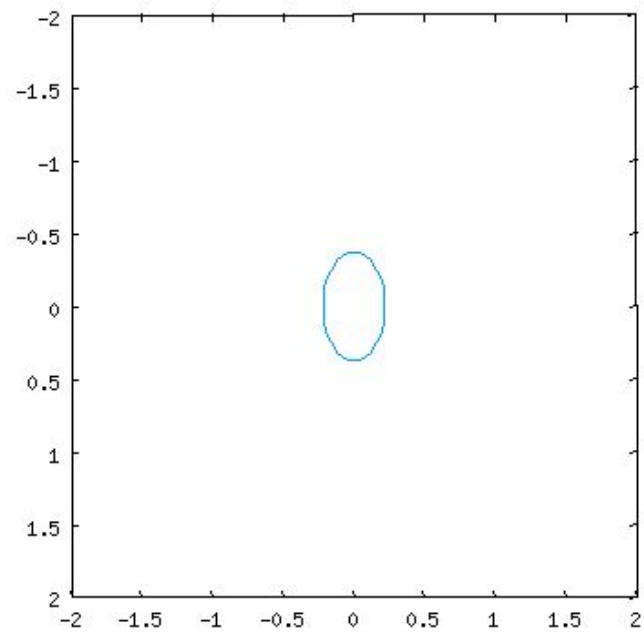
n°iter.=25

1 —



n°iter.=30

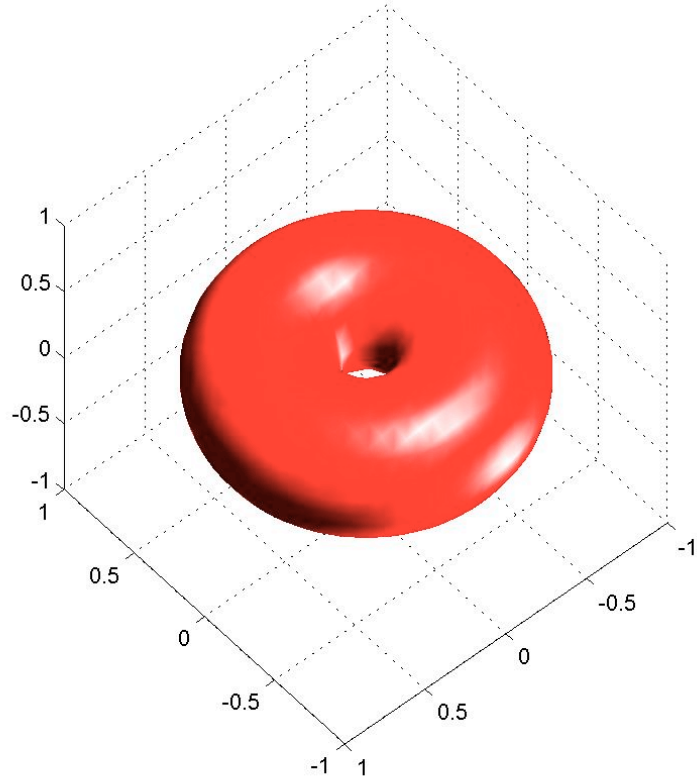
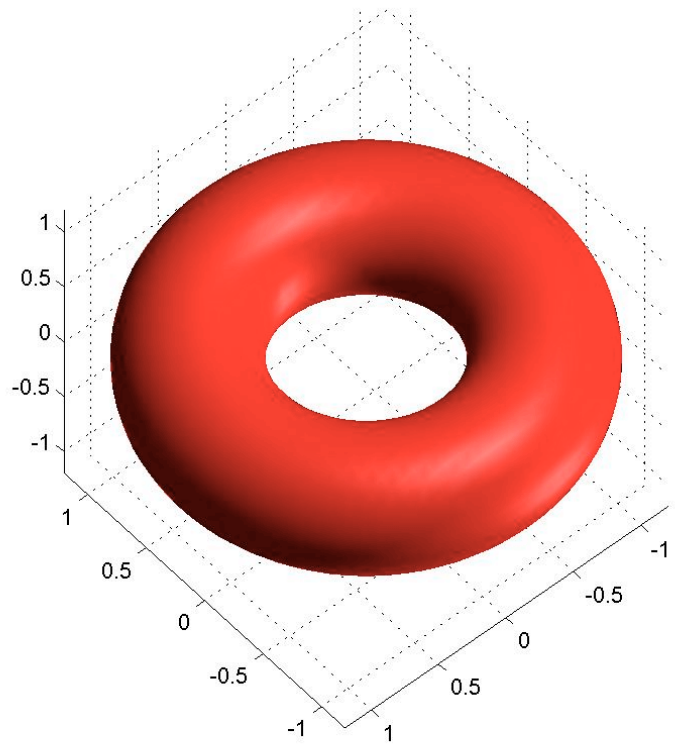
1 —

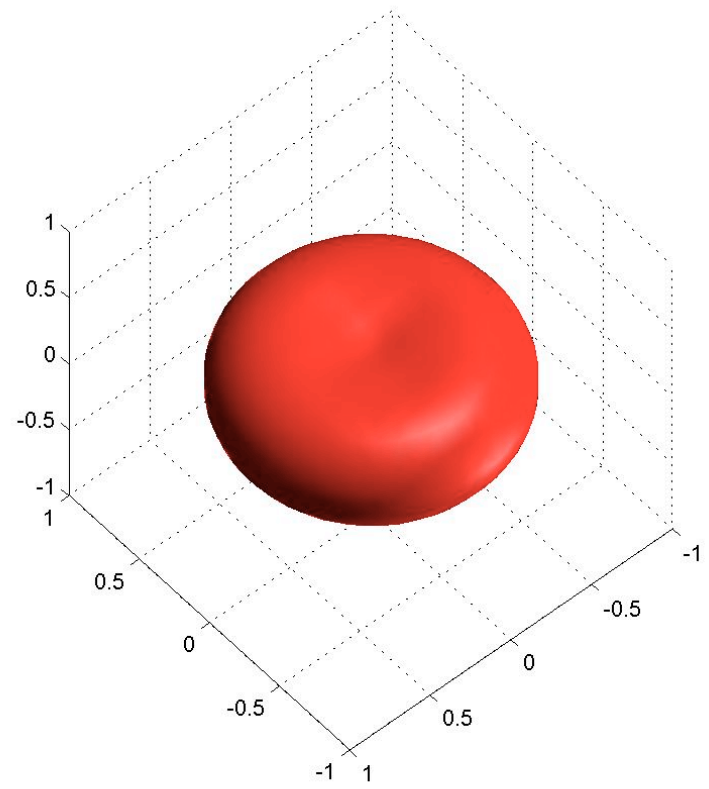
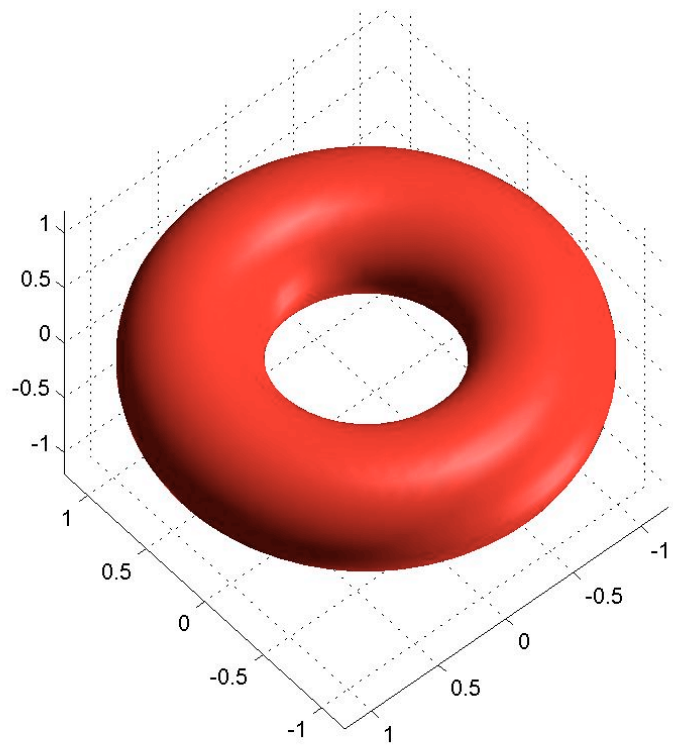


