Esercizio 1 ESONERO BIANCO

$$\varphi(t) = (t\cos 3t,\ t\sin 3t) \qquad t \in [0, 2\pi]$$

$x(t) = t\cos 3t$

$y(t) = t\sin 3t$

$$x^2+y^2 = t^2\cos^2 3t + t^2 \sin^3 3t = t^2.$$

a) $(0, -\frac{\pi}{2}) \in \varphi$. $\exists\, t \in [0, 2\pi]$ t.c. $\varphi(t) = (0, -\frac{\pi}{2})$?

$$\begin{cases} t\cos 3t = 0 \\ t\sin 3t = -\frac{\pi}{2}. \end{cases}$$

$t = 0$ $\qquad \cos 3t = 0$

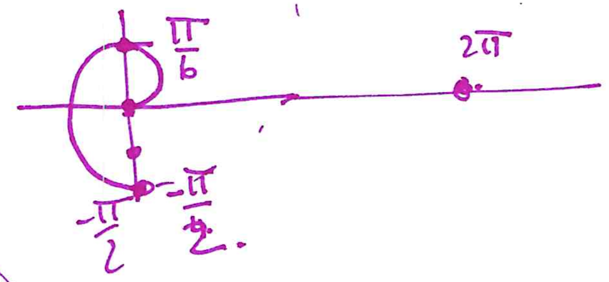
$\downarrow$

$\cancel{3t = \frac{\pi}{2} \to t = \frac{\pi}{6} \to \frac{\pi}{6} = -\frac{\pi}{2}\ ?}$

$3t = \frac{3\pi}{2} \to t = \frac{\pi}{2} \to \frac{\pi}{2}(-1) = -\frac{\pi}{2}$

$3t = \frac{5\pi}{2}$ $\qquad \sin\frac{3\pi}{2}$.

$(0, -\frac{\pi}{2}) = \varphi(\frac{\pi}{2})$ sì!



b) Chiusa? e regolare?

$\Downarrow$

$\varphi(0) = (0,0)$ $\neq$

$\varphi(2\pi) = (2\pi, 0)$.

La curva non è chiusa!

regolare

$\downarrow$

$$\varphi'(t) = (\underbrace{\cos 3t - 3t\sin 3t}_{x'(t)},\ \underbrace{\sin 3t + 3t\cos 3t}_{y'(t)})$$

Verifichiamo che $\varphi'(t) \neq (0,0)$ $\forall t$

$|\varphi'(t)|^2 \neq 0$.

$$|\varphi'(t)|^2 = (\cos 3t - 3t\sin 3t)^2 + (\sin 3t + 3t\cos 3t)^2$$

$$= \cos^2 3t + \sin^2 3t + 9t^2(\sin^2 3t + \cos^2 3t) \cancel{- 6t\cos 3t\sin 3t} \cancel{+ 6t\cos 3t\sin 3t}$$

$$= 1 + 9t^2 > 0$$

φ è <u>regolare</u>.

In alternativa

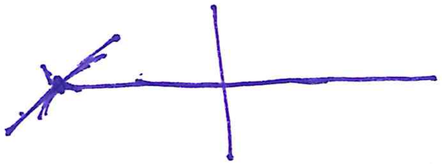
$$\begin{cases} \cos 3t - 3t\sin 3t = 0 \\ \sin 3t + 3t\cos 3t = 0 \end{cases} \quad ?$$

difficili

c) Determinare un vettore tangente in $\varphi\left(\frac{\pi}{3}\right)$

$\Longrightarrow \varphi'\left(\frac{\pi}{3}\right) = (\cos\pi - \pi\sin\pi,\ \sin\pi + \pi\cos\pi) = (-1, -\pi)$. vettore tangente

$(1, \pi)$.



d) $L(\varphi) = \int_0^{2\pi} |\varphi'(t)|\, dt = \int_0^{2\pi} \sqrt{1 + 9t^2}\, dt = \frac{1}{3} \int_0^{6\pi} \sqrt{1+s^2}\, ds.$

$s = 3t \quad ds = 3\,dt$

$s = \sinh x = \frac{e^x - e^{-x}}{2}. \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$

