Lecture 4 A hierarchy of plate models

G. Friesecke, R.D. James, S.M. ARMA 180 (2006), 183-236

$$I^h(y) = \int_\Omega W(\nabla_h y)\,dx$$

$$\nabla_h y = \left(y_{,1} \,\middle|\, y_{,2} \,\middle|\, \tfrac{1}{h} y_{,3}\right)$$

$$J^h(y) = I^h(y) - \int_\Omega f^{(h)} \cdot y$$

$$\int_\Omega f^{(h)} = 0.$$

Q1. If $I^h(y^{(h)}) \sim h^\beta$ what can be said about $y^{(h)}$?

Q2. If $f^{(h)} \sim h^\alpha$ what can be said about (almost) minimizers of $y^{(h)}$?

First three lectures

$\alpha = 2 \Rightarrow \beta = 2 \Rightarrow$ Kirchhoff's plate theory.

$\beta = 2$ borderline case

$\beta < 2 \rightsquigarrow$ non-compactness by wrinkling

$\beta = 2 \rightsquigarrow y^{(h)} \longrightarrow \bar{y}$ isometric immersion

$\beta > 2 \rightsquigarrow y^{(h)} \longrightarrow$ affine isometry.

4/2

<u>Today</u> $\begin{cases} \beta = 0 & (\alpha = 0) \\ \beta = 4. & (\text{which } \alpha \text{ ?}). \end{cases}$

<u>Theorem 6</u> (LeDret – Raoult, $\beta=0$) Suppose

$$c(|F|^2 - 1) \le W(F) \le C(|F|^2 + 1) \qquad (*)$$

Then

(i) If $I^h(y^{(h)}) \le C$, then

$$y^{(h)} \rightharpoonup \bar{y} \quad W^{1,2} \ (\underline{\text{weakly}}), \quad \bar{y}_{,3} = 0$$

$$\liminf I^h(y^{(h)}) \ge \int_S W_{mem}(\nabla' \bar{y})\, dx' = I_{mem}(\bar{y})$$

(ii) $\forall \bar{y} \in W^{1,2}(S;\mathbb{R}^3) \ \exists\, y^{(h)} \to \bar{y} \ W^{1,2}$.

$$\lim I^h(y^{(h)}) \to I_{mem}(\bar{y})$$

<u>Corollary 7.</u> ($\alpha=0$) Suppose $(*)$ and

$$f^{(h)} \rightharpoonup f \quad \text{in } L^2.$$

Then

Almost min. of I^h $\longrightarrow$ minimizer of I_{mem}.

What is W_{mem} ? (Think!)

$G \in \mathbb{R}^{3\times 2}$

$$W_2(G) = \min_{a \in \mathbb{R}^3} W(G|a)$$

$$W_{\text{mem}} = W_2^{\boxed{qc}}$$ quasiconvexification

If $W(\text{Id}) = \min W = 0$ + frame indifference.

$\Rightarrow$ $\underline{W}_{\text{me}}(G) = 0$ $\forall G$ with $G^TG \leq \text{Id}$.

⇝ <u>No resistance to compression.</u>

⇝ Tension-field theory (..., Pipkin).

$\beta = 4$ Very formal expansion → von Kármán plate theory

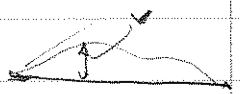
→ Slides

Ciarlet & Destuynder '80 : Formal asymptotics.

$\gtrsim$ 500 papers on vK eqn^s.

<u>Idea behind vK:</u> Linearize Kirchhoff <u>ansatz</u>

$$y^{(h)} = \bar{y}(x') + h x_3 \nu(x'_{\cancel{3}}), \quad \nu = \bar{y}_{,1} \wedge \bar{y}_{,2}$$



$$\bar{y} = \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} h^2 u \\ h v(x') \end{pmatrix}, \quad y_{,1} = \begin{pmatrix} 1 + h^2 u_{,1} \\ 0 \\ h v_{,1} \end{pmatrix}$$

$$\nu = \bar{y}_{,1} \wedge \bar{y}_{,2} \approx e_3 - h \begin{pmatrix} v_{,1} \\ v_{,2} \\ 0 \end{pmatrix}$$

$$A_{ij} = -y_{,ij} \cdot \nu \approx h v_{,ij} \quad (i,j \in \{1,2\})$$

$$\left[(\nabla_h y)^T \nabla_h y \right]_{ij} (x', 0) = \delta_{ij} + h^2 v_{,i} v_{,j} + 2h^2 u_{i,j}$$

