Convergence of a Fast Marching algorithm for a non-convex eikonal equation

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Outline

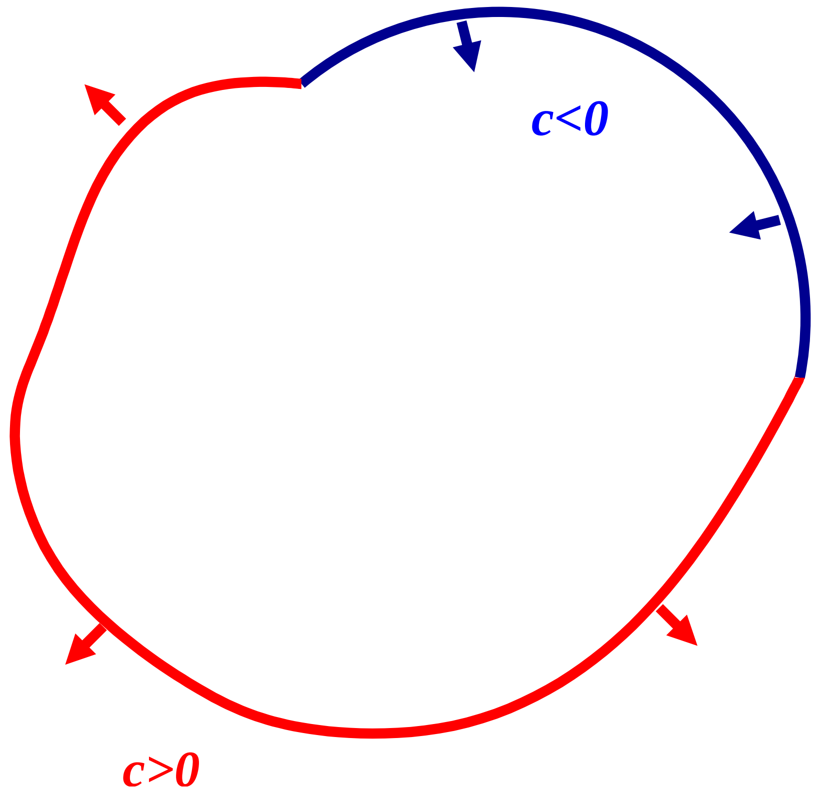
- The model problem
- The Fast Marching Method
- *The non-monotone Fast Marching scheme*
- Convergence result
- Numerical tests

