Eq. del secondo ordine lineari a coef. costanti

I

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$$y'' + by' + cy = f(x).$$

1°) $y'' + by' + cy = 0 \quad \Longmapsto \quad y_0(x) = C_0 y_1(x) + C_1 y_2(x)$

RISOLVERE L'EQ. OMOGENEA, l'eq. caratteristica.

$\lambda^2 + b\lambda + c = 0$ . . . . $\to y_1$, $\to y_2$.

2°) Trovare una soluzione particolare. Metodo delle somiglianze

$f(x)$ è di un "certo tipo" $\longleftrightarrow$ anche y_p sarà dello stesso tipo

$f(x) = P_n(x)$. polinomio

(1) $y'' + 2y' - y = x^2$ Soluzione particolare? RISOLTO

2) $y'' + 2y' = x^2$

3) $y'' = x^2$.

1) $\rightarrow \lambda^2 + 2\lambda - 1 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = \begin{cases} \frac{-2+\sqrt{8}}{2} \rightarrow y_1(x) = e^{\frac{-2+\sqrt{8}}{2}x} \\ \frac{-2-\sqrt{8}}{2} \rightarrow y_2(x) = e^{\frac{-2-\sqrt{8}}{2}x} \end{cases}$$

y_1 e y_2 sono esponenziali $\rightarrow y_0(x)$.

$f(x) = x^2 \longmapsto y_p(x) = ax^2 + bx + c$, trovare a, b e c in modo tale che y_p sia soluzione di (1). $y_p'(x) = 2ax + b$, $y_p'' = 2a$

$$\underbrace{2a}_{y''} + 2\cdot(\underbrace{2ax+b}_{y'}) - \underbrace{(ax^2+bx+c)}_{y} = x^2$$

$$\boxed{-ax^2 + x(-b+4a) + (2a+2b-c) = x^2}$$

$$\begin{cases} -a = 1 \rightarrow a = -1 \\ -b+4a = 0 \rightarrow b = -4 \\ 2a+2b-c = 0 \rightarrow c = -10 \end{cases}$$

$$\boxed{y_p(x) = -x^2 - 4x - 10}$$

DEVONO ESSERE LA STESSA FUNZIONE!

$$y''+2y'=x^2 \longrightarrow y''+2y'=0 \longrightarrow \lambda^2+2\lambda=0 \rightarrow \lambda(\lambda+2)=0$$

$\lambda=0$ e $\lambda=-2$.

$$y_1(x)=C_1e^{0x}=C_1$$

$$y_2(x)=C_2e^{-2x}$$

Quindi in particolare le costanti sono soluzioni di $y''+2y'=0$.

2°) Soluzione particolare $f(x)=x^2$ è un polinomio del secondo ordine

?

$$y_p(x)=ax^2+bx+c. \qquad y_p'(x)=2ax+b \qquad y_p''(x)=2a$$

$$\underbrace{2a}_{y''}+\underbrace{2}_{+2}\cdot\underbrace{(2ax+b)}_{y'}=x^2. \quad \text{NON È POSSIBILE}$$

$$y_p(x)=ax^3+bx^2+cx \Longrightarrow \quad y_p'(x)=3ax^2+2bx+c \qquad y_p''(x)=6ax+2b.$$

$$y_p''+2y_p'=6ax+2b+2(3ax^2+2bx+c)=6ax^2+x(6a+4b)+2b+2c=x^2$$

sono dello stesso tipo.