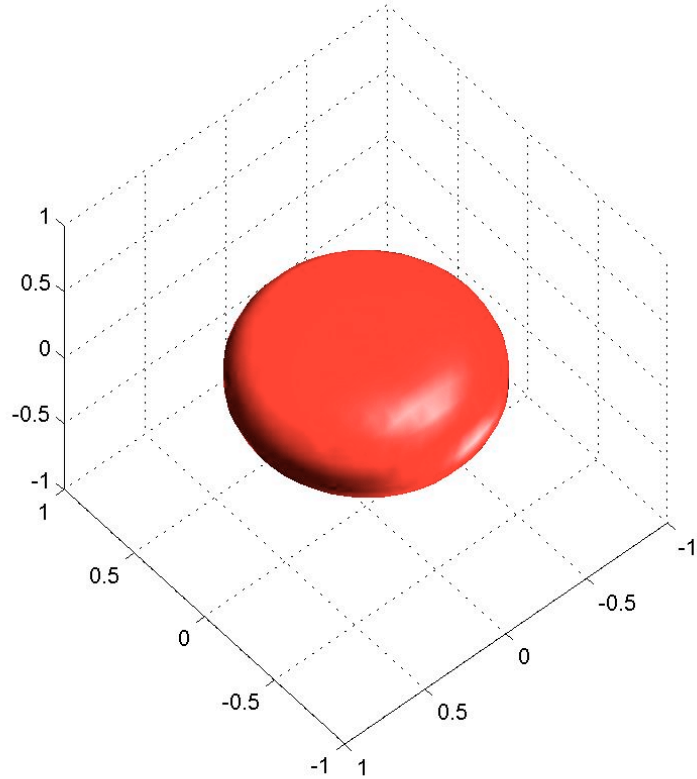
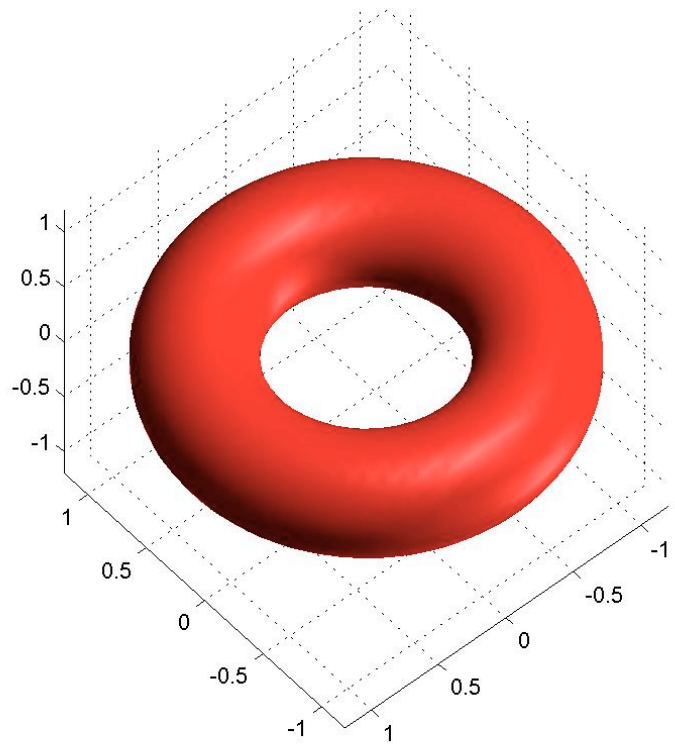
Numerical tests (5)

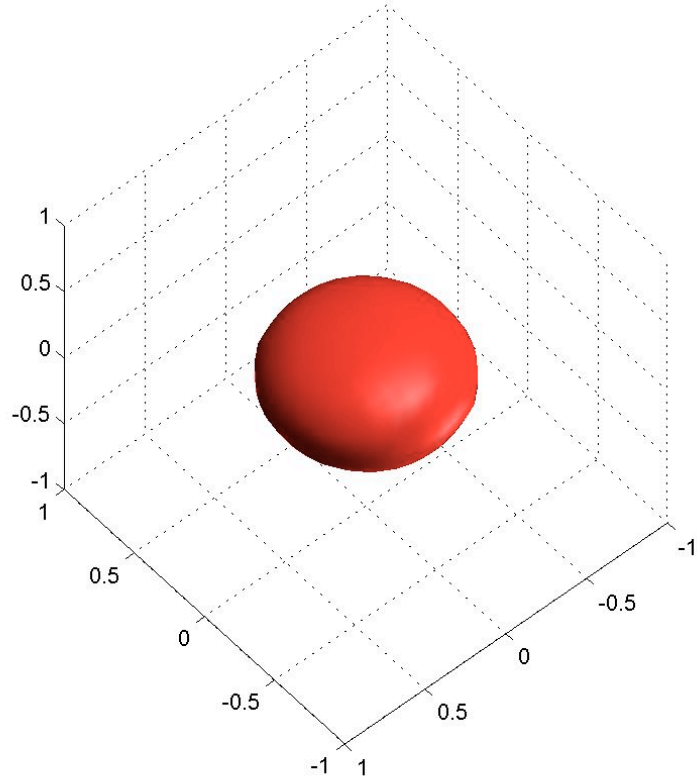
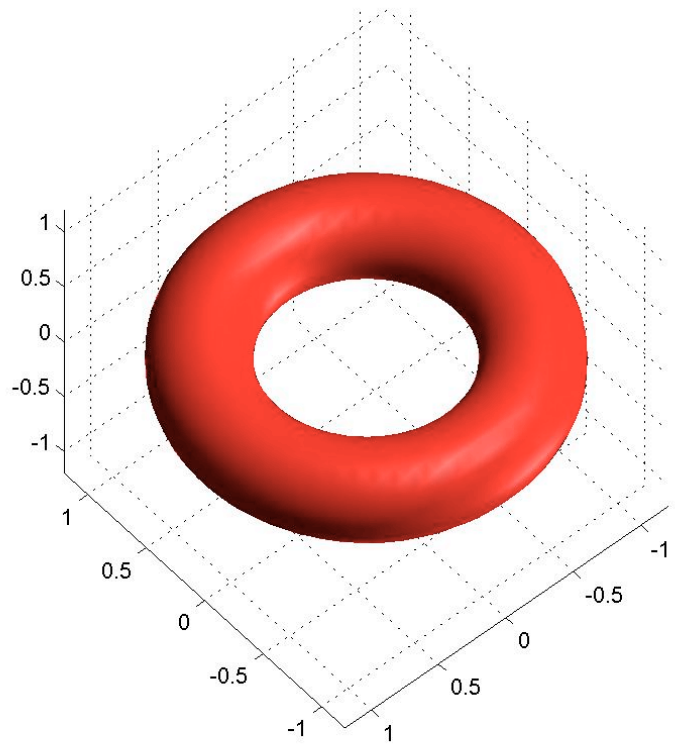
Three-dimensional torus

