A primer in viscosity solutions

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1 Overview

The aim of these notes is to outline the fundamental theory of viscosity solutions with particular emphasis on comparison results.

2 A motivating example

We start by a one dimensional simple Dirichlet problem. We want to solve

$$\begin{cases} |u'| & = 1 \qquad \text{in } (-1,1) \\ u & = 0 \qquad \text{at } -1, 1. \end{cases} \tag{1}$$

Since any C^1 function taking the same value at the endpoints, must have zero derivative somewhere in the interior of the interval, we realize that no classical solution may exist. We therefore look for some weak solution in a suitable sense, and a natural idea is to think of Lipschitz–continuous almost everywhere (a.e.) solutions.

However in this setting solutions are infinite, since they apparently include any sawtooth function with slope ± 1 which vanishes at the endpoints. In addition, taking smaller and smaller teeth, we can construct a sequence of such functions converging to the null function which clearly is not any more solution.

Our aim in what follows is to illustrate a theory, due to to M. Crandall and P.L.L. Lions, which, among other things, will allow to uniquely solve the problem in object and will provide powerful stability results ensuring in particular that uniform limit of solutions is still solutions.

3 Basic material

We will write in what follows the basic first and second order equation as

$$H(x, u, Du) = 0 \tag{2}$$

$$F(x, u, Du, D^2u) = 0, \tag{3}$$

where the state variable x belongs to $\mathbb{R}^N$, or to an open bounded subset Ω of $\mathbb{R}^N$ or to the flat torus $\mathbb{T}^N$, according to the different problems we tackle; u represents the unknown function, Du, D^2u stand for the differential and the Hessian of u, respectively, and H, F are continuous real valued functions defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^N$ and $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$, respectively, with $\mathcal{S}^N$ indicating the space of $N \times N$ symmetric matrices. We will sometimes refer, in what follows, to H in (2) as the *Hamiltonian* of the problem.

In order to give the notion of the weak (super, sub)– solutions in the viscosity sense of the previous equations, the idea is to test pointwise the candidate continuous or semicontinuous functions by means of $\mathcal{C}^1$ or $\mathcal{C}^2$ test functions according on whether the first or second order case is under investigation. We will use the slightly ambiguous term of *regular* to encompass both these setups. To repeat: regular function will mean $\mathcal{C}^1$ or $\mathcal{C}^1$ if (2) or (3) are on the stage, respectively.

Definition 3.1. Given an upper semicontinuous (usc for short) function u, we say that a regular function ψ is *supertangent* to u at some point x_0 if ψ and u coincides at x_0 and ψ is above u locally around x_0. Note that then x_0 is a local maximizer of $u - \psi$.
Similarly, a regular function φ is called *subtangent* to a lower semicontinuous (lsc for short) function v at x_0 if the two functions are equal at x_0 and φ is below v locally around x_0. Accordingly x_0 is a local minimizer of $v - \varphi$.
If in addition x_0 is a strict or global maximizer (resp minimizer) then the supertangent (resp. subtangent) is qualified as *strict* or *global*, respectively.

Remark 3.2. The requirement that test functions ψ, φ and u, v coincide at the point where the test is performed has been introduced just for cosmetic reasons, specifically to underpin the intuition about super/subtangency. It can be dropped without consequences. The key point in the above definition of viscosity test functions is that $u - \psi$, $v - \psi$ attain a local extremum at the point where the test is realized.

The following simple remark will have some utility in what follows.

Remark 3.3. Let u, ψ be usc (resp. lsc) and regular functions, respectively. A regular function φ is supertangent to u (resp. subtangent) at some point x_0 if and only if $\varphi - \psi$ is supertangent (resp. subtangent) to $u - \psi$ at x_0.

Going back to problem (1), we claim that the unique obstruction for the sawtooth function, which is a.e. solution, to also be solution in the viscosity sense is the presence of a local minimizer in $(0, 1)$. In this case, in fact, at any such minimizer the function has a constant subtangent with zero derivative which clearly does not satisfy the supersolution inequality, as required by the definition given above.

