Lecture 3 Kirchhoff's theory - conclusion

3d

$$y : \Omega = S \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbb{R}^3$$

$$I^h(y) = \frac{1}{h} \int_\Omega W(\nabla_h y), \quad \nabla_h y = \left(y_{,1} \,|\, y_{,2} \,|\, \tfrac{1}{h} y_{,3}\right)$$

2d

$$\bar{y} : S \to \mathbb{R}^3$$

$$\mathcal{A} = \left\{ \bar{y} \in W^{2,2}(S, \mathbb{R}^3) : \bar{y}_{,i} \cdot \bar{y}_{,j} = \delta_{ij} \right\}$$

$$\nu = \bar{y}_{,1} \wedge \bar{y}_{,2}$$

$$A_{ij} = -\bar{y}_{,ij} \cdot \nu = \bar{y}_{,i} \cdot \nu_{,j}$$

$$I^{Ki}(\bar{y}) = \begin{cases} \dfrac{1}{24} \displaystyle\int_S Q_2(A) & \text{if } \bar{y} \in \mathcal{A}, \\ +\infty & \text{else} \end{cases}$$

$$Q_3(H) = \frac{\partial^2 W}{\partial F^2}(\mathrm{Id})(H, H) \qquad H \in \mathbb{R}^{3\times 3}$$

$$Q_2(A) = \min_{a \in \mathbb{R}^3} Q_3\left(A + a \otimes e_3 + e_3 \otimes a\right)$$

$$A \in \mathbb{R}^{2\times 2}$$

Theorem 2

(i) If $\limsup_{h\to 0} \frac{1}{h^2} I^{(h)}(y^{(h)}) \le C_1$ then

$$y^{(h)} \to \bar{y} \text{ in } W^{1,2},\ \bar{y}_{,3} = 0,\ \bar{y} \in \mathcal{A}, \quad (1)$$

$$\liminf_{h\to 0} I^{(h)}(y^{(h)}) \ge \bar{I}_{Ki}(\bar{y}). \quad (2)$$

(ii) $\exists\, \bar{y} \in \mathcal{A}$, $\exists\, y^{(h)} \to \bar{y}$ $W^{1,2}$

$$\frac{1}{h^2} I^{(h)}(y^{(h)}) \to \bar{I}(\bar{y}). \quad (3)$$

Pf. (1) + (3) ✓

Need to prove (2).

curvature

Idea: $\nabla_h y^{(h)}(x) \simeq \text{Rotation}(x')\left(Id + h x_3 A(x')\right)$

Last time: $\exists\, R^{(h)} : S \to SO(3)$ piecewise constant

$$\| \nabla_h y^{(h)} - R^{(h)} \|_{L^2} \le C h^2, \quad (4)$$

$$R^{(h)} \to \bar{R} \quad L^2, \quad \bar{R} \in W^{1,2},$$

$$\nabla_h y^{(h)} \to \bar{R} \quad L^2.$$

Set
$$G^{(h)}(x) = \frac{\left(R^{(h)}(x')\right)^T \nabla_h y^{(h)} - Id}{h}$$ 'deviation from rotation'

$$\underset{(4)}{\Longrightarrow} \quad G^{(h)} \rightharpoonup G \quad \text{in } L^2$$

Claim
$$\ell = \liminf_{h \to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)}) \geq \frac{1}{2} \int_\Omega Q_3(G)$$

Proof. Idea: Taylor expansion. But:
$h G^{(h)}$ is not uniformly small (only in L^2)

$$E_h = \left\{ x : |G^{(h)}| \leq \frac{1}{\sqrt{h}} \right\}$$

$$\chi_h = \begin{cases} 1 & \text{in } E_h, \\ 0 & \text{in } \Omega \setminus E_h \end{cases}$$

Then
$$\chi_h \to \chi \quad \text{boundedly a.e}$$
$$\Rightarrow \chi_h G_h \rightharpoonup G \quad L^2$$

$$\ell \geq \liminf \frac{1}{h^2} \int_\Omega \chi_h W(\cancel{R^{(h)}}(Id + h G^{(h)}))$$

$$\geq \liminf \int_\Omega \frac{1}{2} \chi_h Q_3(G^{(h)}) - \int_\Omega \omega(\sqrt{h}) |G^{(h)}|^2$$

$$\geq \frac{1}{2} \int_\Omega Q_3(G),$$

since Q_3 is quadratic, ≥ 0, hence convex.