Applications

- dislocation dynamics
- image processing
- interface motion

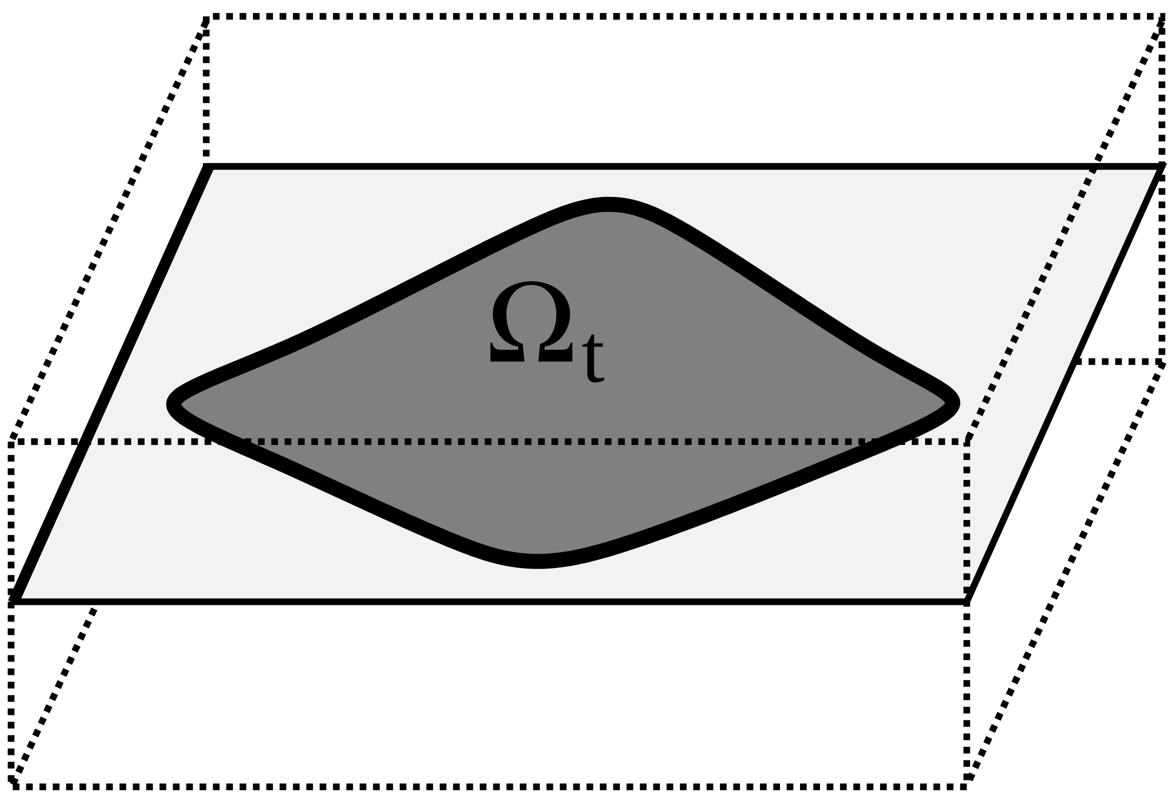
The model problem

A curve in $\mathbb{R}^2$ moves in the normal direction with normal speed $c(x, y, t)$, variable sign velocity.



Reformulation of the dynamics: level set approach

$$u(x,y,t) = \begin{cases} u > 0 & \text{if} (x,y) \in \Omega_t, \\ u = 0 & \text{if} (x,y) \in \Gamma_t = \partial\Omega_t, \\ u < 0 & \text{if} (x,y) \notin \Omega_t \ . \end{cases}$$



Reformulation of the dynamics: level set approach

The function u satisfies

$$\begin{cases} u_t = c(x,y,t)|\nabla u| & \mathbb{R}^2 \times (0,T) \\ u = u^0 & \mathbb{R}^2 \end{cases}$$

in the class of continuous viscosity solutions.

The stationary problem for the monotone eikonal equation

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : u(x,y,t) = 0\} = \{(x,y) \in \mathbb{R}^2 : T(x,y) = t\}$$

where $T(x,y)$ solves the minimum time problem

- $c(x,y) \geq 0$
  $$\begin{cases} c(x,y)|\nabla T(x,y)| = 1 & \mathbb{R}^2 \setminus \Omega \\ T(x,y) = 0 & \Omega \end{cases}$$
  (see Falcone, Giorgi, Loreti)

- $c(x,y,t) \geq 0$
  $$\begin{cases} c(x,y,T(x,y))|\nabla T(x,y)| = 1 & \mathbb{R}^2 \setminus \Omega \\ T(x,y) = 0 & \Omega \end{cases}$$
  (see Vladirmisky)

The present Fast Marching schemes

- $c(x, y) > 0$
  Fast Marching Method
  $[Osher - Sethian]$
- $c(x, y) \geq 0$
  Semi-Lagrangian Fast Marching Methods
  $[Falcone - Cristiani]$
- $c(x, y, t) > 0$
  Ordered Upwind Method
  $[Sethian - Vladimirsky]$

The Finite Difference approximation

Let us write the equation as

$$T_x^2 + T_y^2 = \frac{1}{c^2(x,y)}$$

The standard up-wind FD approximation is

(1)
$$\max(0, T_{i,j} - T_{i-1,j}, T_{i,j} - T_{i+1,j})^2 + \max(0, T_{i,j} - T_{i,j-1}, T_{i,j} - T_{i,j+1})^2 = \left(\frac{\Delta}{c_{i,j}}\right)^2$$

The Finite Difference Approximation

The iterative method is

- consistent
- stable, provided a CFL condition is satisfied
- convergent
- expensive, since it globally works on *all* the grid values at every iteration