Outside these points the tests for solution are satisfied. In fact, locally around any maximizer, denoted by x_0, the function, say u, has on the left slope 1 and on the right -1. Therefore no subtangent can exist. If we denote by ψ a C^1 supertangent at x_0 then the function $u-\psi$ is differentiable in $I := [x_0-\varepsilon, x_0]$, for ε suitably small, and possess a maximizer, global with respect to I, at x_0. This implies

$$1 - \psi'(x_0) = (u - \psi)'(x_0) \geq 0.$$

Arguing similarly in a suitable left neighborhood of x_0, we get

$$-1 - \psi'(x_0) \leq 0$$

which finally shows that the subsolution test

$$|\psi'(x_0)| \leq 1$$

is satisfied. Finally, the function in object is differentiable with derivative 1 or -1 at any point which is not maximizer neither minimizer.

According to the previous remarks, the unique element, in the class of saw-tooth function, which is viscosity solution is the one with slope 1 in $(-1, 0)$ and -1 in $(0, 1)$ and so the unique which does not possess minimizers in the interior of the interval. This function is

$$x \mapsto d(x, \{-1, 1\}) = \min\{|x - 1|, |x + 1|\}.$$

We will show that this is indeed the unique solution to (1) and the above formula can be generalized to any dimension.

We resume the general treatment of the theory and introduce:

Definition 3.4. Given a lsc function u, the *subdifferential* $D^-u(x)$ at any point x is defined as the (possibly empty) set

$$D^-u(x) := \{p : p = D\varphi(x) \mid \varphi \text{ is subtangent to } u \text{ at } x\}$$

and for an usc function v the *superdifferential* $D^+v(x)$ is the (possibly empty) set

$$D^+v(x) := \{p : p = D\psi(x) \mid \varphi \text{ is supertangent to } v \text{ at } x\}$$

The weak derivatives sets above introduced, if not empty, are closed and convex. The next result, at least an half of it, is crucial to show that classical solution are viscosity solutions as well.

Proposition 3.5. *If u is differentiable at x then $D^+u(x) \cap D^-u(x) = \{Du(x)\}$. Conversely, let u be continuous, if both $D^+u(x)$ and $D^-u(x)$ are nonempty then u is differentiable at x and $D^+u(x) \cap D^-u(x) = \{Du(x)\}$.*

Proof. Assume $D^+u(x)$, $D^-u(x)$ simultaneously nonempty, then there are two C^1 functions φ, ψ with

$$\varphi(y) \leq u(y) \leq \psi(y) \text{ locally at } x \text{ and } \varphi(x) = u(x) = \psi(x)$$

then $D\psi(x) = D\varphi(x) =: p$ and

$$\frac{\varphi(y) - \varphi(x) - p \cdot (y - x)}{|y - x|} \leq \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq \frac{\psi(y) - \psi(x) - p \cdot (y - x)}{|y - x|}$$

locally at x. Owing to differentiability of φ and ψ at x, this in turn implies

$$\lim_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} = 0$$

that yields differentiability of u, as well, at x with $Du(x) = p$. This also shows $D^+u(x) \cap D^-u(x) = \{Du(x)\}$.

Conversely, assume that u is differentiable at x, we aim at finding a C^1 superstangent of u at x. The idea is to define it in the form

$$y \mapsto u(x) + Du(x) \cdot (y - x) + g(|y - x|) \tag{4}$$

for a suitable g to be determined. A natural choice is to set

$$g(r) = \max_{y \in \overline{B(0,r)}} \frac{u(y) - u(x) - Du(x) \cdot (y - x)}{|y - x|} \geq 0,$$

where $B(x, r)$ is the open ball with radius r centered at x. It is apparent that g is nondecreasing and continuous for r varying in $(0, 1]$, in addition $\lim_{r \to 0^+} g(r) = 0$ because u is supposed differentiable at x. Therefore g can be extended, keeping continuity, in $[0, 1]$ by setting $g(0) = 0$. Since we have by the very definition

