# Calculus-Unit 1

**Applied Computer Science for AI**

# Blank examination

| Voto finale |
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**Postazione:**

**Cognome:**

**Nome:**

**Matricola:**

**Canale:**

| Esercizio | Punteggio |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| Risp. Mult. | |
| Totale | |

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**Es. 1** [1+2+1 Points] Given the sequence $a_n = \frac{n^2}{3n^2-2}$ for $n \in \mathbb{N}^*$
a) Compute $a_1$ and $a_2$
$a_1 = 1$ and $a_2 = \frac{4}{10}$
b) Prove that the sequence is bounded
*Since $n \geq 1$, $3n^2 - 2 > 0$ and, of course, $n^2 > 0$. Hence $a_n > 0$ so the sequence is bounded from below.*
*On the other hand $\frac{n^2}{3n^2-2} \leq 1$. Indeed*

$$\frac{n^2}{3n^2-2} \leq 1 \Leftrightarrow n^2 \leq 3n^2 - 2 \Leftrightarrow 0 \leq 2n^2 - 2 = 2(n^2-1) \text{ which is true}$$

c) Prove that the sequence is monotone decreasing
*We need to prove that $a_n \geq a_{n+1}$ i.e. $\frac{n^2}{3n^2-2} \geq \frac{(n+1)^2}{3(n+1)^2-2}$ This is equivalent to*

$$n^2 3(n+1)^2 - 2 \geq (n+1)^2(3n^2-2) \Leftrightarrow -2n^2 \geq -2(n+1)^2) \Leftrightarrow (n+1)^2 \geq n^2 \text{ which is true}$$

**Es 2** [3 Points] Given $a \in \mathbb{R}$ and $b \in \mathbb{R}$, let $f(x) = \begin{cases} \frac{\log(1+2x)}{3x} & \text{for } x > 0 \\ a(x+1)^2 + b & \text{for } x \leq 0 \end{cases}$ Determine $a$ and $b$ such that $f$ is differentiable in $\mathbb{R}$.
*In order for $f$ to be differentiable it needs to be continuous. The function is continuous if for every $x \neq 0$. In order to check that $f$ is continuous in 0, we need to see that the limit in zero exists and that it coincides with $f(0)$. By its definition $f(0) = a + b$. The limit exists in zero if both the limit at the left of zero is equal to the limit at the right of zero.*

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\log(1+2x)}{3x} = \lim_{x\to 0^-} \frac{\log(1+2x)}{2x}\frac{2}{3} = \frac{2}{3}$$

*while $\lim_{x\to 0^+} f(x) = a + b$ so the condition for the continuity of $f$ is $a + b = \frac{2}{3}$. The function is differentiable in zero if there exists $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$*

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{\log(1+2x)}{3x} - \frac{2}{3}}{x} = \lim_{x\to 0^+} \frac{\log(1+2x) - 2x}{3x^2}$$

*Using de l'Hopital's rule we can compute the limit of the ratio of the derivatives:*

$$\lim_{x\to 0^+} \frac{\frac{2}{1+2x} - 2}{6x} = \lim_{x\to 0^+} \frac{-2x}{3x(1+2x)} = \lim_{x\to 0^+} \frac{-2}{3(1+2x)} = -\frac{2}{3}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{a(x+1)^2 + b - (a+b)}{x} = \lim_{x\to 0^-} \frac{ax^2 + 2ax}{x} = 2a$$

*So the request is that $2a = -\frac{2}{3}$ and $a + b = \frac{2}{3}$ i.e. $a = -\frac{1}{3}$ and $b = 1$.*

**Es 3** [4 points] Compute the following limit (justify your answer) $\lim_{x\to 1} \dfrac{e^{x^2-1}-1}{\tan(\frac{\pi}{4}x^3)\log(x)}$

We plan to use the special limits.

$$\lim_{x\to 1} \frac{e^{x^2-1}-1}{\tan(\frac{\pi}{4}x^3)\log(x)} = \lim_{x\to 1} \frac{e^{x^2-1}-1}{x^2-1} \cdot \frac{(x-1)}{\log(x)} \cdot \frac{x+1}{\tan(\frac{\pi}{4}x^3)} = 1 \cdot 1 \cdot \frac{2}{1} = 2$$

