Esercizio 1 (polinomio di Taylor)

Verificare che valgono i seguenti sviluppi

1. $\sin^2(x) = x^2 - \frac{x^4}{3} + o(x^5)$ per $x \to 0$

2. $\cos^2(x) = 1 - x^2 + \frac{x^4}{3} + o(x^5)$ per $x \to 0$

3. $\frac{1}{1+e^x} = \frac{1}{2} - \frac{x}{4} + o(x^2)$ per $x \to 0$

4. $\log(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^4)$ per $x \to 0$

5. $e^{\sin(x)} = 1 + x + \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4)$ per $x \to 0$.

Esercizio 2

Studiare le funzioni seguenti e disegnarne un grafico qualitativo.

1. $f(x) = (x-1)^3(2-x)$

2. $f(x) = \frac{x^2}{\log|x| - 1}$ (senza lo studio di f'')

3. $f(x) = \log(x) - \operatorname{arctg}(x-1)$ (senza lo studio di f'')

④ $f(x) = e^{\frac{x-2}{x}}$

⑤ $f(x) = e^{-|x|}\sqrt{x^2-5x+6}$ (senza studio di $f''(x)$)

Svolgimento

Esercizio 1

(2) $\cos^2(x) = 1 - x^2 + \frac{x^4}{3} + o(x^5)$ per $x \to 0$

- $f(x) = \cos^2(x)$

$f(x) = P_5(x) + o(x^5)$

$$P_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = \cos^2(x)$ $\qquad$ $f(0) = 1$

$f'(x) = -2\cos(x)\sin(x)$ $\qquad$ $f'(0) = 0$

$= -\sin(2x)$

$f''(x) = -2\cos(2x)$ $\qquad$ $f''(0) = -2$

$f^{(3)}(x) = 4\sin(2x)$ $\qquad$ $f^{(3)}(0) = 0$

$f^{(4)}(x) = 8\cos(2x)$ $\qquad$ $f^{(4)}(0) = 8$

$f^{(5)}(x) = -16\sin(2x)$ $\qquad$ $f^{(5)}(0) = 0$

$$P_5(x) = 1 - x^2 + \frac{8}{4!} x^4$$

$$P_5(x) = 1 - x^2 + \frac{x^4}{3}$$

$$\cos^2(x) = 1 - x^2 + \frac{x^4}{3} + o(x^5) \quad \text{per } x \to 0$$

$$(\cos(x))^2 = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right)^2$$

Esercizio 2

$$f(x) = e^{\frac{x-2}{x}} = e^{1 - 2/x}$$

① $dom(f) = \mathbb{R} \setminus \{0\}$

② $f \geq 0$

$f(x) > 0 \quad \forall x \neq 0$

③ simmetrie

f ne' pari ne' dispari

④ Comportamento agli estremi del dominio

- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\frac{x-2}{x}} = 0$

  $\frac{x-2}{x} \to -\infty$, $x-2 \to -2$, $x \to 0^+$

  $\lim_{t \to -\infty} e^t = 0$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{\frac{x-2}{x}} = +\infty$

  $\frac{x-2}{x} \to +\infty$

- $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{\frac{x-2}{x}} = e$

  $\lim_{x \to \pm\infty} \frac{x-2}{x} = \lim_{x \to +\infty} \frac{\cancel{x}(1 - 2/x)}{\cancel{x}} = 1$

Asintoto orizzontale $y = e$ per $x \to \pm\infty$

- $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^{\frac{x-2}{x}} = e$

⑤ $f'(x) \geq 0$

$$f'(x) = \left(e^{\frac{x-2}{x}}\right)' = \left(\frac{x-2}{x}\right)' e^{\frac{x-2}{x}} = \frac{2}{x^2} e^{\frac{x-2}{x}}$$

$$\left(\frac{x-2}{x}\right)' = \left(1 - \frac{2}{x}\right)' = \frac{2}{x^2}$$

$f'(x) > 0 \quad \forall x \in \mathbb{R} \longrightarrow f$ strett. crescente

⑥ $f''(x) \geq 0$

$$f''(x) = 2\left(\frac{e^{\frac{x-2}{x}}}{x^2}\right)' = 2\left[(x^{-2})' e^{\frac{x-2}{x}} + \frac{1}{x^2} \cdot \left(e^{\frac{x-2}{x}}\right)'\right]$$

