Singular perturbations and Aubry-Mather theory

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- The Aubry-Mather theory of Hamilton-Jacobi equations
- Improving the PDE approach to Large Deviations via Aubry-Mather theory

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E_ε: a functional depending on the sample paths X_ε
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where $I > 0$ in D is the rate function, i.e.

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- Freidlin-Wentzell: Random perturbations of dynamical systems, Springer, 1994
- Varadhan: Large deviations and applications, SIAM, 1984

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- Show uniqueness for **(HJ)**. Then $I_\varepsilon \to I$ and we have the large deviations result $-\varepsilon \log(E_\varepsilon(x)) \longrightarrow I(x)$ for $\varepsilon \to 0$. Interpreting **(HJ)** as the a control problem, we also have a representation formula for I

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- Barles-Perthame ('90): discontinuous viscosity solutions and half-relaxed limits (estimates for $\|I_\varepsilon\|_\infty$)

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b cannot have equilibria inside $D \Rightarrow$ interesting problems in Large Deviation theory (e.g. Wentzell-Freidlin's theory) are excluded by the viscosity solution approach (Perthame (TAMS '90): the case of a single equilibrium point for b)

A basic problem (Wentzell-Freidlin's book, Ch. IV)

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- $b(x) \cdot n_{ext}(x) < 0$ for $x \in \partial D$ (D is invariant)
- the set Ω_b of the ω-limits of $\dot{x}(t) = b(x(t))$ is a class of equivalence for the quasi-potential

$$V(y,x) = \inf\{\int_0^T \tfrac{1}{2}|\dot{\phi}(s) - b(\phi(s))|^2\, ds : \phi(0) = y, \phi(T) = x, T > 0\}.$$

(i.e. $V(y,x) = V(x,y) = 0$ for $x, y \in \Omega_b$)

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If there a unique y s.t. $V(\Omega_b, y) = \min_{x \in \partial D} V(\Omega_b, x)$, then

$$E_\varepsilon(x) \longrightarrow \varphi(y) \qquad \text{for } \varepsilon \to 0$$

This means that the stochastic trajectories X_ε exit from D close to y

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The invariant measure v^ε associated to the process satisfies the adjoint equation

$$\begin{cases} -\frac{\varepsilon}{2}\Delta v^\varepsilon + div(b(x)v^\varepsilon) = 0 & x \in D \\ \dfrac{\varepsilon}{2}\dfrac{\partial v^\varepsilon}{\partial n}(x) + b(x)\cdot n(x)v_\varepsilon = 0 & x \in \partial D \end{cases}$$

Set $V^\varepsilon = -\varepsilon \log(v^\varepsilon)$, then V^ε is a solution of the singular perturbation problem

$$\begin{cases} -\frac{\varepsilon}{2}\Delta V^\varepsilon + H(x, DV^\varepsilon) = \varepsilon \mathrm{div}(b) & x \in D \\ \dfrac{\partial V^\varepsilon}{\partial n}(x) + 2b(x) \cdot n(x) = 0 & x \in \partial D \end{cases}$$

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Formally $V^\varepsilon \to V$ where V is a solution of

$$\begin{cases} H(x, DV) = 0 & x \in D \\ \dfrac{\partial V}{\partial n}(x) + 2b(x)\cdot n(x) = 0 & x \in \partial D \end{cases}$$

Since Levinson's condition is violated (b has an attractor inside D), no uniqueness and the 3^{rd} step in the PDE approach fails.

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• To understand if the sequence of the solutions V_ε of the 2^{nd} order problems selects a particular solution of the 1^{st} order problem

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$$S_\lambda(x,y) = \inf\{ \quad \int_0^1 \sigma_\lambda(\phi(s),\dot\phi(s))ds : \ \phi \in W^{1,\infty}([0,1],D), \ \phi(0)=x, \phi(1)=y\}$$

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For $\lambda = c$ a non-uniqueness phenomenon appears

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$$\inf_n \{\int_0^1 |\dot{\phi}_n(s)| ds\} \geq \delta > 0 \quad \text{(Euclidean length)}$$

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The main property: There exists a subsolution to $H(x, Du) = c$, which is strict (i.e. $H(x, Du) < c$) out of $\mathcal{A}$.
General Fact: A unique solution to $H(x, Du) = c$+(BC) $\Leftrightarrow$ the value on $\mathcal{A}$ is prescribed, i.e. $\mathcal{A}$ is an uniqueness set for $H(x, Du) = c$

Aubry-Mather theory for the Neumann problem

Recall that we want to study

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• $\mathcal{A}$ is contained in the interior of D. This fact is very important since we can interpret the Neumann boundary condition in standard viscosity sense

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Theorem: u subsolution and v supersolution s.t. $u \leq v$ for $x \in \mathcal{A}$ then

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Corollary: If g is such that $g(y) - g(x) \leq S(x, y)$ for any $x, y \in \mathcal{A}$ then

$$v(x) := \inf_{y \in \mathcal{A}} [g(y) + S(y, x)]$$

is the unique viscosity solution to (HJ) with value g on $\mathcal{A}$.

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  - A subsolution of $H(x, Du)$ is constant on $\mathcal{A}$
  - Ω_b is a uniqueness set for the Neumann problem

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Set $v^\varepsilon(\overline{x}) = 0$ for $\overline{x} \in \mathcal{A}$ (a solution is defined up to a constant). Then

$$v^\varepsilon \to S(\mathcal{A}, \cdot) \qquad \varepsilon \to 0$$

where $S(\mathcal{A}, x) = \min\{S(y, x) : y \in \mathcal{A}\}$.

Remark: Recalling that $S = V$, where V the quasi-potential, the previous theorem implies the Wentzell-Freidlin's large deviations result.

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Recalling the representation formula

$$v(x) := \inf_{y \in \mathcal{A}} \{g(y) + S(y, x)\}$$

we get

$$v(x) = S(\mathcal{A}, x) \qquad \text{for } x \in D.$$

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- With the same method it is possible to study other problems such as the Kamin and Eizenberg singular perturbation problem

$$\begin{cases} -\varepsilon \Delta v_\varepsilon + H(x, Dv_\varepsilon) - \varepsilon c(x) = 0 & x \in D \\ v_\varepsilon(x) = 0 & x \in \partial D \end{cases}$$

where c is non-negative in D.