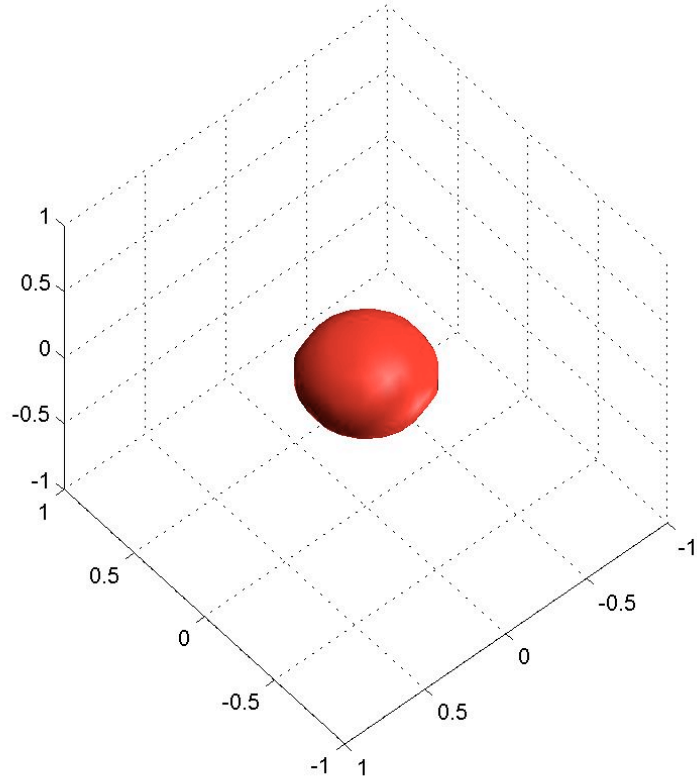
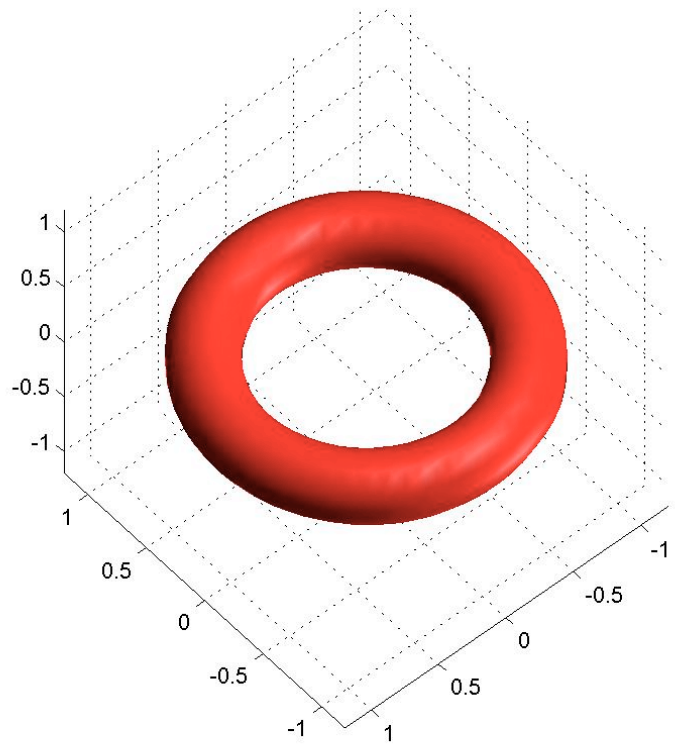
We consider now the evolution of a toroidal surface in three space dimensions. It is known that there exists a **critical ratio** between the two radii of the torus. When above this ratio, the torus evolves towards a circle and extinguishes as such, when below the torus changes topology evolving towards a sphere and collapsing in a point.

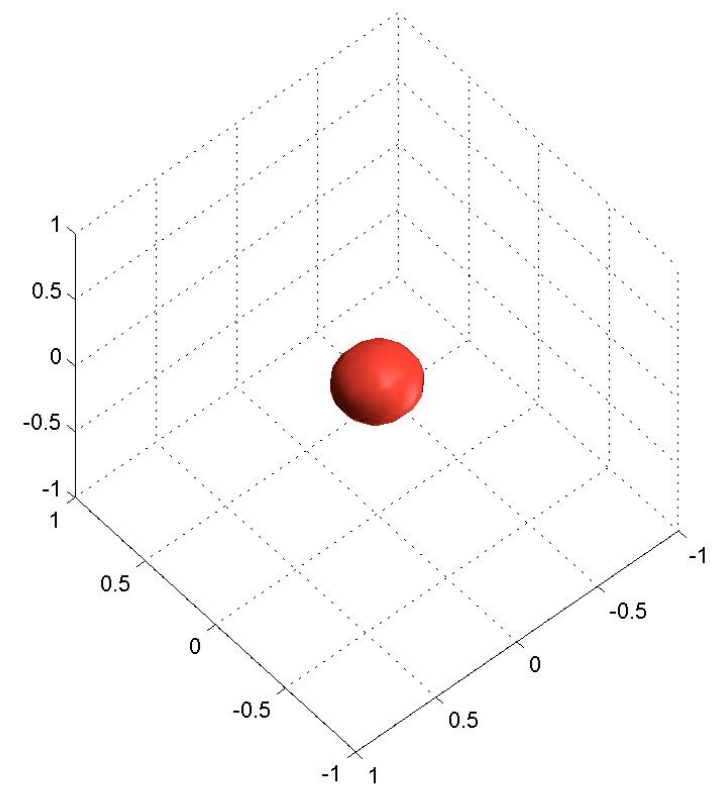
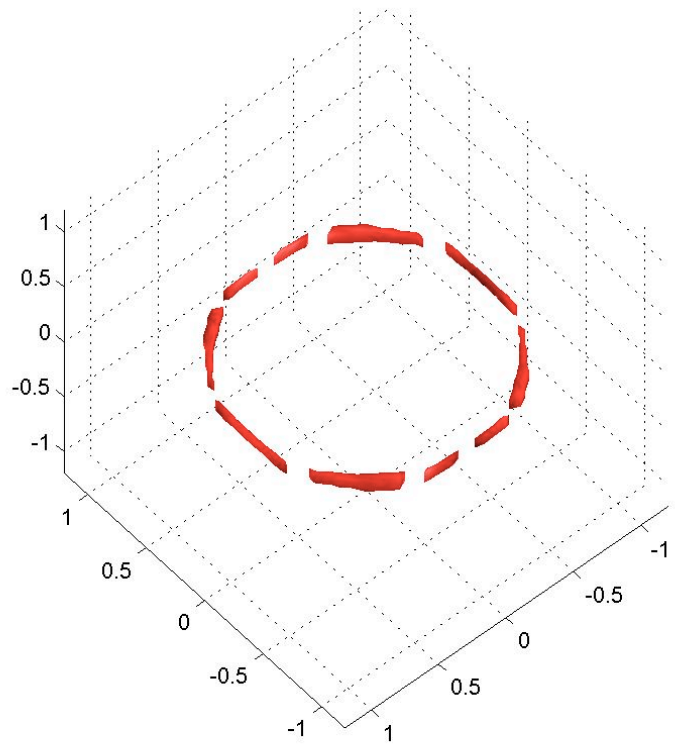