$\cosh^2 x - \sinh^2 x = 1$

$(\cosh x)' = \sinh x$

$(\sinh x)' = \cosh x.$

$ds = \cosh x\, dx$

$\sqrt{1 + \sinh^2 x} = \cosh^2 x.$

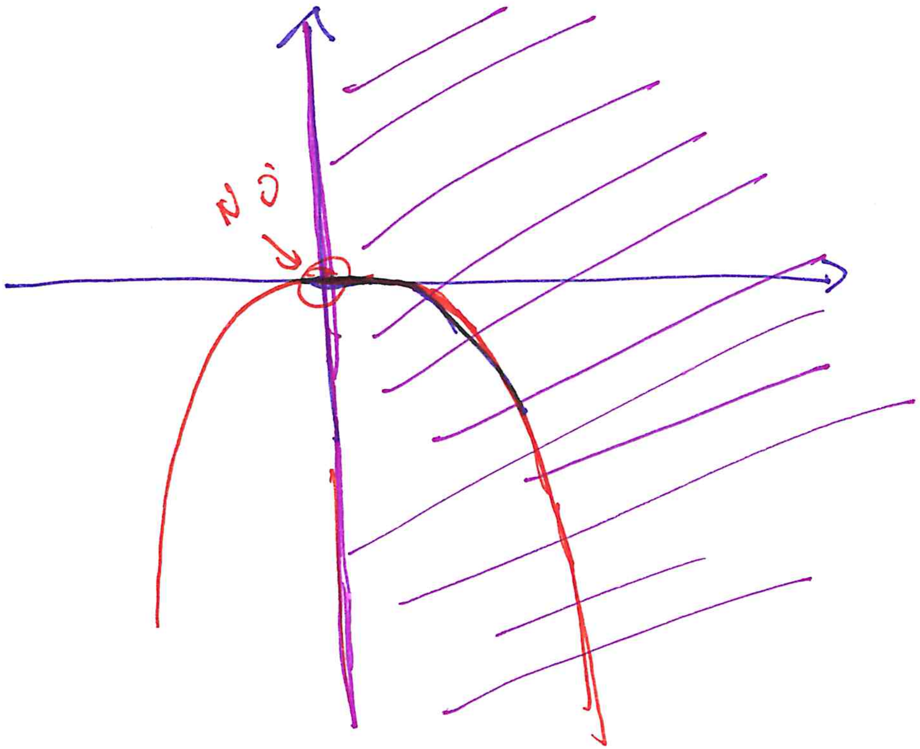
$= \frac{1}{3} \int_{\sim}^{\sim} (\cosh x)^2\, dx = \cdots = \frac{1}{3}\left(\frac{s}{2}\sqrt{s^2+1} + (\sinh)^{-1}(s)\right)\Big|_0^{6\pi}$

$= \pi \sqrt{(6\pi)^2 + 1} + \frac{1}{3}(\sinh)^{-1}(6\pi).$

Esercizio 2 : a) $f(x,y) = \frac{\sqrt{x}}{x^2+y}$ Dominio di esistenza.

$$\boxed{\begin{cases} x \geq 0 \\ x^2+y \neq 0 \end{cases}}$$

$x^2+y=0$ $y=-x^2$.



Dominio è la parte colorata in viola, esclusa la parabola in rosso!

b). $f(x,y) = x+3xy$, $g(x) = \sin(3x)$.

sostituito dalla funzione f.

$h(x,y) = g(f(x,y)) = \sin(3(x+3xy)) = \sin(3x+9xy)$

$k(x,y) = f(x, g(y)) = x + 3x(\sin 3y)$.

