PEPITE E FILONI AURIFERI NELLA TEORIA DEL MATCHING

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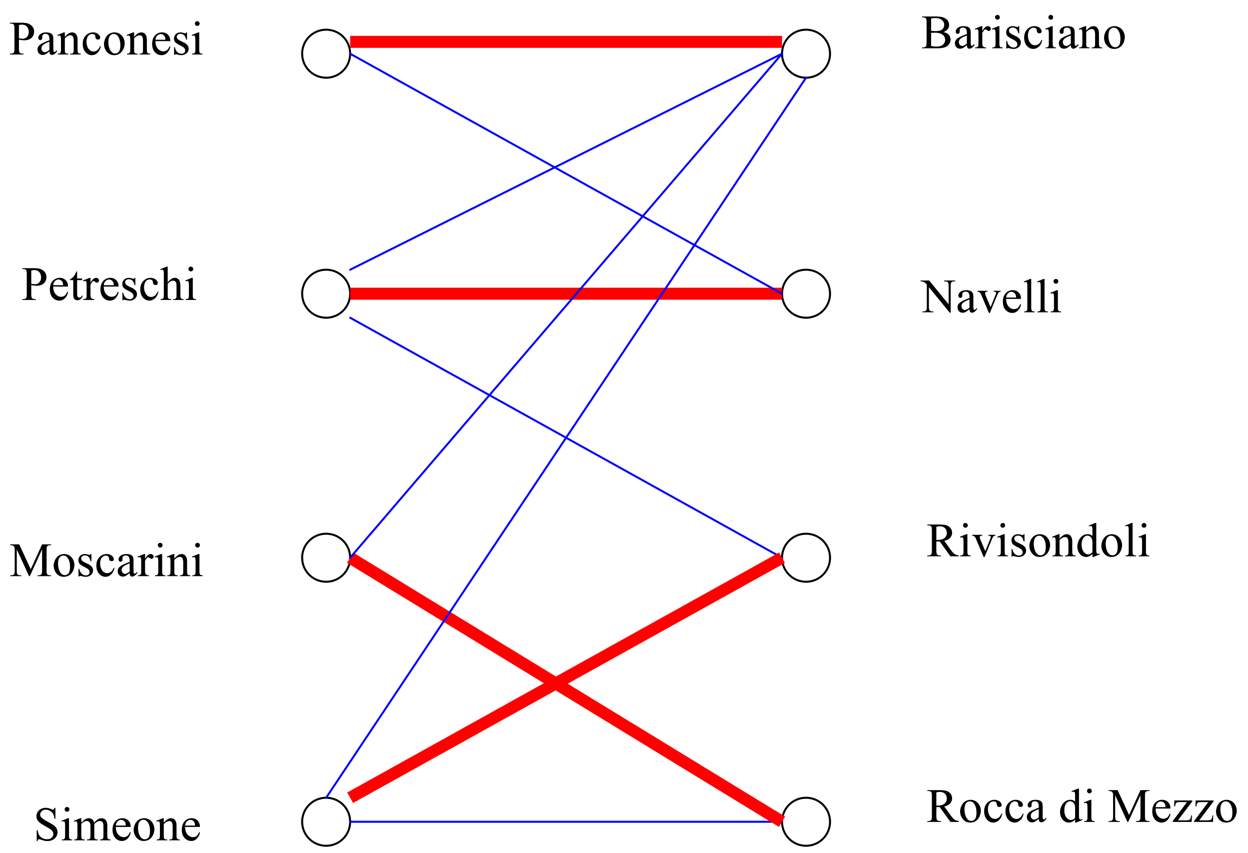
THE MAXIMUM MATCHING PROBLEM

Example 1: Postmen hiring in the province of L'Aquila

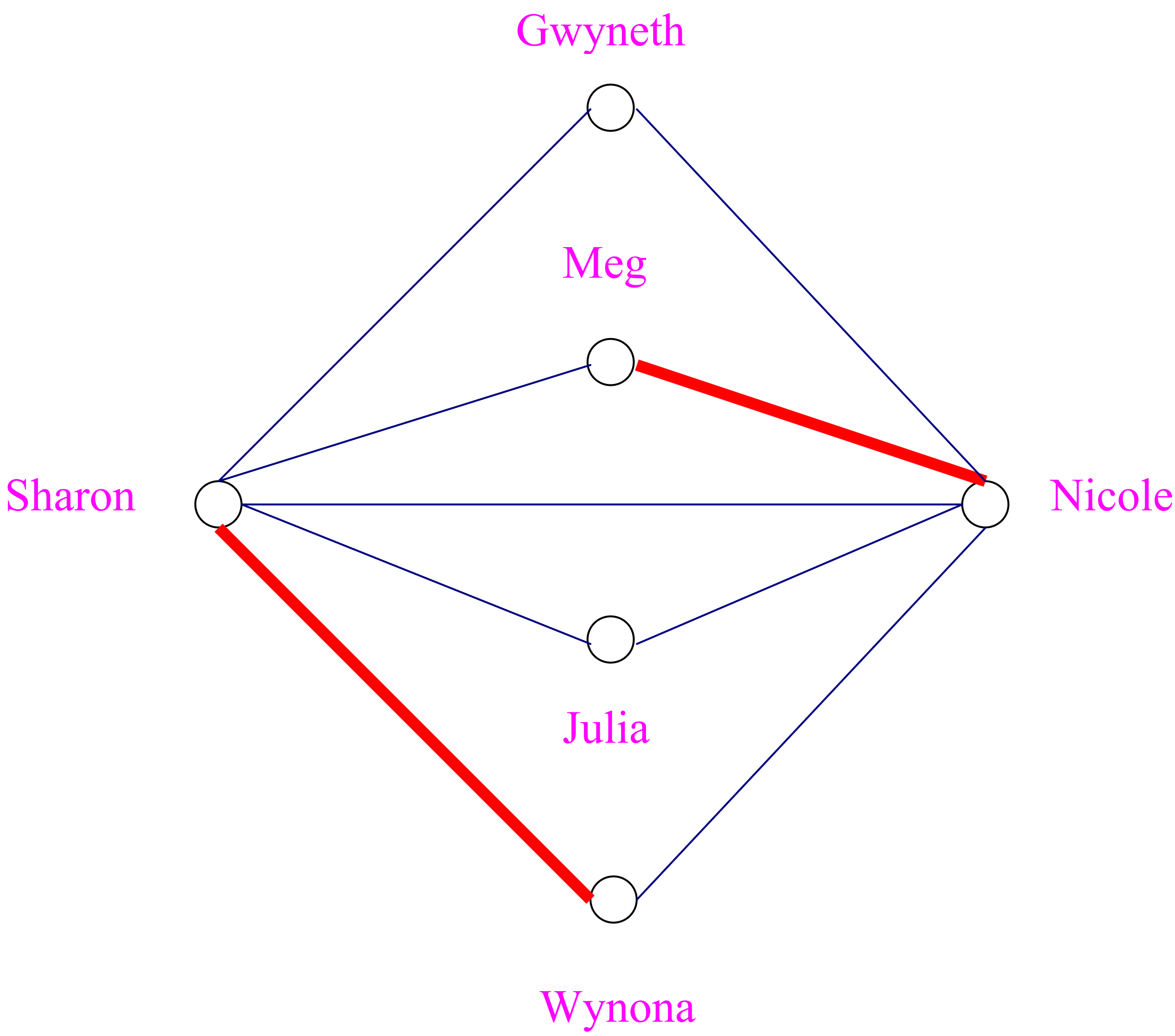


Can each winner be assigned to some place s/he likes?