$$u(y) \leq u(x) + Du(x) \cdot (y - x) + g(|y - x|)\, |y - x| \qquad \text{for } y \in B(0, 1), \tag{5}$$

we actually see that with the above choice of g, the function in (4) is supertangent to u at x. But unfortunately it is not of class C^1. To overcome this difficulty, we consider the antiderivative of g in $[0, 1]$ vanishing at 0, denoted by G. Exploiting the monotonicity and the sign of g, we get the extimate

$$G(2\,r) = \int_0^r g(s)\, ds + \int_r^{2r} g(s)\, ds \geq \int_r^{2r} g(s)\, ds \geq g(r)\, r \tag{6}$$

for $r \in [0, 1/2]$. By combining (5), (6), we further obtain

$$u(y) \leq u(x) + Du(x) \cdot (y - x) + G(2\,|y - x|).$$

for y with $|y - x| < \frac{1}{2}$. The function

$$\psi(y) = u(x) + Du(x) \cdot (y - x) + G(2\,|y - x|)$$

is then supertangent to u at x, and is in addition of class C^1 with

$$\begin{aligned} D\psi(y) &= Du(x) + 2\,g(2\,|y-x|)\,\frac{y-x}{|y-x|} \quad \text{at } y \neq x \\ D\psi(x) &= Du(x). \end{aligned}$$

This shows that $Du(x) = D\psi(x) \in D^+u(x)$. By modifying the definition of g with min in place of max, and accordingly adapting the above argument, we also get $Du(x) \in D^-u(x)$ and conclude the proof. □

We proceed giving the definition of (super, sub)– solutions in the viscosity sense for (3), corresponding notions for equation (2) are obtained with obvious adaptations.

Definition 3.6. An usc function u is called a *viscosity subsolution* of (3) if

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0$$

for any x_0, any regular ψ supertangent to u at x_0.
Similarly, a lsc function v is called *viscosity supersolution* if

$$F(x_0, v(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0$$

for any x_0, any regular φ subtangent to $v$at x_0.
Finally a continuous function u is called viscosity solution if it is at the same time super and subsolution.

This definition can be rephrased, but only for equation (2), using sub and superdifferentials, as follows:

Definition 3.7. An usc function u is a viscosity subsolution of (2) if

$$H(x_0, u(x_0), p) \leq 0 \quad \text{for any } x_0, \text{ any } p \in D^+u(x_0).$$

Similarly, a lsc function v is viscosity supersolution if

$$H(x_0, v(x_0), p) \geq 0 \quad \text{for any } x_0, \text{ any } p \in D^-v(x_0).$$

From now any (sub, super)–solution we will consider are understood to be in the sense of the previous definition , unless otherwise specified. For this reason we will often omit the qualification viscosity, or in the viscosity sense.

A first compulsory step is to show that the above weak notions are consistent with that of classical, i.e. pointwise, solution for sufficiently regular functions. For this, we need to require an additional condition on the F appearing in (3). In the forthcoming definition we are going to employ the natural order relation in $\mathcal{S}^N$, namely, given X, Y in $\mathcal{S}^N$, $X \leq Y$ means that $X - Y$ is nonnegative definite or, in other terms, possess nonnegative eigenvalues, and $X < Y$ means that $X - Y$ is positive definite or, equivalently, possess positive eigenvalues.

Definition 3.8. We say that $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$ is *elliptic* if for any $(x, s, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$

$$F(x, s, p, X) \geq F(x, s, p, Y) \quad \text{whenever } X \leq Y.$$

Even if it is not required in the forthcoming proof, we make also clear that in the development of the theory, we also need to assume $s \mapsto H(x, s, p)$ and $s \mapsto F(x, s, p, X)$ to be nondecreasing for any x, p, X.

