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Esercizio 3
Punti critici e la loro natura cioè se sono pti di max o minimo locale oppure pti sella o nessuno dei 3.

a) $f(x,y) = xy - 3x^2y$

b) $f(x,y) = x e^{x^2y+y^2}$ $\Rightarrow$ $\nabla f(x,y) = \left(e^{x^2y+y^2} + x(2xy)e^{x^2y+y^2},\ x e^{x^2y+y^2}(x^2+2y)\right)$

$\nabla f(x,y) = (0,0)$

$$\nabla f(x,y) = e^{x^2y+y^2}\left(1+2x^2y,\ \underline{x(x^2+2y)}\right) = (0,0)$$

$$\begin{cases} 1+2x^2y = 0 \\ x(x^2+2y) = 0 \end{cases}$$

$\nearrow x = 0 \xrightarrow{1^a eq} 1 = 0$ IMP

$\searrow y = -\frac{x^2}{2} \xrightarrow{1^a eq}$

$$1-x^4=0 \quad \Longleftarrow \quad 1+2x^2\cdot\left(-\frac{x^2}{2}\right)=0$$

$$x^4=1 \begin{cases} \nearrow x=1 \\ \searrow x=-1 \end{cases} \Longrightarrow \begin{matrix} P_0=\left(1,-\frac{1}{2}\right) \\ P_1=\left(-1,-\frac{1}{2}\right) \end{matrix} \quad \text{punti critici}$$

$$D^2f=\begin{pmatrix} \partial_{xx}f & \partial_{xy}f \\ \partial_{xy}f & \partial_{yy}f \end{pmatrix}=e^{x^2y+y^2}\begin{pmatrix} 4xy+(1+2x^2y)2xy & 2x^2+(1+2x^2y)(x^2+2y) \\ 2x^2+(1+2x^2y)(x^2+2y) & 2x+x(x^2+2y)^2 \end{pmatrix}$$

Nei pti

$$D^2f\left(1,-\frac{1}{2}\right)=e^{\sim}\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix} \qquad \det D^2f=e^{\sim}(-4-4)$$

$$=e^{\sim}(-8)<0$$

$$e^{\sim}>0$$

$\left(1,-\frac{1}{2}\right)$ è un pto di sella

$$D^2f\left(-1,-\frac{1}{2}\right)=e^{\sim}\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}=e^{\sim}(-4-4)=e^{\sim}8<0$$

$\left(-1,-\frac{1}{2}\right)$ è un punto di sella.

c) $f(x,y)=(1-x^2)^2(y-1)(y-2)$

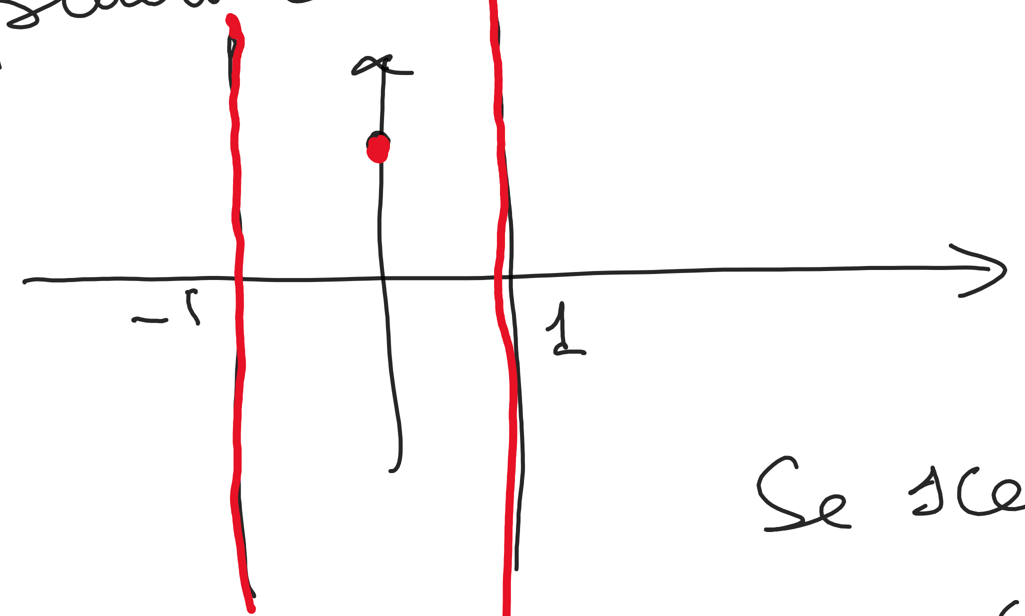
Calcoliamo $\nabla f = \left(2(1-x^2)(-2x)\cdot(y-1)(y-2),\ (1-x^2)^2\cdot(2y-3)\right)$