More generally,

What is the maximum number of winners such that each of them can be assigned to some place s/he likes?

Example 2: Two-bed room assignment in a college



What is the maximum number of two-bed rooms that can be occupied by pairs of compatible girls?

DEF.: A matching in a graph is a set of pairwise nonincident edges

PROBLEM: Find a maximum (cardinality) matching in a graph

- Perfect matchings
- Weighted version

TALK OUTLINE

1. The maximum matching problem

2. Bipartite Matching: the MarriageTheorem and some equivalent theorems

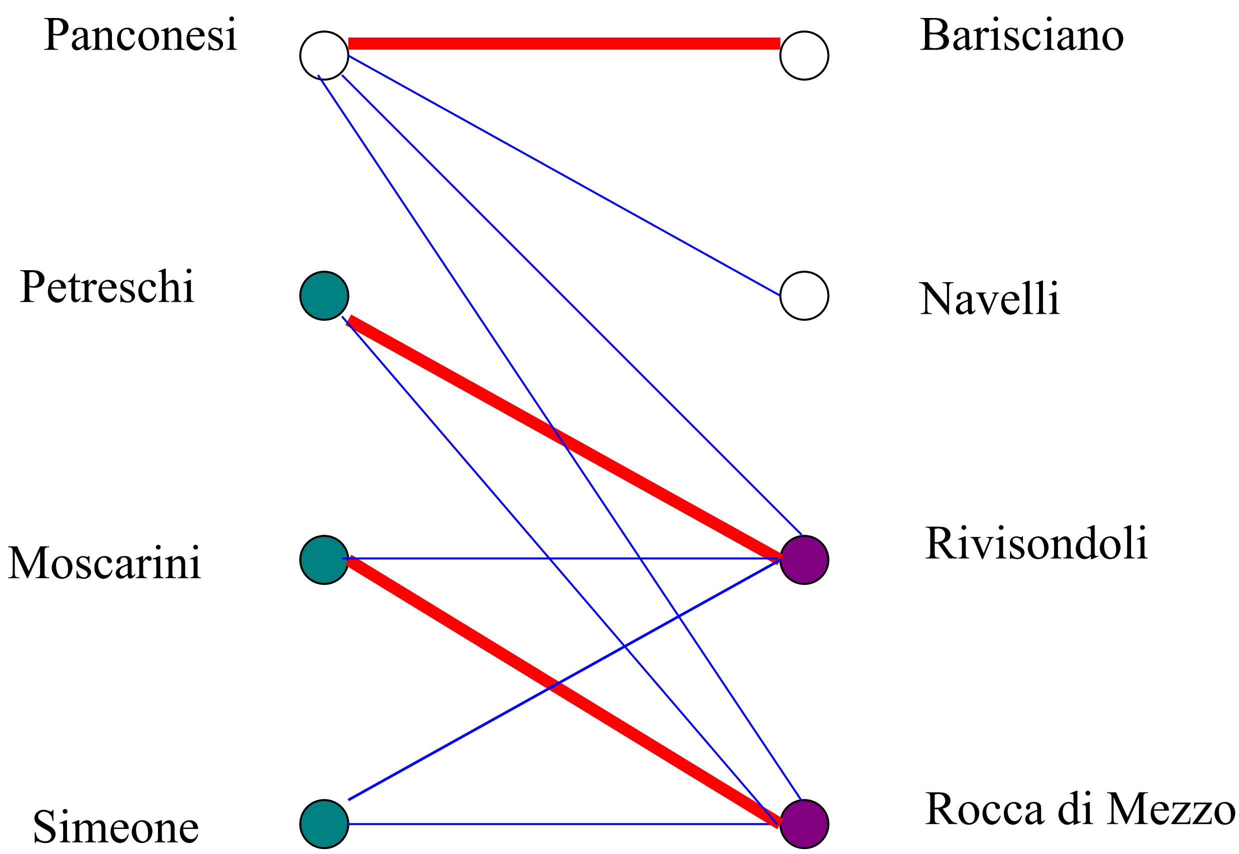
3. Three applications to other branches of mathematics

4. Nonbipartite Matching: The theorems of Tutte and Gallai-Edmonds

5. Matching, polyhedral geometry and linear programming

6. Efficient matching algorithms

THE MARRIAGE THEOREM



A maximum matching

$G = (V, E)$ bipartite graph with sides A and B

$N(S) = \{ y \in B : (x,y) \in E \text{ for some } x \in S\}$ $(S \subseteq A)$

THEOREM: (Frobenius 1917, P. Hall 1935)