Proposition 3.9.

- *If a $\mathcal{C}^1$ function u is a classical (sub, super)–solution to* (2)*, then it is also a viscosity solution.*
- *If a $\mathcal{C}^2$ function u is a classical (sub, super)–solution to* (3) *with F elliptic, then it is also a viscosity solution.*

Proof. To fix our ideas, assume u to be a classical subsolution to (3) and take a regular function ψ supertangent to it at some point x_0, then x_0 is a local maximizer of $u - \psi$, and, of course, $Du(x_0) = D\psi(x_0)$. It follows

$$0 \geq H(x_0, u(x_0), Du(x_0)) = H(x_0, u(x_0), D\psi(x_0)).$$

The subtangent case can be treated in the same way, and so the first item of the assertion is proved.

Let us pass to the second order equation, and assume u to be a classical supersolution to (3). Let ψ be a regular subtangent of u at x_0. Then x_0 is a local minimizer of $u - \psi$ and so $Du(x_0) = D\psi(x_0)$, $D^2u(x_0) \geq D^2\psi(x_0)$. Due to the monotonicity condition on F we have

$$0 \leq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq F(x_0, u(x_0), Du(x_0), D^2\psi(x_0)).$$

The same argument, with easy adjustment, proves all the other cases. The assertion is then proved. $\square$

Next, we show an important principle which can be summarized saying that the supremum of any family of subsolutions is subsolutions and corresponding property holds for infima of families of supersolutions. We first establish this fact for finite families of super/subsolutions, the case of infinite families, which is more involved, will be treated later on. Given a finite family u_i of functions, we set

$$\begin{aligned} \vee_i u_i &= \max_i u_i \\ \wedge u_i &= \min_i u_i \end{aligned}$$

Proposition 3.10. *Let u_i be a finite family of usc subsolutions (resp. lsc supersolutions) to* (2) *or* (3) *in $\mathbb{R}^N$, then $\vee_i u_i$ (resp. $\wedge_i u_i$) is subsolution (resp. supersolution) of the same equation*

Proof. We prove the statement for two usc subsolutions u, v of (2), the argument is basically the same for the the other cases. Fix x and consider $x_n \to x$ then

$$\begin{aligned} \limsup v(x_n) &\leq v(x) \leq u \vee v(x) \\ \limsup u(x_n) &\leq u(x) \leq u \vee v(x), \end{aligned}$$

which implies that $u \vee v$ is usc because of the relation

$$\limsup u \vee v(x_n) = \max\{\limsup u(x_n), \limsup v(x_n)\}.$$

Now, we assume $u \vee v(x) = v(x)$ and that there is a C^1 supertangent ψ to $u \vee v$ at x, then ψ is also supertangent to v at the same point and consequently

$$H(x, u(x), D\psi(x)) \leq 0.$$

This proves the assertion. □

4 Generalized gradient

In this section we introduce some weak differentials for locally Lipschitz–continuous functions. Before entering in the core of the matter, we recall a couple of results we necessitate.

Theorem 4.1. *(Hahn–Banach) Let K, x be a closed convex subset of $\mathbb{R}^N$ and a point of $\mathbb{R}^N \setminus K$, respectively. Then there is $p \in \mathbb{R}^N$ with*

$$p \cdot x > \max\{p \cdot y \mid y \in K\}$$

Theorem 4.2. *(Fubini) Let S, R be two spaces endowed with σ–finite measures μ, ν, respectively. Let $A \subset S \times R$ be measurable with respect the product measure $\mu \times \nu$ then*

$$A_s = \{y \in R \mid (s, y) \in A\} \subset R$$

is ν–measurable for μ–a.e. s, the function $s \mapsto \nu(A_s)$ is μ–measurable and

$$(\mu \times \nu)(A) = \int_S \nu(A_s)\, d\mu(s).$$

We start recalling that any locally Lipschitz-continuous functions is almost everywhere differentiable, in force of Rademacher Theorem. We define the *Clarke Generalized Gradient* at x for a locally Lipschitz function u as:

$$\partial u(x) = co\{p = \lim_n Du(x_n), x_n \to x, \text{u is diff. at } x_n\} \tag{7}$$

