Definizione di limite

$\lim\limits_{n\to+\infty} a_n = \ell \in \mathbb{R}$ se $\underbrace{\forall \varepsilon > 0}$, $\exists N$ t.c. $\forall n \geq N$ $\underline{|a_n - \ell| \leq \varepsilon}$

$N = N(\varepsilon)$

$\lim\limits_{n\to+\infty} a_n = +\infty$ se $\forall M > 0$, $\exists N$ t.c. $\forall n \geq N$ $\boxed{a_n \geq M}$

$N = N(M)$

Algebra dei limiti

$\boxed{\lim\limits_{n\to+\infty} a_n = a \in \mathbb{R}}$ $\boxed{\lim\limits_{n\to+\infty} b_n = b \in \mathbb{R}}$

$$\boxed{\begin{aligned} &\lim_{n\to+\infty} a_n + b_n = a+b \\ &\lim_{n\to+\infty} a_n b_n = ab \\ &\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{a}{b} \end{aligned}}$$

se $b_n \neq 0$ $\forall n$, $b \neq 0$ $\Rightarrow$

$b_n \neq 0$, $b \neq 0$

$$\boxed{\lim_{n\to+\infty} a_n = +\infty} \qquad \boxed{\lim_{n\to+\infty} b_n = +\infty} \qquad \boxed{\lim_{n\to+\infty} c_n = c}$$

$$\lim_{n\to+\infty} a_n + b_n = +\infty \quad , \quad \lim_{n\to+\infty} a_n + c_n = +\infty$$

$$\lim_{n\to+\infty} \frac{c_n}{a_n} = 0 \qquad \lim_{n\to+\infty} c_n \cdot a_n = \begin{cases} +\infty & c > 0 \\ -\infty & \text{se } c < 0 \end{cases}$$

Se $c = 0$ non è possibile stabilire a priori quanto è $\boxed{\lim_{n\to+\infty} a_n c_n}$

Esempi se $a_n = n \to +\infty$, $c_n = \frac{1}{n} \to +\infty$ $\quad \lim_{n\to+\infty} a_n c_n = 1$