G has a perfect matching iff

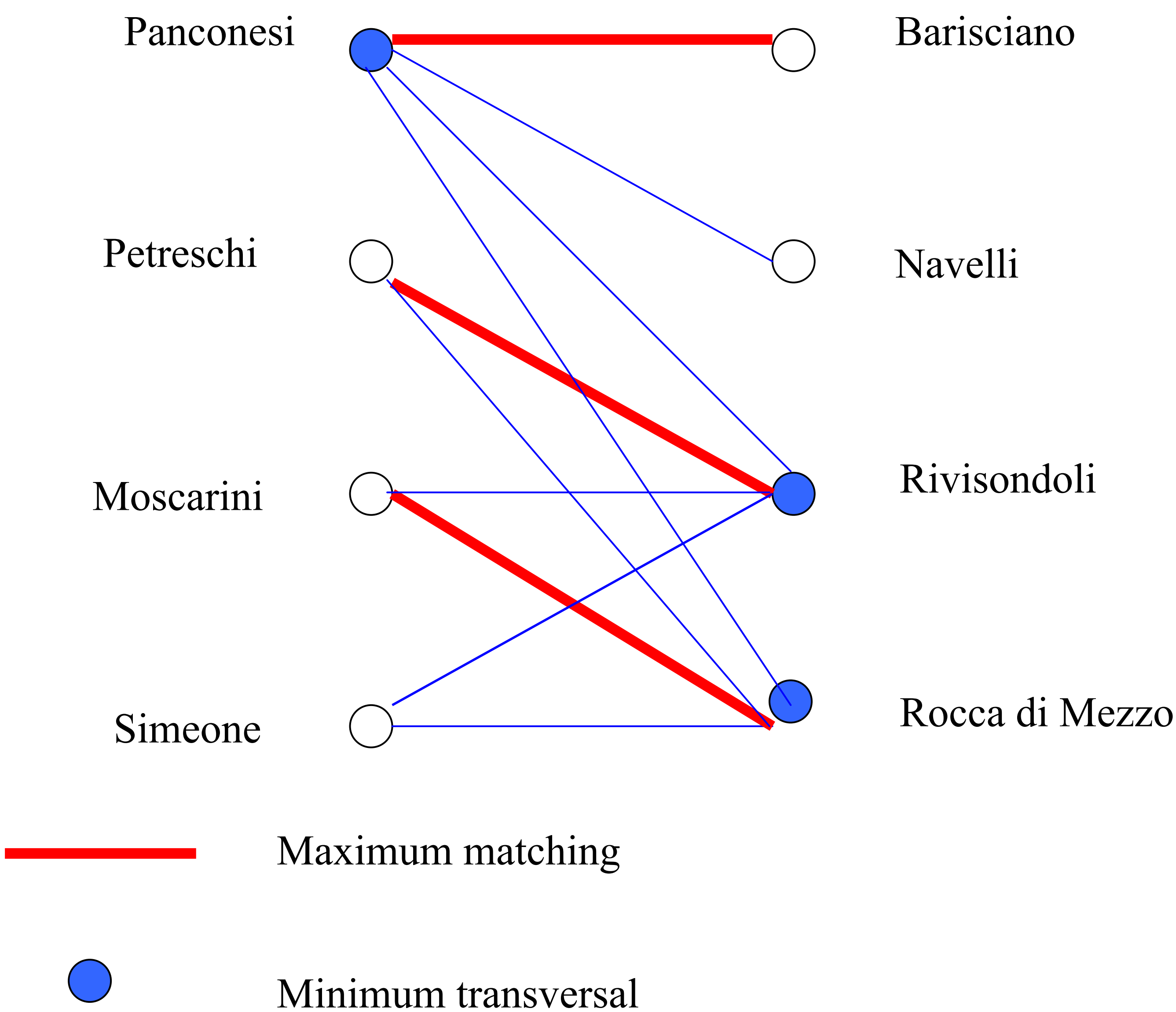
(i) $|A| = |B|$;

(ii) $|S| \leq |N(S)|$, $\forall S \subseteq A$

COROLLARY: Every regular bipartite graph has a perfect matching

KÖNIG–EGERVÁRY's THEOREM

DEF.: A transversal of a graph is a set of nodes that covers all the edges



THEOREM (König, 1931; Egerváry, 1931) :

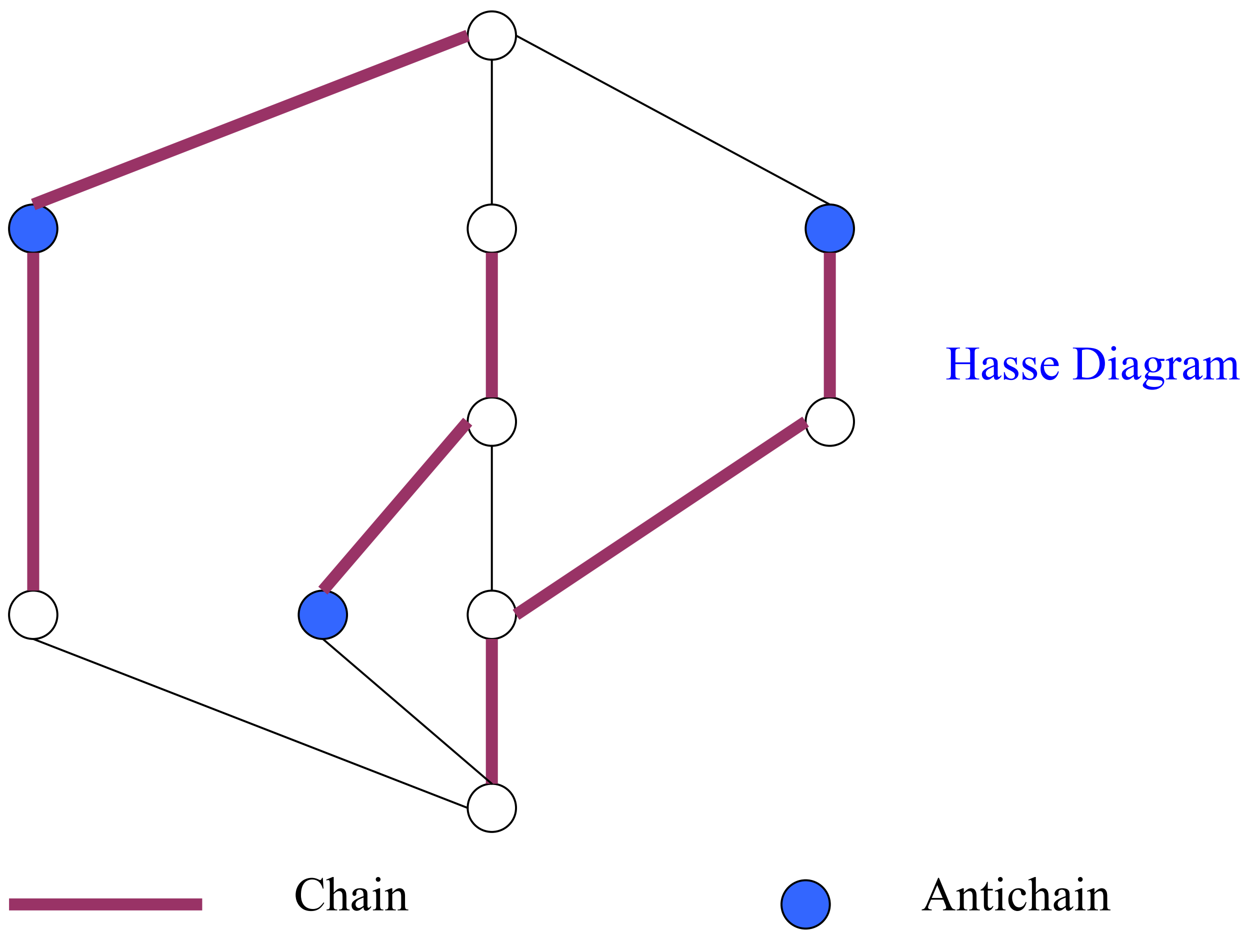
In any bipartite graph, the maximum cardinality of a matching is equal to the minimum cardinality of a transversal

DILWORTH's THEOREM

(P, ≤) finite poset

DEF.: A chain of P is any totally ordered subset of P.

