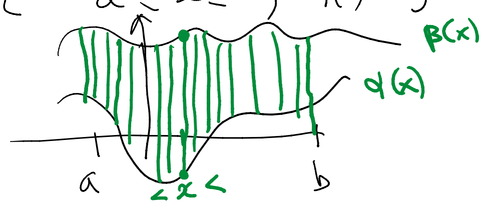
Integrali Multipli

$D = \{ (x,y) \in \mathbb{R}^2 \quad a \leq x \leq b, \ \alpha(x) \leq y \leq \beta(x) \}$

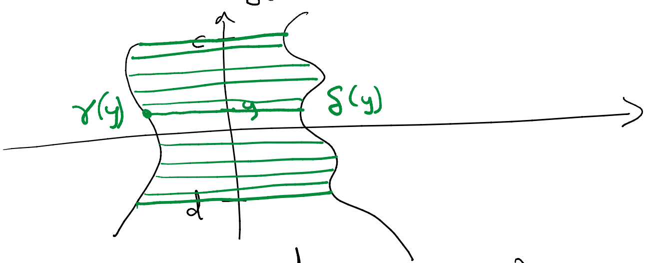
D è normale risp a x, $\exists\, a \in \mathbb{R}, b \in \mathbb{R}$



$$\iint_D f(x,y)\,dx\,dy = \int_a^b \underbrace{\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right)}_{\text{Integrale in } y} dx$$

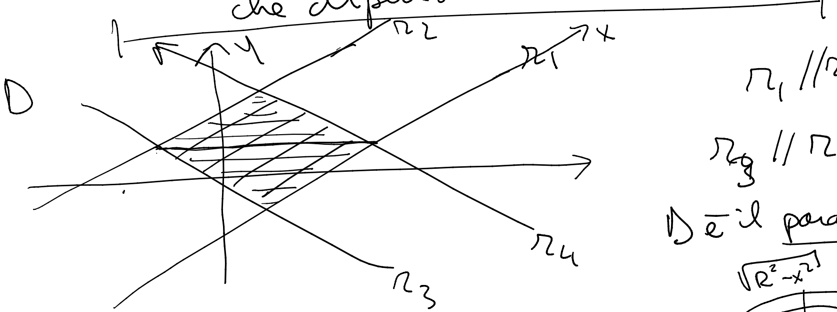
cioè la variabile è y: il cui risultato è una funzione che dipende solo da x

$D = \{ (x,y) \in \mathbb{R}^2, \ \underset{\mathbb{R}}{c} \leq y \leq \underset{\mathbb{R}}{d}, \ \gamma(y) \leq x \leq \delta(y) \}$



$$\iint_D f(x,y)\,dx\,dy = \int_c^d \underbrace{\left(\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx \right)}_{} dy$$

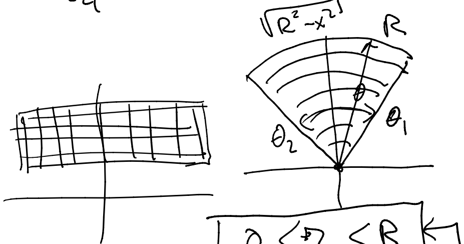
Integrale in x, cioè la variabile è "x", il risultato è una funzione che dipende solo da "y".



$r_1 \parallel r_2$

$r_3 \parallel r_4$

D è il parallelepipedo



$$\boxed{\begin{array}{l} 0 \leq r \leq R \\ \theta_1 \leq \theta \leq \theta_2 \end{array}}$$

"Cambio di variabili"

Caso delle coordinate polari

$$\begin{cases} x = r\cos\theta \\ \ldots \end{cases}$$

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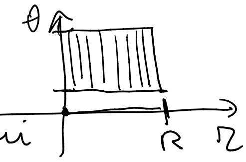
Caso delle coordinate polari