$$\begin{cases} -4x(1-x^2)(y-1)(y-2)=0 \longrightarrow \boxed{x=0},\ \boxed{x=1},\ \boxed{x=-1},\ \boxed{y=1},\ y=2 \\ (1-x^2)^2(2y-3)=0 \longrightarrow \boxed{x=1},\ \boxed{x=-1},\ \boxed{y=\frac{3}{2}} \end{cases}$$

quindi $\forall y\in\mathbb{R}$ i $(1,y)$ e $(-1,y)$ sono punti critici

$\nabla f(1,y)=(0,0)$ x e y

$\nabla f(-1,y)=(0,0)$ x x



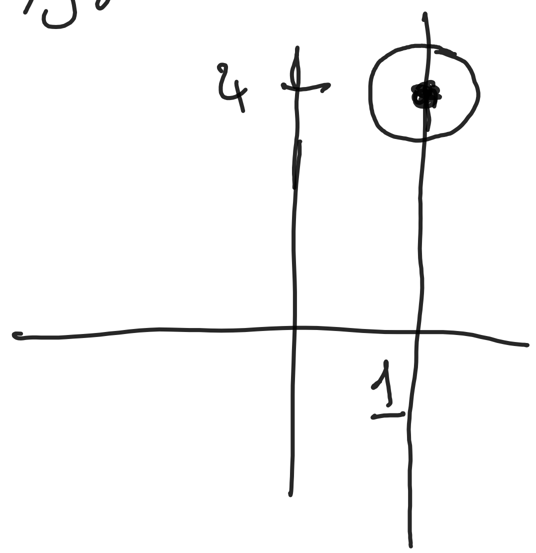
$\nabla f\left(0,\frac{3}{2}\right)=(0,0)$

Se scelgo $y=1$ o $y=2 \Rightarrow x=1$ o $x=-1$ | già visti

$\nabla 0 = \left(-4x(1-x^2)(y-1)(y-2),\ (1-x^2)^2(2y-3)\right)$

$$D^2 f = \begin{pmatrix} -(4-12x^2)(y-1)(y-2) & -4x(1-x^2)(2y-3) \\ -4x(1-x^2)(2y-3) & (1-x^2)^2 2 \end{pmatrix}$$

$$D^2 f(1,y) = \begin{pmatrix} 8(y-1)(y-2) & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \det D^2 f(1,y) = 0$$



$$f(x,y) = \underbrace{(1-x^2)^2}_{\geq 0} \underbrace{(y-1)(y-2)}_{\text{segno ?}}$$

Se $(y_0-1)(y_0-2) > 0 \Longrightarrow$ esiste $\delta > 0$ t.c. $y \in [y_0-\delta, y_0+\delta]$ $(y-1)(y-2) > 0$

In un intorno di $(1, y_0)$

$f(x,y) \geq 0 = f(1,y_0) \Longrightarrow (1, y_0)$ è un punto di minimo locale

Se $(y_1-1)(y_1-2) < 0 \Longrightarrow$ esiste $\delta > 0$ t.c. $y \in [y_1-\delta, y_1+\delta]$

Se $(y_1-1)(y_1 \ldots$ $(y-1)(y-2)<0$

In un intorno di $(1, y_1)$

$$f(x,y) < 0 = f(1,y_1) \Longrightarrow (1,y_1) \text{ è un pto di massimo locale}$$

Funziona esattamente nello stesso modo i punti $(-1,y)$.