Here we have used

$$1 = \lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 1} \frac{e^{x^2-1}-1}{x^2-1}$$

$$1 = \lim_{x\to 0} \frac{\log(x+1)}{x} = \lim_{x\to 1} \frac{\log(x)}{x-1}$$

**Es 4** [1+2+1+2+1 points] Given the function $f(x) = \arctan\left(\frac{x-2}{2x+4}\right)$. Determine:

a) Domain: $\mathbb{R} \setminus \{-2\} = (-\infty, -2) \cup (-2, +\infty)$

b) The limits at the boundary of the domains

$\lim_{x\to -\infty} f(x) = \arctan\left(\dfrac{1}{2}\right)$, $\lim_{x\to -2^-} f(x) = \arctan(+\infty) = \dfrac{\pi}{2}$, $\lim_{x\to -2^+} f(x) = \arctan(-\infty) = -\dfrac{\pi}{2}$,

$\lim_{x\to +\infty} f(x) = \arctan\left(\dfrac{1}{2}\right)$

c) The asymptotes

*At infinity the asymptotes are* $y = \arctan\left(\frac{1}{2}\right)$

d) The derivative

$f'(x) = \frac{1}{1+\left(\frac{x-2}{2x+4}\right)^2} \cdot \left(\frac{8}{(2x+4)^2}\right)$ *which is always positive.*

e) The intervals of monotonicity

*The function is increasing in* $(-\infty, -2)$ *and in* $(-2, +\infty)$.

**Es 5** [2 o -1 points] The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |\log(x)|$

(A) Has a minimum and a maximum (**B**) Has a maximum but no minimum

(X) Has a minimum but no maximum (D) Its minimum is at infinity

**Es 6** [2 o -1 punti] The derivative of $f(x) = \sin(2x)e^{\cos(2x)}$ is:

(A) $-4\cos(2x)\sin(2x)e^{\cos(2x)}$ (X) $2e^{\cos(2x)}(\cos(2x){-}\sin^2(2x))$ (C) $e^{\cos(2x)}(2\cos(2x){+}4\sin^2(2x))$

(D) $e^{\cos(2x)}(2\cos(2x) + \sin(2x))$ (E) None of the previous answers is correct

**Es 7** Let $f : [0, 1] \to \mathbb{R}$ a continuous function. Then

(A)[1/2] The image of $f$ is a closed and bounded interval [X] [F]
(B)[1/2] If $f(0) = f(1)$ then either the maximum of the minimum are reached in the open interval $(0, 1)$ [X] [F]
(C)[1/2] The function reaches all the values between $f(0)$ and $f(1)$. [X] [F]
(D)[1/2] The function reaches only the values between $f(0)$ and $f(1)$. [T] [X]
(E)[1/2] If $f$ is convex in $[0, 1]$, then the graph is below the strait line given by the equation $y = (f(1) - f(0))(x - 1) + f(1)$ [X] [F]

**Es 8** Given the equation $z^6 = 1 + i$ in $\mathbb{C}$

(A) It has 2 solutions in $\mathbb{C}$ [T] [X]
(B) The solutions are on the circle of center 0 and radius $\sqrt{2}$ [X] [F]
(C) There exists a solution in $\mathbb{R}$ [T] [X]
(D) The solutions are at the vertices of an hexagon [X] [F]

**Es 9** [3 o -1 punti] Let $a_n$ be a bounded sequence. Then necessarily

(A) The sequence has a limit [T] [X]. (B) The sequence is monotone [T] [X]

(C) There exists a converging subsequence [X] [F] (D) All subsequences converge [T] [X]

**Es 10** Let $z = \frac{1}{2+3i}$.

(A)[1/2] Then $z_o = \frac{1}{13}(2 - 3i)$ [X] [F]

(B)[1/2] Then $z_o \cdot \overline{z}_o = 13$ [X] [F]

(C)[1/2] $(1 + i)z_o = \frac{1}{13}(5 - i)$ [X] [F]

(D)[1/2]$(z_o)^{-1} = 2i + 3$ [X] [F]