Calculus-Unit 1

Applied Computer Science for AI

Blank examination

Voto finale

Postazione:

Cognome:

Nome:

Matricola:

Canale:

Esercizio	Punteggio
1	
2	
3	
4	
Risp. Mult.	
Totale	

Es. 1 [1+2+1 Points] Given the sequence a_n defined in the following way

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \sqrt{a_n + 1} \end{cases}$$

a) Prove by induction that $a_n \leq \frac{1+\sqrt{5}}{2}$
b) Prove that, if the limit exists, it is equal to $\frac{1+\sqrt{5}}{2}$
c) Prove that the sequence is monotone increasing

Solutions

a)Sstep 1: $a_0 = 1 = \frac{1+1}{2} < \frac{1+\sqrt{5}}{2}$
Step 2: suppose that for some n, $a_n \leq \frac{1+\sqrt{5}}{2}$. We want to prove the inequality for a_{n+1}:
$a_{n+1} = \sqrt{a_n + 1} \leq \sqrt{\frac{1+\sqrt{5}}{2} + 1} = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$, Since

$$(\sqrt{\frac{3+\sqrt{5}}{2}})^2 = \frac{3+\sqrt{5}}{2} \text{ and } (\frac{1+\sqrt{5}}{2})^2 = \frac{1+5+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

This concludes the proof by induction of a)
b) If the limit exists $\lim a_n = l$ and $\lim a_{n+1} = l$ but also $\lim a_{n+1} = \lim \sqrt{a_n + 1} = \sqrt{l+1}$. Bu the limit is unique, hence is needs to satisfy $l = \sqrt{l+1}$ i.e. $l^2 - l - 1 = 0$. This equation has two solutions

$$l_+ = \frac{1+\sqrt{5}}{2} > 0, \text{and } l_- = \frac{1-\sqrt{5}}{2} < 0$$

Since the sequence is positive, if the limit exists it is $l_+ = \frac{1+\sqrt{5}}{2}$
c) Again by induction $a_o = 1 \leq a_1$ and if $a_n \leq a_{n+1}$ then $a_{n+2} = \sqrt{a_{n+1} + 1} \geq \sqrt{a_n + 1} = a_{n+1}$. Q.E.D:

Es 2 [3 Points] Determine the points of discontinuity and of non differentiability of the function $f(x) = |\sin(2x)|$ (justify your answer).

Solution

The function $f(x)$ is the composition of the function $|x|$ and $\sin(2x)$, they are both continuous and the composition of continuous functions is continuous so f is everywhere continuous.
On the other hand the modulus function $|x|$ is not differentiable when the argument is zero, while $\sin(2x)$ is always differentiable. Hence the function f will be non differentiable in the points where $\sin(2x) = 0$ i.e. $2x = k\pi$ for any $k \in \mathbb{Z}$ i.e. $x = \frac{k\pi}{2}$ for any $k \in \mathbb{Z}$ are the points of non differentiability of f

Es 3 [4 points] Compute the following limit (justify your answer) $\lim_{x\to 0^+} \dfrac{1-\cos(2\sqrt{x})}{\log(1+\sin(3x))}$

Solution

$$\lim_{x\to 0^+} \frac{1-\cos(2\sqrt{x})}{\log(1+\sin(3x))} = \lim_{x\to 0^+} \frac{1-\cos(2\sqrt{x})}{(2\sqrt{x})^2} \cdot \frac{4x}{\log(1+\sin(3x))} =$$

$$\lim_{x\to 0^+} \frac{1-\cos(2\sqrt{x})}{(2\sqrt{x})^2} \cdot \frac{\sin(3x)}{\log(1+\sin(3x))} \cdot \frac{3x}{\sin(3x)} \cdot \frac{4}{3} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{4}{3} = \frac{2}{3}$$

We have used that

$$\lim_{x\to 0^+} \frac{1-\cos(x)}{x^2} = \frac{1}{2}, \quad \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{\log(1+x)}{x} = 1.$$