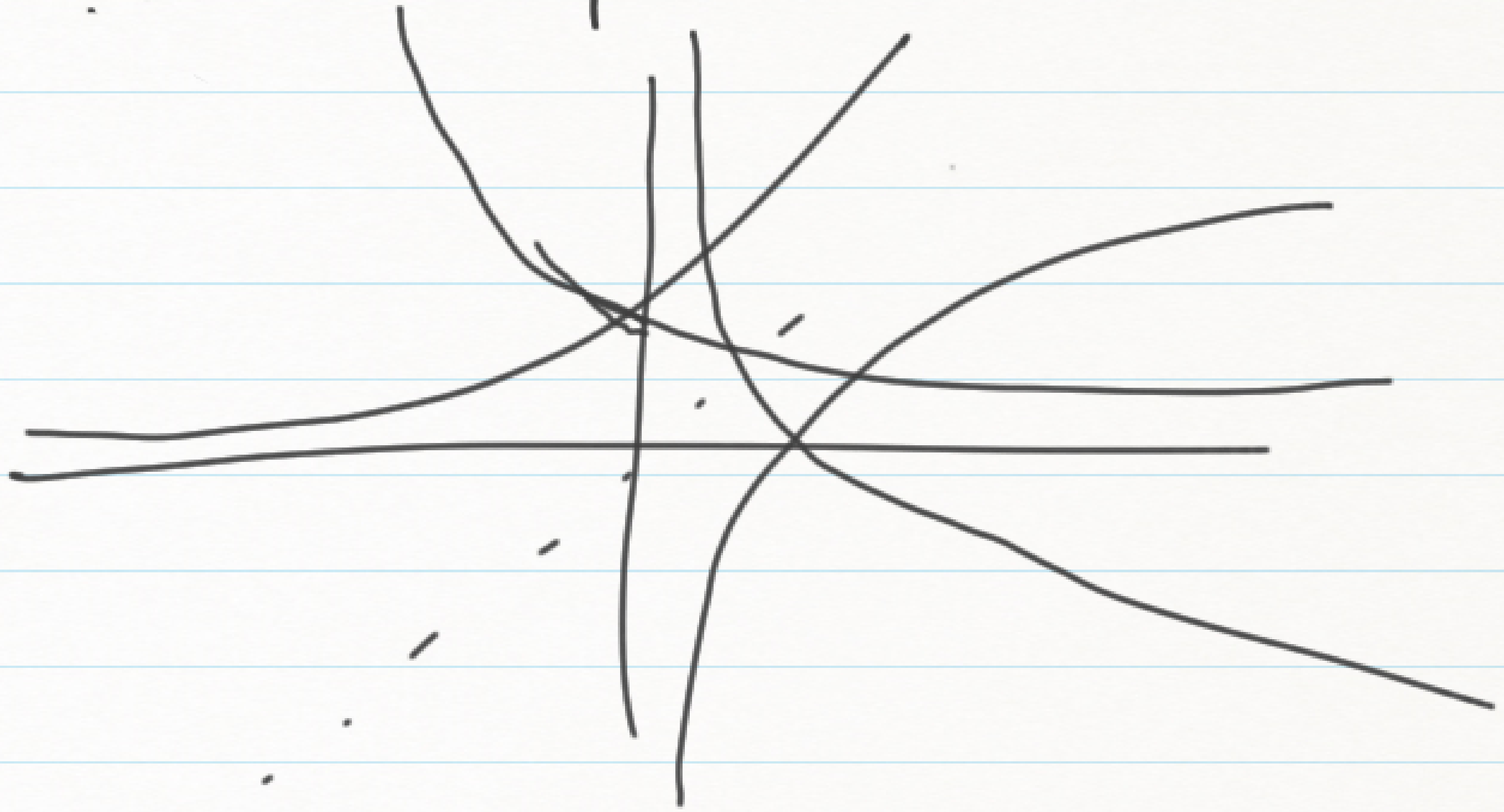
se $a_n = n^2 \to +\infty$, $c_n = \frac{1}{n} \to 0$ $\quad \lim_{n\to+\infty} a_n c_n = \lim_{n\to+\infty} n = +\infty$

se $+\infty \leftarrow a_n = n$, $c_n = \frac{1}{n^2} \to 0$ $\quad \lim_{n\to+\infty} a_n c_n = \lim_{n\to+\infty} \frac{1}{n} = 0$

$\lim_{n\to+\infty} a_n - b_n = ?$

impossibile stabilire a priori qual'è il limite

Limiti da sapere



$\boxed{\alpha > 0}$ $\lim_{n \to +\infty} n^{\alpha} = +\infty$

$\forall M > 0 \quad \exists N$ t.c. $\quad n > N \implies \boxed{n^{\alpha} > M}$

$n > N = M^{1/\alpha} \implies n^{\alpha} > M$

$\lim_{n \to +\infty} n^{-\alpha} = \lim_{n \to +\infty} \frac{1}{n^{\alpha}} = 0$ (per le proprietà sui limiti).

$a > 0$

$\lim_{n \to +\infty} a^{n} = +\infty$ se $a > 1$

$\lim_{n \to +\infty} a^{n} = 0$ se $a < 1$

$\boxed{a > 1}$

$\lim_{n \to +\infty} \log_a n = +\infty$

$\forall M > 0 \quad \exists N$ t.c. $\quad n > N \implies \log_a n > M$ $\quad a$

? $n > N = a^{M} \implies \log_a n > M$

$a < 1$

$\lim_{n \to +\infty} \log_a n = -\infty$

$$\lim_{n\to+\infty} \underset{\substack{\downarrow\\ +\infty}}{n^2} - \underset{\substack{\downarrow\\ +\infty}}{n} = \lim_{n\to+\infty} \overbrace{n^2}^{\uparrow +\infty} \underbrace{\left(1-\frac{1}{n}\right)}_{\substack{\downarrow\\ 1-0=1}} = +\infty$$