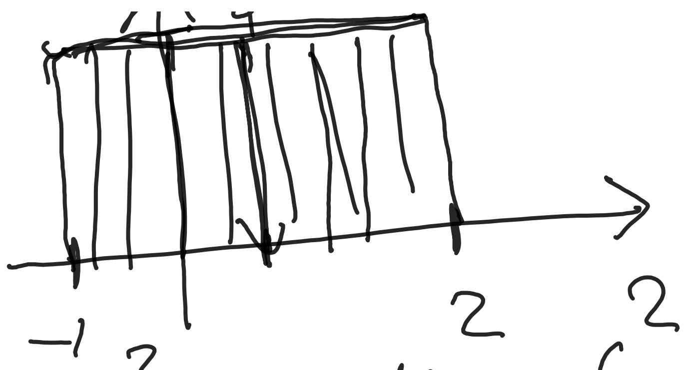
$$D^2 f\left(0,\tfrac{3}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det D^2 f\left(0,\tfrac{3}{2}\right) = 2 > 0$

$\dfrac{\partial^2 f}{\partial x^2} = 1 > 0$

$\Longrightarrow \left(0,\tfrac{3}{2}\right)$ è un pto di minimo.

Esercizio 4 : a) $\iint x+2y \, dx\, dy$ dove $D = [-1,2] \times [0,4]$

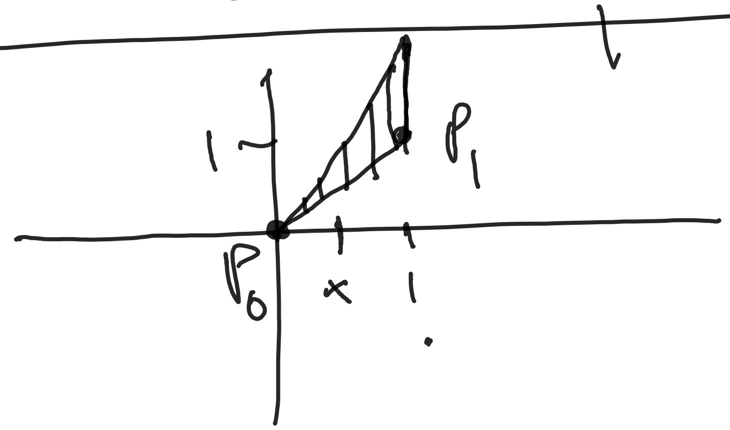


D è normale rispetto a x

$$\iint_D x+2y\,dx\,dy = \int_{-1}^{2}\left(\int_0^4 x+2y\,dy\right)dx = \int_{-1}^{2}\left(xy+y^2\right)\Big|_0^4 dx = \int_{-1}^{2} 4x+16\,dx$$

$$= 2x^2+16x\Big|_{-1}^{2} = 2\cdot 4+16\cdot 2-\left(2\cdot(-1)^2+16\cdot(-1)\right) = 40-(2-16) = 54$$

b) $\iint_T y\,dx\,dy$



D normale risp. a x.

Eq. delle rette tra P_0 e P_1, tra P_0 e P_2

$P_0 = (0,0)$, $P_1 = (1,1)$, $P_0 = (0,0)$, $P_2 = (1,2)$

$P_0 e P_1 \longrightarrow y = x$

$P_0 e P_2 \longrightarrow y = 2x$

$P_1 e P_2 \longrightarrow x = 1$

$$T = \{(x,y) \text{ t.c. } 0 \le x \le 1,\ x \le y \le 2x\}$$

$$\iint_T y\,dx\,dy = \int_0^1 \left(\int_x^{2x} y \, dy \right) dx = \int_0^1 \frac{y^2}{2}\Big|_x^{2x} dx$$

costante (0)