DEF.: An antichain of P is any set of pairwise incomparable elements of P.



THEOREM: (Dilworth, 1950)

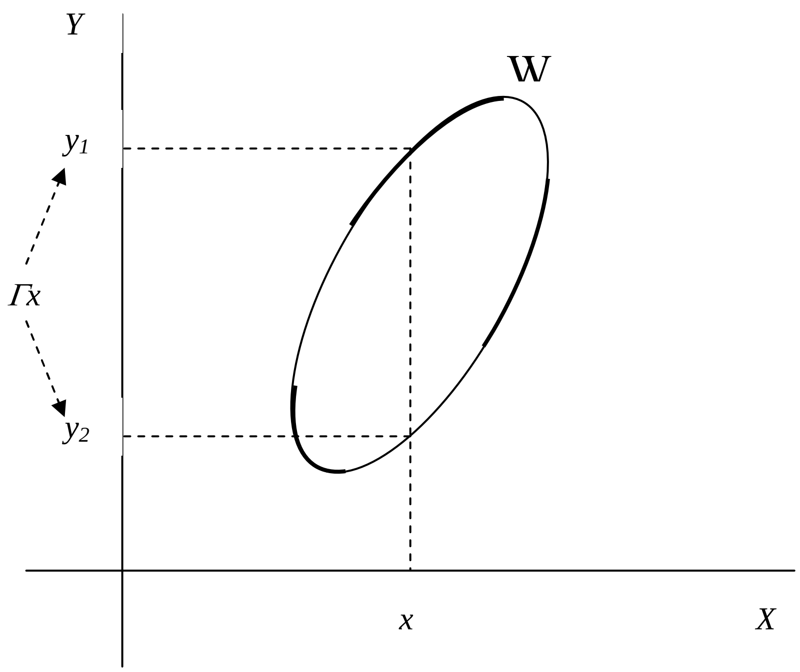
The minimum number of chains into which P can be partitioned is equal to the maximum cardinality of an antichain of P

A THEOREM ON CLOSED CURVES IN THE PLANE

C continuous closed curve (Jordan curve) in the plane

Assumption:
There exist an open interval X of the *x*-axis and an open interval Y of the *y*-axis such that:
(i) for each $a \in$ X, the vertical line $x = a$ intersects ***C*** in two points;
(ii) for each $b \in$ Y, the horizontal line $y = b$ intersects ***C*** in two points.

THEOREM: (Berge, 1962)
There is a ***C'*** $\subseteq$ ***C*** such that:
(i) for each $a \in$ X, the vertical line $x = a$ intersects ***C'*** exactly in one point;
(ii) for each $b \in$ Y, the horizontal line $y = b$ intersects ***C'*** exactly in one point.



EGÉRVÁRY-BIRKHOFF-VON NEUMANN's THEOREM

DEF.: A bistochastic matrix is a square real nonnegative matrix where the entries of each row and of each column add up to 1.

DEF.: A permutation matrix is a square binary matrix where each row and each column has exactly one entry equal to 1.
Obviously, any permutation matrix is bistochastic

$$\begin{bmatrix} 1/4 & 0 & \mathbf{3/4} \\ \mathbf{1/2} & 1/2 & 0 \\ 1/4 & \mathbf{1/2} & 1/4 \end{bmatrix} = 1/2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1/4 & 0 & \mathbf{1/4} \\ 0 & \mathbf{1/2} & 0 \\ \mathbf{1/4} & 0 & 1/4 \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{1/4} & 0 & 0 \\ 0 & \mathbf{1/4} & 0 \\ 0 & 0 & \mathbf{1/4} \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 1/4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THEOREM: (Egérváry 1931, Birkhoff 1946, Von Neumann 1953)

Every bistochastic matrix B is a convex combination of (a finite number of) permutation matrices,
i.e., there exist permutation matrices $P_1, \dots, P_r$ and nonnegative reals $\alpha_1, \dots, \alpha_r$, with $\alpha_1 + \dots + \alpha_r = 1$, such that

$$B = \alpha_1 P_1 + \dots + \alpha_r P_r .$$

HAAR MEASURES IN COMPACT TOPOLOGICAL GROUPS

Γ compact topological group

$\mathcal{C}(\Gamma)$ set of all continuous functions $f : \Gamma \to \mathbf{R}$ (= reals)

DEF.: An invariant integration is a functional $L : \mathcal{C}(\Gamma) \to \mathbf{R}$ having the following properties:

(a) $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ (linearity)

(b) $f \geq 0 \Rightarrow L(f) \geq 0$ (monotonicity)

(c) if $\mathbf{i}$ is the identity function, then $L(\mathbf{i}) = 1$ (normalization)

(d) if $s, t \in \Gamma$ and $f, g \in \mathcal{C}(\Gamma)$ are such that

$$g(x) = f(s\,x\,t), \qquad \forall\, x \in \Gamma,$$

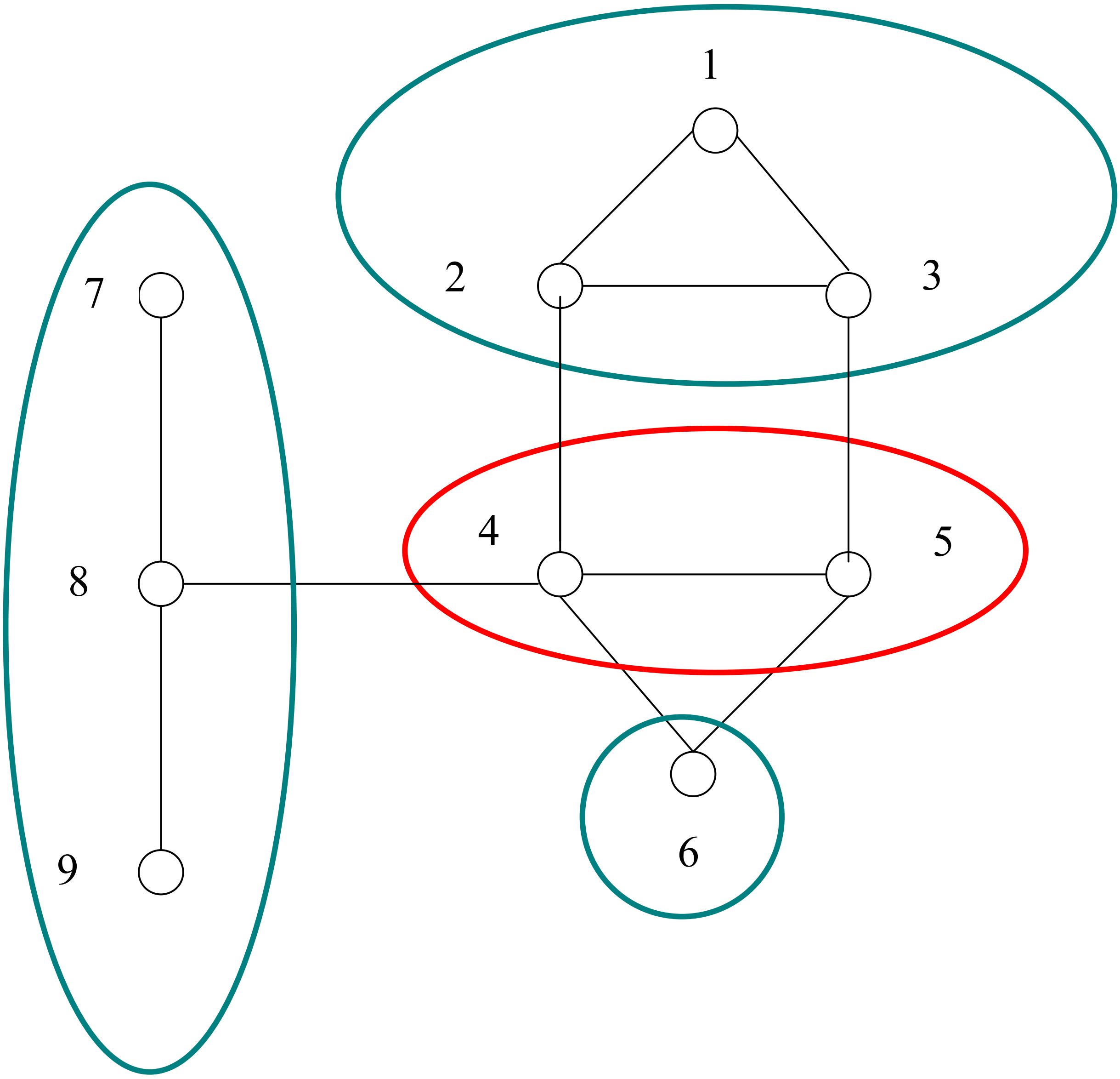
then $L(g) = L(f)$ (double translation invariance)