$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}$$

$\boxed{\theta_1 \leq \theta \leq \theta_2}$



Regola del cambio di variabili da coordinate cartesiane a coordinate polari

$$D = \{(x,y) \in \mathbb{R}^2, \quad h(x,y) \leq 0, \quad k(x,y) \geq 0, \dots\}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$$

$$D = \tilde{D} = \{(\theta, r), \quad h(r\cos\theta, r\sin\theta) \leq 0, \quad k(r\cos\theta, r\sin\theta) \geq 0\}$$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{\tilde{D}} f(\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})\, r \, dr\, d\theta}$$

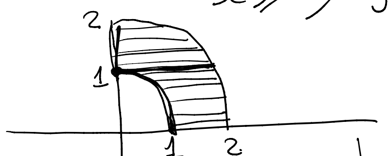
cartesiano $\longrightarrow$ fatta la sostituzione $\longrightarrow$ nuovo integrale

Una volta fatta la sostituzione, l'integrale in $dr\,d\theta$ si svolge esattamente come si sarebbe fatto se le variabili si chiamassero $dx\,dy$.

In particolare

Se $\tilde{D}$ è "normale" rispetto a una delle due variabili possiamo usare la formula di riduzione per calcolarlo.

$$\iint_D x^2+y^2\,dx\,dy \quad \text{dove} \quad D = \{(x,y) \text{ t.c. } 1 \leq x^2+y^2 \leq 4, \; x \geq 0, \; y \geq 0\}$$



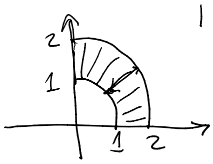
$0 \leq y \leq 1, \quad \sqrt{1-y^2} \leq x \leq \sqrt{4-y^2}$

$1 \leq y \leq 2, \quad 0 \leq x \leq \sqrt{4-y^2}$

$x^2+y^2 = r^2$

$1 \leq r^2 \leq 4$

$\Downarrow$



$r\cos\theta \geq 0 \implies \cos\theta \geq 0$

$r\sin\theta \geq 0 \implies \sin\theta \geq 0$

$\implies 0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 2$

$$D = \tilde{D} = \{(r,\theta), \text{ t.c. } 1 \leq r \leq 2, \; 0 \leq \theta \leq \frac{\pi}{2}\}$$

Scritto D in termini di coordinate cartesiane chiamando il dominio $\tilde{D}$

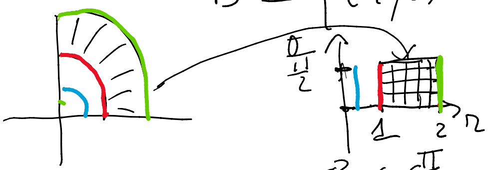
$$\iint_D x^2+y^2\,dx\,dy = \iint_{\tilde{D}} r^2 \cdot r\,dr\,d\theta = \iint_{\tilde{D}} r^3\,dr\,d\theta$$

$$f(x,y) = x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$\tilde{\ }$ $1 \leq r \leq 2 \quad 0 \leq \theta \leq \frac{\pi}{2}\}$

$$f(x,y)=x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2$$

$$\tilde{D}=\{(r,\theta)\ \text{t.c.}\ 1\leq r\leq 2,\ 0\leq\theta\leq\tfrac{\pi}{2}\}$$



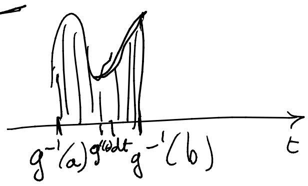
Vi potete dimenticare da dove viene il nuovo integrale

$$\iint_{\tilde{D}} r^3\,dr\,d\theta=\int_1^2\left(\int_0^{\frac{\pi}{2}} r^3\,d\theta\right)dr=\int_1^2\left(r^3\theta\right)\Big|_0^{\frac{\pi}{2}}dr=$$

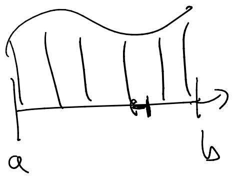
$$=\int_1^2\frac{\pi}{2}r^3\,dr=\frac{\pi}{2}\cdot\frac{r^4}{4}\Big|_1^2=\frac{\pi}{8}(16-1)=\frac{15}{8}\pi$$