Adaptive time–stepping

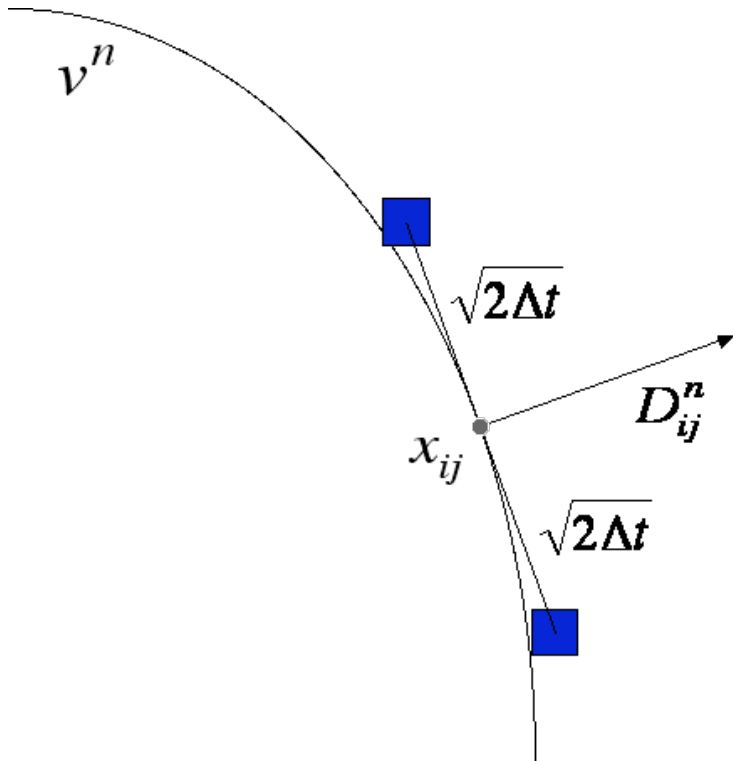
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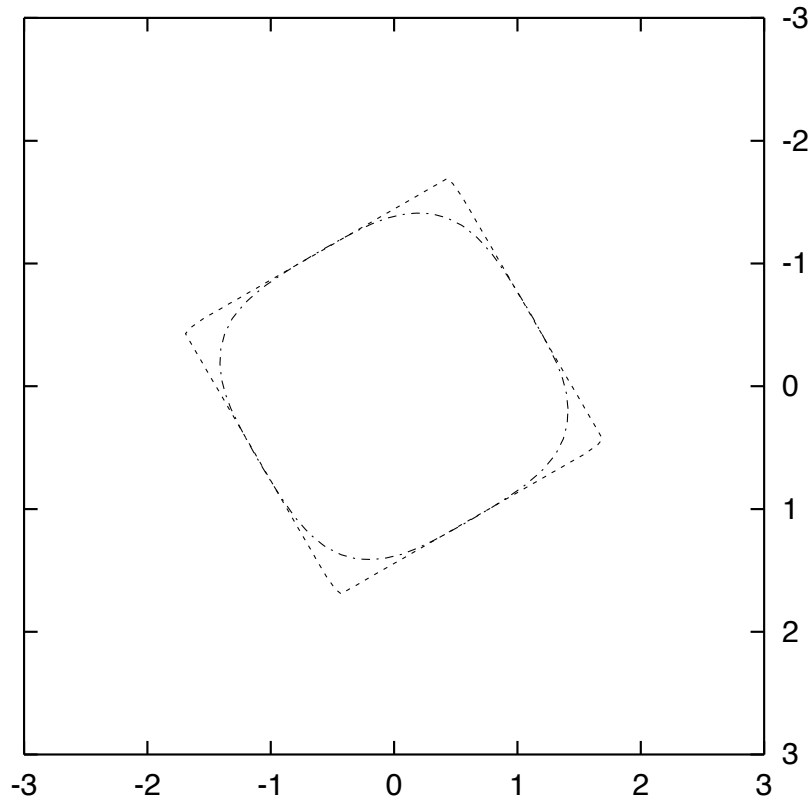
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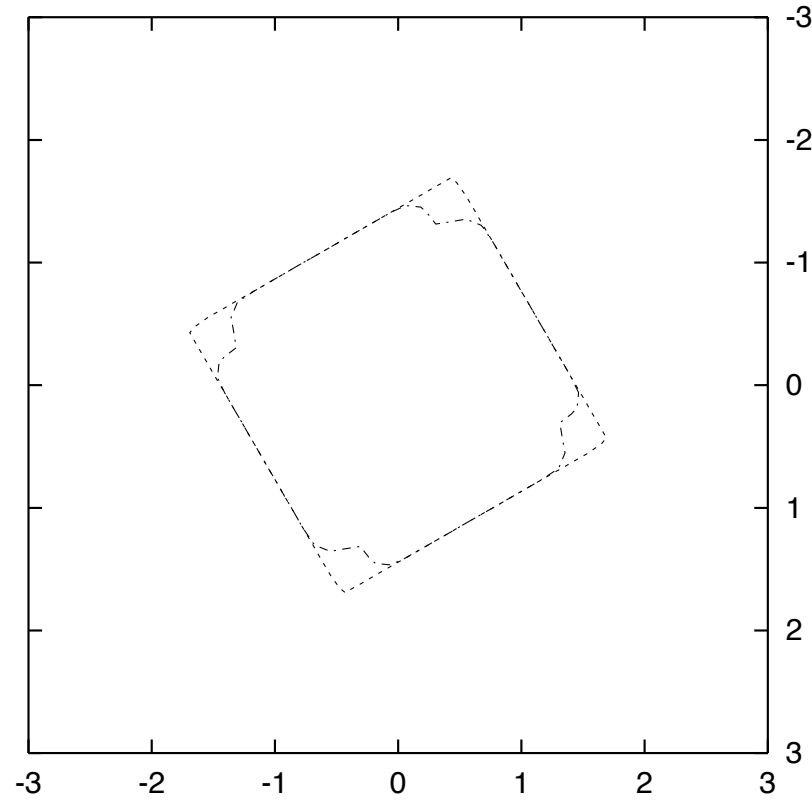
- Large time–step schemes do not allow in general to efficiently detect small scales (e.g. corners) when the level–set function is nonsmooth
- Reducing time step in order to resolve small structures leads to a drop in efficiency as soon as the solution is smoothed out
- Resolution of the scheme is related to both space and time step, but our first attempt is only towards time–step adaptivity



- The numerical domain of dependence is made of two regions of reconstruction which are $2\sqrt{2\Delta t}$ apart
- This is precisely what causes the smaller scales to be underresolved
- In some sense, the problem is accuracy rather than stability



10 time steps, $\Delta t = 0.01$



1 time step, $\Delta t = 0.1$

- Avoiding any such "hole" in the numerical domain of dependence would require the parabolic CFL condition $\Delta t \sim \Delta x^2$

- Asymptotically, "holes" are filled at a given time T under the weaker condition:

$$\Delta t = o\left(T^{2/3} \Delta x^{2/3}\right)$$

- The behaviour of the scheme at fixed time step improves for large times (or in the limit case of computing the regime solution)

We follow a strategy based on the estimation of the **local (time–) truncation error** as used for ODEs:

- **Time–step derefinement** is performed assuming the solution is **smooth** and the error introduced on a single time step is estimated by

$$\epsilon_{\Delta x, \Delta t}(x_j, t_n) \sim C_1 \Delta t^{3/2} + C_2 \Delta x^r + C_3 \Delta t^{1/2} \Delta x^q$$

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- **Time–step refinement** is performed assuming the solution is only **Lipschitz continuous** and the error introduced on a single time step is

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- We assume **space discretization error is negligible**

We compute the $(n + 1)$ -th time step with **1 step Δt** and **2 steps $\Delta t/2$** and estimate the local truncation error by:

$$\epsilon_{\Delta t} \sim \frac{v_{\Delta t} - v_{\Delta t/2}}{1 - \frac{1}{\sqrt{2}}}$$

in smooth conditions (\rightarrow **derefinement**), and by

$$\epsilon_{\Delta t} \sim \frac{v_{\Delta t} - v_{\Delta t/2}}{\sqrt{2} - 1}$$

for Lipschitz solutions (\rightarrow **refinement**).

- The **L^∞ norm** of this estimate is computed on a suitable set S (a **"narrow band"** around the interface)

The local error estimate is compared with a [refinement](#) and a [derefinement](#) threshold:

$$\left(1 - \frac{1}{\sqrt{2}}\right) \tau_D \Delta x < \left\| \frac{v_{\Delta t} - v_{\Delta t/2}}{|Dv|} \right\|_{L^\infty(S)} < (\sqrt{2} - 1) \tau_R \Delta x$$

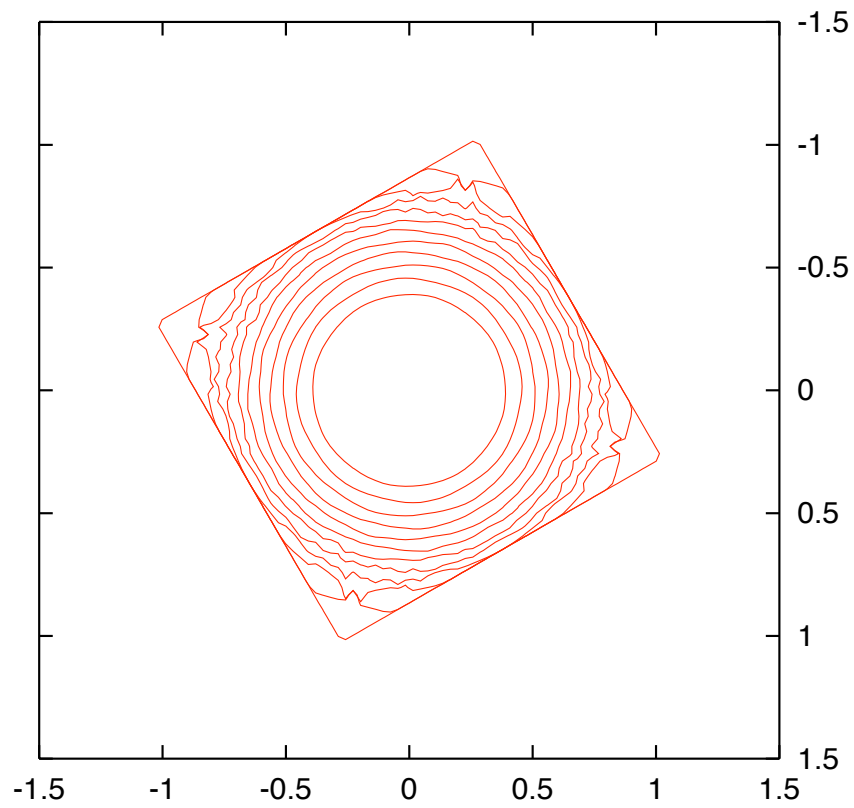
- The [renormalization](#) by $|Dv|$ allows the adaptation to be independent of the function v_0 (i.e. "geometric")
- The thresholds are set proportional to Δx in order to take into account the [parabolic CFL condition](#) in nonsmooth conditions