$$\boxed{\lim_{n\to+\infty} n + \sin n = +\infty}$$

<u>Prop</u>: $\lim_{n\to+\infty} a_n = +\infty$ e $(b_n)_{n\in\mathbb{N}}$ è una successione limitata $\Longrightarrow \lim_{n\to+\infty} a_n + b_n = +\infty$

$$\forall M>0 \quad \exists N \text{ t.c. } \quad n>\boxed{N} \Longrightarrow n+\sin n > M$$

$$N = M+2 \qquad\qquad 2+\sin n > 0$$

$$\lim_{n\to+\infty} \frac{n^2+2n}{3n+4}$$

$$\lim_{n\to+\infty} a_n = \infty$$

$$\lim_{n\to+\infty} b_n = \infty$$

$$\boxed{\begin{array}{c} +\infty-\infty \\ \dfrac{\infty}{\infty} \\ 0\cdot\infty \end{array}}$$

NON NOTE A PRIORI

NON è possibile stabilire a priori quanto vale $\lim_{n\to+\infty} \frac{a_n}{b_n}$

Viceversa $\boxed{\frac{0}{\infty}=0} \Leftarrow \boxed{\begin{array}{l} \lim_{n\to+\infty} a_n = 0 \\ \lim_{n\to+\infty} b_n = \infty \end{array}}$

$$\Downarrow$$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$$

$$\lim_{n\to+\infty} \frac{n^2+2n}{3n+4} = \lim_{n\to+\infty} \frac{n^2(1+\frac{2}{n})}{n(3+\frac{4}{n})}$$

$$= \lim_{n\to+\infty} \underset{\substack{\downarrow \\ +\infty}}{n} \underbrace{\left(\frac{1+\frac{2}{n}}{3+\frac{4}{n}}\right)}_{\substack{\downarrow \\ \frac{1}{3}>0}} = +\infty$$

$$\lim_{n\to+\infty} \underbrace{\sqrt{n+2}}\ - \underbrace{\sqrt{n+3}}$$

$$\underline{(a-b)(a+b)=a^2-b^2}$$

$$\boxed{\sqrt{n+2}-\sqrt{n+3}} = \left(\sqrt{n+2}-\sqrt{n+3}\right)\frac{\left(\sqrt{n+2}+\sqrt{n+3}\right)}{\left(\sqrt{n+2}+\sqrt{n+3}\right)} = \frac{(n+2)-(n+3)}{\sqrt{n+2}+\sqrt{n+3}} = \boxed{\frac{-1}{\sqrt{n+2}+\sqrt{n+3}}} \quad \forall n\in\mathbb{N}$$

$$\lim_{n\to+\infty} \sqrt{n+2}-\sqrt{n+3} = \lim_{n\to+\infty} \frac{-1}{\underbrace{\sqrt{n+2}}_{+\infty}+\underbrace{\sqrt{n+3}}_{+\infty}} = 0$$

$$a = a\frac{b}{b}$$

$$\boxed{\lim_{n\to+\infty} \frac{n!}{n^n} = 0}$$

$$n! = 1.2\cdot\ldots\cdot n$$

$$n^n = n.n\cdot\ldots\cdot n$$

$$\forall \varepsilon>0 \quad \exists N=N(\varepsilon) \qquad 0<\frac{n!}{n^n}<\varepsilon$$

$$n>N(\varepsilon) \qquad \frac{1}{n}\cdot\boxed{\frac{2}{n}\cdot\ldots\cdot\frac{n}{n}} \le \frac{1}{n} \le \varepsilon$$

$$\frac{2}{n}<1 \qquad \frac{3}{n}<1 \qquad n>\frac{1}{\varepsilon}$$

$$\lim_{n\to+\infty} \frac{2^n}{n!} = 0$$

$$\frac{2\cdot \ldots \cdot 2}{1\cdot 2 \cdot \ldots \cdot n}$$

$$\frac{2}{1}\cdot\frac{2}{2}\cdot\underbrace{\boxed{\frac{2}{3}\cdot\frac{2}{4}\cdots}}_{\leq 1}\frac{2}{n} \leq 2\cdot 1\cdot \boxed{\leq 1}\cdot\frac{2}{n}$$

