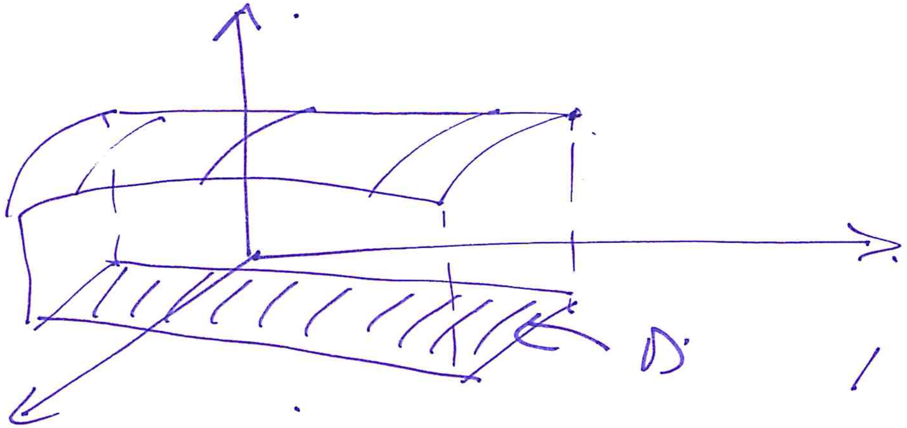
3/12/21

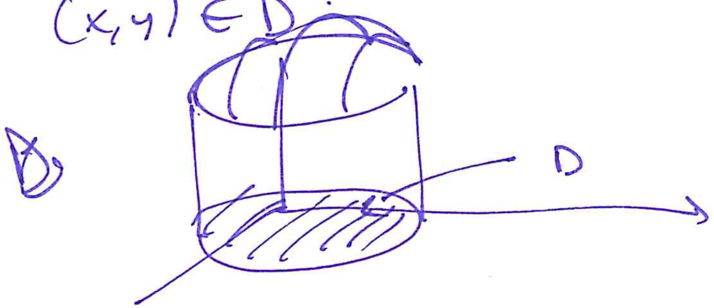
Integrali doppi



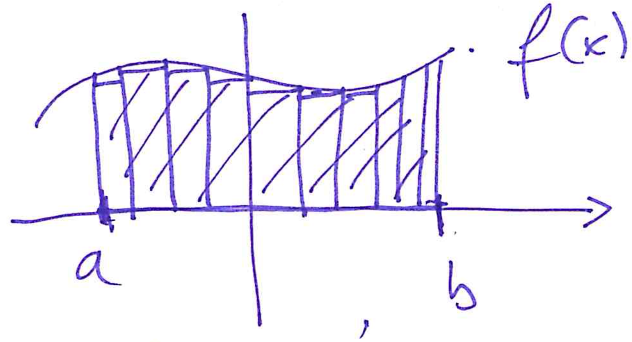
Calcolare il volume del sottografico.

Il volume compreso tra il grafico di $z=f(x,y) \geq 0$, il piano xy $(z=0)$ e ristretto a un dominio $D \subset \mathbb{R}^2$

$(x,y) \in D$.



$$\boxed{\iint_D f(x,y)\,dx\,dy.}$$



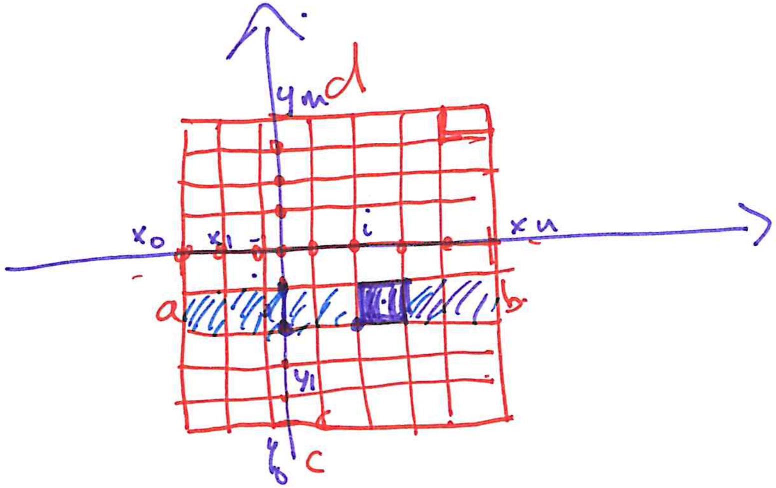
$f(x) \geq 0$.