THEOREM: (von Neumann, 1934; Rota and Harper, 1971)

For every compact topological group Γ there exists an invariant integration

CONSEQUENCE:
Existence of a Haar measure on locally compact topological groups

TUTTE's THEOREM



G = (V, E) arbitrary graph

odd(G) = no. of odd components of G

THEOREM: (Tutte, 1947)
G has a perfect matching if and only if

$$\mathrm{odd}(G - S) \leq |S|, \quad \forall\ S \subseteq V$$

GALLAI-EDMONDS' STRUCTURE THEOREM

DEF.: A matching is near-perfect if exactly one vertex is left exposed

DEF.: A graph G is factor-critical if, for each $v \in V$, the graph $G - v$ has a perfect matching.

THEOREM: (Gallai, 1963, 1964; Edmonds, 1965)

Let

$G = (V, E)$ arbitrary graph

D = set of all vertices that are exposed in some maximum matching

$A = N(D)$; $C = V - (D \cup A)$;

M = any maximum matching of G.

Then:

(a) G(C) has a perfect matching;

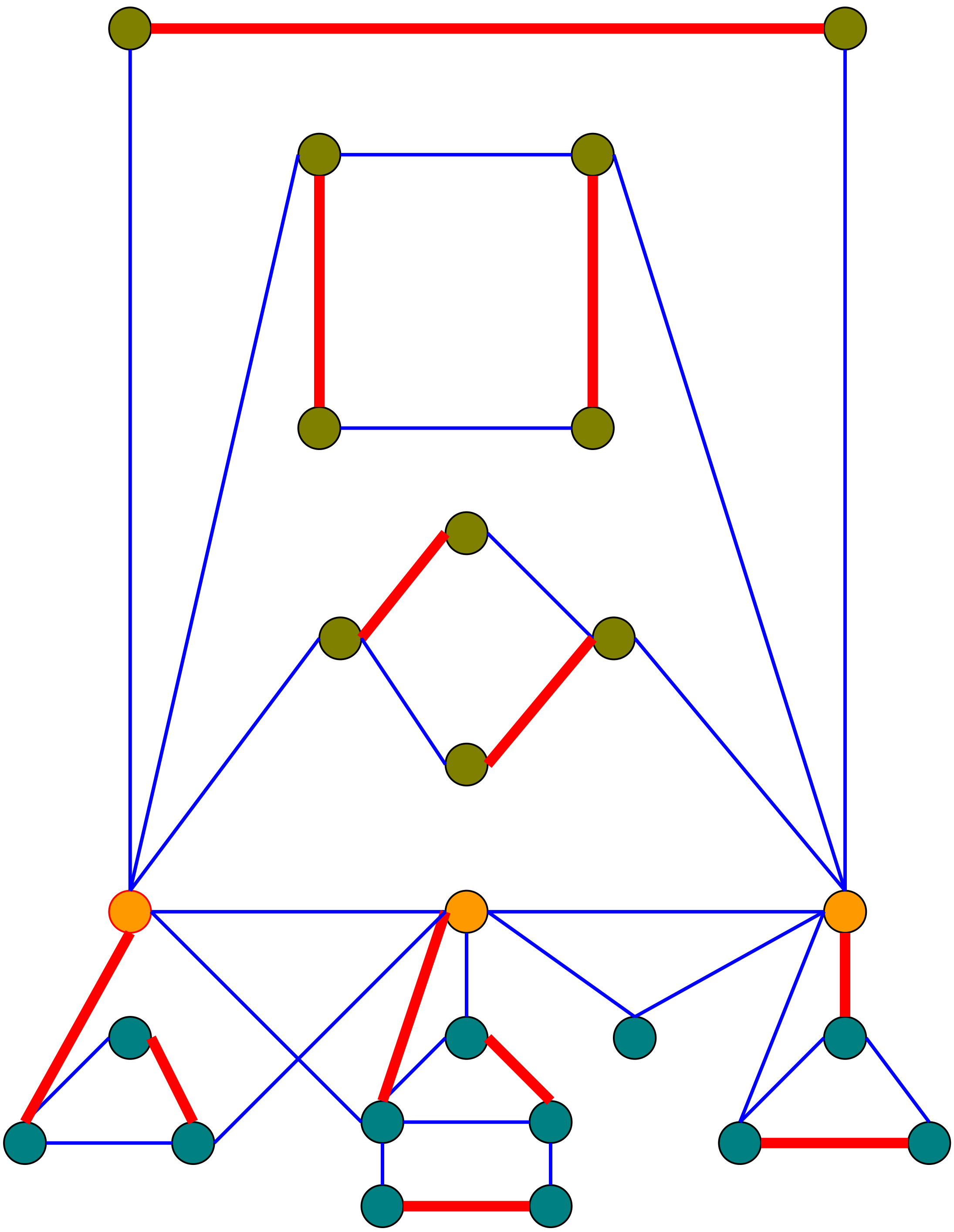
(b) all the components of G(D) are factor-critical, and M induces a near-perfect matching in each of them;

(c) each vertex in A is matched in M to some vertex in D, and no two vertices of A are matched to vertices lying in the same component of G(D).