$$= 2e^{\frac{x-2}{x}}\left[-2\frac{1}{x^3} + \frac{2}{x^4}\right]$$

$$f''(x) = 4e^{\frac{x-2}{x}} \cdot \frac{-x+1}{x^4}$$

$f''(x) \geqslant 0$

$\cdot \ -x+1 \geqslant 0 \qquad x \leqslant 1$

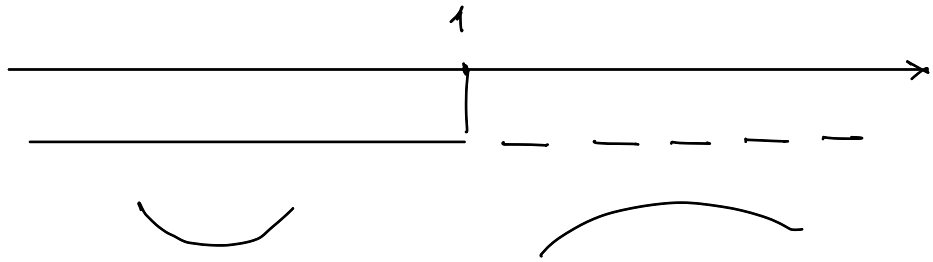
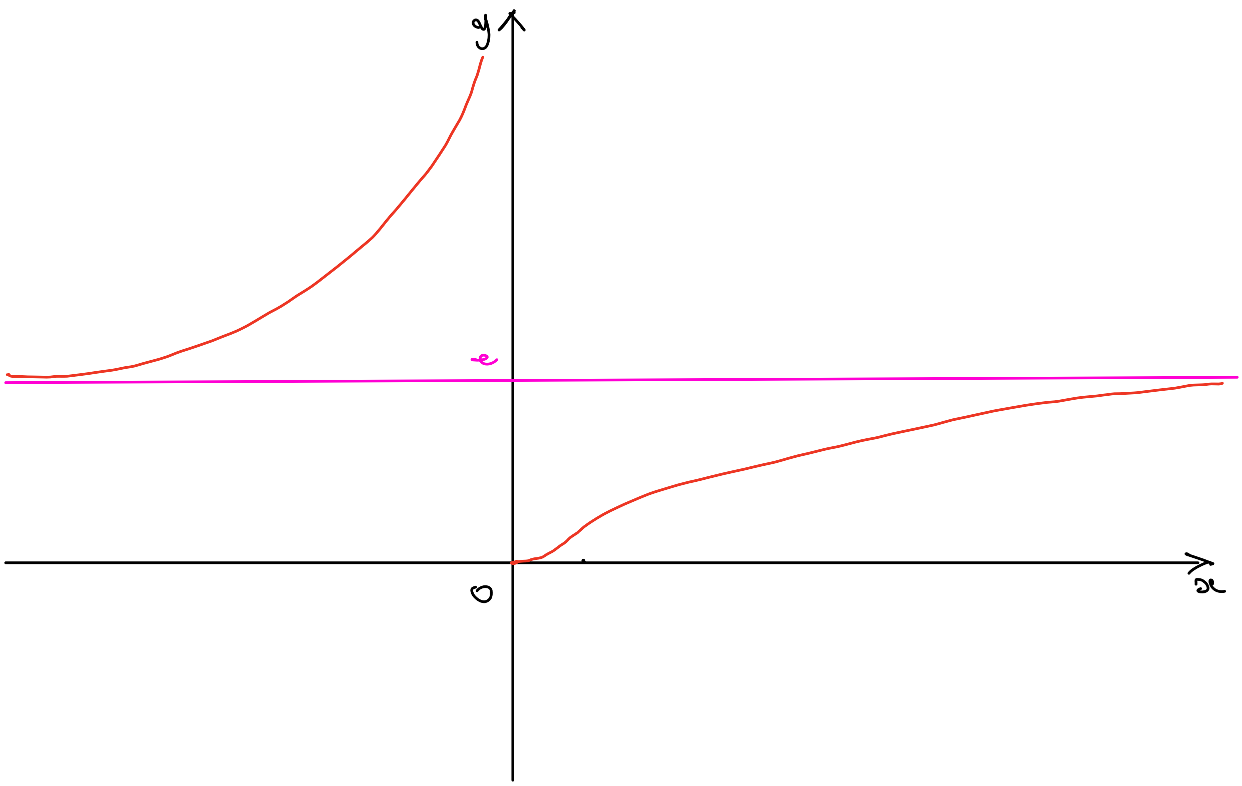