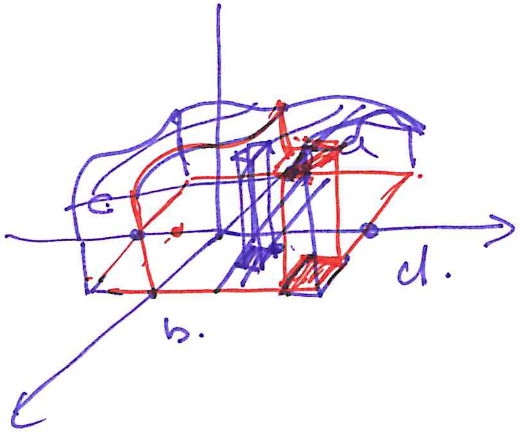
$$\int_a^b f(x)\,dx = \text{Area del sottografico}$$

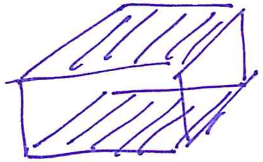
$$\| $$

$$F(b)-F(a). \qquad F'(x)=f(x)$$

1° Caso D sia un rettangolo: $D = [a,b] \times [c,d]$ $f(x,y)$ II







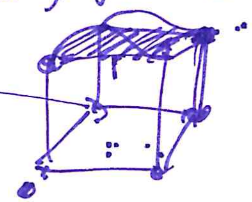
$a = x_0 < x_1 < x_2 < \dots < x_n = b$ $\longrightarrow$ $x_{i+1} - x_i$

$c = y_0 < y_1 < y_2 < \dots < y_m = d$ $\longrightarrow$ $y_{j+1} - y_j$

V_{ij} $f(x_i, y_i)$.

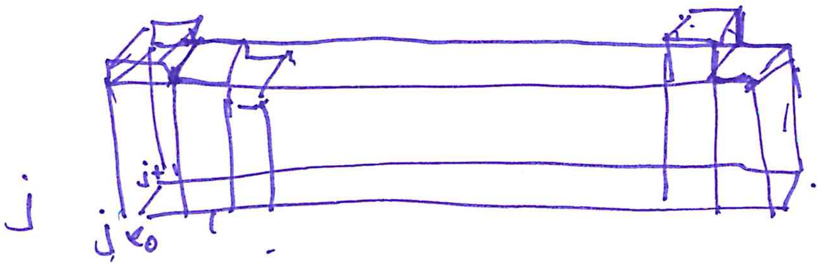
Cerchiamo il volume di un parallelepipedo che approssimi il volume del sottografico nel quadrato $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$; altezza sarà approssimativamente $f(\bar{x}, \bar{y})$ con $(\bar{x}, \bar{y}) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$

p.e.: $(\bar{x}, \bar{y}) = (x_i, y_i)$. $V_{ij} = (y_{j+1} - y_j)(x_{i+1} - x_i) \cdot f(x_i, y_j)$.



Sommiamo tra loro tutti i volumi di questi parallelepipedi in modo da approssimare il volume del sottografico.

$$\tilde{V}_{n,m} = \sum_{j=0}^{m-1}\left(\sum_{i=0}^{n-1} V_{ij}\right) = \sum_{j=0}^{m-1}\left(\sum_{i=0}^{n-1} f(x_i,y_j)(x_{i+1}-x_i)(y_{j+1}-y_j)\right).$$

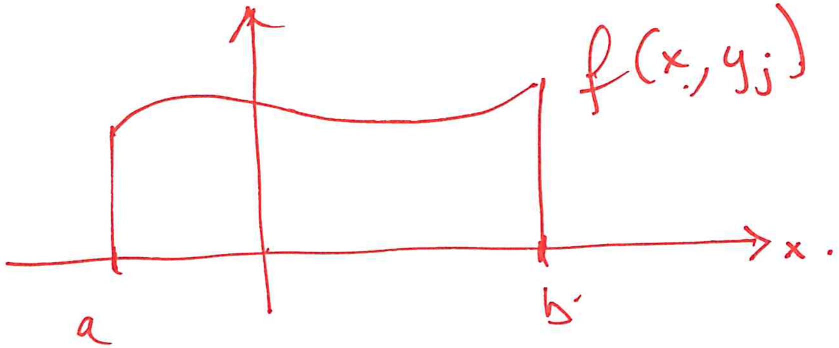


somma di Riemann.