The Fast Marching Method, $c > 0$

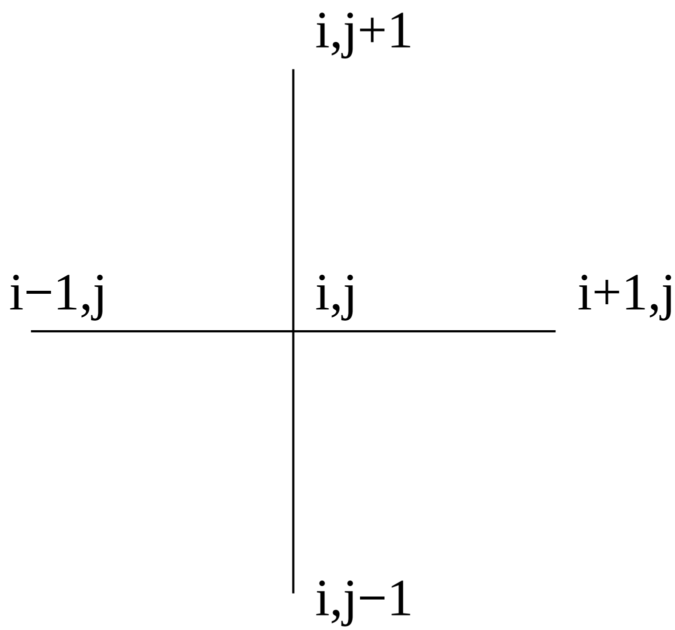
Main Idea (Tsitsiklis (1995), Sethian (1996))
Processing the values on the nodes in a special order one can compute the solution in just 1 iteration.

This special order is obtained introducing a *NARROW BAND* which locates the front.

Just the nodes in the NB are computed at every iteration, in this way the "natural" ordering corresponds to the increasing values of T.

The Fast Marching Method

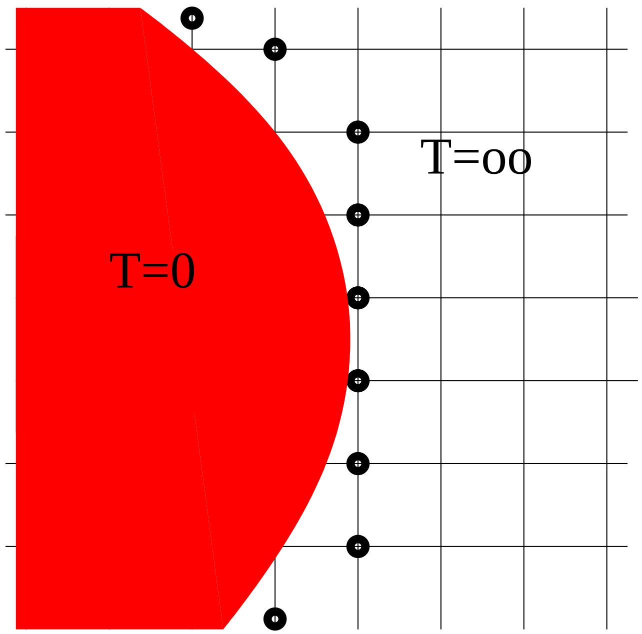
Def. We define **neighborhood of the node** $x_{i,j}$ the set
$V(i,j) \equiv \{(l,m) \in \mathbb{Z}^2 \ such \ that \ |(l,m)-(i,j)| = 1\}$.



The Fast Marching Method

Def. We define **Narrow Band** the set

$$NB \equiv V(E) \backslash E, \text{ where } E = \{(i,j);\ (x_i, y_j) \in \Omega\}.$$



The Fast Marching Method, $c > 0$

Inizialization

1. All the nodes which belong to the initial front configuration are labeled as ACCEPTED and they are given the value $T = 0$.
2. The initial narrow band is defined, these nodes are labeled NB and they are given the value $T = \frac{\Delta}{c}$.
3. All the remaining nodes are labeled as FAR and they are given the value $T = +\infty$

The Fast Marching Method, $c > 0$

Main Cycle

1. Among the NB nodes take the one which has minimal T value (let us call A this node).
2. A is labeled ACCEPTED and it is removed from the narrow band.
3. The neighboring nodes to A are included in the narrow band.
4. We (re)compute the value T in the neighboring nodes to A by the explicit evaluation of Eq.**1** , selecting the largest possible solution to the quadratic equation.
5. If the narrow band is not empty, go back to 1.

Non monotone FMM method

Some important modifications to the classical scheme

- our new *narrow band* is 'DOUBLE' : the set of nodes which are going to be reached by the front *and* the nodes just reached by the front
- we force the speed c to be exactly zero on the boundaries of the regions where the speed change sign so that the evolution of the front in each region can be considered completely separated

Non monotone FMM method

- in the evaluation of Eq.**1** we take into account only the nodes already accepted $\rightarrow$ no CFL condition!
- we introduce an auxiliary discrete function

$$\theta^n_{i,j} = \begin{cases} 1 & \text{if } (x_i, y_j) \in \Omega_{t_n} \\ -1 & \text{otherwise.} \end{cases}$$