Grafico qualitativo



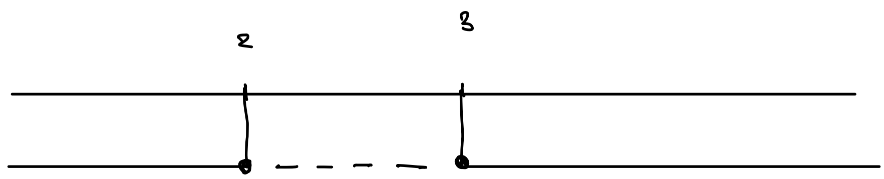
⑤ $f(x) = e^{-|x|}\sqrt{x^2-5x+6}$

① dom $(f) = (-\infty, 2] \cup [3, +\infty)$

$x^2 - 5x + 6 \geq 0$

$x^2 - 3x - 2x + 6 = x(x-3) - 2(x-3)$

$= (x-2)(x-3)$



② $f(x) \geq 0$

$e^{-|x|} > 0 \quad \forall x \in \mathbb{R}$

$\sqrt{x^2-5x+6} \geq 0 \quad \forall x \in \text{dom}(f)$

$= 0 \quad$ sse $\quad x = 2, x = 3$

③ limiti agli estremi del dominio

- $\lim_{x \to 2^-} f(x) = 0$
- $\lim_{x \to 3^+} f(x) = 0$
- $\lim_{x \to +\infty} f(x)$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\sqrt{x^2-5x+6}}{e^x}$$

$$= \lim_{x \to +\infty} \frac{x\sqrt{1-5/x+6/x^2}}{e^x} = 0$$

0

- $\lim_{x \to -\infty} f(x)$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{x^2-5x+6}}{e^{|x|}} =$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{1-5/x+6/x^2}}{e^{|x|}} = 0$$

$\sqrt{x^2} = |x|$

0

④ $f'(x) \geq 0$

$$f(x) = e^{-|x|}\sqrt{x^2-5x+6}$$

$$f'(x) = (e^{-|x|})'\sqrt{x^2-5x+6} + e^{-|x|}\left(\sqrt{x^2-5x+6}\right)'$$

$$f'(x) = e^{-|x|}\left(-\frac{x}{|x|}\sqrt{x^2-5x+6} + \frac{1}{2}\cdot\frac{2x-5}{\sqrt{x^2-5x+6}}\right)$$

$|x|$ non è derivabile in $x = 0$.

$$f'(x) = \frac{e^{-|x|}}{2\sqrt{x^2-5x+6}}\left(2x-5-\frac{x}{|x|}(2x^2-10x+12)\right)$$

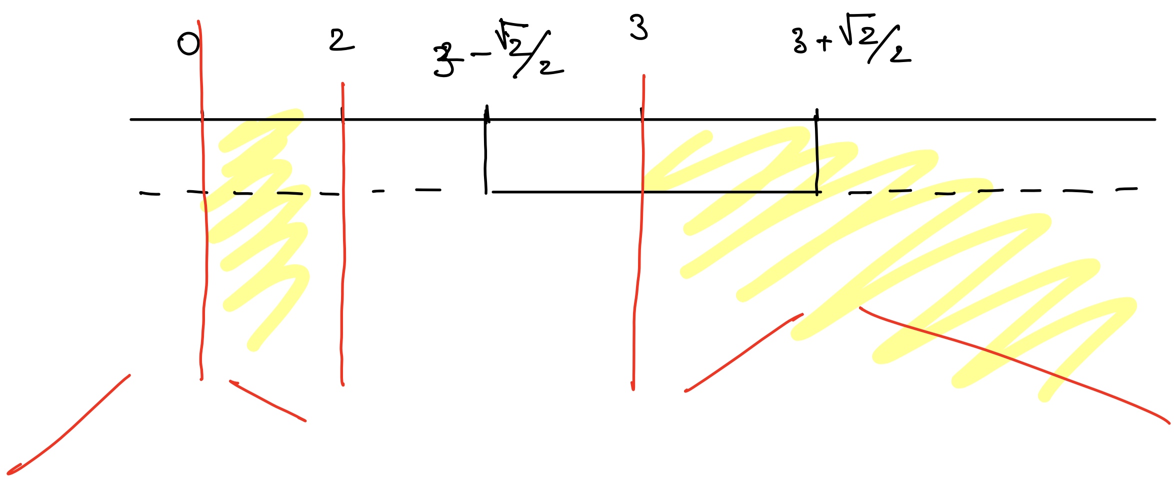
sempre > 0 in $(-\infty,2)\cup(3,+\infty)$