Numerical tests

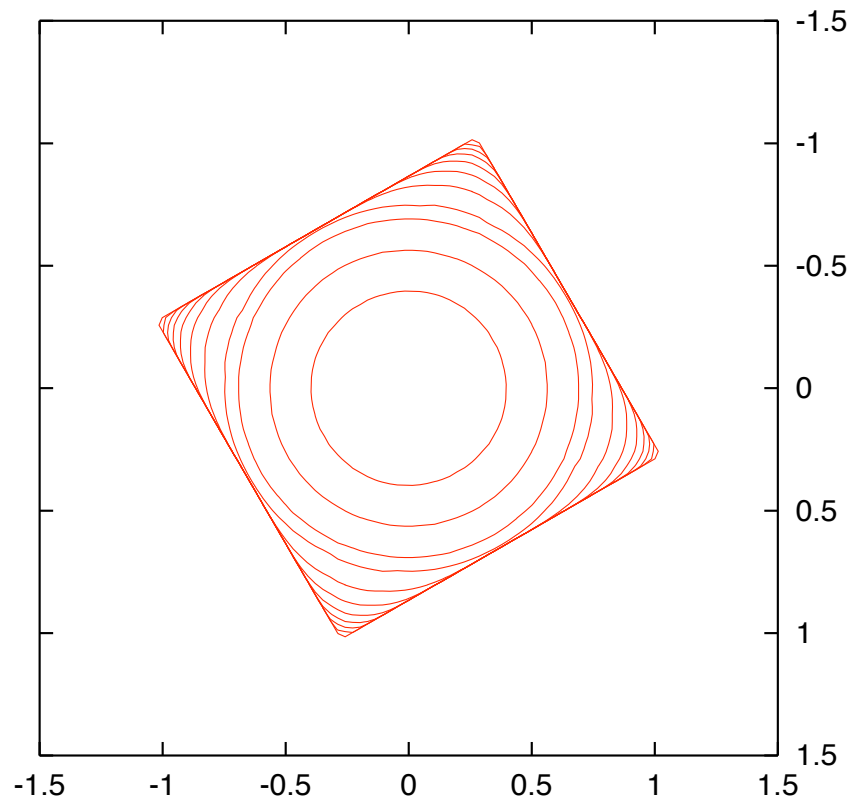
L^∞ errors in a neighbourhood of the level curve for the shrinking circle:

Δx	Adaptive Δt	Fixed Δt
$1.428 \cdot 10^{-1}$	$1.581 \cdot 10^{-3}$	$1.066 \cdot 10^{-3}$
$7.070 \cdot 10^{-2}$	$3.872 \cdot 10^{-4}$	$2.587 \cdot 10^{-4}$
$3.517 \cdot 10^{-2}$	$1.039 \cdot 10^{-4}$	$6.725 \cdot 10^{-5}$

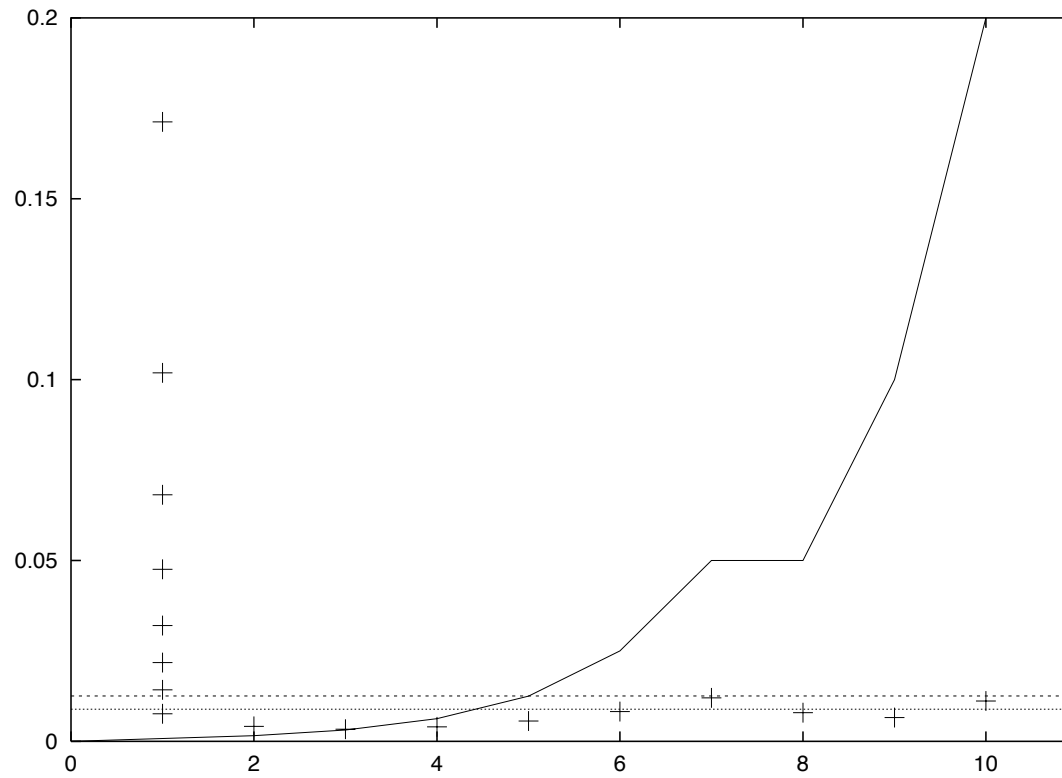
In this situation the fixed step scheme performs slightly better than the adaptive scheme with the same number of steps. However, we expect adaptivity to become crucial in nonsmooth situations, as in the evolution of a Lipschitz level set function.



Fixed time step, $t \in [0, 0.3]$



Adaptive time step, $t \in [0, 0.3]$



Adapted time step, truncation error and thresholds, $t \in [0, 0.3]$

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The case of codimension 2 (curves in \mathbb{R}^3)

In this case the framework is similar, but:

- We use the Ambrosio-Soner approach: the curve is interpreted as **zero level curve of a non-negative function** (e.g. using the squared distance from the manifold Γ_0 as an initial condition)

The case of codimension 2 (curves in \mathbb{R}^3)

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- We use the Ambrosio-Soner approach: the curve is interpreted as **zero level curve of a non-negative function** (e.g. using the squared distance from the manifold Γ_0 as an initial condition)
- The Soner-Touzi formula has a generalized form (in particular, there appears a minimization on **all the directions orthogonal to Dv**)

The evolution equation describing the **codimension–k MCM** reads:

$$\begin{cases} v_t = F_k(D^2v, Dv) = \inf_{\nu} [\text{trace}(D^2v P^{\nu})] \\ v(x, 0) = v_0(x) \end{cases} \quad \text{with } v_0 = 0 \text{ on } \Gamma_0$$