$$\leq 2\cdot\frac{2}{n} = \frac{4}{n} < \varepsilon$$

$$n > \frac{4}{\varepsilon}$$

$x > 0$

$$\lim_{n\to+\infty}\frac{x^n}{n!} = 0$$

$$0 < \frac{x^n}{n!} = \underbrace{\boxed{\frac{x\cdot x\cdot x \cdots x}{1\cdot 2\cdot 3 \cdots [x]}}}_{\frac{x^{[x]}}{[x]!}} \cdot \underbrace{\boxed{\frac{x \cdots x}{([x]+1)\cdots}}}_{\leq 1}\frac{x}{n} \leq \underbrace{\frac{x^{[x]}}{[x]!}\cdot x}_{K}\frac{1}{n} < \varepsilon$$

$$\Updownarrow$$

$$n \geq \boxed{\frac{K}{\varepsilon}}$$

$\forall n$

$[x] = n$ t.c. $n \leq x < n+1$

parte intera

$[102{,}3] = 102$

Teoremi di confronto

Teorema di permanenza del Segno Se $\lim_{n\to+\infty} a_n = a > 0 \implies \exists N$ t.c. $\forall n > N$, $a_n > 0$

$\forall \varepsilon > 0\ \exists N$ t.c. $\forall n > N$

$-\varepsilon < a_n - a < \varepsilon \implies \boxed{a_n > a - \varepsilon}$

scegliamo $\varepsilon = \frac{a}{2} < a$

$\Downarrow$

$\exists N = N\left(\frac{a}{2}\right)$ t.c. $\forall n > N$

$\Downarrow$

$a_n > a - \frac{a}{2} = \frac{a}{2} > 0$

$\Updownarrow$

Teorema di confronto Sia (a_n), $\lim_{n\to+\infty} a_n = a$ e t.c. $a_n \geq 0 \implies a \geq 0$

$\Downarrow$

Se (b_n), $\lim_{n\to+\infty} b_n = b$ e $a_n \geq b_n \implies \lim_{n\to+\infty} a_n \geq \lim_{n\to+\infty} b_n$

$\Updownarrow$

Teorema dei Carabinieri: Siano tre successioni tali che $a_n \leq b_n \leq c_n$

se $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = l \implies \lim_{n\to+\infty} b_n = l$

$a_n = n + \sin n$

$$\underset{\downarrow \atop +\infty}{n-1} \le n + \sin n \le \underset{\downarrow \atop +\infty}{n+1}$$

$$1 + \frac{1}{n+1} = \frac{n+2}{n+1}$$

$$1 + \frac{1}{n} = \frac{n+1}{n}$$

$n($

$$\frac{1}{3} \longleftarrow \frac{n-1}{3n+2} \le \frac{n+\cos n}{3n-2\sin n} \le \frac{n+1}{3n-2} \longrightarrow \frac{1}{3}$$

$$\frac{n+\cos n}{3n-2\sin n} \to \frac{1}{3}$$

Existe ed è finito $\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n} = e$

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n}\right)^n}$$

$a_n = \left(1+\frac{1}{n}\right)^n$

Dimostriamo $a_{n+1} > a_n$ $\quad \boxed{\frac{a_{n+1}}{a_n}} \overset{?}{>} 1.$

$$= \left(\frac{(n+2)n}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right)$$

$$(n+1)^2 - 1 = n^2+2n+1-1 = n^2+2n$$

$n+2 = (n+1)+1.$

$n = (n+1)-1.$

$$= \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right) = \left(1-\frac{1}{(n+1)^2}\right)^n\left(1+\frac{1}{n+1}\right)$$