Let

$G'' = 2\times 2$ submatrix of G

$$G = \begin{pmatrix} G'' & \vdots \\ \cdots & \cdot \end{pmatrix}$$

$$Q_3(G) = Q_3(\operatorname{sym} G) \geq Q_2(\operatorname{sym} G'').$$

<u>Proposition 4</u> (Identification of limiting strain).

$$\operatorname{sym} G''(x) = \Pi_0(x') + x_3 A(x')$$

<u>Prop. 4</u> $\Rightarrow$ $$\int_{-1/2}^{1/2} Q_2(\operatorname{sym} G''(x))\,dx_3$$

$$= Q_2(\Pi_0) + \underbrace{\int_{-1/2}^{1/2} x_3^2}_{=1/12}\; Q_2(A)$$

$\Rightarrow$ Thm 2.

<u>Proof of Prop. 4</u>. Formal argument $\quad i,j \in \{1,2\}$

$$\nabla_h y^{(h)} = R^{(h)}(x')\left(Id + hG^{(h)}\right)$$

$$y^{(h)}_{,j} = R^{(h)}(x')\left(Id + hG^{(h)}\right)e_j$$

compatibility

$$\frac{1}{h} y^{(h)}_{,j3} = R^{(h)}(x')\; G^{(h)}_{,3}\; e_j$$

$$\left(\frac{1}{h} y^{(h)}_{,3}\right)_{,j} = R^{(h)}\; G^{(h)}_{,3}\; e_j$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\nu_{,j} = (\bar{R}e_3)_{,j} = \bar{R}\; G_{,3}\, e_j$$

$$\underbrace{\bar{R}e_i \cdot \nu_{,j}}_{A_{ij}} = e_i \cdot \bar{R}^T \nu_{,j} = e_i \cdot G_{,3} e_j = G_{ij,3}$$

Rigorous proof: Use differences

$$H^{(h)}(x', x_3) = (\mathcal{G}^{(h)})''(x', x_3+s) - (\mathcal{G}^{(h)})''(x', x_3)$$

$$R^{(h)} H^{(h)} e_j = \frac{1}{h}\left[y^{(h)}_{,j}(x', x_3+s) - y^{(h)}_{,j}(x', x_3) \right]$$

$$= \Big[\underbrace{\int_{x_3}^{x_3+hs} \frac{1}{h} y^{(h)}_{,3}(\sigma)\, d\sigma}_{\to\ v(x')} \Big]_{,j}$$

$$\downarrow L^2 \qquad \downarrow L^2$$

$$\bar{R}\, H e_j = (s v)_{,j}$$

$$\Rightarrow \quad H_{ij} = s\, \overline{R e_i} \cdot v_{,j} = s A_{ij}$$

$\Rightarrow$ Prop. 4. $\boxed{}$

Convergence of minimizers

$$\mathcal{J}^{(h)}(y^{(h)}) = \int_\Omega W(\nabla_h y) - \int_\Omega f^{(h)} \cdot y^{(h)}$$

$f^{(h)} : S' \to \mathbb{R}^3$ body force

Suppose
$$\int_{S} f^{(h)} dx' = 0$$

$$\frac{f^{(h)}}{h^2} \rightharpoonup f \quad \text{in } L^2.$$

Set
$$J_{K_i}(\bar{y}) = I_{K_i}(\bar{y}) - \int_S f \cdot \bar{y}$$

Cor. 5
$$\inf \frac{1}{h^2} \mathcal{J}^{(h)} \to \min J_{K_i}$$

Moreover if $y^{(h)}$ is a minimizing sequence in the sense that
$$\frac{1}{h^2} \mathcal{J}^{(h)}(y^{(h)}) - \inf \frac{1}{h^2} \mathcal{J}^{(h)} \to 0$$

then for a subsequence
$$y^{(h)} \to \hat{y} \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3)$$
and $\hat{y}$ minimizes J_{K_i}.

Proof of Corollary 5

Assume for a moment

$$I^h(y^{(h)}) \le C h^2 \qquad (*)$$

Rest follows from Thm. 2 by 'general nonsense'
(" Γ-convergence implies convergence of minimizer")

$$y^{(h)} \to \hat{y} \quad \text{in } W^{1,2},\ \hat{y} \in \mathcal{A}.$$

$$\int_\Omega f^{(h)} \cdot y^{(h)} \to \int_S f \cdot \hat{y}$$

$$J_{K_i}(\hat{y}) \le \liminf_{h\to 0} \frac{1}{h^2} J^{(h)}(y^{(h)}) = \liminf \left(\frac{1}{h^2} \inf J^{(h)}\right)$$
$$= e$$

Consider any competitor $\bar{y} \in \mathcal{A}$

$\exists\, y^{(h)} \to \bar{y} \quad W^{1,2}$

$$J_{K_i}(\bar{y}) = \lim_{h\to 0} \frac{1}{h^2} J^{(h)}(\bar{y}^{(h)}) \ge e \ge J_{K_i}(\hat{y}).$$

Hence $\hat{y}$ is minimizing and $J_{K_i}(\hat{y}) = e$ (take $\bar{y} = \hat{y}$).

