A variational approach to the macroscopic electrodynamics of hard superconductors

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Joint work with Annalisa Malusa

Outline

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Superconductivity

In many materials, the resistance drop to an unmeasurably small value if the sample is cooled down at a temperature below its critical temperature T_c

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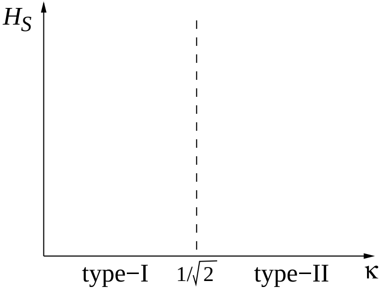

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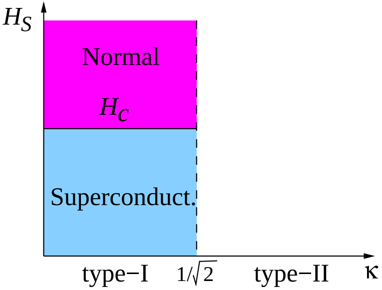

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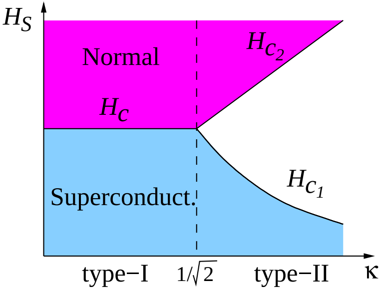

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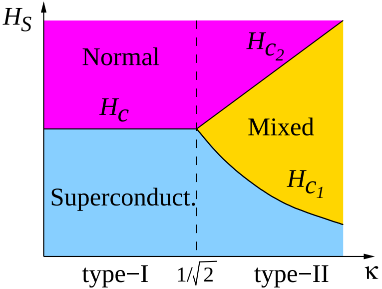

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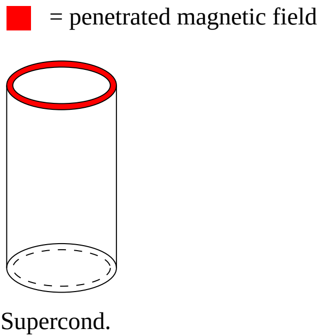
Penetration of external magnetic field



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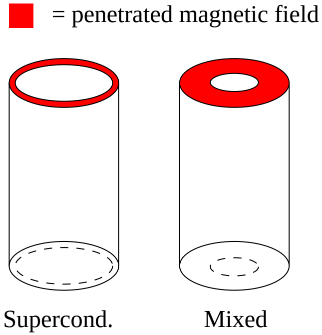

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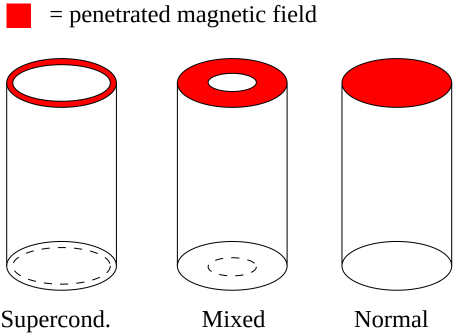

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- Mixed state: partial penetration in the bulk.
- Normal conducting phase: full penetration in the bulk.

Advantages of type-II superconductors

Technological advantages of type-II superconductors in mixed state:

- high T_c (up to $135K$)
- superconductivity properties with large magnetic fields
- current flow in the bulk of the sample (not only in thin layers)

Models for type-II superconductors

- Microscopic: BCS theory (Bardeen, Cooper, Schrieffer 1957), quantum mechanical description.

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- Microscopic: BCS theory (Bardeen, Cooper, Schrieffer 1957), quantum mechanical description.
- Mesoscopic: Ginzburg-Landau model (1950); formation of vortices (filaments), Abrikosov (1957).
- Macroscopic: Bean's model (1962), critical state model for the description of macroscopic electrodynamics.

Bean's model

Bean's model (C.P. Bean, 1962) for type-II hard superconductors: exists a critical current J_c such that:

- $|\vec{J}| = J_c$ in the region penetrated by the magnetic field;
- $\vec{J} = 0$ otherwise.