$$\begin{cases} 6a = 1 \\ 6a + 4b = 0 \\ 2b + 2c = 0 \end{cases} \qquad \left.\begin{array}{l} a = \frac{1}{6} \\ b = \frac{-6}{4} a = -\frac{1}{4} \\ c = -b = \frac{1}{4} \end{array}\right\} \Longrightarrow y_p = \frac{1}{6} x^3 - \frac{1}{4} x^2 + \frac{1}{4} x$$

$$y(x) = \frac{1}{6} x^3 - \frac{1}{4} x^2 + \frac{1}{4} x + c_1 + c_2 e^{-2x}$$

$$y'' = x^2 \longrightarrow \underset{\text{eq. omogenea}}{y'' = 0} \longrightarrow \lambda^2 = 0 \longrightarrow \lambda = 0 \longrightarrow y_1(x) = e^{0x} = 1$$

$$y_2(x) = x$$

$$y_0(x) = c_0 + c_1 x$$

Se $y_p(x) = ax^2 + bx + c \longrightarrow y_p'(x) = 2ax + b, \quad y_p'' = 2a$

$2a = x^2$

IMPOSSIBILE

$\times x^2$

$$y_p(x) = ax^4 + bx^3 + cx^2 \longrightarrow y_p'(x) = 4ax^3 + 3bx^2 + 2cx$$

$$y_p''(x) = \underbrace{12ax^2 + 6bx + 2c}_{\text{Pol. 2° ordi}} = \underbrace{x^2}_{\text{pol. 2° ord.}}$$

$12a = 1 \qquad a = \frac{1}{12}$

$6b = 0 \qquad b = 0 \qquad y_p(x) = \frac{1}{12}x^4.$

$2c = 0 \qquad c = 0$

$$\boxed{y(x) = \frac{1}{12}x^4 + c_1 x + c_2 \qquad \forall\, c_1 \text{ e } c_2 \in \mathbb{R}}$$

Risolvere: $y''(x) = x^2. \longrightarrow y'(x) = \frac{x^3}{3} + c_1 \longrightarrow y(x) = \frac{x^4}{12} + c_1 x + c_2.$

VI

$1°)\ y'' + 2y' + 2y = e^{3x}$.

$2°)\ y'' + 2y' - 15y = e^{3x}$

$3°)\ y'' - 6y' + 9y = e^{3x}$.

$\longrightarrow y(x) = \frac{1}{17} e^{3x} + C_0 e^{-x} \cos x + C_1 e^{-x} \sin x$

$\forall C_0 \in \mathbb{R} \ \forall C_1 \in \mathbb{R}$

Eq. omogenea.

$1°)\ y'' + 2y' + 2y = e^{3x} \longrightarrow y'' + 2y' + 2y = 0 \longrightarrow \lambda^2 + 2\lambda + 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2}.$$

$$\lambda = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i2}{2} = -1 \pm i.$$

$y_0(x) = C_0 e^{-x} \cos x + C_1 e^{-x} \sin x$

Sol. particolare $\longrightarrow \boxed{y_p(x) = \frac{1}{17} e^{3x}}$

$f(x) = e^{3x}$. $\longmapsto y_p(x) = a e^{3x}$. $\longrightarrow y_p'(x) = 3a e^{3x}$

$y_p''(x) = 9a e^{3x}$.

Polinomio di ordine 0 x exp.

$9a e^{3x} + 2(3a e^{3x}) + 2(a e^{3x}) = e^{3x} a (9 + 6 + 2) = e^{3x}$. $\longrightarrow 17a = 1$.

$\Downarrow$

$a = \frac{1}{17}$.

STESSA FUNZIONE

$$y'' + 2y' - 15y = e^{3x}.$$

$$\Downarrow$$

$$y'' + 2y' - 15y = 0 \longrightarrow \lambda^2 + 2\lambda - 15 = 0$$

Eq. om

$$\lambda = \frac{-2 \pm \sqrt{4 + 60}}{2} = \begin{cases} \dfrac{-2+8}{2} = 3 \\[2ex] \dfrac{-2-8}{2} = -5 \end{cases}$$

$$y_0(x) = C_0 e^{3x} + C_1 e^{-5x} \longrightarrow \text{sol. particolare}$$

$$f(x) = e^{3x} \longrightarrow \cancel{y_p(x) = a e^{3x}}$$

IMP.