It is easy to check that $\partial u(x)$ is convex compact valued. If it reduces to a singleton at some point, say x, then u is *strictly differentiable* at x, in the sense that $Du(x)$ does exist and

$$Du(x_n) \to Du(x) \quad \text{for any } x_n \text{ converging to } x \text{ where } u \text{ is differentiable.}$$

Moreover, the following continuity property can be proved.

Proposition 4.3. *Assume*

$$x_n \to x, \; p_n \in \partial u(x_n), \; p_n \to p$$

then

$$p \in \partial u(x).$$

If u is in addition convex then the Clarke generalized gradient is nothing but the usual subdifferential of convex analysis, which in turn coincide with the subdifferential we have previously defined. A convex function admits at any point global linear subtangents, accordingly, given x, we have

$$u(y) \geq u(x) + p \cdot (y - x) \quad \text{for any } y\text{, any } p \in \partial u(x) = D^- u(x) \tag{8}$$

Let us point out that the presence of the convex hull in the definition of generalized gradient is essential to keep, in the nonsmooth setting, the usual variational property that if a point x_0 is a local maximizer or minimizer of u then $0 \in \partial u(x_0)$, as fully proved in the next result.

Proposition 4.4. *Let u be locally Lipschitz–continuous then $0 \in \partial u(x_0)$ at any of its local maximizer or minimizer.*

Proof. Since, by the very definition of generalized gradient, $\partial(-u) = -\partial u$ and x_0 is a local minimizer of $-u$ provided it is local maximizer of u, it is enough to show the assertion for minimizers.

Given such a point x_0, assume, for purposes of contradiction, that there is $a \in \mathbb{R}^N$ with

$$\max\{p \cdot a \mid p \in \partial u(x_0)\} < 0,$$

so that

$$Du(y) \cdot a < -\alpha \tag{9}$$

for all differentiability points of u belonging to a suitable ball B centered at x_0 and a constant $\alpha > 0$.

Having denoted by Π the hyperplane orthogonal to a passing through x_0, we look at the N–dimensional Lebesgue measure on $\mathbb{R}^N \sim \Pi \times \Pi^\perp$, where $\perp$ means orthogonal, as the product of the $N-1$–dimensional Lebesgue measure, denoted by $\mathcal{L}^{N-1}$, on Π and the 1 dimensional measure, denoted by $\mathcal{L}^1$, on $\Pi^\perp$. Further, we indicate by D the subset of $\mathbb{R}^N$ where u is not differentiable. According to Fubini Theorem, we have

$$0 = \int_\Pi \mathcal{L}^1(D_x)\, d\mathcal{L}^{N-1}(x),$$

and we consequently deduce that for $\mathcal{L}^{N-1}$–a.e. $x \in \Pi$, $\mathcal{L}^1(D_x) = 0$. We can consequently find a sequence of lines

$$x_n + t\,a \qquad \text{for } t \in \mathbb{R}.$$

with x_n in $\Pi \cap B$ converging to x_0, made up by differentiability points of u, up to a set of vanishing 1–dimensional measure.

There is $h > 0$ such that $x_n + h\,a \in B$ for n sufficiently large, so that, taking into account (9), we get

$$u(x_n + h\,a) - u(x_n) = \int_0^h Du(x + ta) \cdot a\, dt \leq -h\,\alpha.$$

Sending n to $+\infty$, we deduce

$$u(x_0 + h\,a) - u(x_0) \leq -h\,\alpha,$$

in contrast with x_0 being a minimizer of u in B. We must therefore have

$$\max\{p \cdot a \mid p \in \partial u(x_0)\} \geq 0 \qquad \text{for any } a \in \mathbb{R}^N,$$

and the assertion is therefore obtained in force of Hahn–Banach Theorem. $\square$

The proof of previous result is not at all trivial since it makes essential use of Fubini Theorem. The role of it is specifically to prove the existence of a dense family of lines where the Lipschitz function u under exam is differentiable up to a set of 1–dimensional Lebesgue measure. This in turn allows to put in relation the variation of u on the line and a suitable integral containing the differential of u in the integrand. The issue is somehow slippery, and we devote next Remark to cast some light on it.