$$= \sum_{j=0}^{m-1}\left(\sum_{i=0}^{n-1} f(x_i,y_j)(x_{i+1}-x_i)\right)(y_{j+1}-y_j) \longrightarrow \sum_{j=0}^{m-1}\left(\int_a^b f(x,y_j)\,dx\right)(y_{j+1}-y_j)$$

$n \to +\infty$

$$\int_a^b f(x,y_j)\,dx.$$



$$g(y) = \int_a^b f(x,y)\,dx.$$

... parametro.

$$= \sum_{j=0}^{m-1}\left(\int_a^b f(x,y_j)\,dx\right)(y_{j+1}-y_j) = \sum_{j=0}^{m-1} g(y_j)(y_{j+1}-y_j) \xrightarrow{m\to+\infty} \int_c^d g(y)\,dy.$$

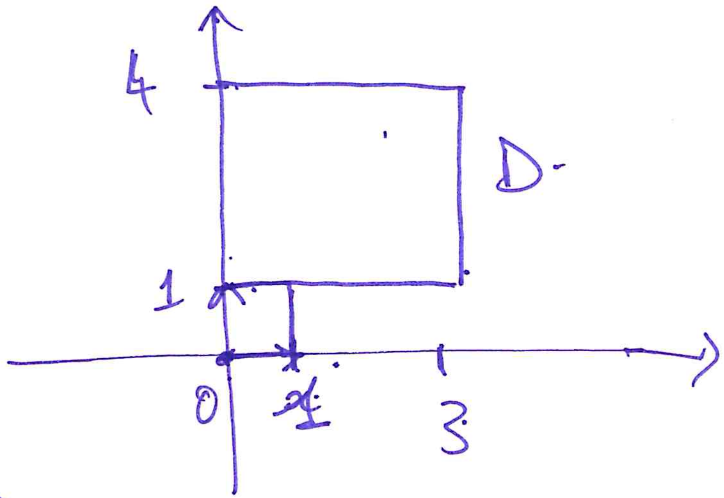
$$\tilde{V}_{n,m} \xrightarrow{(n,m)\to+\infty} V = \int_c^d g(y)\,dy = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{g(y)} dy.$$

Se $D=[a,b]\times[c,d]$, la formula di riduzione:

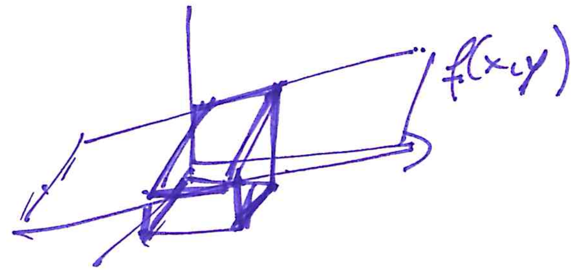
$$V = \underbrace{\iint_D f(x,y)\,dx\,dy}_{\text{Concetto nuovo}} = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{\substack{\text{integrale della funzione } h(x)=f(x,y) \\ \text{quindi si può usare il Teorema Fondam.} \\ \text{del Calcolo Integrale}}} dy.$$

$f(x,y) = x+2y \qquad D=[0,3]\times[1,4]$

$$\iint_D x+2y\,dx\,dy = \int_1^4\left(\int_0^3 x+2y\,dx\right)dy =$$



$$\left(\int_0^3 x+\underbrace{2y}_{\text{pontica}}\,dx = \frac{x^2}{2}+2y\,x\Big|_0^3 = \frac{9}{2}+6y = g(y)\right).$$



$$= \int_1^4 \frac{9}{2}+6y\,dy = \frac{9}{2}y+3y^2\Big|_1^4 = \left(\frac{9}{2}\cdot 4+3\cdot 16\right)-\left(\frac{9}{2}+3\right).$$