$$\left((\nabla_h y)^T \nabla_h y \right)_{ij} (x, \cancel{x}_3) = h^2 v_{,i} v_{,j} \overset{2h^2 u_{i,j}}{+} 2h x_3 A_{ij}$$

$$= h^2 \left(\nabla v \otimes \nabla v + 2 x_3 \nabla^2 v \right)_{ij}$$

$$\frac{1}{h^4}\int W(\nabla_h y) \approx \frac{1}{2}\int Q_2\left(\mathrm{sym}\nabla' u + \frac{\nabla' v \otimes \nabla' v}{2}\right)$$

$$+ \frac{1}{24}\int Q_2\left(\nabla'^2 v\right).$$

$$= I_{vK}(u,v)$$

Now add u

// Remarks Q_2 quadratic → linear constitutive relation }
I_{vK} not quadratic → geometric nonlinearity }
Q_2 instead of Q_3 only certain components of the strain enter

Theorem 8. Spse $I^h(y^{(h)}) \leq Ch^4$.

Then

(i) (crack-free lower bd.)
∃ constants $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$

s.t.

(1) $$\tilde{y}^{(h)} := \left(\bar{R}^{(h)}\right)^T y^{(h)} - c^{(h)} \to \bar{y} = \begin{pmatrix} x' \\ 0 \end{pmatrix} \text{ in } W^{1,2}$$

(2) $$v^{(h)} := \frac{1}{h}\int_{-1/2}^{1/2} \tilde{y}_3^{(h)}\, dx_3 \to v \quad W^{1,2}, \ v \in W^{2,2}$$

(3) $$u^{(h)} = \frac{1}{h^2}\int \begin{pmatrix} \tilde{y}_1^{(h)} - x_1 \\ \tilde{y}_2^{(h)} - x_2 \end{pmatrix} dx_3 \to u \quad W^{1,2}$$

$$\liminf \frac{1}{h^4} I^h(y^{(h)}) \geq I_{vK}(u,v)$$

(ii) upper bound. Given $u \in W^{1,2}(S;\mathbb{R}^2)$, $v \in W^{2,2}(S)$
∃ $\tilde{y}^{(h)}$ s.t. (1) − (3) hold an

$$\lim \frac{1}{h^4} I^h(y^{(h)}) = I_{vK}(u,v).$$

<u>GF, RDJ, SM</u> CRAS 335 (2002), 201-206

R. Monneau ARMA 165 (2003), 1-84.

<u>Ingredients of proof.</u>

① (Scaled) rigidity estimate

② Convergence of $v^{(h)}$ and $\underline{\mathrm{sym}\nabla u^{(h)}}$

③ Identification of limiting strain G

$$G^{(h)} = \frac{(R^{(h)})^T \nabla_h y^{(h)} - Id}{h^2} \to G$$

① <u>Theorem</u> Let

$$E_h = \int_\Omega \mathrm{dist}^2(\nabla_h y^{(h)}, SO(3))$$

and suppose that

$$h^{-2} E_h \to 0 \qquad (*)$$

Then $\exists \; R^{(h)} : S' \to SO(3)$ s.t.

$$\| \nabla_h y^{(h)} - R^{(h)} \|_{L^2}^2 \le C E_h, \qquad (4)$$

$$\| \nabla R^{(h)} \|_{L^2}^2 \le C h^{-2} E_h. \qquad (5)$$

<u>Proof.</u> "As before". Turn difference quotient estimate into ∇-estimate by convolution (and projection back to $SO(3)$; for this one uses $(*)$ and $n=3$). Some care needed near $\partial\Omega$.

② **Lemma 10** Assumptions + notation as in Thm. Then $\exists\, R^{(h)} : S \to SO(3)$

$$\|\nabla_h y^{(h)} - R^{(h)}\|_{L^2} \leq C h^2, \qquad (6)$$

$$\|R^{(h)} - Id\|_{W^{1,2}} \leq C h \qquad (7)$$

$$\frac{R^{(h)} - Id}{h} \rightharpoonup A \quad W^{1,2}, \quad A^T = -A \qquad (8)$$

$$2 \,\mathrm{sym}\, \frac{R^{(h)} - Id}{h^2} \to A^2 \quad L^p, \ \forall p < \infty \qquad (9)$$

$$v^{(h)} \to v \quad W^{1,2}, \quad v \in W^{2,2} \qquad (10)$$

$$u^{(h)} \rightharpoonup u \quad W^{1,2} \qquad (11)$$

$$A = \begin{pmatrix} 0 & 0 & -v_{,1} \\ 0 & 0 & -v_{,2} \\ v_{,1} & v_{,2} & 0 \end{pmatrix} \qquad (12)$$