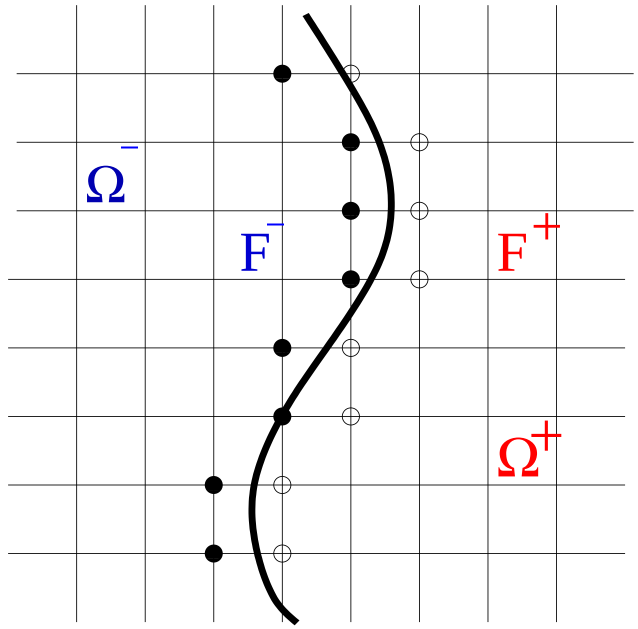
Non monotone FMM method

Def. We define two different **narrow bands**:

$$F_+^n = \{(i,j)\ s.t.\exists(l,m) \in V(i,j)\ s.t.\ \theta^n_{l,m} = -1,\ \theta^n_{i,j} = 1, \}$$

$$F_-^n = \{(i,j)\ s.t.\exists\ (l,m) \in V(i,j)\ s.t.\ \theta^n_{l,m} = 1,\ \theta^n_{i,j} = -1\}$$

We define **front** the set $F^n = F_+^n \cup F_-^n$.



Non monotone FMM algorithm

Inizialization

- *Initialization of the matrix* θ^0
  $$\theta^0_{i,j} = \begin{cases} 1 & (i,j) \in \Omega_0 \\ -1 & (i,j) \notin \Omega_0 \end{cases}$$
- *Initialization of the time on the front*
  $T^0_{i,j} = 0$ for all $(i,j) \in F^0$

Non monotone FMM algorithm

Main Cycle

- computation of $T^n_{i,j},\ \forall (i,j) \in F^{n-1}$

  if $c^{n-1}_{i,j} > 0$ *and* $i,j \in F^{n-1}_{-}$ *compute* $T^n_{i,j}$ *by the explicit valuation of Eq.***1** *using the nodes from* F^{n-1}_{+}.

  if $c^{n-1}_{i,j} < 0$ *and* $i,j \in F^{n-1}_{+}$ *compute* $T^n_{i,j}$ *by the explicit valuation of Eq.***1** *using the nodes from* F^{n-1}_{-}.

- $t_n = \min \left\{ T^n_{i,j},\ (i,j) \in F^{n-1} \right\}$.

- Initialization of new accepted point

  $NA^n = \{(i,j) \in F^{n-1},\ T^{n-1}_{i,j} = t_n\}$

Non monotone FMM algorithm

- Re-initialization of θ^n

$$\theta^n_{i,j} = \begin{cases} -1 & \text{if} \quad (i,j) \in NA^n \text{ and } \quad \theta^{n-1}_{i,j} = 1 \\ 1 & \text{if} \quad (i,j) \in NA^n \text{ and } \quad \theta^{n-1}_{i,j} = -1 \end{cases}$$

- Re-initialization of T^n on F^n

Convergence result

Theorem

Let $c(x, y, t)$ be globally Lipschitz continuous in space and time, the initial set Ω_0 be with piece wise smooth boundary and $\theta_\Delta(x, t)$ be an appropriate extension of the discrete function $\theta_{i,j}^n$ over all the continuous space,then