~~$z(g(x),y)$~~

$z(x,y) = f(g(x),y) = \sin 3x + 3 \cdot \sin 3x \cdot y = \sin 3x + 3y \sin 3x$

c) $f(x,y) = x e^{xy^2} \longrightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$

$$\frac{\partial f}{\partial x} = 1 \cdot e^{xy^2} + x \cdot e^{xy^2} \cdot y^2 = e^{xy^2}(1 + xy^2).$$

$$\frac{\partial f}{\partial y} = x\, e^{xy^2} (2xy) = 2x^2y\, e^{xy^2}.$$

$$\nabla f = e^{xy^2}(1 + xy^2,\ 2x^2y) \equiv \left(e^{xy^2}(1+xy^2),\ e^{xy^2} 2x^2 y\right).$$

d) Eq. del piano tangente in $(0,0)$ di $f(x,y) = x^2 - y^2 + 2xy$.

$$z = \underbrace{\frac{\partial f}{\partial x}(x_0,y_0)}\,(x - \underset{\uparrow \atop 0}{x_0}) + \underbrace{\frac{\partial f}{\partial y}(x_0,y_0)}\,(y - \underset{\uparrow \atop 0}{y_0}) + \underbrace{f(x_0,y_0)}$$

$f(0,0) = 0$

$\frac{\partial f}{\partial x} = 2x + 2y \longrightarrow \frac{\partial f}{\partial x}(0,0) = 0$

$\boxed{z = 0}$ eq. del piano tangente

$\frac{\partial f}{\partial y} = -2y + 2x \qquad \frac{\partial f}{\partial y}(0,0) = -0 + 0 = 0$

Nel pto $(1, 0) \longrightarrow f(1,0) = 1.$

$z = 2(x-1) + 2(y-0) + 1.$

$\frac{\partial f}{\partial x}(1,0) = 2.$

$\boxed{z = 2x - 1} \longrightarrow$ piano tangente.

$\frac{\partial f}{\partial y}(1,0) = 2.$

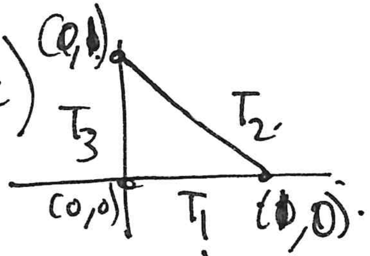
Esercizio 3 a) Punti critici e loro natura:

$$f(x,y)=x^2-y^2 \longrightarrow \nabla f=(2x,-2y)=(0,0) \iff (x,y)=(0,0)$$ unico punto critico

$$D^2 f=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \qquad \det D^2 f=-4 \rightarrow (0,0)$$ pt. di sella

b) $f(x,y)=xe^{xy}$, $\nabla f=(1+xy,\ x^2)e^{xy}=(0,0) \iff \begin{cases} 1+xy=0 \\ x^2=0 \end{cases} \rightarrow x=0 \rightarrow 1=0$ IMP.

f non ha punti critici.

c) 

$$f(x,y)=xy-x$$

$y=0 \rightarrow f(x,0)=-x \searrow f(0,0)=0,\ f(0,1)=-1$
$0\le x\le 1$

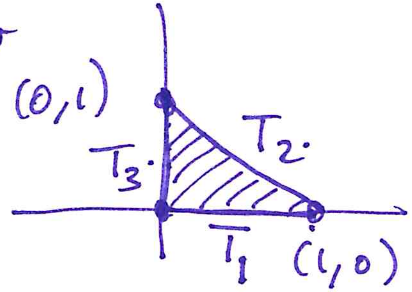
$$f(x,-x+1)=x(-x+1)-x=-x^2=g(x)$$

$T_2 \longrightarrow \begin{cases} y=-x+1 \\ 0\le x\le 1 \end{cases}$, $g'(x)=-2x\le 0 \implies g(x)\searrow$, $g(1)=-1$, $g(0)=0$.

$T_3 \rightarrow \{x=0,\ 0\le y\le 1\} \rightarrow f(0,y)=0$

$\max_T f=0 \quad \min_T f=-1.$

$f(x,y) = xy - x$ massimo e minimo assoluto nel triangolo



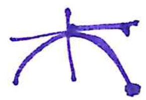
1) $\nabla f = (y-1, x) = (0,0)$ $\begin{cases} y-1=0 \\ x=0 \end{cases}$ $y=1$, $x=0$. $f(0,1)=0$

2) $T_1 \rightarrow \begin{cases} y=0 \\ 0 \le x \le 1 \end{cases} \rightarrow f(x,0) = -x$ $\searrow$ $\begin{array}{l} f(0,0)=0 \\ f(1,0)=-1 \end{array}$

monotona decrescente $x \in [0,1]$

$T_2 \rightarrow \begin{cases} y=-x+1 \\ 0 \le x \le 1 \end{cases} \rightarrow f(x,-x+1) = x(-x+1) - x = -x^2$ $\searrow$ $\begin{array}{l} f(0,1)=0 \\ f(1,0)=-1 \end{array}$

$0 \le x \le 1$



la retta passante per i punti (0,1) e (1,0).