Remark 4.5. For a locally Lipschitz–continuous function u, we have the relation

$$u(y) - u(x) = \int_0^T \frac{d}{dt} u(\xi(t))\, dt$$

for any x, y, any Lipschitz–continuous curve ξ with $\xi(0) = x$, $\xi(T) = y$.

Under the above assumptions on u and ξ, in fact, it can be proved that the composition $u \circ \xi$ inherits Lipschitz–continuity and so by Rademacher Theorem $\frac{d}{dt}u(\xi(t))$ do exist for a.e t. However, one should be careful when writing down the apparently equivalent formula

$$u(y) - u(x) = \int_0^T Du(\xi(t)) \cdot \dot{\xi}(t))\, dt,$$

since the relation $\frac{d}{dt}u(\xi(t)) = Du(\xi(t)) \cdot \dot{\xi}(t)$, on which it is based, is surely true if both u and ξ are differentiable, and thus is indeed the case for ξ at a.e. t. But such regularity is not guaranteed for u at any $\xi(t)$ for the reason that the support of the curve has vanishing N–dimensional Lebesgue measure, and so Rademacher Theorem does not give information. However we can prove:

Proposition 4.6. *Given a Lipschitz–continuous curve ξ defined in some interval $[0, T]$, there is a measurable map $p : [0, T] \to \mathbb{R}^N$ such that*

$$\frac{d}{dt}u(\xi(t)) = p(t) \cdot \dot{\xi}(t) \qquad \text{for a.e. } t. \tag{10}$$

holds true.

Proof. We will use in the proof the relation

$$\limsup_{\substack{y \to x \\ h \to 0^+}} \frac{u(y + h\, q) - u(y)}{h} = \max\{p \cdot q \mid p \in \partial u(x)\} \tag{11}$$

which holds true for any x and q, see [3]. We know that for a.e. t both $t \mapsto u(\xi(t))$ and $t \mapsto \xi(t)$ are differentiable, for such a t we have exploiting differentiability of $t \mapsto u(\xi(t))$

$$\frac{d}{dt}u(\xi(t)) = \lim_{h \to 0} \frac{u(\xi(t + h)) - u(\xi(t))}{h}$$

and at the same time exploiting differentiability of $t \mapsto \xi(t)$ plus Lipschitz continuity of u

$$\lim_{h \to 0} \frac{u(\xi(t + h)) - u(\xi(t) + h\, \dot{\xi}(t))}{h} = 0$$

which implies

$$\frac{d}{dt}u(\xi(t)) = \lim_{h \to 0^+} \frac{u(\xi(t) + h\, \dot{\xi}(t)) - u(\xi(t))}{h}$$

and taking into account (11)

$$\frac{d}{dt}u(\xi(t)) \leq \max\{p \cdot \dot{\xi}(t) \mid p \in \partial u(x)\}. \tag{12}$$

We further have

$$-\frac{d}{dt}u(\xi(t)) = \lim_{h\to 0^+} \frac{u(\xi(t) + h\,(-\dot{\xi}(t))) - u(\xi(t))}{h}$$

and consequently

$$\frac{d}{dt}u(\xi(t)) \geq -\max\{p\cdot(-\dot{\xi}(t)) \mid p \in \partial u(x)\} = \min\{p\cdot\dot{\xi}(t) \mid p \in \partial u(x)\} \tag{13}$$

Bearing in mind that $\partial u(\xi(t))$ is convex, we directly deduce (10) from (12) and (13). The fact that $p(\cdot)$ is in addition measurable depends on some selection results, we skip the detail here. □