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Anisotropy of J_c, due to Cu-O planes, structure of defects, etc: exists $\Delta \subset \mathbb{R}^3$ compact convex containing a neighborhood of 0 s.t.

- $\vec{J} \in \partial\Delta$, in the region penetrated by the magnetic field;
- $\vec{J} = 0$ otherwise.

Macroscopic electrodynamics

PROBLEM: given a superconductor $Q \subset \mathbb{R}^3$ in an external field $\vec{H}_S(t)$, find the internal magnetic field $\vec{H}(x,t)$ and the electric field $\vec{E}(x,t)$. (Boundary condition: $\vec{H} = \vec{H}_S$ on ∂Q.)

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- anisotropic conductor: $\vec{E}(\vec{J}) = A\,\vec{J}$, $A =$ resistivity tensor

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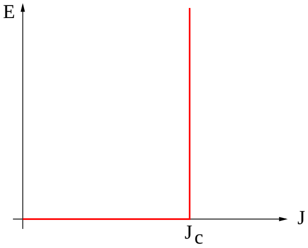

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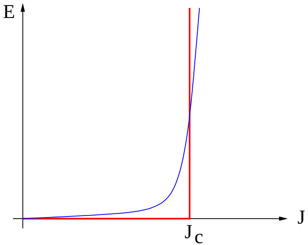

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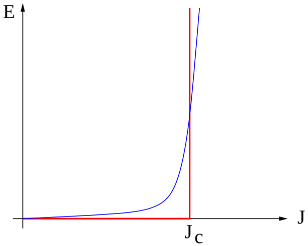

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The electric field is determined using the additional condition $\vec{E} || \vec{J}$.

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- In the limit as $p \to \infty$: $\vec{E}(\vec{J}) \in \partial I_\Delta(\vec{J})$
$$\partial I_\Delta(\vec{J}) = \begin{cases} \{0\}, & \text{if } \vec{J} \in \text{interior of } \Delta, \\ \{\lambda\, D\rho_\Delta(\vec{J});\ \lambda \geq 0\}, & \text{if } \vec{J} \in \partial\Delta, \\ \emptyset, & \text{if } \vec{J} \notin \Delta \end{cases}$$
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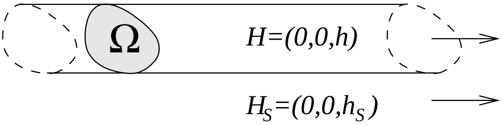
Rigorous giustification by Γ-convergence.

Cylindrical symmetry



- $Q = \Omega \times \mathbb{R}$, cylinder with cross-section $\Omega \subset \mathbb{R}^2$, smooth;
- $\vec{H}_S(t) = (0, 0, h_S(t))$ directed along the axis of the cylinder.

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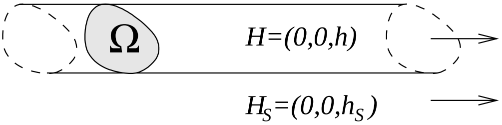

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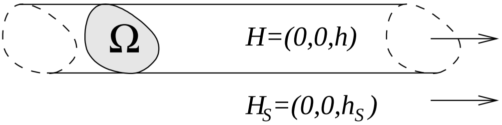

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$\Longrightarrow$ $\vec{J} = \text{curl}\, \vec{H} = (\partial_{x_2} h, -\partial_{x_1} h, 0)$

Remark: $\vec{J} \in \Delta \iff Dh \in K$, where $K \subset \mathbb{R}^2$ is the rotation of the section $z = 0$ of Δ.

Quasistatic evolution

Time discretization in $[0, T]$: $\delta t = T/n$, $t_i = i\delta t$, $\vec{H}_i = \vec{H}(t_i)$, $\vec{E}_i = \vec{E}(t_i)$.

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$\Longrightarrow$ admits the variational formulation

$$J_p(h) = \int_\Omega \frac{1}{p} [\rho(Dh)]^p + \frac{\mu_0}{2c\delta t}(h-h_i)^2, \qquad h \in h_s(t_{i+1}) + W_0^{1,p}(\Omega)$$

i.e., h_{i+1} is the unique minimum point of J_p in $h_s(t_{i+1}) + W_0^{1,p}(\Omega)$.
$\rho = \rho_K =$ gauge function of $K \subset \mathbb{R}^2$.