$$\Downarrow$$

$$\cancel{y_p'' + 2y_p' - 15y_p = 0} \neq e^{3x}$$

$\times x$

$$y_p(x) = a x e^{3x} \qquad \text{troviamo } a?$$

$$y_p'(x) = a e^{3x} + 3axe^{3x} = e^{3x}(3ax + a)$$

$$y_p''(x) = 3ae^{3x} + (9ax + 3a)e^{3x} = (9ax + 6a)e^{3x}.$$

$$y_p''(x) + 2y_p' - 15y_p = (9ax + 6a)e^{3x} + 2(3ax + a)e^{3x} - 15(axe^{3x}) = e^{3x}$$

mettendo la x raccogliendo $\rightarrow e^{3x}[9ax + 6ax - 15ax + 6a + 2a] = e^{3x}.$

$8ae^{3x} = e^{3x} \longrightarrow 8a = 1$ cioè $a = \frac{1}{8} \to y_p(x) = \frac{1}{8} x e^{3x}$.

$$y(x) = \frac{1}{8} x e^{3x} + C_0 e^{3x} + C_1 e^{-5x} \quad \forall C_1, C_2 \in \mathbb{R}$$

$$y'' - 6y' + 9y = e^{3x} \longrightarrow y'' - 6y' + 9y = 0 \longrightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Downarrow$$

$$(\lambda - 3)^2 = 0$$

$$\Downarrow$$

$\boxed{\lambda = +3}$ 1 sol.

$$y_0(x) = C_1 e^{3x} + C_2 x e^{3x} = e^{3x}(C_1 + C_2 x)$$

sol. particolare $f(x) = e^{3x} \longrightarrow$

$\cancel{y_p(x) = a e^{3x}}$ Imp. $\to$ sol. omogenea

$\cancel{y_p(x) = a x e^{3x}}$. IMPOSSIBILE

$y_p(x) = a x^2 e^{3x}$

mi aspetto che i termini in x^2 e in x dovranno sparire

$$y_p'(x) = 2axe^{3x} + 3ax^2 e^{3x} = (2ax + 3ax^2)e^{3x}$$

$$y_p''(x) = (2a + 6ax)e^{3x} + (6ax + 9ax^2)e^{3x} = e^{3x}(9ax^2 + 12ax + 2a).$$

$$e^{3x}\Big[\underbrace{9ax^2 + 12ax + 2a}_{y''} \underbrace{- 12ax - 18ax^2}_{-6y'} + \underbrace{9ax^2}_{9y}\Big] = e^{3x} \longrightarrow 2a e^{3x} = e^{3x}$$

$2a = 1 \qquad a = \frac{1}{2}$.

$y(x) = e^{3x}\left(\frac{1}{2}x^2 + C_2 x + C_1\right) \; \forall C_i \in \mathbb{R}$:

$$y'' + 2y' + y = \cos x \longrightarrow y(x) = \frac{1}{2}\sin x + c_1 e^{-x} + c_2 x e^{-x}$$

1°) $y'' + 2y' + y = 0 \longrightarrow \lambda^2 + 2\lambda + 1 = 0$

$$(\lambda + 1)^2 = 0 \qquad \lambda = -1$$

$$y_0(x) = c_1 e^{-x} + c_2 x e^{-x}$$

2°) Soluzione particolare, $f(x) = \cos x$.

$$y_p(x) = a\cos x + b\sin x \qquad a?\ ,\ b?$$

$$y_p'(x) = -a\sin x + b\cos x\ , \quad y_p'' = -a\cos x - b\sin x$$

$$\cancel{-a\cos x} \cancel{-b\sin x} + 2(-a\sin x + b\cos x) + \cancel{a\cos x} + \cancel{b\sin x} = \cos x$$

$$-2a\sin x + 2b\cos x = \cos x$$

$$-2a = 0 \longrightarrow a = 0$$

$$+2b = 1 \qquad b = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}\sin x$$