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{x}{1} = \frac{y-1}{-1} = -x = y-1$$

$$y = -x+1$$

$$T_3 \to \left\{ \begin{array}{l} x=0 \\ 0\le y\le 1 \end{array} \right\} \quad f(0,y)=0$$

$$\max_{T} f = 0 \qquad \min_{T} f = -1.$$

Esercizio 4 $\quad y' = \frac{x^2}{y^2} = x^2 \cdot \frac{1}{y^2}$ (Variabili separabili)

a). $y(x) = x$ è soluzione? Calcolare il termine di sinistra

$\boxed{y'(x) = 1}$, poi calcolare il termine di destra

$\boxed{\frac{x^2}{y^2} = \frac{x^2}{x^2} = 1.}$ $\underset{x \neq 0}{\Longrightarrow}$ $y(x)$ è soluzione. $(0, +\infty)$.

e

$(-\infty, 0)$.

$y' = g(x) \cdot f(y)$ con $g(x) = x^2$, $f(y) = \frac{1}{y^2}$.

b). $f(y) = 0$ se $\frac{1}{y^2} = 0$ è impossibile.

$y' = \frac{x^2}{y^2} \longrightarrow \boxed{y' y^2 = x^2.} \xrightarrow{\text{Integriamo}} \int y^2 y' \, dx = \int x^2 \, dx = \frac{x^3}{3} + C.$

$$\int y^2(x) y'(x) \, dx = \int s^2 \, ds = \frac{s^3}{3} = \frac{y(x)^3}{3}.$$

$S = y(x)$
$ds = y'(x) dx$

$\Downarrow$

$\frac{y(x)^3}{3} = \frac{x^3}{3} + C. \longrightarrow \boxed{Eq}$

Trovare y in funzione di x.

$\Downarrow$

$y^3 = x^3 + 3C.$

$\boxed{y(x) = \sqrt[3]{x^3 + 3C}. \quad \forall C \in \mathbb{R}}$

c) $\boxed{y(2)=-3.}$ $\longrightarrow$ $y(x)=\sqrt[3]{x^3+3c}$

Incognita è c

calcolare $y(2)=\sqrt[3]{\underbrace{8+3c}}=-3.$

elevando al cubo $\downarrow$

$8+3c=(-3)^3=-27$

$\underbrace{3c}=-27-8=-35$

$\Longrightarrow$ La soluzione del Pb di Cauchy è

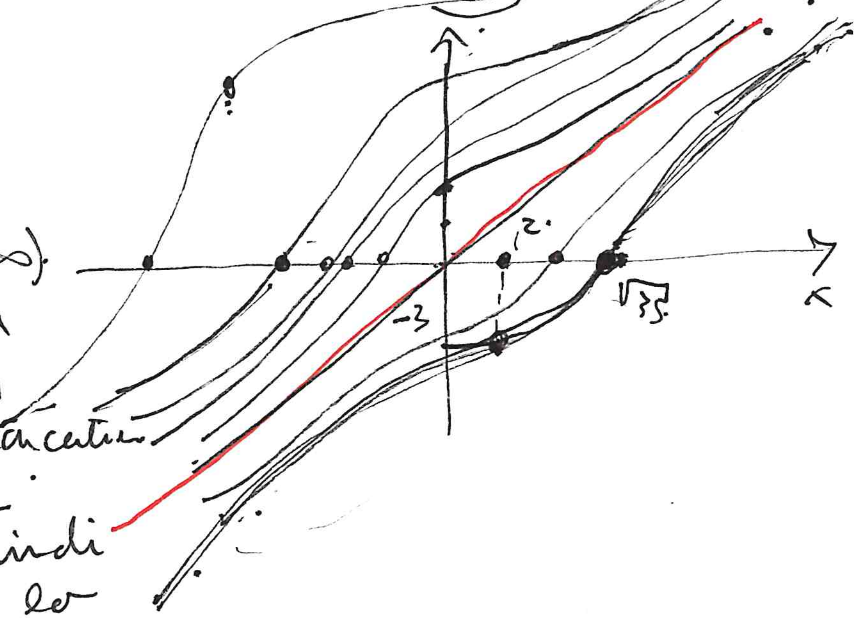
$y(x)=\sqrt[3]{x^3-35}.$ in $(-\infty,\sqrt[3]{35})$