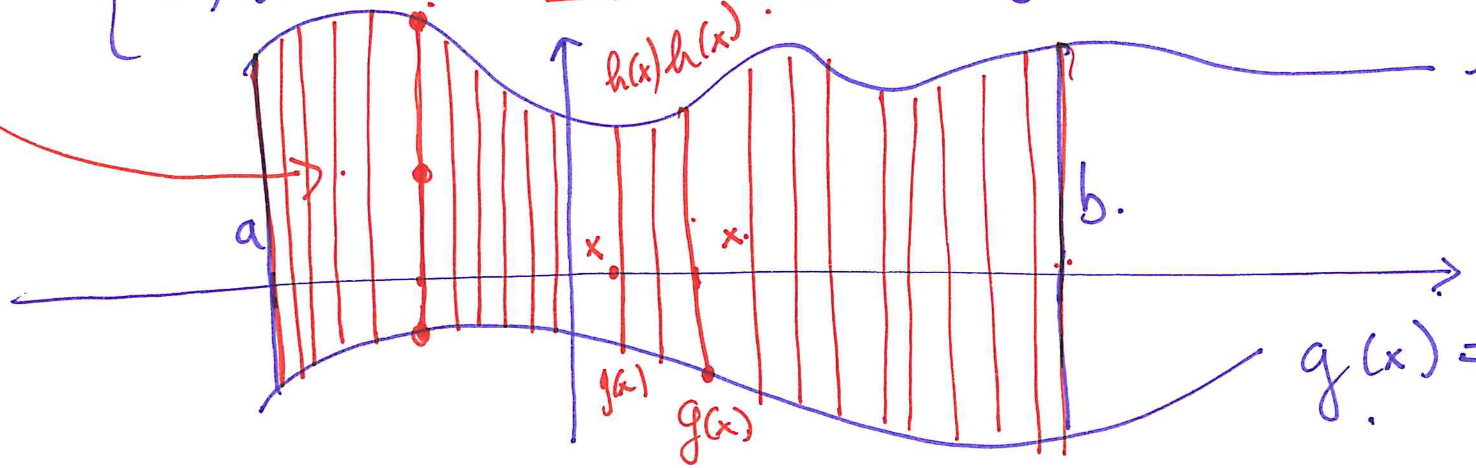
$$= \frac{27}{2}+3\cdot 15 = \frac{27}{2}+45 = \frac{117}{2}\ u^3$$

$u=[0,$

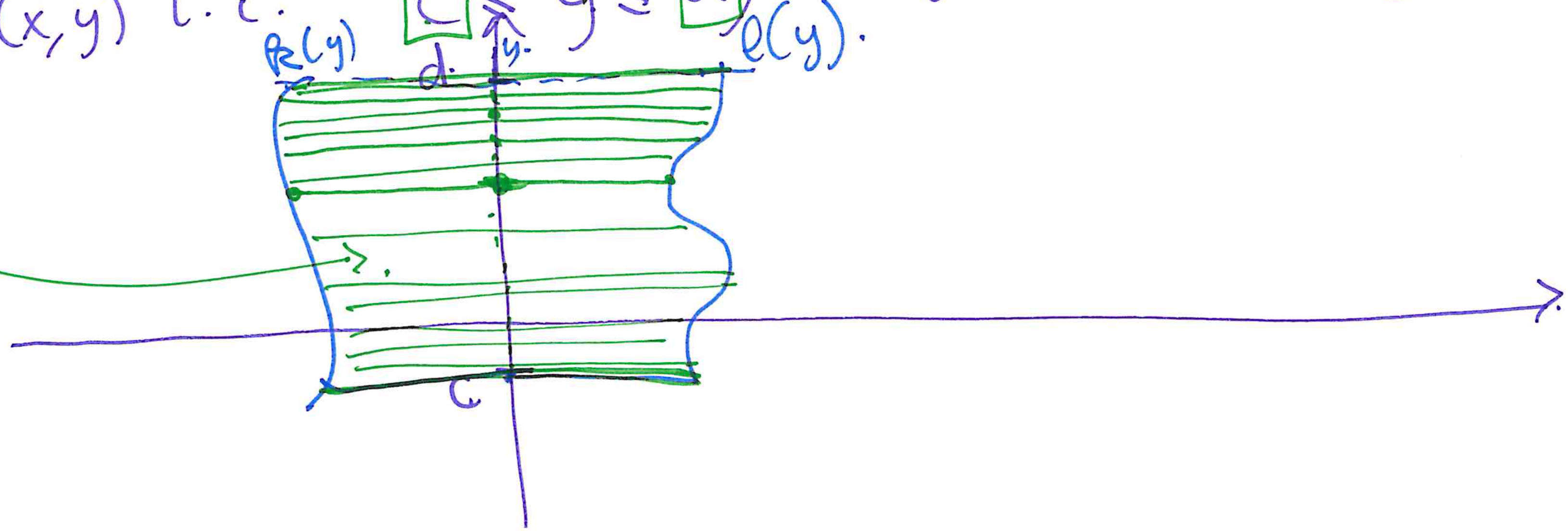
Domini Normali

~~Def~~ DEFINIZIONE $D \subset \mathbb{R}^2$ $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, $d \in \mathbb{R}$