Parentesi

$$\int_a^b f(x)\,dx=\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\cdot g'(t)\,dt$$



$$x=g(t)$$
$$dx=g'(t)\,dt$$



$$x=a\rightarrow\quad a=g(\bar{t})\Rightarrow\bar{t}=g^{-1}(a)$$

In analogia

$$\iint_D f(x,y)\,dx\,dy=\iint_{\tilde{D}} f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\rightarrow dx\,dy=r\,dr\,d\theta$$

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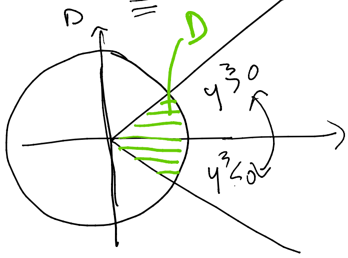
$$\iint_D y^3\,dx\,dy,\quad D=\{(x,y)\ \text{t.c.}\ x^2+y^2\leq 1,\ x\geq|y|\}$$

$x^2+y^2=r^2\rightarrow$ fa sospettare che sia una buona idea usare le coordinate polari

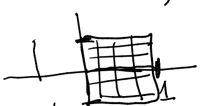


$$\tilde{D}=\{(r,\theta)\ \text{t.c.}\ 0\leq r\leq 1,\ -\tfrac{\pi}{4}\leq\theta\leq\tfrac{\pi}{4}\}$$

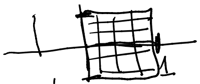
$x=|y|$: $x=y\rightarrow r\cos\theta=r\sin\theta\rightarrow\frac{\pi}{4}=\theta$; $x=-y\rightarrow r\cos\theta=-r\sin\theta\rightarrow\theta=-\frac{\pi}{4}$

$$\iint_D y^3\,dx\,dy=\iint_{\tilde{D}}(r\sin\theta)^3\cdot r\,dr\,d\theta=\iint_{\tilde{D}} r^4\sin^3\theta\,dr\,d\theta$$

r e θ sono le nuove variabile, usiamo la formula di riduzione.



di riduzione.



$$0=\int_0^1\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^4\sin^3\theta\,d\theta\right)dr=\int_0^1 r^4\underbrace{\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin^3\theta\,d\theta\right)}\,dr=\int_0^1 r^4\,dr\cdot\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin^3\theta\,d\theta$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin^3\theta\,d\theta=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}\sin\theta\,d\theta=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin\theta\,d\theta-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2\theta\sin\theta\,d\theta$$

$t=\cos\theta$

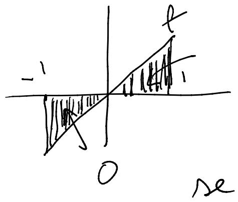
$dt=-\sin\theta\,d\theta$

$$=-\cos\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}+\frac{\cos^3\theta}{3}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=0+0=0$$

$$\int\cos^2\theta\sin\theta\,d\theta=-\int t^2\,dt=-\frac{t^3}{3}$$

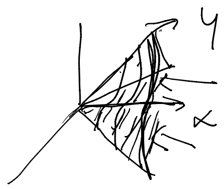
$\sin^3\theta=\underbrace{\sin^2}\sin\theta$ $\qquad$ $\sin^2\theta=1-\cos^2\theta$

$\cos\frac{\pi}{4}=\cos\left(-\frac{\pi}{4}\right)$



$$\int_{-1}^{1}f(x)\,dx=\int_{-1}^{0}f(x)\,dx+\int_0^1 f(x)\,dx$$

se f è dispari $=-\int_0^1 f(x)\,dx+\int_0^1 f(x)\,dx$

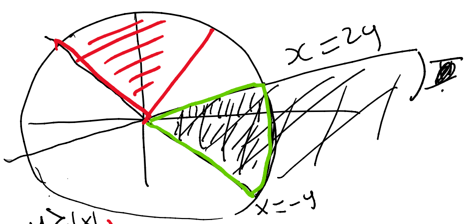


$\longrightarrow 0$ $\qquad \iint_D y^3\,dx\,dy=0$

perché D è simmetrico rispetto all'asse delle x, quindi è simmetrico in y e $f(x,y)=y^3$ è una funzione dispari in y. $\Longrightarrow \iint_D y^3\,dx\,dy=0$.