X

$$\begin{cases} y'' + 2y' + y = \cos x \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \longrightarrow$$

$$y(x) = \frac{1}{2}\sin x + c_1 e^{-x} + c_2 x e^{-x} \longrightarrow y(0) = \frac{1}{2}\sin 0 + c_1 e^0 + c_2 \cdot 0 e^0 = c_1 = 1$$

$$y'(x) = \frac{1}{2}\cos x - \underset{\substack{\uparrow \\ c_1 = 1}}{1} e^{-x} + c_2 \underset{\substack{\uparrow \\ x}}{e^{-x}} - c_2 x \underset{\substack{\uparrow \\ e^{-x}}}{e^{-x}} \longrightarrow y'(0) = \frac{1}{2}\cos 0 - \cancel{e^0} + c_2 e^0 - c_2 0 e^0$$

$$= \frac{1}{2} - 1 + c_2 = 0$$

$$\Downarrow$$

$$c_2 = \frac{1}{2}.$$

La sol. del problema di Cauchy

$$\boxed{y(x) = \frac{1}{2}\sin x + e^{-x} + \frac{1}{2} x e^{-x}.}$$

$$\begin{cases} y'' + 3y' + 2y = x e^{-2x} \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$$

1°) $\boxed{y'' + 3y' + 2y = 0}$

2°) $y_p(x) = ?$

3°) Pb. di Cauchy

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, \ \lambda = -2.$$

$$\Longrightarrow y_0(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

$f(x) = x\,e^{-2x}$ polinomio di grado 1 $\times\ e^{-2x}$

$$\Downarrow$$

$$y_p(x) = x(ax+b)\,e^{-2x} = (ax^2+bx)\,e^{-2x}$$

$$y_p' = (2ax+b)e^{-2x} + (-2ax^2-2bx)e^{-2x} = e^{-2x}\left(-2ax^2 + 2(a-b)x + b\right).$$

$$y_p'' = e^{-2x}\left(-4ax + 2(a-b)\right) + e^{-2x}\left(4ax^2 - 4(a-b)x - 2b\right).$$

$$= e^{-2x}\left[4ax^2 - 8ax + \cancel{8}4bx + 2a - 4b\right]$$

$$y_p'' + 3y_p' + 2y_p = e^{-2x}\left[4ax^2 - 8ax + 4bx + 2a - 4b - 6ax^2 + 6(a-b)x + 3b + 2ax^2 + 2bx\right]$$

$$= e^{-2x}\cdot x$$

$$e^{-2x}\left[x(-8a+4b+6a-6b+2b) + 2a - 4b + 3b\right] = x\,e^{-2x}.$$

$$e^{-2x}\left[x(-2a) + 2a - b\right] = x\,e^{-2x}$$

$$\begin{cases} -2a = 1 \\ 2a - b = 0 \end{cases} \qquad a = -\tfrac{1}{2}. \quad b = -1.$$

$$y_p(x) = \left(-\tfrac{1}{2}x^2 - x\right)e^{-2x}.$$

$y(x) = C_1 e^{-x} + C_2 e^{-2x} + (-\frac{1}{2}x^2 - x) e^{-2x}$

~~$y(x) = C_1 e^{-x} + e^{-2x}(C_2 - \frac{1}{2}x^2 - x)$~~ $\longrightarrow y(0) = C_1 + C_2 = 0$.

$\downarrow$

$y'(x) = -C_1 e^{-x} - 2C_2 e^{-2x} + (-x - 1)e^{-2x} - 2(-\frac{1}{2}x^2 - x)e^{-2x}$

$y'(0) = -C_1 - 2C_2 - 1 = 2$.

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + 2C_2 = -3 \end{cases} \qquad \begin{matrix} C_1 = -C_2 = 3 \\ C_2 = -3 \end{matrix}$$

$-C_2 + 2C_2 = -3$

$y(x) = 3e^{-x} - 3e^{-2x} + (-\frac{1}{2}x^2 - x)e^{-2x} = 3e^{-x} - e^{-2x}(\frac{1}{2}x^2 + x + 3)$

– Curve, F + variabili → Ins di li, Ins di def, Derivate parziali, Ottimizzazioni / Eq. → 1° ordine lineare, 1° ordine sep. di variabili, 2° ordine.