Short SO_3-subgroups in simple Lie groups and associated quasielliptic geometries

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Let G be a simple compact Lie group without center. An SO_3-subgroup S of G is called *short* if the dimensions of irreducible components of the adjoint representation of S in $\mathfrak{g} = \mathrm{Lie}\, G$ do not exceed 5. It is easy to find all such subgroups. Their numbers (up to conjugacy) in the exceptional groups are given in the following table:

G_2	F_4	E_6	E_7	E_8
1	2	2	4	2

Any such subgroup gives rise to a presentation of $\mathfrak{g}$ in the form

$$\mathfrak{g} = \mathfrak{su}_3(J) + \mathrm{Der}\, J, \tag{1}$$

where J is a (in general non-associative) algebra with involution and the commutation in $\mathfrak{g}$ subjects the following rules:

- the commutation in Der J is usual;
- the commutator of a derivation D and a matrix $X \in \mathfrak{su}_3(J)$ is obtained by applying D to the entries of X;
- the commutator of two matrices is their usual commutator (lying in $\mathfrak{u}_3(J)$) projected onto $\mathfrak{su}_3(J)$ plus some derivation which can be explicitly written in terms of J.

All this is known (sometimes in another language) due to works of Tits (1962), Vinberg (1966), Kantor (1972), Allison (1978, 1979, 1991) and Seligman (1988). The algebras with involution appeared it this way were axiomatically characterized by Allison under the name of *structurable algebras*. Structurable algebras with trivial involution are just Jordan algebras.

A *triad* in G is a triple $\{s_1, s_2, s_3\}$ of involutions of the same conjugacy class $X \subset G$ such that $s_1 s_2 s_3 = e$. For example,

$$s_1 = \mathrm{diag}(-1, 1, 1), \quad s_2 = \mathrm{diag}(1, -1, 1), \quad s_3 = \mathrm{diag}(1, 1, -1)$$

constitute a triad in SO_3. Thereby any SO_3-subgroup $S \subset G$ generates a triad in G (contained in S). Comparing classifications comes to the following miraculous theorem: the (conjugacy classes of) triads in G are in one-to-one correspondence with the (conjugacy classes of) short SO_3-subgroups. It is a challenge to find a conceptual proof of this theorem.

Any triad $\{s_1, s_2, s_3\} \subset G$ provides the symmetric space X with a structure of *"quasielliptic plane"* over the corresponding structurable algebra J. Namely, as soon as a short SO_3-subgroup containing the triad is fixed, the tangent space of X at s_1 is canonically identified with $J+J$ due to (1). Further, one can define "lines" in X as subsets of the form

$$\mathrm{pol}(x) = \{y \in X : (x,y) \text{ is conjugate to } (s_1,s_2)\} \quad (x \in X),$$

which are connected totally geodesic submanifolds of dimension equal to dim $J = \frac{1}{2}$dim X. There is a polarity, i.e a one-to-one correspondence between points and lines preserving incidence. Two lines in general position intersect in a finite number of points called the *degree* of the quasielliptic plane (and two points in general position belong to the same number of lines). For $G = SO_3$, one thus obtains the ordinary elliptic plane over $\mathbb{R}$ (of degree 1).

All symmetric spaces of inner type of the exceptional groups admit a structure of quasielliptic plane. Moreover, all of them but $E_7/(D_6 \times A_1)$ admit only one such structure, while the latter admits two. In particular, the groups F_4, E_6, E_7, E_8 are the automorphism groups of quasielliptic planes over $\mathbb{O}, \mathbb{O}\otimes\mathbb{C}$, $\mathbb{O}\otimes\mathbb{H}$, $\mathbb{O}\otimes\mathbb{O}$, respectively. The degrees of these planes are 1, 1, 3, 135. This "proves" Rosenfeld's conjecture (1956) on the existence of projective planes over $\mathbb{O}\otimes\mathbb{H}$ and $\mathbb{O}\otimes\mathbb{O}$.