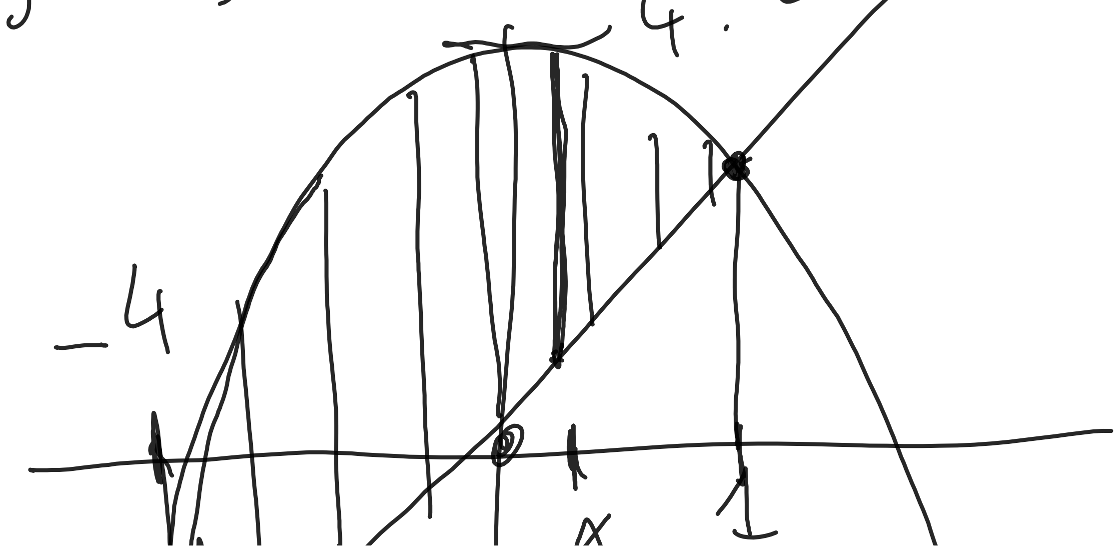
variabile è diversa dalla variabile di integrazione (x)

$$= \int_0^1 2x^2 - \frac{x^2}{2}\,dx = \int_0^1 \frac{3}{2}x^2\,dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}$$

c) $\iint_D xy\,dx\,dy$ dove $D = \{(x,y),\ 3x \le y \le 4 - x^2\}$

$y = 3x$

$y = 4 - x^2$



Trovare i punti di intersezione tra la retta e la parabola

$$\begin{cases} y = 3x \\ y = 4 - x^2 \end{cases} \Rightarrow 4 - x^2 = 3x \qquad x = 1$$

$$x^2 + 3x - 4 = 0 \Rightarrow x = -4$$

$$\iint_D xy\,dx\,dy = \int_{-4}^{1}\left(\int_{3x}^{4-x^2} xy\,dy\right)dx = \int_{-4}^{1} x\frac{y^2}{2}\Big|_{3x}^{4-x^2} dx$$

$$= \int_{-4}^{1} \frac{x}{2}(4-x^2)^2 - \frac{x}{2}\,9x^2\,dx = \frac{1}{2}\int_{-4}^{1} x(4-x^2)^2 - 9x^3\,dx$$

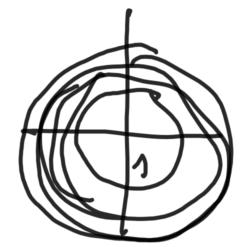
$$t = 4 - x^2$$
$$dt = -2x\,dx$$
$$-\frac{1}{2}dt = x\,dx$$
$$\int x(4-x^2)^2\,dx$$
$$= -\frac{1}{2}\int t^2\,dt = -\frac{1}{6}t^3$$
$$-1(4-x^2)^3$$