5 Viscosity and almost everywhere solutions

We directly deduce from the definition of generalized gradient:

Proposition 5.1. *Let u, ψ be locally Lipschitz continuous and $\mathcal{C}^1$, respectively, then*

$$\partial(u-\psi)(x) = \partial u(x) - D\psi(x) \qquad \textit{for any } x$$

We derive from Propositions 4.4, 5.1

Corollary 5.2. *Let u be locally Lipschitz–continuous then*

$$D^+u(x) \cup D^-u(x) \subset \partial u(x) \qquad \textit{for any } x.$$

Notice that, in contrast to what happens for the sub superdifferential above defined which can be empty, $\partial u(x) \neq \emptyset$ for any x.

Proposition 5.3. *Let H be convex in the second argument. For a locally Lipschitz function u the following conditions are equivalent in reference to equation* (2)*:*

i. *u is viscosity subsolution.*

ii. *u is a.e. subsolution.*

iii. *For all x, all $p \in \partial u(x)$, $H(x,u(x),p) \leq 0$.*

Proof. **i.** $\Rightarrow$ **ii.** If u is as in the statement, then by Proposition 3.5

$$Du(x) \in D^+u(x) \quad \text{at any } x \text{ where } u \text{ is differentiable,}$$

accordingly, if u is subsolution in the viscosity sense then it is also a.e. subsolution.

ii. $\Rightarrow$ iii. We proceed assuming u to be an a.e. subsolution. We take a point x and $p \in \partial u(x)$, by the very definition of generalized gradient

$$p = \sum_i \lambda_i p_i,$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, $p_i = \lim_k Du(x_k^i)$, $\lim_k x_k^i = x$, for any i. We derive exploiting the continuity of H

$$H(x, u(x), p_i) = \lim_k H(x_k^i, u(x_k^i), Du(x_k^i)) \leq 0 \quad \text{for any } i,$$

and by the convexity assumption

$$H(x, u(x), p) = H\left(x, u(x), \sum_i \lambda_i p_i\right) \leq \sum_i \lambda_i H(x, u(x), p_i) \leq 0,$$

as claimed.
iii. $\Rightarrow$ i. The implication directly comes from Corollary 5.2. $\square$

If we ask more on the differential structure of u, we can state a more general result and skip the assumption of convexity for H.

Proposition 5.4. *Assume u to be a Lipschitz function, with $\partial u(x) = D^- u(x)$ for any x. Then u is an a.e. subsolution to $H(x, u, Du) = 0$ if and only it is a viscosity subsolution.*
If $\partial u(x) = D^+ u(x)$ for any x then u is an a.e. supersolution to (2) *if and only it is a viscosity supersolution.*

Proof. If $\partial u(x) = D^- u(x)$ for any x then, according to Proposition 3.5, u is differentiable at some x_0 if and only it admits a supertangent at such point. It is hence clear that the notions of a.e. and viscosity subsolution coincide. Same argument works, with obvious adaptations, for the part of the statement about supersolutions. $\square$

We provide an application of previous results to the generalization of Problem (1) to any dimension. It is called Eikonal equation coupled with Dirichlet homogeneous boundary condition. Given an open bounded set $\Omega \subset \mathbb{R}^N$, we consider the problem

$$\begin{cases} |Du| & = 1 \quad \text{in } \Omega \\ \quad u & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{14}$$

For an upper semicontinuous subsolution u, the boundary condition must be understood as follows:

$$\limsup_{y \to x} u(y) = 0 \qquad \text{for } x \in \partial\Omega,\ y \in \Omega.$$

In the forthcoming proof we will exploit the property that the signed distance $d^\#(\cdot, C)$ to any convex subset C of $\mathbb{R}^N$ is a convex function, where

$$d^\#(x, C) = \begin{cases} d(x, C) & \text{if } x \notin C \\ -d(x, \partial C) & \text{if } x \in C \end{cases}$$

Proposition 5.5. *Any upper semicontinuous subsolution of the problem* (14) *with* Ω *convex, is Lipschitz–continuous with Lipschitz constant* 1.