Proof of (*) $\quad \inf J^h \le J^h(\mathrm{id}) \le C h^2$
$\Rightarrow J^{(h)}(y^{(h)}) \le C h^2$.

$$I^h(y^{(h)}) \le J^h(y^{(h)}) + \int f^{(h)} \cdot (y^{(h)} - \mathrm{const}^{(h)})$$

$x \le 1 + x^2$

$$\le C h^2 + C h^2 \, \| y^{(h)} - c^{(h)} \|_{L^2}$$
$$\le C h^2 + C h^2 \, \| y^{(h)} - c^{(h)} \|_{L^2}^2$$
$$\| y^{(h)} - c^{(h)} \|_{L^2}^2 \le C_1 \| \nabla_h y^{(h)} \|^2 \le C_1 \left(I^h(y^{(h)}) + 1 \right)$$

What is special about scaling $\frac{1}{h^2} I^{(h)}$?

<u>Borderline case</u>

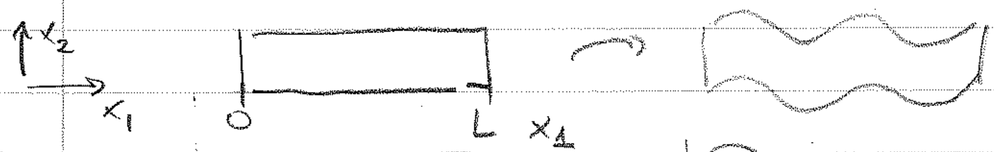
(1) No compactness if $\frac{1}{h^2} I^{(h)} \to \infty$

(2) If $\frac{1}{h^2} I^{(h)} \to 0$ then $y^{(h)} \to$ <u>affine map</u>

(this follows from Thm 2, $A = 0 \Rightarrow \bar{y}$ affine!)

Ad (1) Loss of compactness by <u>wrinkling</u>

<u>Euler-Bernoulli deformations</u> Consider 2d domain



$$y^{(h)}(x_1, x_2) = \int_0 \underbrace{\begin{pmatrix} \cos\Theta_h(s) \\ \sin\Theta_h(s) \end{pmatrix}}_{\tau_h} ds + h x_2 \underbrace{\begin{pmatrix} -\sin\Theta_h(x_1) \\ \cos\Theta_h(x_1) \end{pmatrix}}_{\nu_h}$$

$$\nabla_h y^{(h)} = \left(\tau_h + h x_2 \nu_h' \mid \nu_h \right)$$

$$\nu_h' \cdot \tau_h = -\Theta_h' = -\kappa$$

$$\left((\nabla_h y^{(h)})^T (\nabla_h y^{(h)}) \right) = \begin{pmatrix} 1 - 2h x_2 \kappa & 0 \\ 0 & 1 \end{pmatrix} + O(h^2)$$

$$\int W(\nabla_h y^{(h)}) \sim h^2 \int_0^L \kappa^2 \, dx_1 .$$

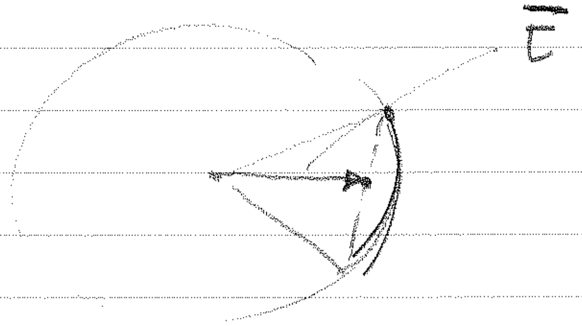
Now take
$$\Theta_h(x_1) = \Theta(h^{-\alpha} x_1).$$
$\uparrow$ 1-periodic

$$\int W(\nabla_h y^{(e)}) \sim C_1 h^{2-2\alpha}$$

On the other hand

$$y^{(e)} \longrightarrow x_1 \overline{\tau} = \overline{y}(x_1)$$

$$\overline{\tau} = \int_0^1 \begin{pmatrix} \cos \Theta(s) \\ \sin \Theta(s) \end{pmatrix} ds, \quad |\overline{\tau}| < 1$$



Moral. $y^{(e)}$ "almost isometry"

but $\overline{y}$ is too short.