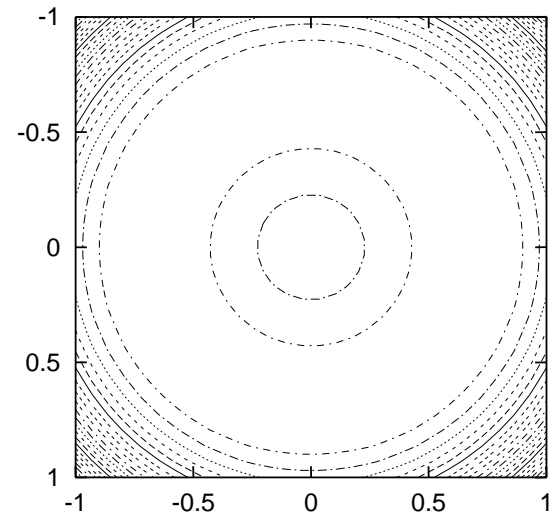
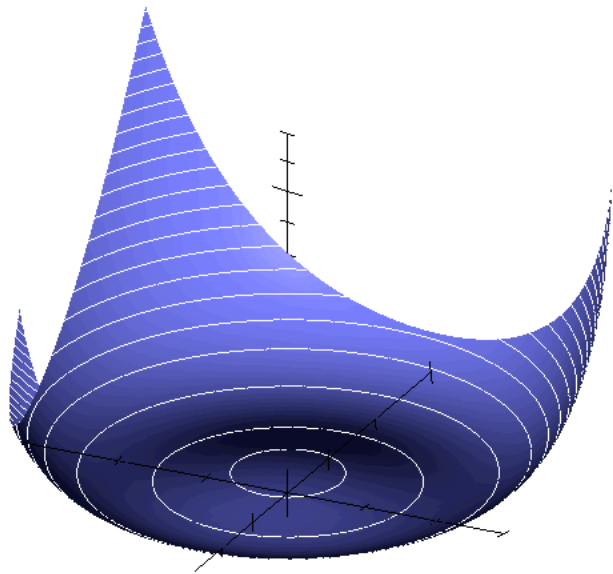
with $|\nu| = 1$, $\nu \cdot Dv(x) = 0$ and P^{ν} an orthogonal projection. The related **Soner–Touzi representation formula** is

$$u(x, t) = \inf_{\mu(\cdot)} (\mathbb{E} \{u(y_{\mu}(x, t, 0), 0)\})$$

where $\mu(\cdot)$ plays the role of a "control":

$$\begin{cases} dy_{\mu}(x, t, s) = \sqrt{2}\mu(t-s)d\widehat{W}(s) & s \in (0, t] \\ y_{\mu}(x, t, 0) = x \end{cases}$$

with $|\mu(\cdot)| = 1$, and $\mu(t-s) \cdot Dv(y_{\mu}(x, t, s)) = 0$.



The **fully discrete scheme** has the same structure as for the codimension–1 case, unless for a minimization:

$$v_j^{n+1} = \min_{\nu \in S^2} \left\{ \frac{1}{2} I[v^n](x_j + \sqrt{2\Delta t} \nu) + \frac{1}{2} I[v^n](x_j - \sqrt{2\Delta t} \nu) + \frac{1}{\varepsilon} (D_j[v^n] \cdot \nu)^2 \right\}$$

where:

- $I[v](\cdot)$ is a **numerical reconstruction**
- $D_j[v]$ is a centered–difference **approximation of the gradient**
- The constraint $\nu \cdot Dv = 0$ has been treated by **penalization**

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- Following a curve made of stationary points is an inherently ill-conditioned operation
- The ε -sublevel set gets thicker and thicker because of the parabolic behaviour of the equation
- It is analytically known that the fattening phenomenon cannot be avoided (Bellettini, Novaga, Paolini '98).

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Post-processing of the solution (1)

Rather than follow the zero-level set, we move on Γ_t from a point ξ_j to the next point ξ_{j+1} by constrained minimization:

$$v^n(\xi_{j+1}) = \min_{|\xi - \xi_j| = \Delta\xi} v^n(\xi)$$

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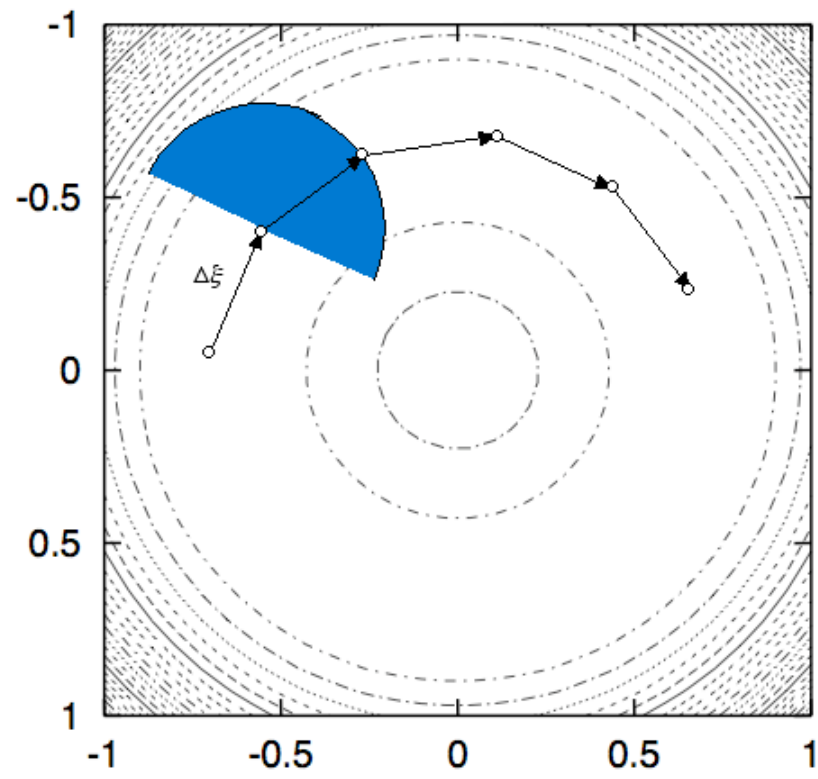
- The admissible displacement $\xi_{j+1} - \xi_j$ is required to satisfy the restriction $(\xi_{j+1} - \xi_j) \cdot (\xi_j - \xi_{j-1}) \geq 0$ in order to avoid changing direction on Γ_t

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- The step $\Delta\xi$ is of the order of Δx



Post-processing of the solution (2)

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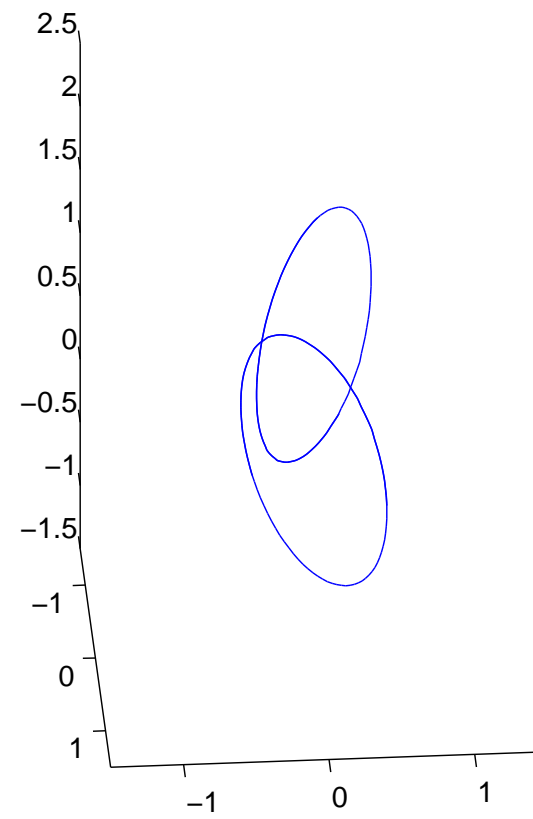
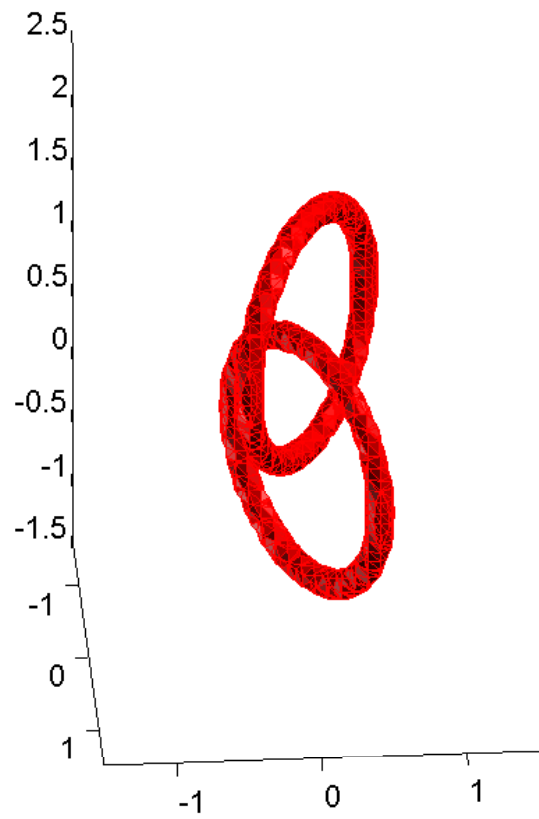
- As a general rule to check numerical schemes, double points should be "cut" keeping acute angles connected
- Although in our case fattening corresponds to an "almost constant" region for the solution, the post-processing phase tends to select the physically relevant evolution of the curve