$$D_x = \{ (x,y) \text{ t.c. } \boxed{a} \leq x \leq \boxed{b},\ g(x) \leq y \leq h(x) \} \rightarrow \text{risp } x$$



$$D_y = \{ (x,y) \text{ t.c. } \boxed{c} \leq y \leq \boxed{d},\ k(y) \leq x \leq \ell(y) \} \rightarrow \text{risp. } y$$



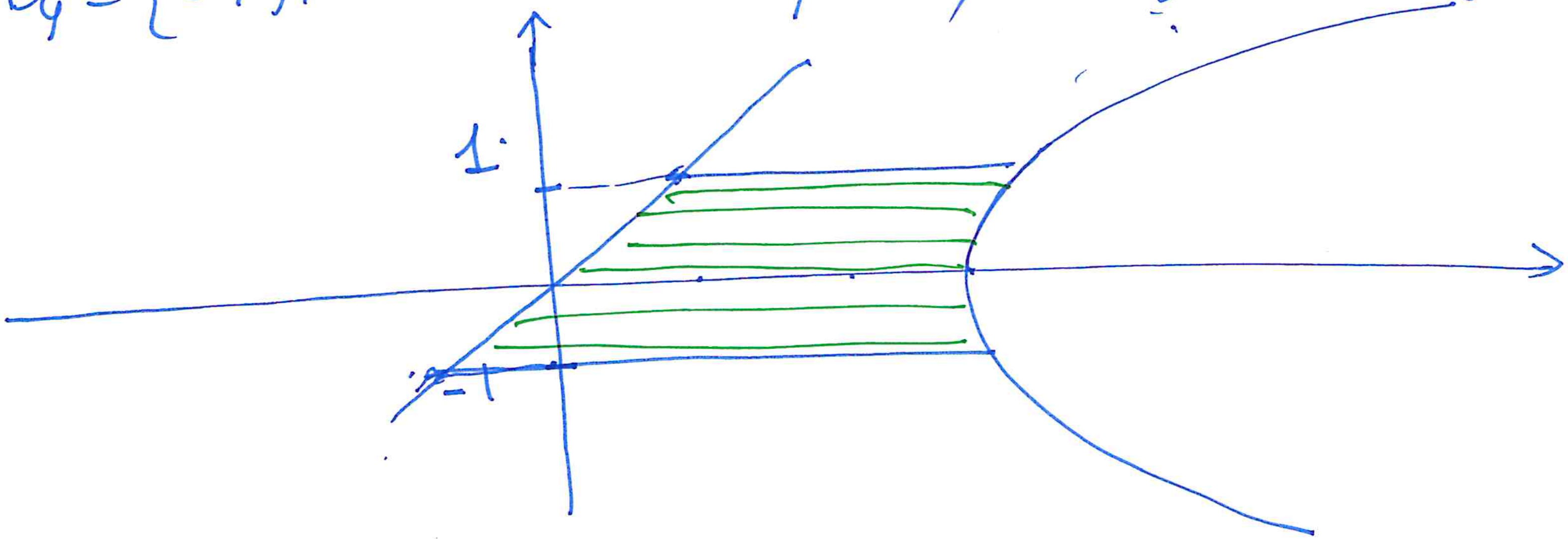
Formula di Riduzione:

$$\iint_{D_y} f(x,y)\,dx\,dy := \int_c^d \left(\int_{k(y)}^{\ell(y)} f(x,y)\,dx \right) dy$$

$D_y = \{(x,y)$ t.c. $\boxed{c} \le y \le \boxed{d}$, $k(y) \le x \le \ell(y)\}$.

