Polinomio di Taylor

f n volte derivabile in x_0, o in un intorno di x_0

$$T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f^3(x_0)(x-x_0)^3 + \cdots + \frac{1}{n!} f^n(x_0)(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

$$T_n(x_0) = f(x_0) \quad , \quad T_n^K(x_0) = f^k(x_0) \qquad k = 1, \ldots, n$$

<u>Teorema</u> Se f è n volte derivabile in un intorno di x_0 allora

$$f(x) = T_n(x-x_0) + o((x-x_0)^n) \qquad \left[\text{cioè} \quad \lim_{x \to x_0} \frac{f(x) - T_n(x-x_0)}{(x-x_0)^n} = 0\right.$$

f	T_n
$f(x) = e^x$	$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} \qquad x_0 = 0$
$f(x) = \sin x$	$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad x_0 = 0$
$f(x) = \cos x$	$T_{2n}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^n \frac{x^{2n}}{2n!}$
$f(x) = \log(1+x)$	$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{k!} x^k = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} x^k$

$$f(x)=(1+x)^{\alpha}$$

$$f(0)=1$$

$$f'(x)=\alpha(1+x)^{\alpha-1} \qquad f'(0)=\alpha$$

$$f''(x)=\alpha(\alpha-1)(1+x)^{\alpha-2} \qquad f''(0)=\alpha(\alpha-1)$$

$$f'''(x) \qquad f'''(0)=\alpha(\alpha-1)(\alpha-2)$$

$$T_n(x)=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\dots+\frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}x^n.$$

$\alpha=3$

$$(1+x)^3=1+3x+3x^2+x^3$$

$$1+3x+\frac{3\cdot 2}{2}x^2+\frac{3\cdot 2\cdot 1}{3!}x^3$$

$$\lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha$$

$$(1+x)^{\alpha}=1+\alpha x+o(x)$$

$$(1+x)^{\alpha}-1=\alpha x+o(x)$$

$$\frac{(1+x)^{\alpha}-1}{x}=\alpha+\frac{o(x)}{x}\longrightarrow\alpha.$$

$$\lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x}.$$

$$\frac{\sin x}{x} = 1 + o(x)$$

$$\frac{\log\left(\frac{\sin x}{x}\right)}{x} = \frac{\log(1+o(x))}{o(x)} \cdot \frac{o(x)}{x} \longrightarrow 0$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + o(x^2)$$

$$\frac{\log\left(1-\frac{x^2}{6}+o(x^2)\right)}{x} = \underbrace{\boxed{\frac{\log\left(1-\frac{x^2}{6}+o(x^2)\right)}{-\frac{x^2}{6}+o(x^2)}}}_{1} \cdot \overbrace{\frac{\left(-\frac{x^2}{6}+o(x^2)\right)}{x}}^{\to 0}$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$-\frac{1}{6} = \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} = \boxed{\frac{\log\left(1-\frac{x^2}{6}+o(x^2)\right)}{-\frac{x^2}{6}+o(x^2)}} \cdot \frac{-\frac{x^2}{6}+o(x^2)}{x^2}$$

$\to 1$

$\to -\frac{1}{6}$

<u>Teorema</u> (Resto di Lagrange) f è $n+1$ volte derivabile in un intorno di x_0.

allora esiste $c \in (x_0, x)$, $(c \in (x, x_0))$ t.c.

$$f(x) = T_n(x) + \frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}$$

<u>Dimostrazione</u> Usa il Teorema di Lagrange per la funzione

$$g(x) = f(x) - T_n(x).$$

Metodo di Newton

<u>Risolvere</u> $f(x)=0$. $\exists\ a$ e b t.c. f continua in $[a,b]$

1) $f(a) \cdot f(b) < 0 \longrightarrow \boxed{\exists\ \alpha \text{ t.c. } f(\alpha)=0}$

2) $f'(x)$ non cambia segno in (a,b) $\quad$ α è unico
$\quad f''(x)$ non cambia segno in (a,b)

Due casi A) $f(a)\,f''(a) > 0$ $\qquad$ oppure B) $f(b)\,f''(b) > 0$

$f(a)>0 \quad f''(a) < 0 \longrightarrow f''(b)<0 \ ,\ f(b)<0$

$f''(a)>0$

A) $\longrightarrow \begin{cases} x_0 = a \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \end{cases}$ $\qquad$ B $\longrightarrow \begin{cases} x_0 = b \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \end{cases}$

<u>Teorema</u> allora $x_n \longrightarrow \alpha$ t.c. $f(\alpha)=0$

$f(x_0) = 0$

$x_0 = a$ consideriamo la retta tangente in $(x_0, f(x_0))$

$$y = f(x_0) + f'(x_0)(x - x_0) \implies \underline{f(x_0) + f'(x_0)(x - x_0) = 0}$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