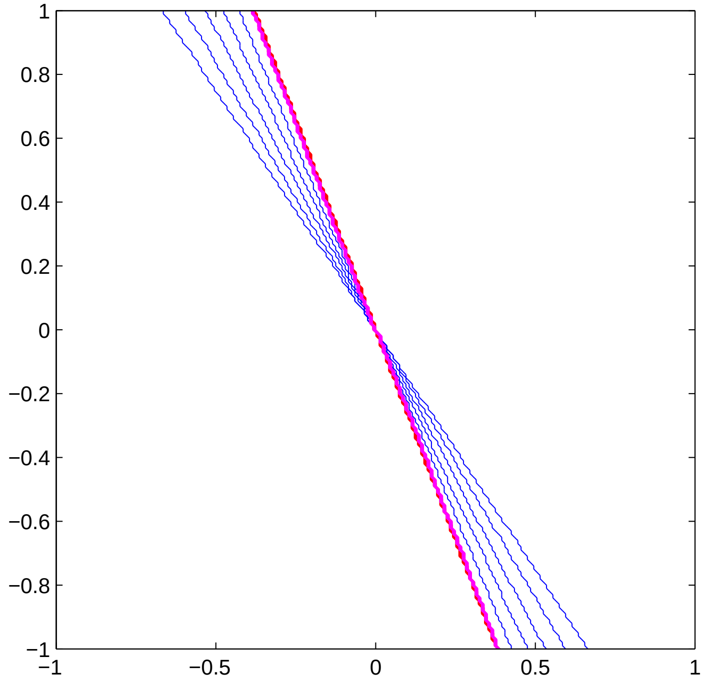
$$\theta(x, t) = \lim_{\Delta \to 0} \theta_\Delta(x, t)$$

is a viscosity discontinuous solution of the problem

$$\begin{cases} \theta_t = c(x, y, t)|\nabla \theta| & \mathbb{R}^2 \times (0, T) \\ \theta = 1_{\Omega_0} - 1_{\Omega_0^c} & \mathbb{R}^2, \end{cases}$$

A rotating line

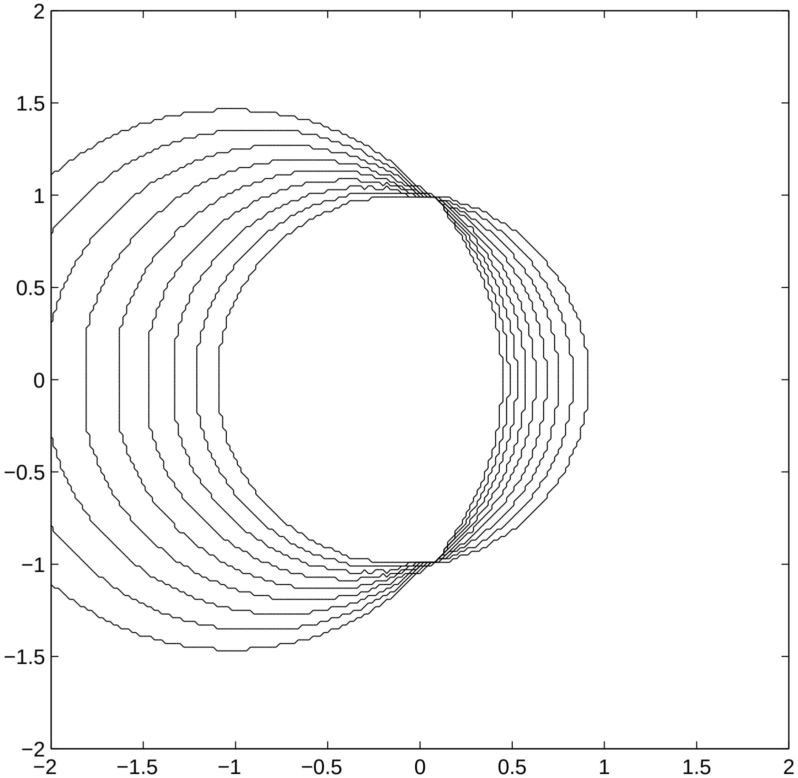
$$c(x, y, t) = x$$



Δ	L_1-error
0.08	0.102
0.04	0.0576
0.02	0.0304
0.01	0.0160

A propagating circle

$$c(x, y, t) = 0.1t - x$$



Δ	L_1-error
0.08	0.4992
0.04	0.2784
0.02	0.1288
0.01	0.0582

Numerical tests: evolution of two circles

Speed $c(x,y,t) = 1 - t$

Increasing (left) and decreasing (right) evolution of two circles

Open Problems

- extension of the FMM non monotone scheme to non local speed
- convergence for the FMM non monotone non-local scheme

References

- Sethian, J. A.: Level Set Methods, Evolving interfaces in Geometry, Fluid Mechanics, Computer Vision, and Material Science. Cambridge University Press (1996)
- E. Carlini, E. Cristiani, N. Forcadel *A non-monotone FM scheme modeling dislocation dynamics*. Submitted to Proceedings on ENUMATH 2005.
- E. Carlini, M. Falcone, N. Forcadel, R. Monneau *Convergence of a Fast Marching algorithm for a non-convex eikonal equation.* In preparation.