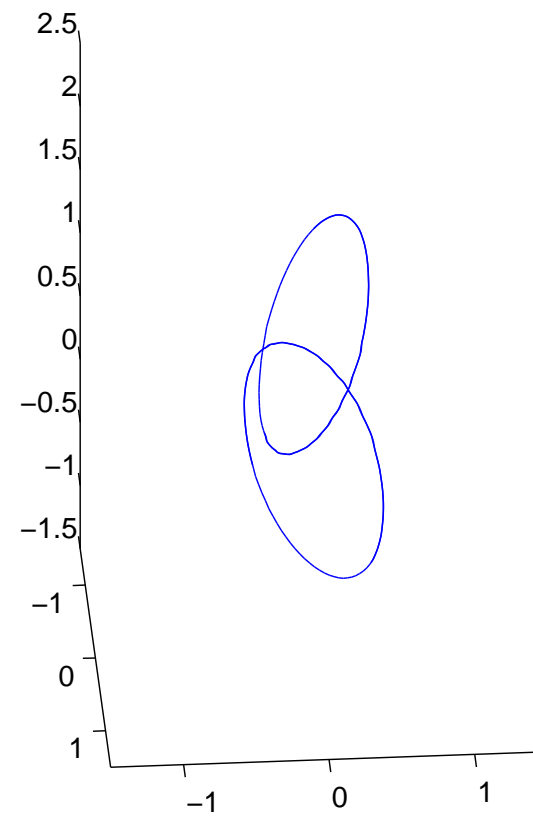
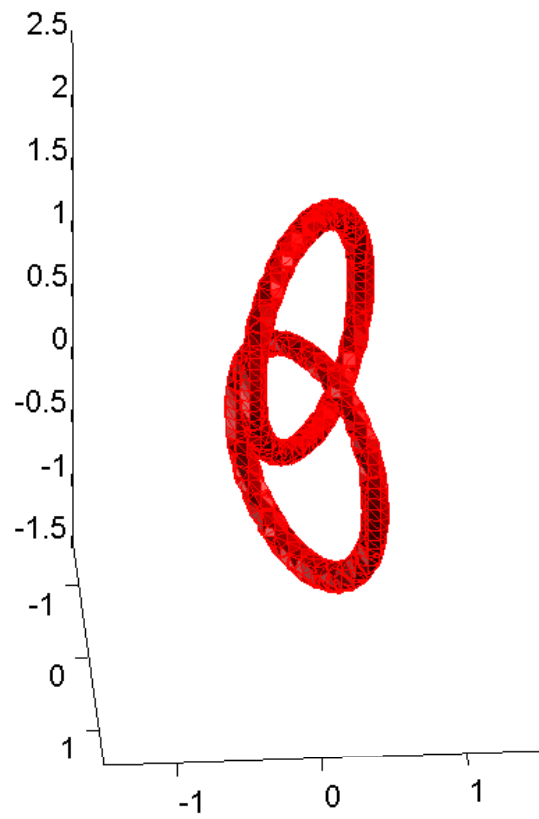
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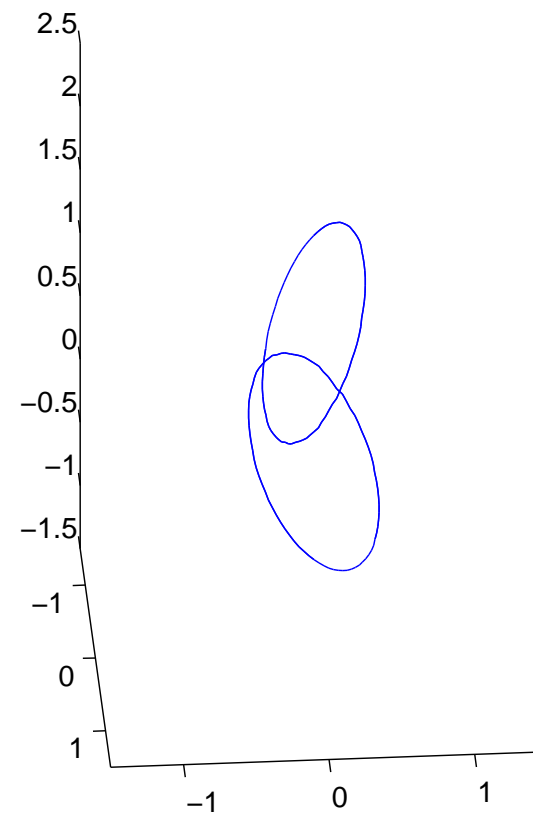
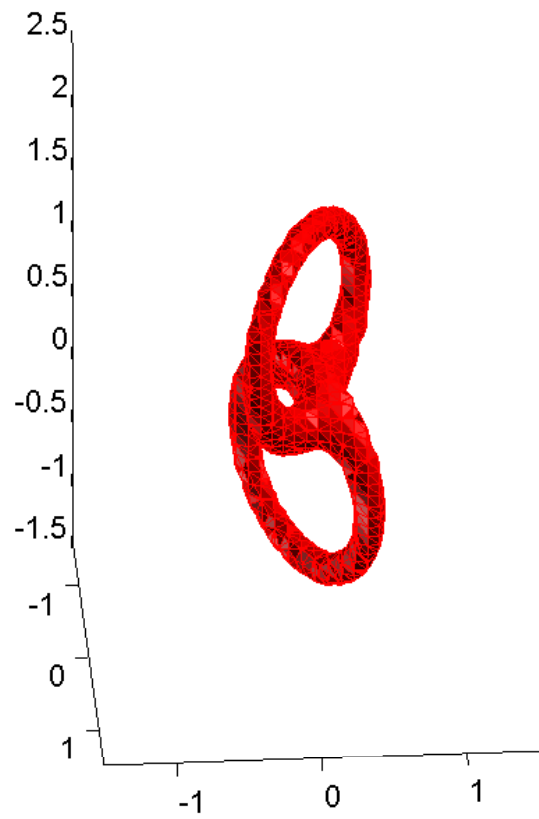
Numerical tests

