Sharp-interface limits in variational models for solid-solid phase transitions

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Abstract

The analysis of phase transitions leads naturally to noncovex variational problems which, in general, do not have a minimizer; minimizing sequences typically develop very fine oscillations, which can be physically interpreted as mixtures between the several phases. Introducing a singular perturbation which penalizes the transition regions between the different phases one obtains existence, and a finite (but small) length scale for the oscillations. The analysis of the qualitative behavior of low-energy states requires understanding of an appropriate vanishing-regularization limit; the appropriate mathematical notion of convergence being Γ convergence.

In the case of fluid-fluid phase transitions, which was hystorically one of the first examples of Γ convergence and is tightly linked to the development of the abstract theory, Modica and Mortola have shown that one obtains a minimal-interface criterion, i.e., that asymptotically the problem is equivalent to minimizing the surface area between the phases.

In the setting of elasticity, as appropriate to solid-solid phase transitions, two main differences arise. Firstly, the nonconvex term in the energy depends on a gradient field, i.e., a differential constraint of the form "curl equal zero" must be satisfied. Secondly, due to invariance under superimposed rotations, the zero-set of the energy density is not discrete but continuous, i.e., instead of a finite set $\{A_1, \dots A_k\}$ of energy-minimizing pure phases one has a set of the form $SO(n)A_1 \cup \dots \cup SO(n)A_k \subset \mathbb{R}^{n\times n}$. For these reasons, the elastic problem has proved significantly harder than the one on fluids, and has been up to now understood only in two dimensions.

The course will begin with an introduction to the problem of solid-solid phase transitions, aiming at motivating the structural assumptions to be made later, and at providing concrete examples of physical relevance. Then I shall briefly review Γ convergence in the prototypical case of liquid-liquid phase transitions, and explain the main differences with the case at hand.

I shall then turn to the case of elasticity, focussing mainly on the two-dimensional case. A first problem I shall address is how to derive compactness, i.e., how to characterize the class of limiting deformations. It turns out that the gradient structure renders the problem very rigid, and only straight interfaces are possible in the limit. A second issue will be to extend the rigidity result from the limiting case to the case of finite (but small) ε, i.e., obtaining quantitative estimates on the distance of low-energy states from the possible limiting deformations. This derivation will clarify the significance of the restriction to two dimensions, and which difficulties are expected in higher dimensions. Finally, I will discuss how Γ-convergence to a sharp-interface model can be obtained, in the two-dimensional, two-well case.

Material will be mainly taken from the following references. The last three are available as preprints from

`http://analysis.math.uni-duisburg.de/`.

References

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