$D_1=\{(x,y)$ t.c. $x^2+y^2\le 1$, $x\ge -y$, $x\le 2y\}$

$\iint_{D_1} y^3\,dx\,dy\neq 0$



$D_2=\{(x,y)$ t.c. $x^2+y^2\le 1$, $y\ge|x|\}$

↑ ESERCIZIO

$D_2 = \{(x,y)$ t.c. $x + \dots \}$

$\rightarrow \iint_{D_2} y^3 \, dx\, dy \neq 0$ ↑ ESERCIZIO

Caso generale di Cambiamento di Variabile

$$\iint_D f(x,y)\, dx\, dy \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$$

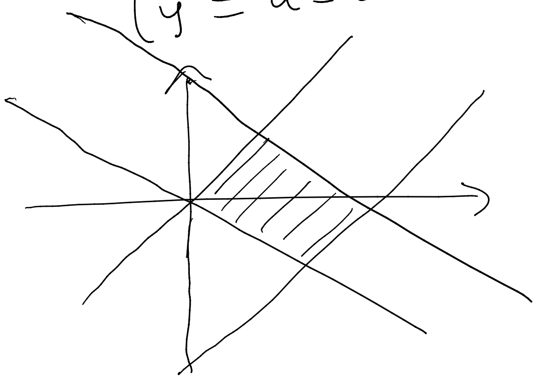
L'idea è di fare una sostituzione in modo che il nuovo integrale sia più semplice da fare, precisamente magari nelle nuove coordinate il dominio è normale.

$$(x,y) \longrightarrow (u,v)$$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \longleftrightarrow \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

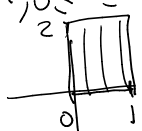
Es.:

$$\begin{cases} x = u+v \\ y = u-v \end{cases} \longrightarrow \begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases}$$



$D = \{(x,y)$ t.c. $0 \leq x+y \leq 2,\ 0 \leq x-y \leq 4\}$

$\tilde{D} = \{(u,v)$ t.c. $0 \leq u \leq 1,\ 0 \leq v \leq 2\}$



$x+y = 2u \qquad 0 \leq 2u \leq 2 \longrightarrow 0 \leq u \leq 1$

$x-y \qquad 0 \leq 2v \leq 4 \longrightarrow 0 \leq v \leq 2$

$$\iint_D f(x,y)\, dx\, dy = \iint_{\tilde{D}} f(x(u,v), y(u,v)) \cdot J(u,v)\, du\, dv$$

$$J(u,v) = \left| \det \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} \right|$$

$J(u,v)$ jacobiano

Dobbiamo verificare che nel caso del

Dobbiamo verificare che nel caso del cambiamento a coordinate polari la Jacobiano è proprio "r".

$$\iint_D x y^2 \, dx\,dy$$

$$D = \{(x,y), \quad 0 \le x+y \le 2, \quad 0 \le x-y \le 4\}$$

$$\boxed{\begin{cases} x = u+v \\ y = u-v \end{cases}} \longleftrightarrow \boxed{\begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases}}$$

$x = x(u,v)$

$\partial_u x = \partial_u (u+v)$

$$\tilde{D} = \{(u,v), \quad 0 \le u \le 1, \quad 0 \le v \le 2\}$$

$$\left| \det \begin{pmatrix} \partial_u (u+v) & \partial_v (u+v) \\ \partial_u (u-v) & \partial_v (u-v) \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = |1\cdot(-1) - 1\cdot 1|$$

$$= |-2| = 2$$

$$\partial_u (u+v) = 1 + 0$$

$$\iint_D x y^2 \, dx\,dy = \iint_{\tilde{D}} \underbrace{(u+v)}_{x} \underbrace{(u-v)^2}_{y^2} \underbrace{2}_{J} \, du\,dv$$