THE GALLAI-EDMONDS SETS D, A, C : A GROUP PHOTO

any maximum matching

D A C



LINEAR PROGRAMMING

DEF.: Linear program (LP) : optimization (maximization or minimization) of a linear function of $n \geq 1$ real variables (objective function), subject to a finite system of linear inequalities or equations (constraints) on these variables

DEF.: Feasible solution: real n-vector satisfying all constraints

DEF.: Feasible region: set of all feasible solutions

DEF.: Optimal solution: feasible solution that optimizes the objective function over the feasible region

DEF.: Polyhedron: the set of all solutions to a finite system of linear inequalities. Polytope: bounded polyhedron

REMARK: The feasible region of any LP is a polyhedron. Hence an LP amounts to the optimization of a linear function over a polyhedron.

DEF.: Intermediate point of a polyhedron P: any $x \in P$ with the property that there exist $y, z \in P$, $y \neq z$, such that x is an interior point of the segment [y,z], i.e. , there is an $0 < \alpha < 1$ such that $x = \alpha\, y + (1 - \alpha)\, z$.

DEF.: Extreme point of P: any point of P that is not intermediate.

FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING:
If a linear function has a finite optimum in a polyhedron P, then among the optimal solutions there is always at least one extreme point.

MAXIMUM WEIGHT MATCHING: A BINARY LP FORMULATION

w_{ij} weight of edge (i,j)

$$x_{ij} = \begin{cases} 1, & (i,j) \text{ matching edge} \\ 0, & \text{else} \end{cases}$$

(M)

$$\max \sum_{(i,j)\in E} w_{ij} x_{ij}$$

s.t.

$$\sum_{j\in N(i)} x_{ij} \le 1, \quad i \in V$$

$$x_{ij} \in \{0,1\}, \qquad (i,j) \in E$$

(FM)

$$0 \le x_{ij} \ (\le 1), \qquad (i,j) \in E$$

(FM) is an ordinary linear program

INTEGRALITY AND HALF-INTEGRALITY PROPERTIES

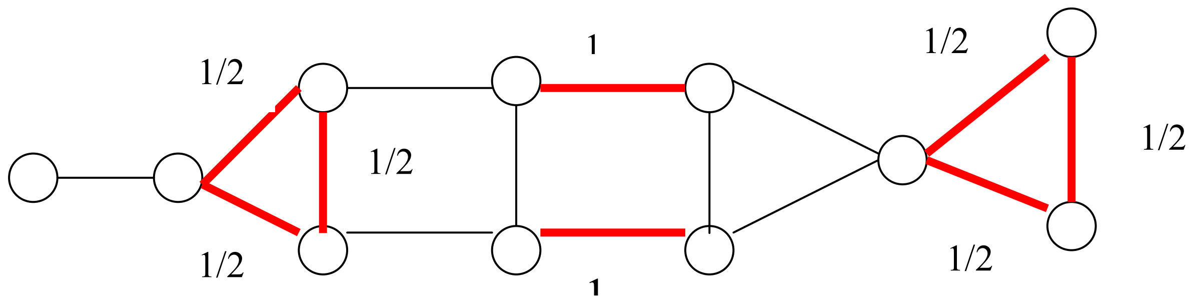
P(G) feasible polytope of (FM)

THEOREM: (Heller and Tompkins, 1956)
If G is bipartite, then every extreme point of P(G) is binary.

COROLLARY:
If G is bipartite, there exists some binary optimal solution to (FM). Such solution is clearly optimal also for (M).

DEF.: For an arbitrary graph G, a (basic) 2-matching of G is any collection of disjoint edges and odd cycles.

THEOREM: (Balinski, 1970)
If G is an arbitrary graph, then every extreme point x of P(G) is half-integral, i.e., its components are in {0, 1, ½ }.



COROLLARY:
If G is an arbitrary graph, then there exists some half-integral optimal solution to (FM).

PERSISTENCY THEOREM: (Balas, 1981)
In the unweighted case there are some half-integral optimal solution $\bar{x}$ to (FM) and some optimal solution x* to (M) such that:

$$\bar{x}_{ij} = 0 \; or \; 1 \Rightarrow x^*_{ij} = \bar{x}_{ij}$$

LINEAR FINITE OPTIMIZATION = LP

$S \subseteq R^n$ finite; $[S]$ convex hull of S: set of all convex combinations of the points in S

linear function $c(x) = c_1 x_1 + \ldots + c_n x_n$

Then one has:

$$\min \{ c(x) : x \in S \} = \min \{ c(x) : x \in [S] \}$$

(notice that the r.h.s. is an LP).

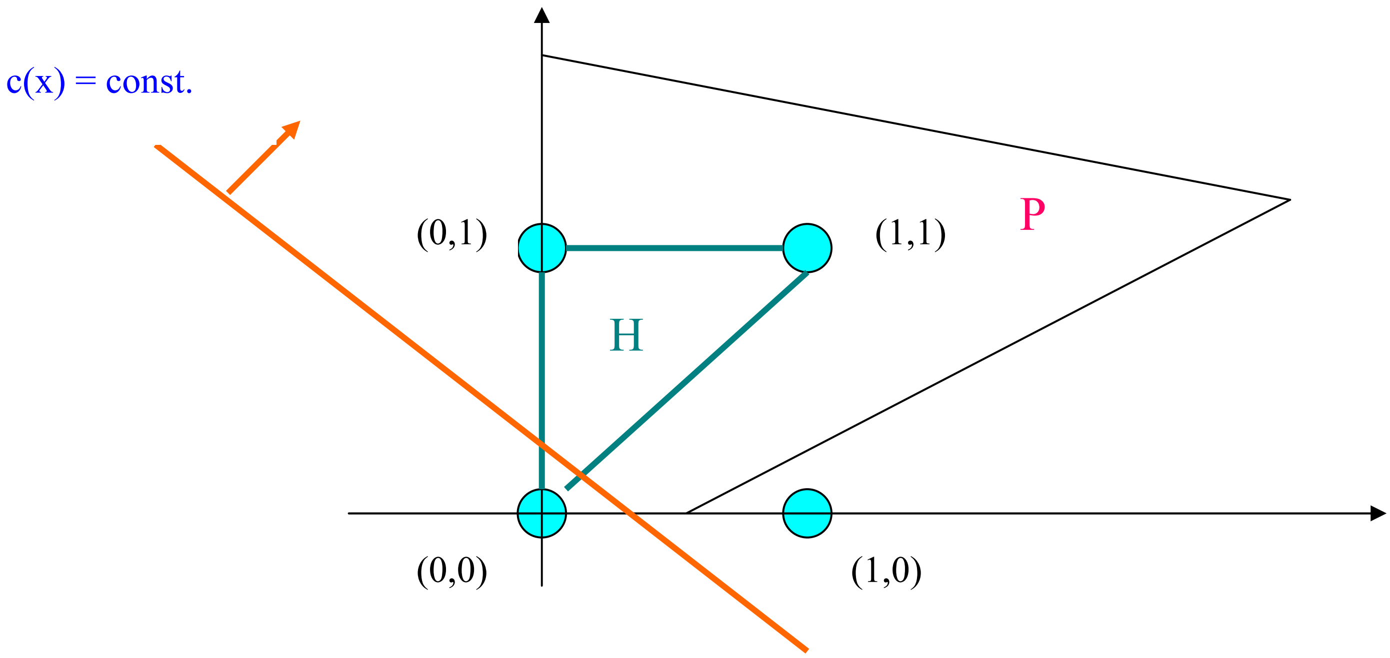
Proof:
Fundamental Theorem of Linear Programming and Ext $[S] \subseteq S$.