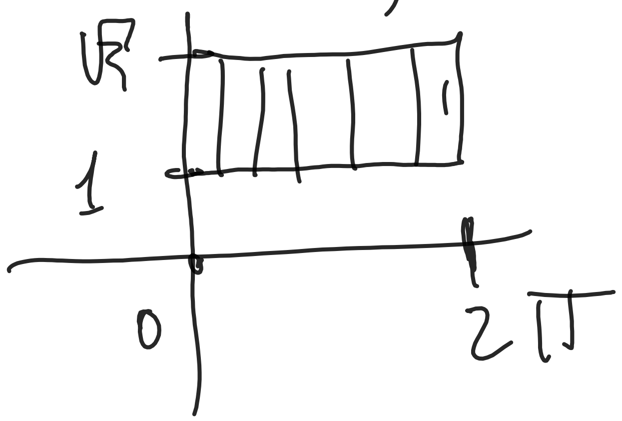
$$= \frac{1}{2}\left[-\frac{1}{6}(4-x^2)^3 - \frac{9}{4}x^4\right]_{-4}^{1}$$

$$= \frac{1}{4}\left[-\frac{1}{3}\cdot 3^3 - \frac{9}{2} - \left(+\frac{1}{3}(12)^3 - 9\cdot 4^3\right)\right]$$

$$= \frac{1}{4}\left[-9 - \frac{9}{2} - 4^3\cdot 3^2 + 9\cdot 4^3\right] = -\frac{27}{8}$$

d) $\iint_D e^{x^2+y^2}\,dx\,dy$ $\qquad D = \{(x,y) \quad 1 \le x^2+y^2 \le 3\}$





$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x}\end{cases} \Leftrightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta\end{cases}$$

$$\tilde{D} = \{(x,y) \quad 0 \le \theta \le \cancel{2\pi}, \quad 1 \le r \le \sqrt{3}\}$$

$$\iint_D e^{x^2+y^2}\,dx\,dy = \iint_{\tilde{D}} e^{r^2}\,\underset{\uparrow \text{polari}}{\underline{r}}\,dr\,d\theta = \int_0^{2\pi}\left(\int_1^{\sqrt{3}} e^{r^2}\,r\,dr\right)d\theta$$