- $x > 0$ $\quad$ (e $x \in dom(f)$)

$$2x-5-(2x^2-10x+12) = -(2x^2-12x+17) \geq 0$$

$$x_{1,2} = \frac{6 \pm \sqrt{36-34}}{2} = \frac{6 \pm \sqrt{2}}{2}$$

$$x_{1,2} = 3 \pm \frac{\sqrt{2}}{2}$$

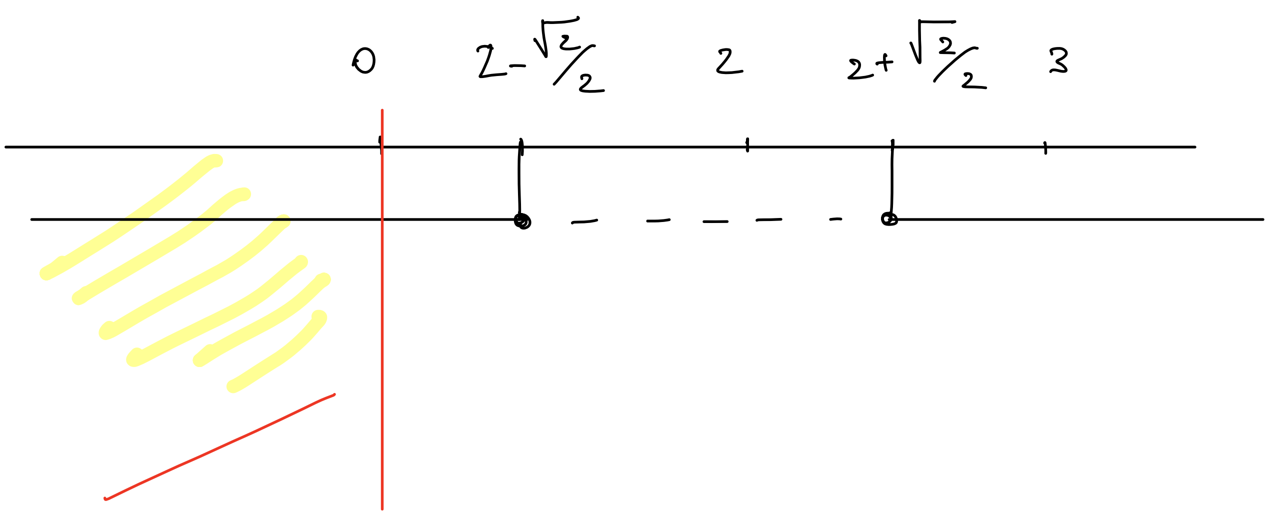


• $x < 0$

$$2x - 5 + (2x^2 - 10x + 12) \geq 0$$

$$2x^2 - 8x + 7 \geq 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 14}}{2} = 2 \pm \frac{\sqrt{2}}{2}$$