<u>Dimostrazione</u> $x_n \to \alpha$

$\boxed{\text{Se}}$ $x_n \to c$ allora $\boxed{x_{n+1} \to c}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow c - \frac{f(c)}{f'(c)} = c$$

$$\Downarrow$$

$$f(c) = 0$$

$$\Updownarrow$$

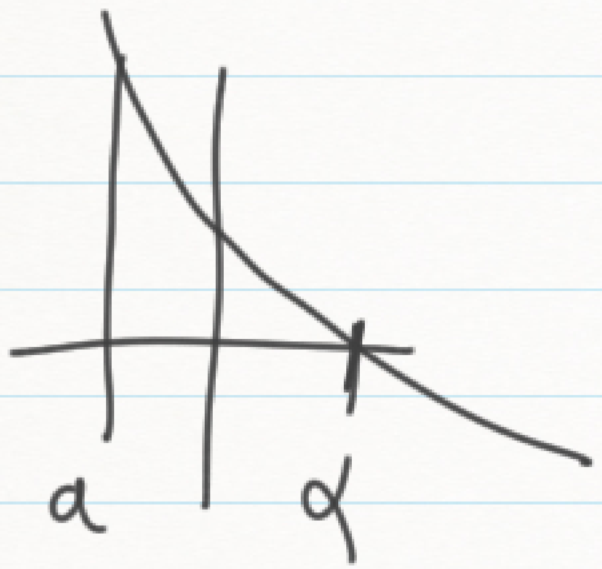
$$c = \alpha$$

<u>usato continuità di f e f'</u>

Dimostriamo che x_n è monotona $\qquad \boxed{f(a)\, f''(a) > 0}$ oppure $f(b)\, f''(b) > 0$

scegliamo $f(a) > 0$ e $f''(a) > 0 \quad \Longrightarrow \quad f''(x) > 0 \quad \forall x \in (a, b)$

$$f'(x) < 0 \quad \forall x \in (a, b)$$



$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$a < \alpha < b$$

$$g'(x) = 1 - \frac{f'(x) \cdot f'(x) - f(x)\, f''(x)}{f'(x)^2}$$

$$= \frac{f'(x)^2 - f'(x)^2 + f(x)\, f''(x)}{f'(x)^2} = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$\forall x \in (a, \alpha) \qquad g'(x) > 0 \quad \Longrightarrow \quad g(x) \nearrow$

$x_0 = a \qquad x_1 > x_0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} > x_0$$

$$x_2 = g(x_1) > g(x_0) = x_1$$

$$\Downarrow$$

$$x_3 = g(x_2) > g(x_1) = x_2$$

$$\lim_{n \to +\infty} x_n = \alpha .$$

$\{x_n\}$ è monotona crescente

(Piccolo "buco")

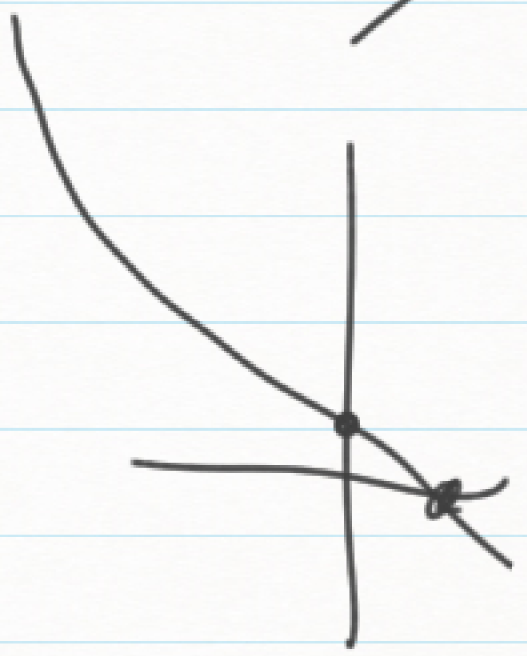
Esempio $\quad f(x) = e^{-x} - x \quad$ dove si annulla $f(x)$

$$\lim_{x \to -\infty} e^{-x} - x = +\infty$$

$$\lim_{x \to +\infty} e^{-x} - x = -\infty$$

$$f'(x) = -e^{-x} - 1 < 0$$

$$f''(x) = e^{-x} > 0$$



$$f(0) = 1 - 0 = 1 > 0$$

$$f(1) = e^{-1} - 1 < 0$$

$$f(0) f''(0) > 0$$

$$\begin{cases} x_0 = 0 \\ x_{n+1} = x_n - \dfrac{e^{-x_n} - x_n}{-e^{-x_n} - 1} = x_n + \dfrac{e^{-x_n} - x_n}{e^{-x_n} + 1} \end{cases}$$

$$x_1 = \frac{1}{1+1} = \frac{1}{2}$$

$$x_2 = \frac{1}{2} + \frac{e^{-\frac{1}{2}} - \frac{1}{2}}{e^{-\frac{1}{2}} + 1} = \frac{1}{2} + \frac{2 - \sqrt{e}}{2 + 2\sqrt{e}} = 0{,}5663\ldots$$

$$x_3 = x_2 + \frac{e^{-x_2} - x_2}{e^{-x_2} + 1} = 0{,}5671\ldots$$

$$x_4 = 0{,}5671\ldots$$

$$e^{-x_4} - x_4 \approx 10^{-17}$$