Es 4 [1+2+1+2+1 points] Given the function $f(x) = e^{\frac{x}{x^2-1}}$. Determine:

a) Domain:

Solution

Condition $x^2 - 1 \neq 0$ i.e. Def$f = \mathbb{R} \setminus \{1, -1\}$

b) The limit at the boundary of the domains

Solution $\lim_{x\to-\infty} e^{\frac{x}{x^2-1}} = e^0 = 1$, $\lim_{x\to+\infty} e^{\frac{x}{x^2-1}} = e^0 = 1$, $\lim_{x\to-1^-} e^{\frac{x}{x^2-1}} = e^{-\infty} = 0$, $\lim_{x\to-1^+} e^{\frac{x}{x^2-1}} = e^{\infty} = \infty$, $\lim_{x\to1^-} e^{\frac{x}{x^2-1}} = e^{-\infty} = 0$, $\lim_{x\to1^+} e^{\frac{x}{x^2-1}} = e^{\infty} = \infty$

c) The asymptotes:

Solution $y = 0$ at ∞ and $-\infty$, $x = -1$ and $x = 1$.

d) The derivative

Solution $f'(x) = \frac{-(1+x^2)}{(x^2-1)^2} e^{\frac{x}{x^2-1}}$

e) The intervals of monotonicity

Solution The derivative is always non positive (≤ 0) so in each interval of existence the solution is montone decreasing i.e. it is decreasing in $(-\infty, -1)$, in $(-1, 1)$ and it is decreasing in $(1, +\infty)$.

Es 5 [2 o -1 points] The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |e^{-x^2} - \frac{1}{2}|$

(X) **Has a minimum and a maximum** (**B**) Has a maximum but no minimum

(C) Has a minimum but no maximum (D) Its minimum is at infinity

Es 6 [2 o -1 punti] The derivative of $f(x) = \arctan(\frac{2x}{x-2})$ is:

(A) $\frac{1}{1+x^2} \cdot \frac{-4}{(x-2)^2}$ (B) $\frac{1}{1+\left(\frac{2x}{x-2}\right)^2}$ (C) $\frac{-4}{(x-2)^2}$

(X) $\mathbf{\frac{-4}{5x^2-4x+4}}$ (E) None of the previous answers is correct

Es 7 Let $f : [0,2] \to \mathbb{R}$ continuous such that the image of f is $[0,2]$. Then

(A)[1/2] The function $g(x) = f(x) - x$ has at least a zero in $[0,2]$ [X] [F]

(B)[1/2] The function is tangent to the bisector [T] [X]

(C)[1/2] f has a maximum and a minimum [X] [F]

(D)[1/2] $\exists x_o \in (0,1)$ and $x_1 \in (1,2)$ such that $f(x_o) = f(x_1)$ [T] [X]

Es 8 Given the equation $(z+i)^4 = 1$ in $\mathbb{C}$

(A) It has 2 solutions [T] [X]

Es 9 [3 o -1 punti] The $\lim\limits_{n \to +\infty} \dfrac{-e^{2n} + 2n^4 + \ln(n^2-1)}{n \sin n + 2e^{2n} + \sqrt{3}}$ equals

(A) 1 (**B**) $\frac{1}{2}$ (C) $+\infty$

(D) $-\infty$ (E) The limit does not exist (X) None of the previous answers is correct

Es 10 The function $f : [a,b] \to \mathbb{R}$ is differentiable. Say which of the following holds true

(A)[1/2] If $f(a) = f(b)$ then the maximum of f is 0 [T] [X]

(B)[1/2] If f is convexe then the deirvative of f is increasing [X] [F]

(C)[1/2] If $f(b) > f(a)$, then f is increasing in (a,b). [T] [X]

(D)[1/2] If $f(x) = 2f(a) + b(x-a)$, then $f(a) = 0$ [X] [F]

(E)[1/2] There exists an $x \in (a,b)$ such that $f'(x) = \frac{f(a)-f(b)}{a-b}$ [X] [F]