Convergence

Theorem (G.C. - A. Malusa)

$u_p \in h_s(t_{i+1}) + W_0^{1,p}(\Omega)$: *unique minimum point of* J_p, $p \geq 1$.
$u_\infty \in h_s(t_{i+1}) + W_0^{1,1}(\Omega)$: *unique minimum point of*

$$J(u) = \int_\Omega I_K(Du) + (u - h_i)^2 , \qquad u \in h_s(t_{i+1}) + W_0^{1,1}(\Omega).$$

Then, for every $q > 1$, (u_p) *converges to* u_∞ *in weak-*$W^{1,q}$.

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Remark: variational formulation proposed by Badía-López (2002) starting from physical considerations.

Necessary conditions - Electric field

Theorem (Dual function)

$\exists$ a non-negative continuous function v_i such that

$$-\operatorname{div}(v_i\, D\rho(Dh_{i+1})) = h_i - h_{i+1} \qquad in\ \Omega.$$

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Techniques developed in
G.C., Malusa: to appear in Trans. Amer. Math. Soc.
Isotropic case ($K =$ ball):
Cannarsa, Cardaliaguet, G.C., Giorgieri: Calc. Var. 2005

Selected references

- Badía, López, Phys. Rev. B 2002, J. Low Temp. Phys. 2003, J. Appl. Phys. 2004: anisotropic Bean's model
- Barrett, Prigozhin, Nonlinear Anal. 2000, preprint 2005: isotropic Bean's model, variational inequalities
- Brandt *et al.*, Phys. Rev. B 1996 and 2000: numerical and experimental data

Candidate solution $u_\infty = h_{i+1}$

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Solution of the minimum problem:

$$h_{i+1}(x) = \begin{cases} d(x) + h_s(t_{i+1}), & \text{if } x \in \Omega^+ = \{h_i > d\}, \\ -d^-(x) + h_s(t_{i+1}), & \text{if } x \in \Omega^- = \{h_i < -d^-\}, \\ h_i(x), & \text{if } x \in \Omega^0 = \Omega \setminus (\Omega^+ \cup \Omega^-). \end{cases}$$

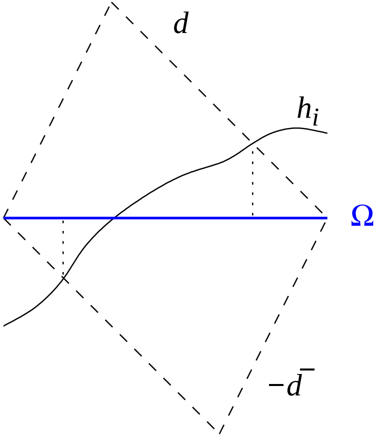
$$h_{i+1}(x) = [h_i(x) \vee (-d^-(x) + h_s(t_{i+1}))] \wedge (d(x) + h_s(t_{i+1}))$$

1D heuristics

$$K = [-1, 2]$$

$$J(h) = \int_\Omega |h - h_i|^2 + I_K(Dh)$$

$$h = 0 \text{ on } \partial\Omega$$



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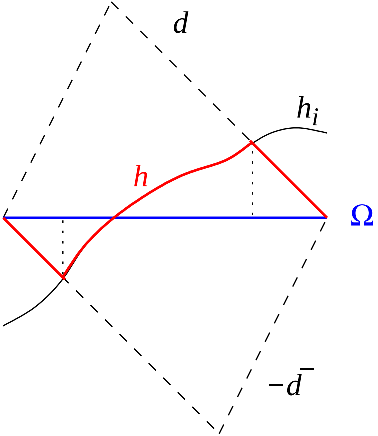

Decomposition of Ω in transport rays

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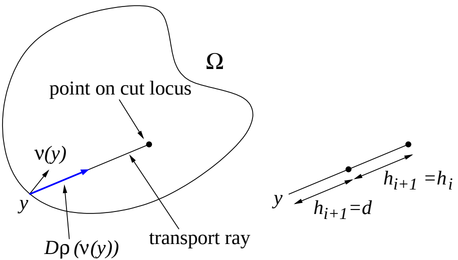
Example: $h_i(y) > 0$.