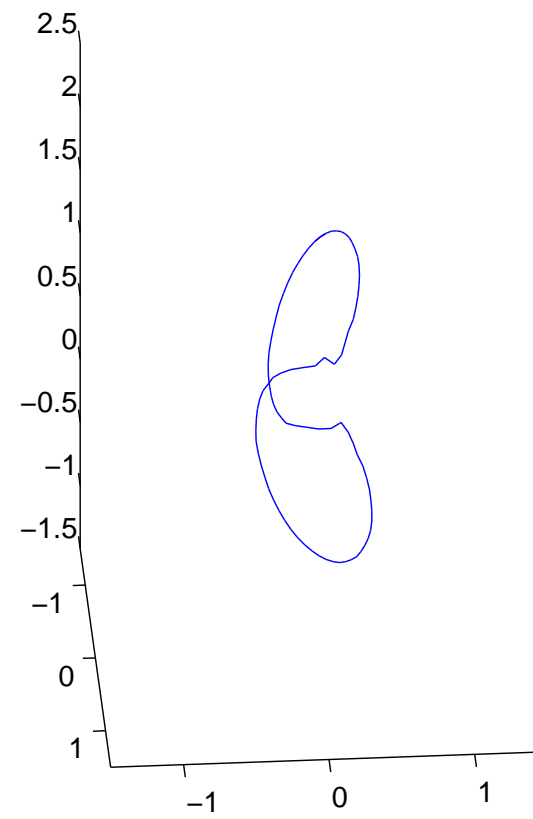
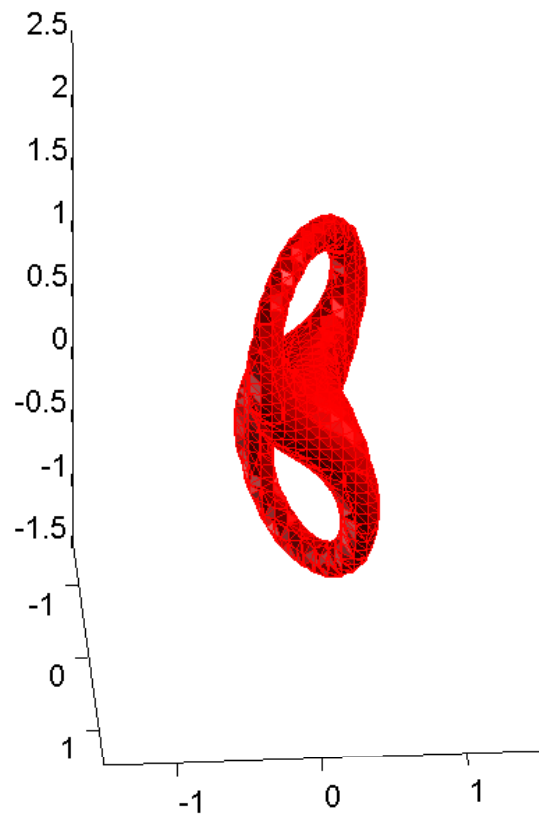
Evolution of two linked circles

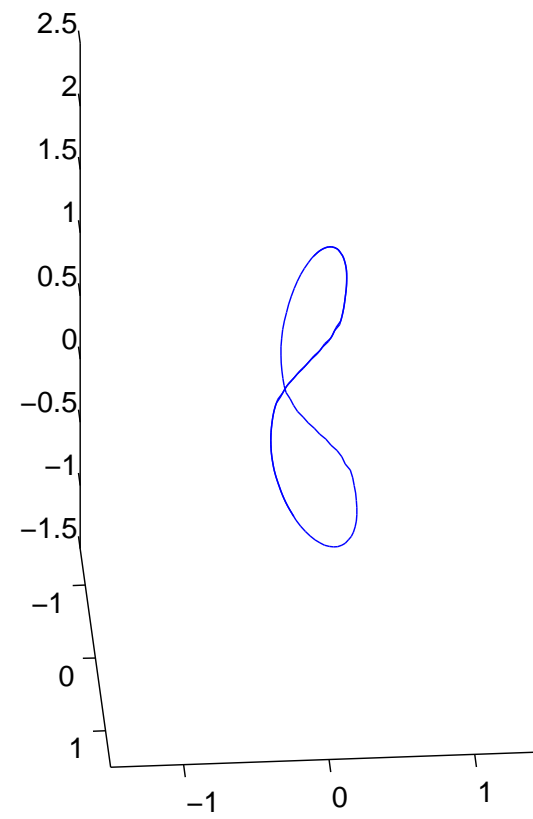
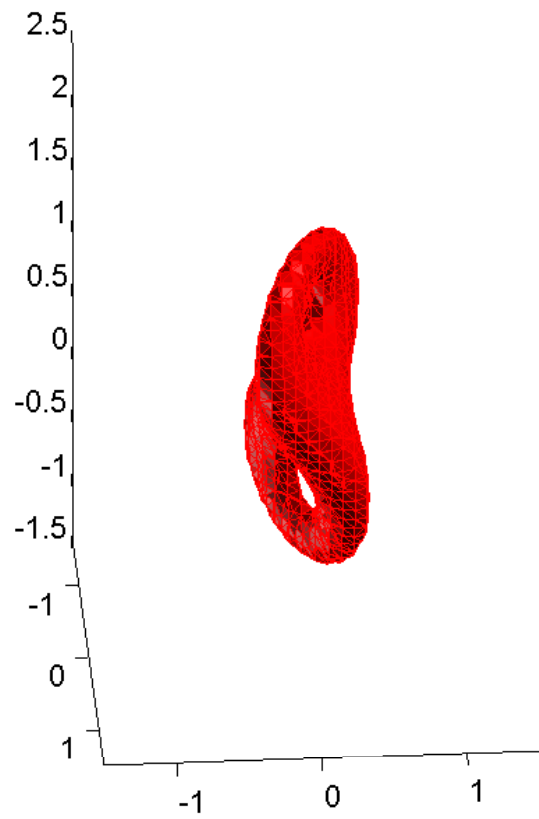
In this test we follow the evolution of two linked circles in \mathbb{R}^3 . In this case by tracking the ε -level set we are unable to resolve the ambiguity arising after the double point generation, due to fattening. On the contrary, post-processing singles out the meaningful solution

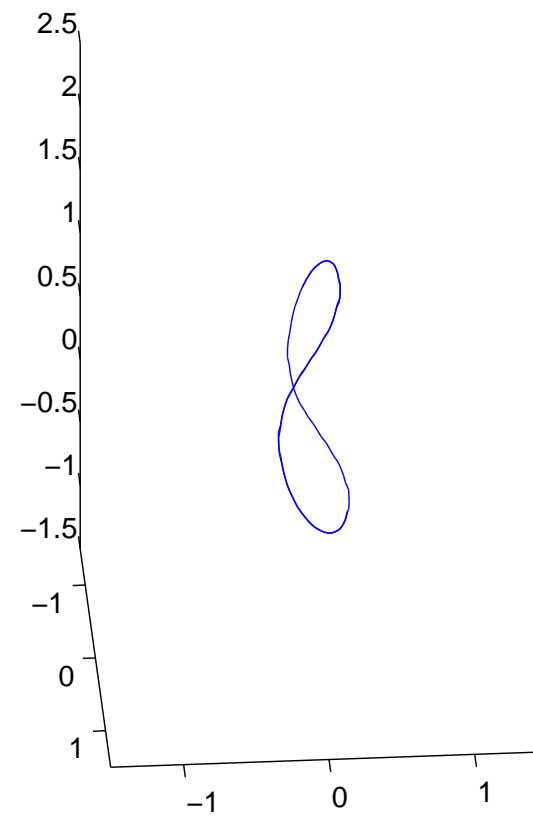
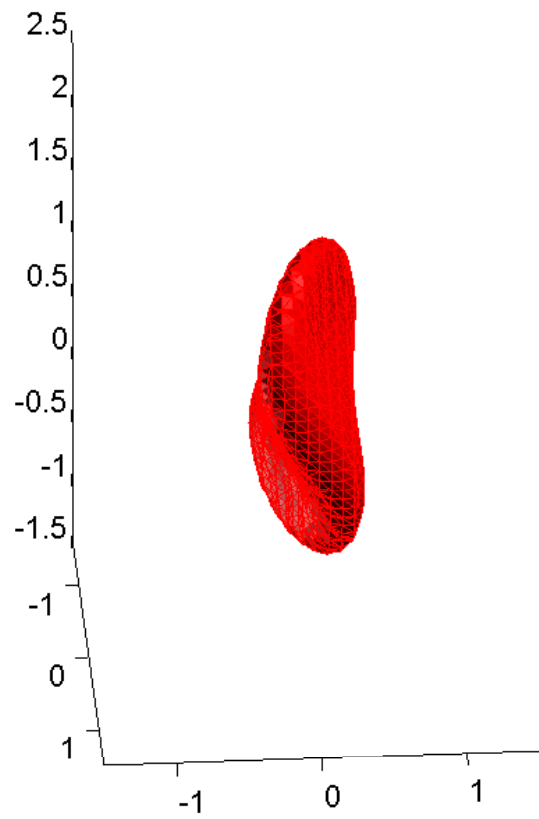












Evolution of a helical curve

We consider the evolution of a helix in three dimensions, with periodic conditions. The Mean Curvature Flow preserves the shape of the curve, but causes its straightening towards the axis. The figures also compare the ε -level set with the post-processed curve