Many important combinatorial optimization problems (clique number, chromatic number, set covering, knapsack, travelling salesman, and so on) can be formulated as binary linear programs

$$\min \{ c(x) : x \in P \cap \mathbf{B}^n \},$$

(P polyhedron; $\mathbf{B} = \{ 0,1 \}$; $c(x)$ linear).

In view of the above, such binary LP can be formulated as the ordinary LP $\min \{ c(x) : x \in H \equiv [P \cap \mathbf{B}^n] \}$



THE MATCHING POLYTOPE

A fundamental question in polyhedral combinatorics is to give an explicit representation of the polytope H = $[P \cap \mathbf{B}^n]$ as the solution set of some finite system of linear inequalities

DEF.: Matching polytope: convex hull of all (binary) feasible solutions to (M)

Let $S \subseteq V$; E(S) = set of all edges having both their endpoints in S

THEOREM: (Edmonds, 1965)

The matching polytope is precisely the set of all real solutions x to the following system of linear inequalities:

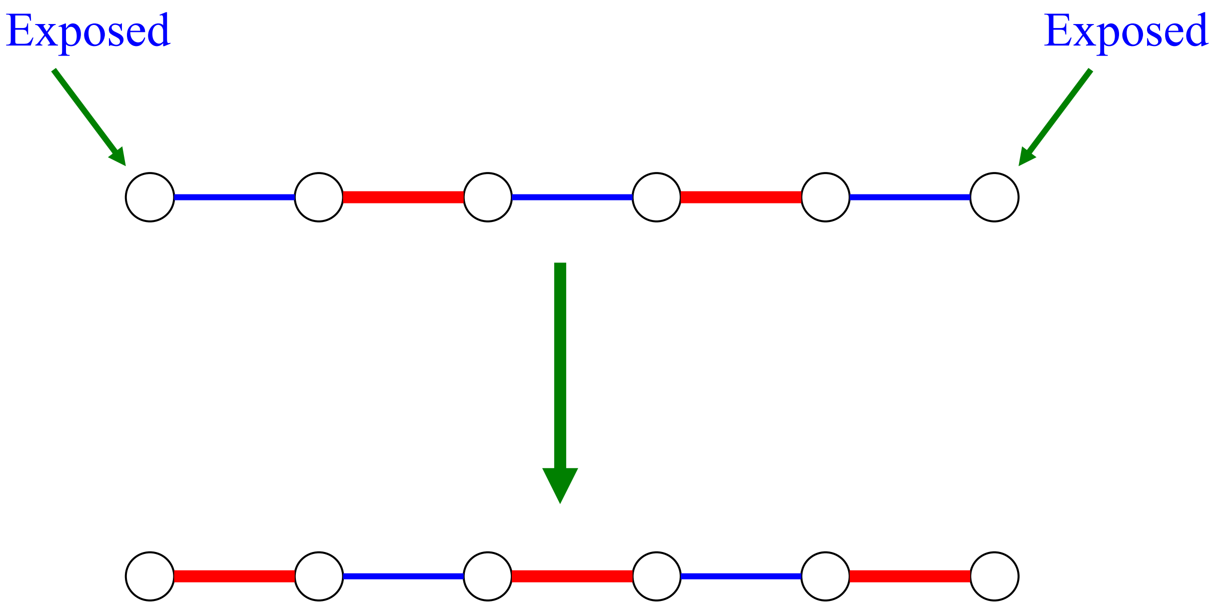
$$\sum_{j \in N(i)} x_{ij} \leq 1, \quad i \in V$$

$$\sum_{(i,j) \in E(S)} x_{ij} \leq \frac{|S|-1}{2}, \qquad \forall\, S \subseteq V, \quad 3 \leq |S| \quad \text{odd}$$