$\nu(y)$ = inward Euclidean normal of $\partial\Omega$ at y
$l(y)$ = length of the transport ray

$\Longrightarrow$ on each transport ray apply the 1D-heuristics.

Quasistatic evolution

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  $\Longrightarrow$ solution of the minimization problem
  $$\min\left\{\int_\Omega \frac{\mu_0}{2}\,|h - h_i|^2 + \delta t\, I_K(Dh);\ \ h \in h_S(t_{i+1}) + W_0^{1,1}(\Omega)\right\}$$

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- Explicit formula for monotone external field:
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  2. h_S monotone decreasing in $[0, T]$:
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The limit $\delta t \to 0$

For $\delta t = T/n$, $n \in \mathbb{N}^+$, construct h_i as above and define $h^n(x, t) = h_i(x)$, for $t \in [t_i, t_{i+1})$

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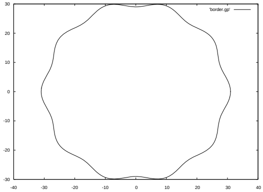
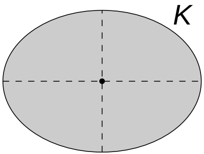
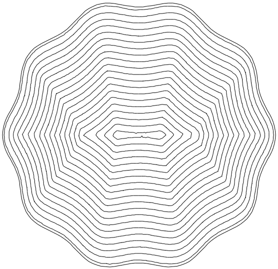
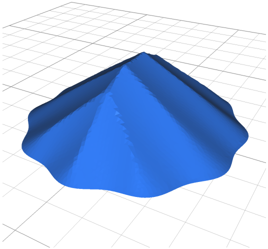
$\Longrightarrow$ the internal magnetic field can be explicitly computed if h_S is piecewise monotone.

Example



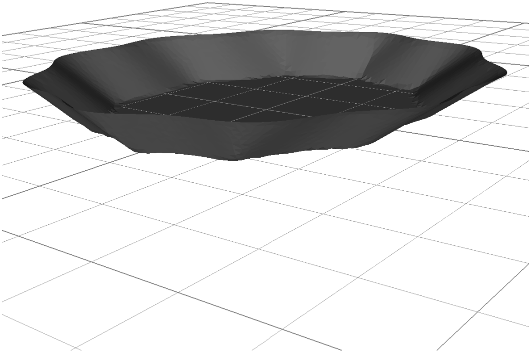


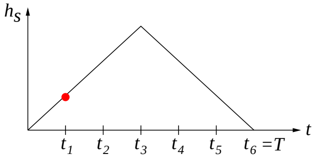




The section Ω, the constraints set K; Level sets and 3D-plot of the distance d.

Example: plot of h





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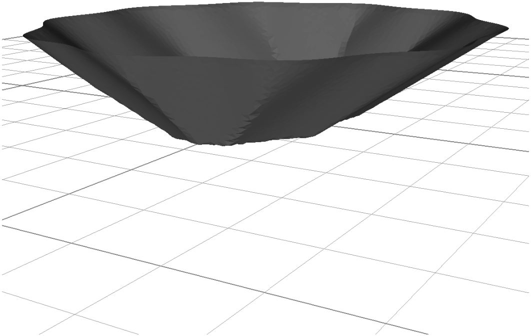

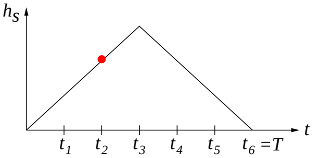

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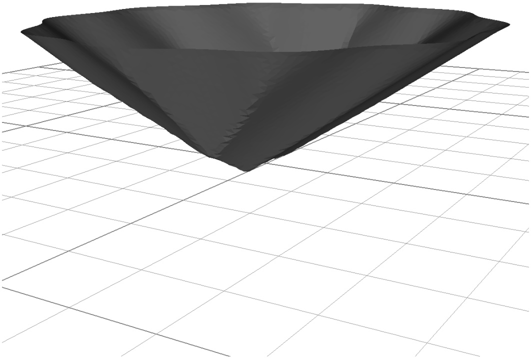

