Lecture 5: Overview and open problems

Summary of limiting theories for $\beta > 2$

$$I_\beta = \underbrace{\frac{1}{2}\Lambda(\beta)\int_S Q_2\left(\operatorname{sym}\nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\right)}_{\text{membrane energy}}$$

$$\underbrace{+\frac{1}{24}\int_{S'} Q_2\left((\nabla')^2 v\right)}_{\text{bending energy}}$$

$$\Lambda(\beta) = \begin{cases} +\infty & 2 < \beta < 4 \quad \text{linearized von} \\ 1 & \beta = 4 \quad \text{vK} \\ 0 & \beta > 4 \quad \text{biharmonic eq}^{n} \end{cases}$$

$2 < \beta < 4$ constraint:

$$\operatorname{sym}\nabla' u + \frac{\nabla' v \otimes \nabla' v}{2} = 0.$$

$$\Longleftrightarrow \quad \det(\nabla')^2 v = 0$$

if there are no boundary conditions for u

Gauss curvature $K = 0$

Classification / approximation of $W^{2,2}$ $K=0$ surfaces

B. Kirchheim — www.mis.mpg.de Lecture Notes MPI

P. Pakzad — J. Diff. Geom. 66 (2004), 47 - 69

$0 < \beta < 2$ 'No man's land'

S. Conti: $0<\beta<1$

$$J_\beta = \min_{\Omega} \int -f \cdot y$$

subject to $W_{\text{mem}}(\nabla' y) = 0$

$$(\Leftrightarrow) \ (\nabla' y)^T \nabla y \leq Id$$

(short maps).

Special exponents β:

(i) $\beta = 1 \rightsquigarrow$ Blisters.

Thm. (BenBelgacem, Conti, DeSimone, M.
ARMA 164 (2002), 1-37)

Suppose $y_0^{(h)}(x) = \begin{pmatrix} \lambda x' \\ h x_3 \end{pmatrix} \quad \frac{1}{2} < \lambda < 1$

then

$$c(\lambda) h \leq \inf_{y = y_0 \text{ on } \partial\Omega} \int_\Omega W(\nabla_h y) \leq C(\lambda) h.$$

Open: What is the Γ-limit of $\frac{1}{h} \int_\Omega W(\nabla_h y)$?

(unusual for Γ-convergence.)

Strong dependence on boundary conditions.

(ii) $\beta = 5/3$ ⇝ packaging of thin sheets

T. Witten et al., S. Venkataramani,
S. Conti & F. Maggi.

Do demonstration with two paper cones
and ridge in between.

⟶.

Problem 1 Explore the range $1 \le \beta < 2$.

Problem 2. What's a good theory for
small but finite h?

{ Membrane too floppy
{ Kirchhoff too stiff.

Physicists: 2d approximation which still
involves h e.g.

$$\int_{\Omega'} \overline{\Phi}(\nabla y) + \frac{1}{12} h^2 \int Q(A)$$

Consistent w/ above table but in
what sense does it provide a better/
optimal approximation
(⇝ refined notion of Γ-convergence ?)
Γ-development is not good enough

Other topics

- What about theories which are not on the list of asymptotic limits ?

- Shells : "local" Friesecke-James-M.-Mora CRAS
"global": flexibility vs. rigidity
(convex compact surfaces have no infinitesimal isometries).

- Beyond the variational framework:
a) equilibria

$2d \to 1d$ M.-Mora-Schultz
$3d \to 1d$ M.-Mora
$3d \to 2d$ von Kármán: M.-Pakzad

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Kirchhoff wide open
b) dynamics
Formal Hamiltonian approach by Marsden et al. (→ Šverák's lectures)

- Multiwell materials

$$W \sim \mathrm{dist}\left(\bigcup_{i=1}^{m} SO(n)U_i\right)$$

3d, 2 wells $SO(3) \cup SO(3)\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}$

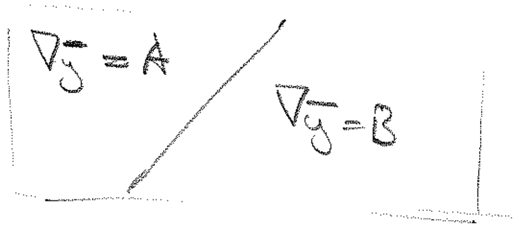
3d incompatible $\left(\sum_{i=1}^{3} (1-\lambda_i)\left(1-\frac{\lambda_1\lambda_2\lambda_3}{\lambda_i}\right) > 0\right.$

($\Longrightarrow$ no planar interface $\nabla u = A'$ / $\nabla u = B'$ exists)

2d compatible

$\lambda_1 < 1 < \lambda_2$ ($\rightsquigarrow$ 2 rank-1 connections in 2d)

$y^{(h)} \longrightarrow \bar{y}$



2d limit

(SIAM J. Math. Anal. 38 (2006), 468–477)

Chaudhuri–M.

$\liminf_{h\to 0} \frac{1}{h} I^h(y^{(h)}) > 0$

$\exists\ \tilde{y}^{(h)}$ $\limsup_{h\to 0}$ —— $< \infty$

Γ-limit open

partial results by P. Hornung.

- Error estimates for numerical analysis