$$x_{ij} \geq 0, \qquad (i,j) \in E$$

MATCHING ALGORITHMS

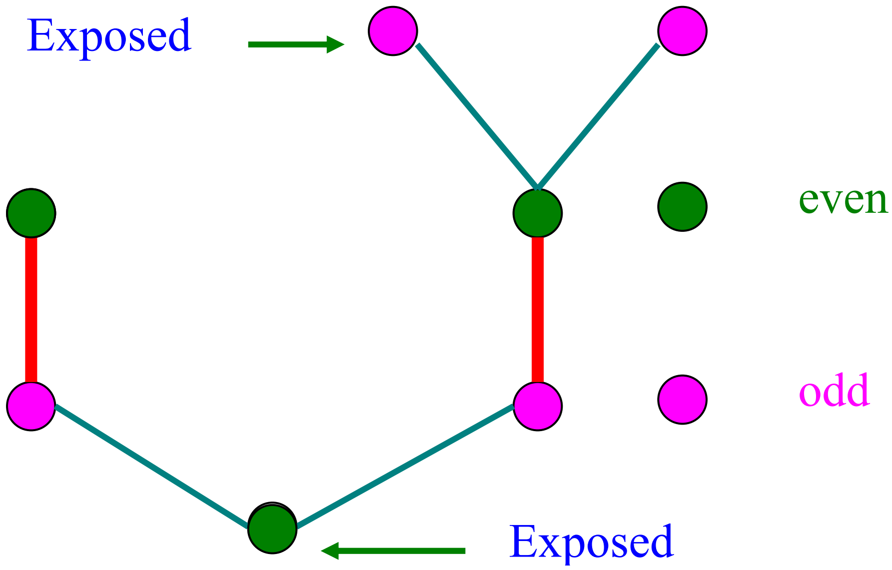
Augmenting path



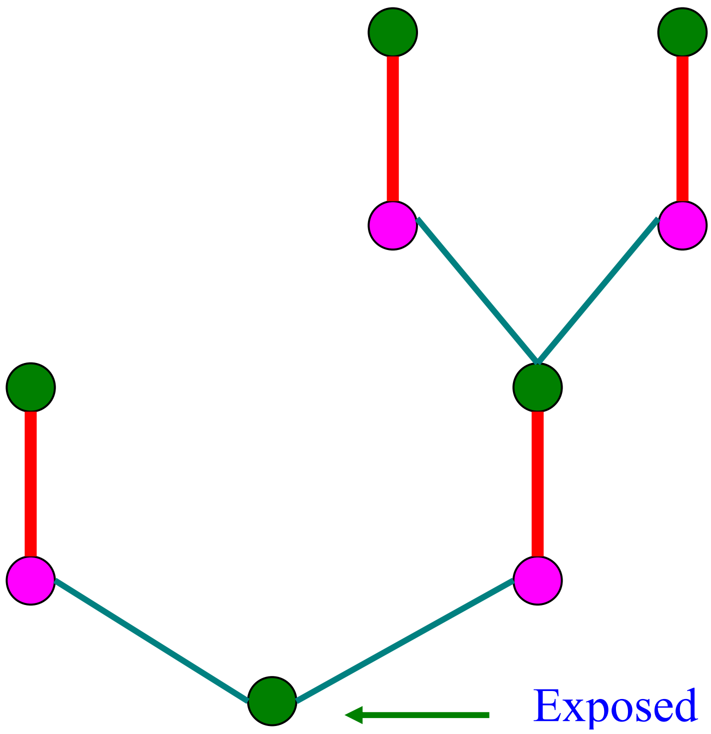
THEOREM: (Petersen,1891;Berge,1957;Norman and Rabin,1959)
A matching is maximum iff it has no augmenting path

BIPARTITE MATCHING

IDEA: Starting from an exposed vertex, grow an alternating tree



(a) Augmenting path found

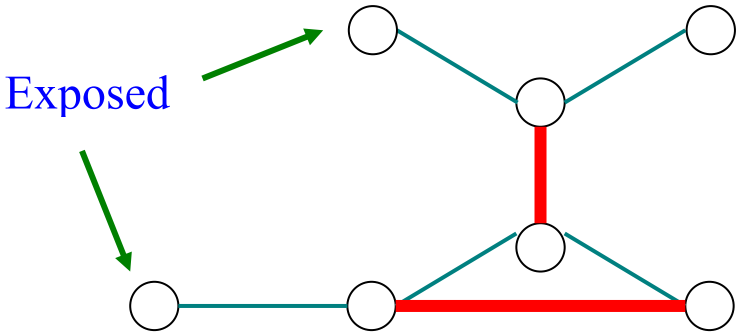


(b) Hungarian tree: no augmenting path from exposed vertex

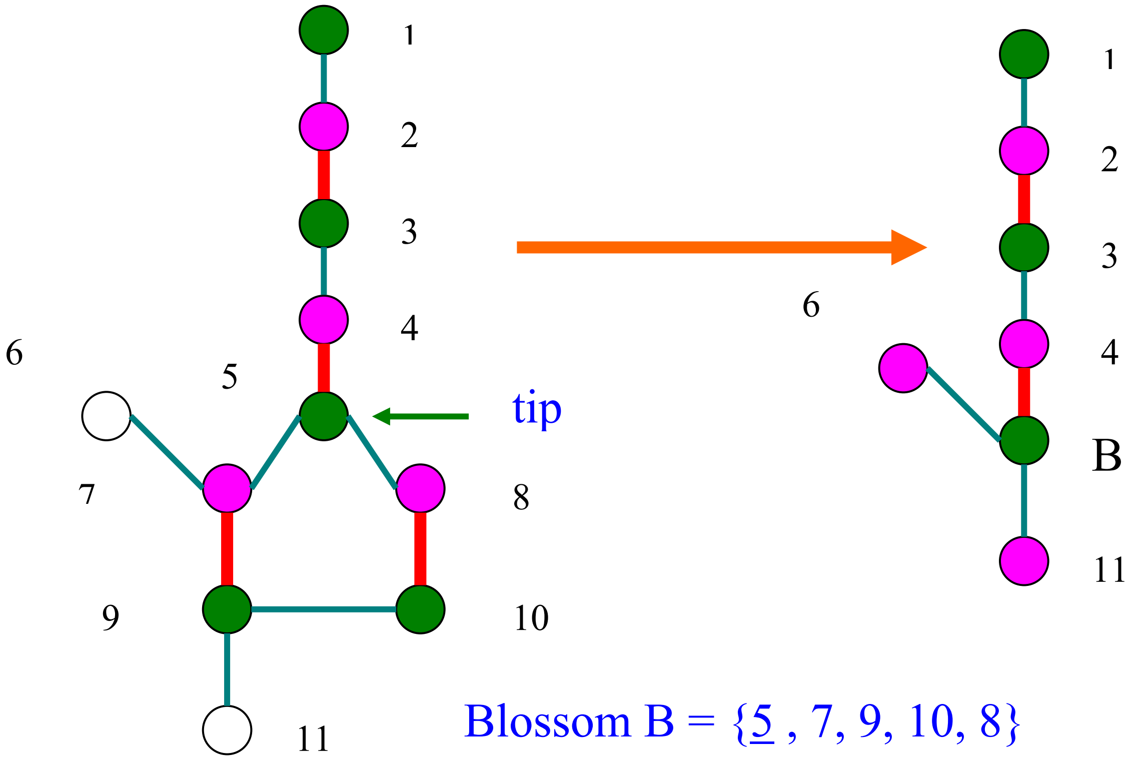
NONBIPARTITE MATCHING

EDMONDS' BLOSSOM ALGORITHM

PROBLEM: Odd cycles



SOLUTION: Blossom shrinking



Blossom B = {5 , 7, 9, 10, 8}

THEOREM: (Edmonds, 1965)
If G' is obtained from G by shrinking a blossom B, then G has an augmenting path iff G' does

- Basis for Edmonds' nonbipartite matching algorithm

BEST KNOWN MATCHING ALGORITHMS

$G = (V, E)$; $n = |V|$; $m = |E|$;

- MAXIMUM MATCHING

Bipartite:

$O(n^{1/2} m)$ (Hopcroft and Karp, 1973)
$O(n^{3/2} (m/\log n)^{1/2})$ (Alt, Blum, Mehlhorn, Paul, 1991)
$O(n^{1/2} (m + n) (\log (1 + n^2/m))/\log n)$ (Feder and Motwani, 1991)

Nonbipartite:

$O(n^{1/2} m)$ (Micali and Vazirani, 1980)

- MAXIMUM WEIGHT MATCHING

Bipartite:

$O(mn + n \log n)$ (Fredman and Tarjan, 1987)

Nonbipartite:

$O(mn + n^2 \log n)$ (Gabow, 1990)

A RETROSPECTIVE VIEW