e in $(\sqrt[3]{35},+\infty)$

$x^3-35=0 \Longrightarrow x^3=35.$

$x=\sqrt[3]{35}>2.$

mancata 2 quindi non la posso considerare.



d) $2\in(-\infty,\sqrt[3]{35})$ $(\sqrt[3]{35},+\infty)$

$\Downarrow$

y è soluzione in $(-\infty,\sqrt[3]{35})$

$y(x)=\sqrt[3]{x^3-35}$

$$\begin{cases} y'=\dfrac{x^2}{y^2} \\ y(2)=-3. \end{cases}$$

Esercizio 5 : $y'' + 3y = 2x$.

a) $y(x) = \frac{1}{3}x$ è soluzione ?

Calcoliamo $y'(x) = \frac{1}{3}$
$y''(x) = 0$.

$\Rightarrow 0 + 3\left(\frac{1}{3}x\right) = x \overset{?}{\neq} 2x$. $\boxed{\text{NO}}$

La stessa funzione ?

b). $y'' + 3y = 2x$ → l'insieme delle soluzioni

b1) $y'' + 3y = 0$ $\xrightarrow[\text{eq. caratteristica}]{}$ $\boxed{\lambda^2 + 3 = 0 \;\Downarrow\; \lambda^2 = -3 \to \lambda = i\sqrt{3}}$

$y_0(x) = C_0 \cos(\sqrt{3}x) + C_1 \sin(\sqrt{3}x)$.

b2) $y_p(x)$. con il meto delle somiglianze $f(x) = 2x$ polinomio di grado 1

$\Downarrow$

dato che i polinomi non sono soluzione di $\boxed{y'' + 3y = 0}$ $y_p(x) = ax + b$ "

$y_p'(x) = a$, $y_p'' = 0$ $\Rightarrow 0 + 3(ax + b) = 2x$
$3ax + 3b = 2x$

$\begin{cases} 3a = 2 & a = \frac{2}{3}. \\ 3b = 0 & \to b = 0 \end{cases}$

$\boxed{y_p(x) = \frac{2}{3}x}$ → L'ins. delle soluzione $\boxed{y(x) = C_0 \cos\sqrt{3}x + C_1 \sin\sqrt{3}x + \frac{2}{3}x}$ in tutto $\mathbb{R}$

c). Trovare la soluzione che verifica $y(0)=0$, $y'(0)=1$.

$$y(0)=C_0\cos\sqrt{3}\cdot 0+C_1\sin\sqrt{3}\cdot 0+\frac{2}{3}\cdot 0=C_0=0 \implies C_0=0.$$

$$y'(x)=\sqrt{3}\,C_1\cos\sqrt{3}\,x+\frac{2}{3} \implies y'(0)=C_1\sqrt{3}+\frac{2}{3}=1.$$

$$C_1\sqrt{3}=1-\frac{2}{3}=\frac{1}{3}.$$

$$C_1=\frac{1}{3\sqrt{3}}=\frac{\sqrt{3}}{9}.$$

La soluzione del pb. di Cauchy $\begin{cases} y''+3y=2x \\ y(0)=0 \\ y'(0)=1 \end{cases} \implies \boxed{y(x)=\frac{\sqrt{3}}{9}\sin\sqrt{3}\,x+\frac{2}{3}x}$

d) Det. se $\exists$ una soluzione limitata

$\lim_{x\to+\infty} y(x)=+\infty$ $\forall$ soluzione

$\lim_{x\to-\infty} y(x)=-\infty$ $\forall$ soluzione.

$C_0\cos\sqrt{3}\,x+C_1\sin\sqrt{3}\,x$ è limitata