$c \in \mathbb{R}$, $d \in \mathbb{R}$

$D_y = \{(x,y)$ t.c. $-1 \le y \le 1$, $y \le x \le y^2+3\}$



$$\iint_{D_y} x^2+y\,dx\,dy.$$

..

$$\iint_D xy\,dx\,dy = \int_0^1 \underbrace{\left(\int_{-x^2}^{x^2+1} xy \qquad dy\right)} dx = \qquad \left(D = \{0 \le x \le 1, -x^2 \le y \le x^2+1\right.$$

$$\int_{-x^2}^{x^2+1} xy\,dy \underset{\substack{\uparrow \\ \text{TFI}}}{=} x\frac{y^2}{2}\Bigg|_{-x^2}^{x^2+1} = x\frac{(x^2+1)^2}{2} - x\frac{(-x^2)^2}{2} = \frac{x}{2}(x^4+2x^2+1) - \frac{x^5}{2}$$

$$= x^3 + \frac{x}{2} \rightarrow \text{funzione di } x.$$

$$= \int_0^1 x^3 + \frac{x}{2}\,dx = \frac{x^4}{4} + \frac{x^2}{4}\Bigg|_0^1 = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

NUMERO, né x né y.

La Formula di Riduzione

$$\iint_{D_x} f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

$x \neq y$.

x parametro.

$$D_x = \{(x,y) \; a \leq x \leq b,\, g(x) \leq y \leq h(x)\}$$

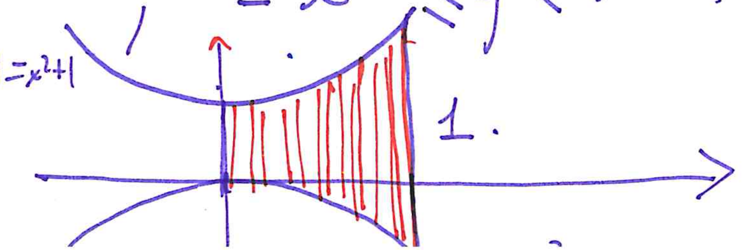
dipende solo da x
(la $y \to h(x)$ e $y \to g(x)$)

$$:= \int_a^b dx \int_{g(x)}^{h(x)} f(x,y)\,dy.$$

(scrittura alternativa)

$$D = \{(x,y) \text{ t.c. } 0 \leq x \leq 1,\; -x^2 \leq y \leq x^2+1\}.$$

$$\iint_D xy\,dx\,dy = ?$$



$$\iint_{D_y} x^2+y\,dx\,dy = \int_{-1}^{1}\left(\int_{y}^{y^2+3} x^2+y\quad dx\right)dy.$$

$$= \int_{-1}^{1}\left(\frac{x^3}{3}+yx\right)\Bigg|_{y}^{y^2+3} dy = \int_{-1}^{1} \frac{1}{3}(y^2+3)^3 + y(y^2+3) - \left(\frac{y^3}{3}+y^2\right)\ dy$$

la x è sparita

$$=\int_{-1}^{1} \frac{1}{3}\left(y^6+3\cdot y^4\cdot 3+3y^2\cdot 3^2+3^3\right)+y^3+3y-\frac{y^3}{3}-y^2\,dy.$$

$$=\int_{-1}^{1} \frac{1}{3}y^6+3y^4+\frac{2}{3}y^3+8y^2+3y+9\,dy = \frac{1}{21}y^7+\frac{3}{5}y^5+\frac{2}{12}y^4+\frac{8}{3}y^3+\frac{3}{2}y^2+9y\Bigg|_{-1}^{1}$$

$$=\frac{2}{21}+\frac{6}{5}+\frac{16}{3}+18$$

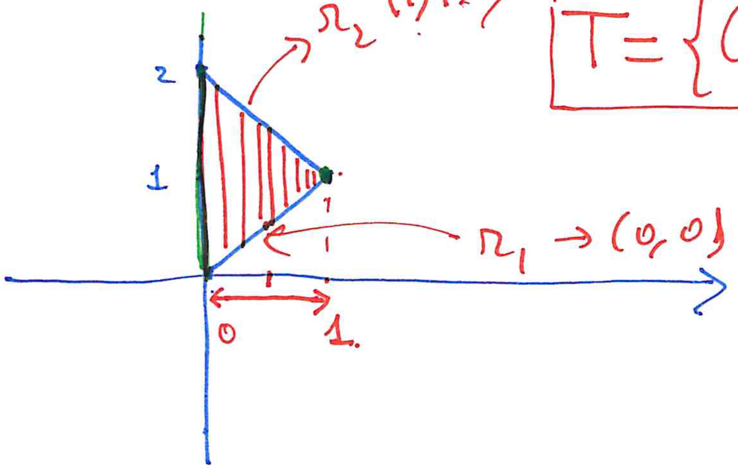
T il triangolo di vertici $(0,0)$, $(1,1)$, $(0,2)$.

$$\iint_T x e^y \, dx\, dy = \int_0^1 \left(\int_x^{-x+2} x e^y \, dy \right) dx$$

$$T = \{ (x,y) \text{ t.c. } 0 \le x \le 1, \quad x \le y \le -x+2 \}$$

costanti

funzioni



r_2 $(1,1)$, $(0,2)$.