sviluppo

$$\boxed{\begin{aligned} &(u+v)(u^2 - 2uv + v^2) = \\ &= u^3 - 2u^2 v + u v^2 + \\ &\quad + v u^2 - 2 u v^2 + v^3 \\ &= u^3 - u^2 v - u v^2 + v^3 \end{aligned}}$$

$$= \int_0^1 \left(\int_0^2 (u+v)(u-v)^2 \cdot 2 \, dv \right) du$$

$$= 2 \int_0^1 \left(\int_0^2 u^3 - u^2 v - u v^2 + v^3 \, dv \right) du = 2 \int_0^1 \left. u^3 v - u^2 \frac{v^2}{2} - \frac{u v^3}{3} + \frac{v^4}{4} \right|_0^2 du$$

$$= 2 \int_0^1 2u^3 - 2u^2 - \frac{8}{3} u + 4 \, du = 2 \left(\frac{2u^4}{4} - \frac{2u^3}{3} - \frac{8}{3} \frac{u^2}{2} + 4u \right)\Big|_0^1$$

$$= 2 \left(\frac{1}{2} - \frac{2}{3} - \frac{4}{3} + 4 \right) = 2\left(\frac{1}{2} + 2\right) = 5$$

Calcolo dello Jacobiano per le coordinate polari

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longrightarrow \begin{aligned} &\partial_r x = \cos\theta, \quad \partial_\theta x = r\sin\theta \\ &\partial_r y = \sin\theta, \quad \partial_\theta y = r\cos\theta \end{aligned}$$

$$\Bigg| \begin{pmatrix} \cos\theta & -r\sin\theta \\ \end{pmatrix} \Bigg| = | r\cos^2\theta + r\sin^2\theta | = r$$

$$\left|\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\right| = \left|r\cos^2\theta + r\sin^2\theta\right| = r$$

<u>Suggerimento</u> $D=\{(x,y) : a\le h(x,y)\le b,\ \ c\le g(x,y)\le d\}$

è <u>possibile</u> che il cambio di coordinate $\begin{cases}u=h(x,y)\\ v=g(x,y)\end{cases}$ possa semplificare l'integrale perché

$$\tilde{D}=\{(x,y),\ a\le u\le b,\ c\le v\le d\}$$

$\tilde{D}$ è <u>un rettangolo</u>

(*) <u>Bisogna vedere</u> se è possibile passare da $\begin{cases}u=h(x,y)\\ v=g(x,y)\end{cases} \longleftrightarrow \boxed{\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}}$

risolvere il sistema a "x" e a "y"

Questo è possibile solo se lo Jacobiano è diverso da zero. cioè se

$$\det\begin{pmatrix}\frac{\partial h}{\partial x} & \frac{\partial h}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\neq 0 \quad \forall (x,y) \text{ in } D.$$

Il fatto che sia "possibile" non implica che sia <u>realizzabile</u>.

Se (*) è verificato allora vale la pena tentare il cambio di variabili.

Se $\exists(\bar{x},\bar{y})$ all'interno del dominio tale che

$$\det\begin{pmatrix}\frac{\partial h}{\partial x}(\bar{x},\bar{y}) & \frac{\partial f}{\partial y}(\bar{x},\bar{y})\\ \frac{\partial g}{\partial x}(\bar{x},\bar{y}) & \frac{\partial g}{\partial y}(\bar{x},\bar{y})\end{pmatrix}=0$$

allora <u>non</u> posso fare la sostituzione allora **non** posso fare la sostituzione

$$\begin{cases} u = h(x,y) \\ v = g(x,y) \end{cases}.$$

~~1 Curve~~ | ~~2 Esercizi Funzioni~~ | ~~1 Integrali~~

~~1 Esercizio 4 o 5 domande con difficoltà crescente~~

~~a) facilissima $\to$ 3~~

~~b) facili $\to$ 2~~

~~c) facili $\to$ 2~~

~~d) Difficili $\to$~~