$$= \int_0 \frac{1}{2}\int_1 e^t dt \, d\theta$$

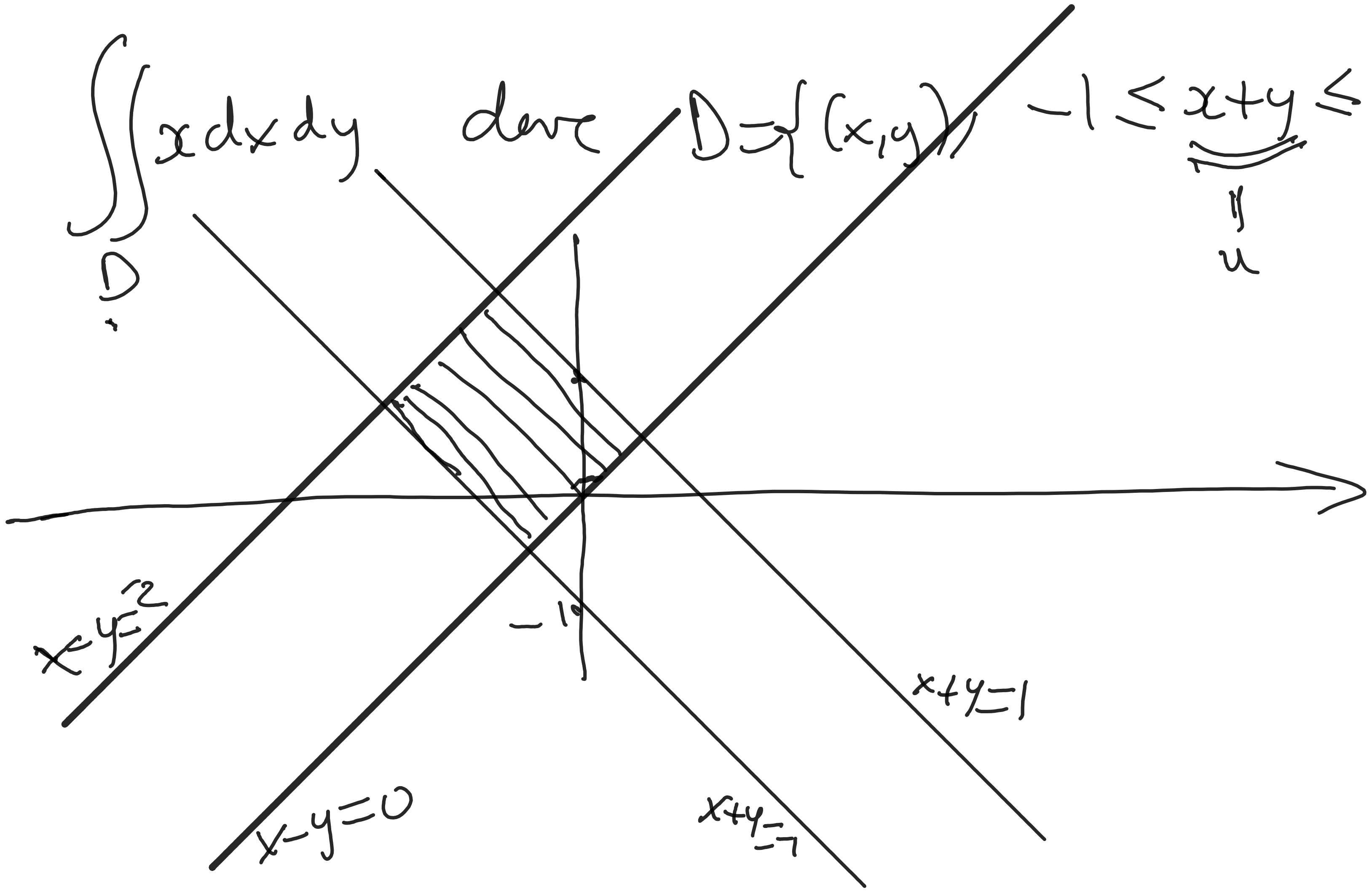
$t = r^2$

$dt = 2r\,dr \longrightarrow \frac{1}{2}dt = r\,dr$

$$= \int_0^{2\pi} \frac{1}{2}(e^3 - e)\,d\theta = \frac{2\pi}{2}(e^3 - e)$$

$$= \pi(e^3 - e)$$

e) $\iint_D x\,dx\,dy$ dove $D = \{(x,y),\ -1 \le \underbrace{x+y}_{=u} \le 1,\ -2 \le \underbrace{x-y}_{=v} \le 0\}$



$x+y=-1$

$y=-x-1$

$x+y=1$

$y=-x+1$

$x-y=-2$

$y=x+2$

$x-y=0$

(1) $\begin{cases} u = x+y \end{cases} \longrightarrow \tilde{D} = \{(u,v),\ -1 \le u \le 1,\ -2 \le v \le 0\}$

$\begin{cases} v = x - y \end{cases}$

↓ risolvere il sistema in modo di ottenere x e y in termini di u e v.

$$\begin{cases} u + v = 2x \\ u - v = 2y \end{cases} \longrightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

$$J = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

$$\iint_D x \, dx\, dy = \iint_{\tilde{D}} \frac{1}{2}(u+v) \cdot \frac{1}{2} \, du\, dv = \frac{1}{4} \iint_{\tilde{D}} u + v \, du\, dv$$

$$= \frac{1}{4} \int_{-1}^{1} \left(\int_{-2}^{0} u + v \quad dv \right) du = \frac{1}{4} \int_{-1}^{1} uv + \frac{v^2}{2} \Big|_{-2}^{0} \quad du$$

$$= \frac{1}{4} \int_{-1}^{1} 2u - 4 \, du = \frac{1}{4} \left(u^2 - 2u \right) \Big|_{-1}^{1} = \frac{1}{4} (1 - 2 - 1 - 2)$$

-4 $)$ 2 7

-1

$= -\frac{4}{4} = -1.$