Proof. Step 1

Thm. $\Rightarrow$ (6) and $\|\nabla R^{(h)}\| \leq C h$

Hence $\|R^{(h)} - \mathrm{const}^{(h)}\|_{W^{1,2}} \leq C h$,

Wlog const $= Id$ (by choice of $R^{(h)}$).

$$\longrightarrow L^p_{loc}.$$

<u>Step 2</u> $A^{(h)} = \frac{R^{(h)} - Id}{h}$, $A^{(h)} \xrightarrow[(7)]{} A$ $W^{1,2}$

$$Id = (Id + hA^{(h)})^T (I + hA^{(h)}) = Id + h((A^{(h)})^T + A^{(h)}) + h^2 (A^{(h)})^T A^{(h)})$$

$$2\,\mathrm{sym}\,A^{(h)} = A^{(h)} + (A^{(h)})^T = -h((A^{(h)})^T A^{(h)}) = h(A^{(h)})^2$$

$$\Longrightarrow \quad A + A^T = 0 \quad \text{a.e.} \quad (9)$$

<u>Step 3</u> (6) + (8)

$$\Rightarrow \frac{\nabla_h y^{(h)} - Id}{h} \longrightarrow A \quad L^2$$

$$v^{(h)}_{,i} = \frac{1}{h}\int y^{(h)}_{3,i} \longrightarrow A_{3i} \quad L^2. \quad (*)$$

<u>Step 4.</u> (6) + (9)

$$\left\| \mathrm{sym}\, \frac{\nabla_h y^{(h)} - Id}{h^2} \right\|_{L^2} \le C.$$

$$\Longrightarrow \| \mathrm{sym}\, \nabla' u^{(h)} \|_{L^2} \le C.$$

<u>Korn</u> (+ normalization) $\Longrightarrow$ (11).

<u>Step 5</u> $(*) \Rightarrow A_{3i} = v_i$, $(11) \Rightarrow A_{12} = 0$.

③ 1 Set $G^{(e)} = \dfrac{(R^{(e)})^T \overline{\nabla_h \tilde y^{(e)}} - Id}{h^2}$

By (6) $G^{(h)} \to G \quad L^2.$

Lemma 11 The 2x2 submatrix G'' satisfies

$$G''(x', x_3) = G_0(x') + x_3 G_1(x')$$

where

$$\operatorname{sym} G_0 = \operatorname{sym} \nabla u + \frac{1}{2} \nabla v \otimes \nabla v$$

$$G_1 = -(\nabla')^2 v.$$

Lemma 11 $\Rightarrow$ Thm. 8 (i) : Careful Taylor expansion (as before).

Rem ([illegible]) Lemma 11 proves that strain varies linear in x_3.

Proof of Lemma : $x_3 G_1 \rightsquigarrow$ as before

$$H^{(e)}(x', x_3) = G^{(e)}(x', x_3 + s) - G^{(e)}(x')$$

$$\left(R^{(e)} H^{(e)}\right)_{ij} = \left(\frac{1}{\,} \int_{x_3}^{x_3+s} \frac{1}{h^2} \tilde y^{(e)}_{i,3} \right)_{,j}$$

$$\begin{tikzcd} R \arrow[d] & \overline{\left(\frac{1}{h} \nabla_h \tilde y^{(e)}\right)}_{i3} \arrow[d, "L^3"] \\ Id & A_{i3} \end{tikzcd}$$

$$H_{ij} = s A_{i3,j} = -s v_{,ij}.$$

For G_0 study

$$(R^{(e)})^T \int_{-1/2}^{1/2} G^{(e)}_{ij}\, dx_3 = \int_{-1/2}^{1/2} \left(\frac{\nabla_h y^{(e)} - R^{(e)}}{h^2} \right)_{ij}$$

$$= (\nabla' u)_{ij} + \left(\frac{R^{(e)} - Id}{h^e} \right)_{ij}$$

Now take sym and let $h \to 0$.

☒

<u>Upper bound</u> Use ansatz + compare for

Q_2 vs. Q_3.