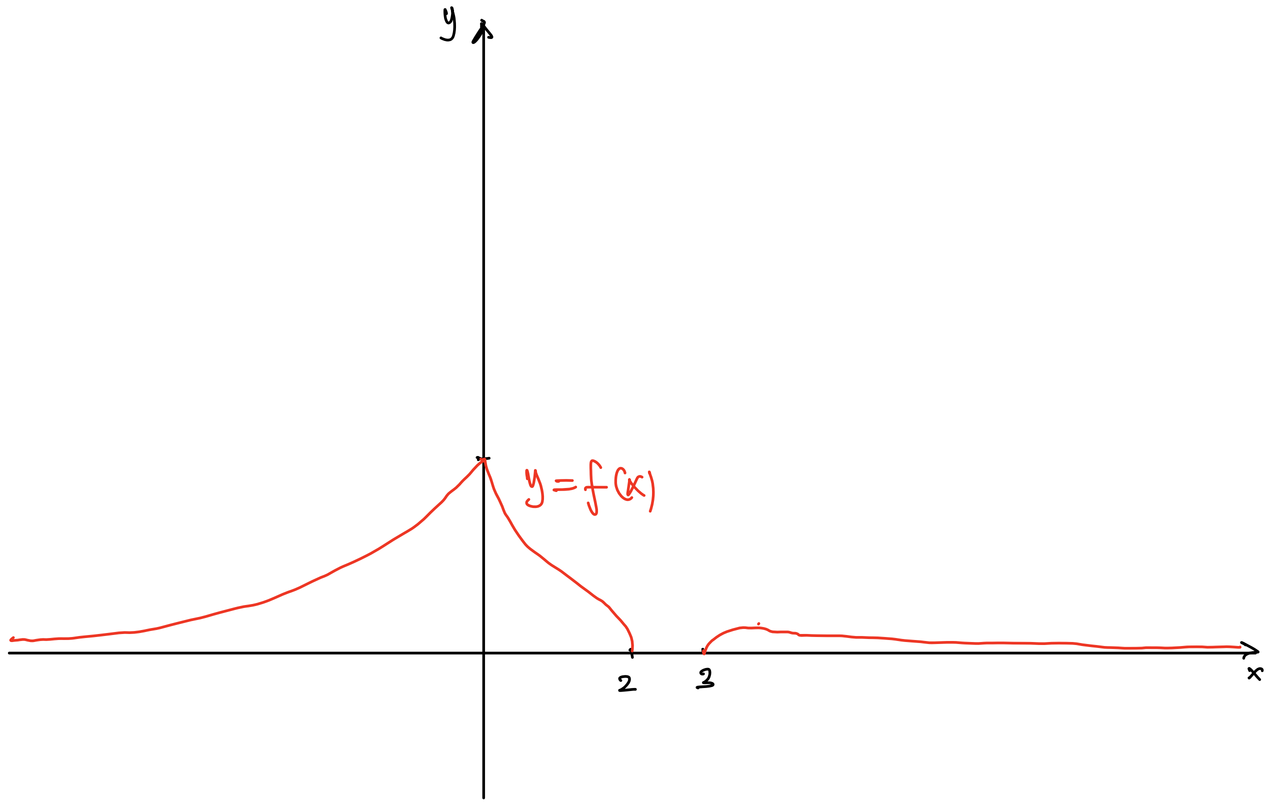


Punti di max relativo: $x = 0$, $x = 3 + \sqrt{2}/2$

$$f(0) = \left(e^{-|x|} \sqrt{x^2 - 5x + 6} \right)\Big|_{x=0} = \sqrt{6}$$

$$f\left(3 + \frac{\sqrt{2}}{2}\right) \cong 0{,}027$$

Grafico qualitativo



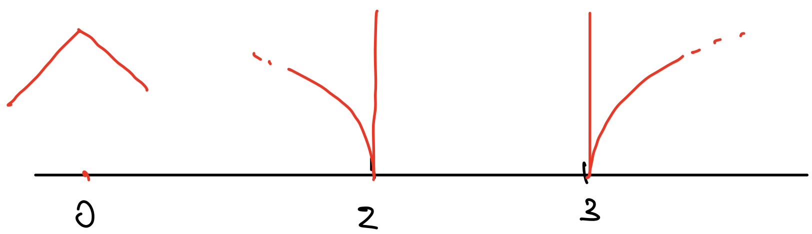
▷ Discutere la derivabilità di $f(x)$ in

$x = 0$; $x = 2$; $x = 3$

R: Non è derivabile in nessuno di questi 3 pti

$$f'(0^-) = \frac{7\sqrt{6}}{2} \qquad f'(0^+) = -\frac{17\sqrt{6}}{2}$$

$$\text{"} f'(2^-) = -\infty \text{"} \qquad \text{"} f'(3^+) = +\infty \text{)}$$



$$f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x} \overset{H}{=} \lim_{x \to 0^-} f'(x)$$

$$= \lim_{x \to 0^-} \frac{e^{-|x|}}{2\sqrt{x^2-5x+6}} \left(2x - 5 + (2x^2 - 10x + 12) \right)$$

$$= \frac{7}{2\sqrt{6}}$$

$$f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} \overset{H}{=} \lim_{x \to 0^+} f'(x)$$

$$= \lim_{x \to 0^+} \frac{e^{-|x|}}{2\sqrt{x^2-5x+6}} \left(2x - 5 - (2x^2 - 10x + 12) \right)$$

$$= -\frac{17}{2\sqrt{6}}$$

$$f'(2^-) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} \overset{H}{=} \lim_{x\to 2^-} f'(x)$$

$$= \lim_{x\to 2^-} \underbrace{\frac{e^{-|x|}}{2\sqrt{x^2-5x+6}}}_{+\infty} \underbrace{\left(2x-5-(2x^2-10x+12)\right)}_{-1}$$

$$= -\infty$$

$$f'(3^+) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} \overset{H}{=} \lim_{x\to 3^+} f'(x)$$

$$= \lim_{x\to 3^+} \underbrace{\frac{e^{-|x|}}{2\sqrt{x^2-5x+6}}}_{+\infty} \underbrace{\left(2x-5-(2x^2-10x+12)\right)}_{1}$$

$$= +\infty$$

5 $f''(x) \geq 0$

Il calcolo della derivata seconda è complicato e andava specificato che non era richiesto.