Questo conto è giusto ma inutile non permette di concludere la monotonia

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n} \cdot \left(1 + \frac{1}{n+1}\right) \geq \left(1 - \frac{n}{(n+1)^2}\right)\left(1 + \frac{1}{n+1}\right) = \left(\frac{(n+1)^2 - n}{(n+1)^2}\right)\left(1 + \frac{1}{n+1}\right)$$

$$= \left(\frac{n^2+n+1}{(n+1)^2}\right)\left(\frac{n+2}{n+1}\right) \neq$$

$$\boxed{(1+x)^n \geq 1 + nx} \qquad \forall x > -1$$

Così invece funziona

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^{-1}} \geq \left(1 - \frac{(n+1)}{(n+1)^2}\right) \cdot \frac{n+1}{n} = \left(1 - \frac{1}{n+1}\right)\left(\frac{n+1}{n}\right)$$

$$= \frac{n}{n+1} \cdot \left(\frac{n+1}{n}\right) = 1$$

$$\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n}} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n+1}} \cdot \left(1 + \frac{1}{n}\right)^{-1}$$

$$\boxed{\lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

$$e^x e^y = e^{x+y}$$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \simeq 2{,}71 \ldots$$

imaginale

$$\boxed{\log x = \log_e x}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Più in generale vale che

Se a_n t.c. $\lim_{n\to+\infty} a_n = +\infty \implies \boxed{\lim_{n\to+\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e}$

$a_n \to a \neq 1$
$b_n \to b \neq \infty$
$\lim_{n\to+\infty}(a_n)^{b_n} = a^b$

$$\lim_{n\to+\infty}\left(\frac{n}{3+n}\right)^{5n} = \frac{1}{e^{15}}$$

$5n = \frac{n}{3}\cdot 3\cdot 5$

$$\left(\frac{n}{3+n}\right)^{5n} = \left(\frac{1}{\left(\frac{3}{n}+1\right)}\right)^{5n} = \frac{1}{\left[\left(\frac{1}{\frac{n}{3}}+1\right)^{\frac{n}{3}}\right]^{3\cdot 5}}$$

$$\lim_{n\to+\infty}\left(\frac{n^2}{4+n^2}\right)^{6n^2} = \lim_{n\to+\infty}\left(\frac{1}{\frac{4}{n^2}+1}\right)^{6n^2} = \lim_{n\to+\infty}\boxed{\left(\frac{1}{\frac{1}{\frac{n^2}{4}}+1}\right)^{\frac{n^2}{4}}}^{\cdot 24} = \frac{1}{e^{24}} = e^{-24}$$

$$\lim_{n\to+\infty}\left(\frac{n}{2+n}\right)^{n^2} = 0$$

$$\left(\frac{n}{2+n}\right)^{n^2} = \left(\frac{1}{\frac{2}{n}+1}\right)^{n^2} = \frac{1}{\left[\left(\frac{1}{\frac{n}{2}}+1\right)^{\frac{n}{2}}\right]^{2n}}$$

$2n \longrightarrow +\infty$

$\frac{2}{n} = \frac{1}{\frac{n}{2}}$

$\downarrow$
e

$$\lim_{n\to+\infty}\left(1+\frac{2}{n^2}\right)^n = \lim_{n\to+\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}}}_{\to e}\right]^{\overbrace{\frac{2}{n}}^{\to 0}} = e^0 = 1.$$

ordine degli infiniti e degli infinitesimi

$\lim_{n\to+\infty} a_n = \infty$

$\lim_{n\to+\infty} b_n = \infty$

Se $\lim_{n\to+\infty}\left|\frac{a_n}{b_n}\right| = +\infty$

$\Downarrow$

ordine di infinito di (a_n) è maggiore di b_n

$\lim_{n\to+\infty}\left|\frac{a_n}{b_n}\right| = 0$

ordine di infinito di (a_n) è inferiore all'ordine di infinito di b_n

$\lim_{n\to+\infty}\left|\frac{a_n}{b_n}\right| = \underbrace{l}_{\in\mathbb{R}} \neq 0 \implies a_n$ e b_n hanno lo stesso ordine di infinito.

$$\boxed{\lim_{n\to+\infty}\frac{\log n}{n^\alpha} = 0}$$

ordine ∞ di $\log n$ è inferiore a l'ordine d'infinito di n^α $\forall\,\alpha>0$

ordine ∞ di a^n, $a>1$ è superiore a l'ordine d'infinito di n^α $\forall\,\alpha>0$