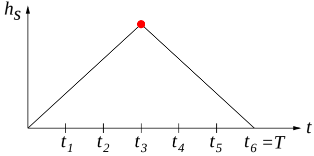

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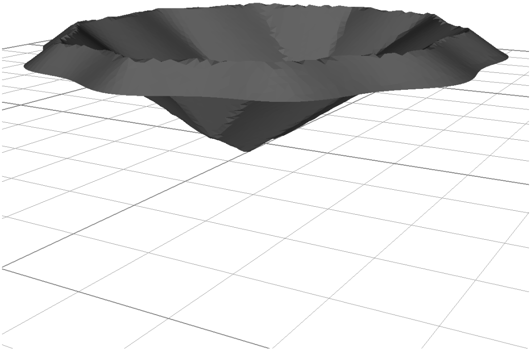

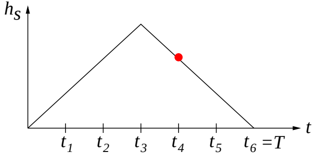

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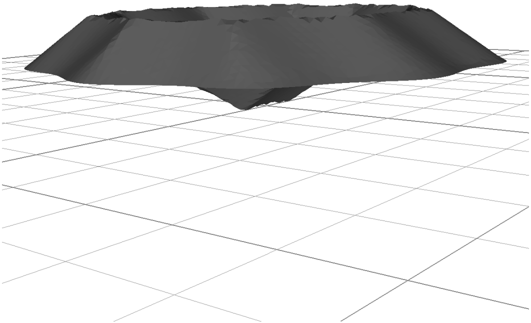

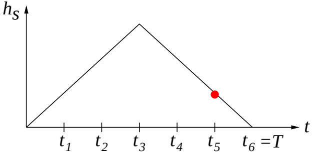

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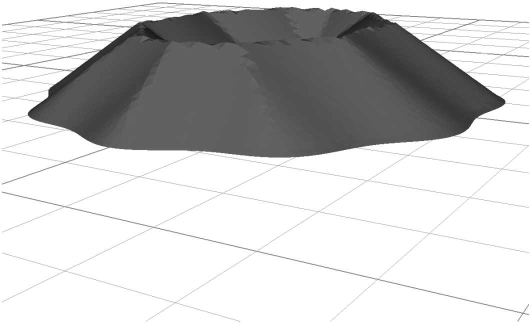
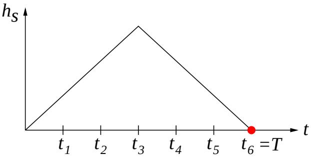
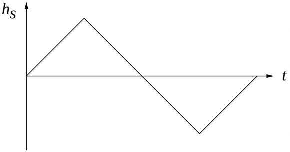
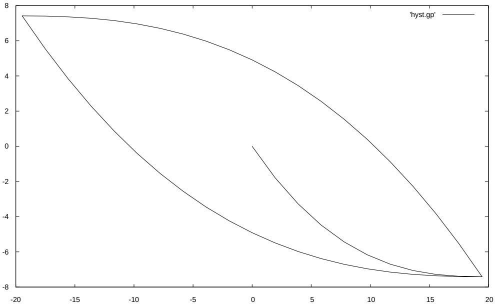




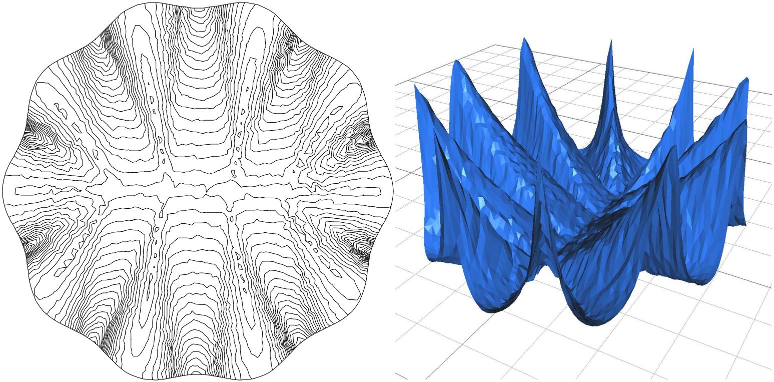
Hysteresis loop





Hysteresis loop: magnetization $\vec{M} = \langle \vec{H} \rangle - \vec{H}_S$ versus external field $\vec{H}_S$.

Example: plot of w



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