$r_1 \to (0,0)$ e $(1,1) \to y = x$.

$$\boxed{\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}}$$

$$\frac{x}{1} = \frac{y-2}{-1}$$

$$\boxed{y + x = 2}$$

$$y = -x + 2.$$

$$= \int_0^1 \left(x e^y \Big|_x^{-x+2} \right) dx$$

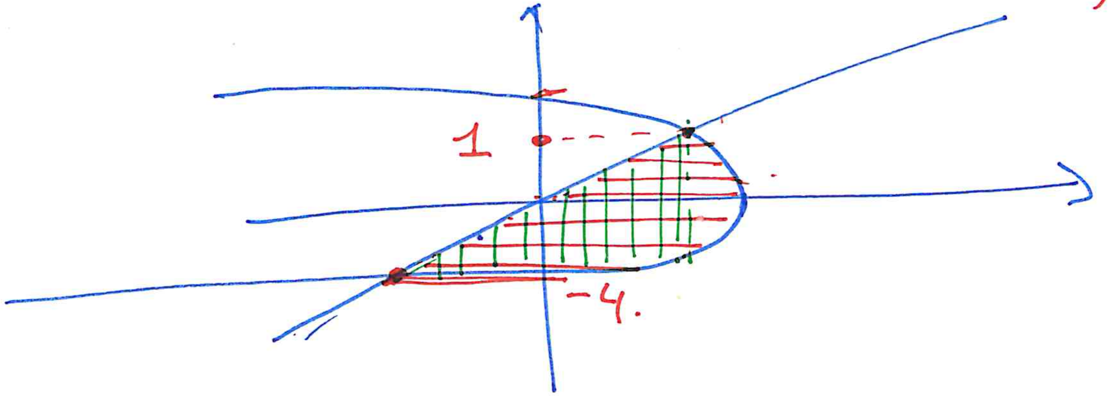
$$= \int_0^1 x e^{-x+2} - x e^x \, dx.$$

$$= \left\{ -x e^{-x+2} - x e^x \Big|_0^1 - \left(\int_0^1 -e^{-x+2} - e^x \, dx \right) \right. = -x e^{-x+2} - x e^x - e^{-x+2} + e^x \Big|_0^1$$

$$= -e - e - e + e - (-e^2 + 1)$$

$$= -2e + e^2 - 1 = e^2 - 2e - 1$$

$$\iint_D xy\,dx\,dy \qquad D = \{(x,y) \text{ t.c. } 3y \le x \le 4-y^2\} = \{(x,y) : -4 \le y \le 1,\ 3y \le x \le 4-y^2\}$$



Trovare i punti di intersezione tra $\begin{cases} x = 3y \\ x = 4-y^2 \end{cases} \Rightarrow 3y = 4 - y^2$

$$y^2 + 3y - 4 = 0$$

$$y = \frac{-3 \pm \sqrt{9+16}}{2}$$

$$y = \frac{-3 \pm 5}{2} = \begin{cases} 1 \\ -4 \end{cases}$$

$$\iint_D xy\,dx\,dy = \int_{-4}^{1} \left(\int_{3y}^{4-y^2} xy\,dx \right) dy$$

$$= \int_{-4}^{1} \left(\frac{x^2}{2} y \right)_{3y}^{4-y^2} dy = \int_{-4}^{1} \Big(\underbrace{\frac{y}{2}(4-y^2)^2 - \frac{y}{2}(3y)^2}_{\text{non c'è più } x\,!} \Big) dy = \int_{-4}^{1} \frac{y}{2}(16 - 8y^2 + y^4) - \frac{9}{2} y^3\,dy$$

$$\int_{-4}^{1} \frac{1}{2}y^5 - \frac{17}{2}y^3 + 8y \, dy = \frac{1}{12}y^6 - \frac{17}{8}y^4 + 4y^2 \Big|_{-4}^{1} = \frac{1}{12} - \frac{17}{8} + 4 - \left(\frac{1}{4}(-4)^6 - \frac{17}{8}(-4)^4 + 4(-4)^2 \right)$$

$$\ldots$$