Proof. Let v be an upper semicontinuous subsolution and M a positive constant, then

$$u := v \vee -M \tag{15}$$

is still a subsolution satisfying homogeneous boundary conditions by Proposition 3.10, with the extra property of being bounded from below. We take $\varepsilon > 0$ such that

$$\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) \leq \varepsilon\} \neq \emptyset,$$

note that this set is convex because Ω is so and the signed distance from a convex set is convex. Since ε is a parameter devoted to go to 0, we can assume, without loosing generality, that $\frac{M}{\varepsilon} > 1$. We claim that u is Lipschitz–continuous with constant $\frac{2M}{\varepsilon}$ in Ω_ε; to prove it, we fix any $x_0 \in \Omega_\varepsilon$ and consider the function

$$\psi(x) = u(x_0) + \frac{2M}{\varepsilon}\,|x - x_0|.$$

We first exclude a maximizer of $u-\psi$ in $\overline{\Omega}$ to be in $\partial\Omega$, in fact, if such a maximizer, denoted by $y \in \partial\Omega$, does exist then

$$u(y) - \psi(y) = -u(x_0) - \frac{2M}{\varepsilon}\,|y - x_0| \geq u(x_0) - \psi(x_0) = 0$$

so that we derive

$$u(x_0) \leq -\frac{2M}{\varepsilon}\,|y - x_0| \leq -2M$$

which is impossible in view of (15). We proceed also excluding a maximizer, still denoted by y, to be $\Omega \setminus \{x_0\}$; in fact if this were the case then ψ should be a viscosity test function from above to u at y of class C^1 in a neighborhood of y, so that

$$1 \geq |D\psi(y)| = \frac{2M}{\varepsilon} > 1$$

which cannot be. Consequently, the unique maximizer of $u - \psi$ in $\overline{\Omega}$ must be at x_0, so that

$$u(x) - \psi(x) \leq u(x_0) - \psi(x_0) = 0 \quad \text{for } x \in \Omega$$

or, in other terms

$$u(x) - u(x_0) \leq \frac{2M}{\varepsilon} |x - x_0| \quad \text{for } x \in \Omega.$$

This shows the claimed Lipschitz continuous character of u in Ω_ε (recall that x_0 was taken in Ω_ε).

We proceed showing that u has actually Lipschitz constant 1 in Ω_ε. We consider a pair of points x, y in this set and the segment joining them, parametrized as $\xi(t) = t\,x + (1-t)\,y$, $t \in [0,1]$. It entirely lies in Ω_ε which is convex. Owing to Proposition 10.1, we have

$$u(y) - u(x) = \int_0^1 \frac{d}{dt}\, u(\xi(t))\, dt = \int_0^1 p(t) \cdot \dot{\xi}(t)\, dt$$

for suitable $p(t) \in \partial u(\xi(t))$. By Proposition 5.3 $|p(t)| \leq 1$, so that

$$|u(y) - u(x)| \leq \int_0^1 |p(t)|\, |\dot{\xi}(t)|\, dt \int_0^1 |\dot{\xi}(t))|\, dt = |x - y|.$$

Summing up, we have proved that u is Lipschitz continuous with Lipschitz constant 1 in Ω_ε, for any ε sufficiently small. Letting ε go to 0, we get the 1–Lipschitz continuity of u in Ω, and thanks to the boundary condition,is continuous up to the boundary.

To conclude the proof, we take $M > \operatorname{diam}\Omega$, where diam is the diameter, then, for x in Ω and y in $\partial\Omega$, we get

$$u(x) = u(x) - u(y) \geq -|x - y| \geq -\operatorname{diam}\Omega > -M,$$

so that for this choice of M, we have $u = v$. Hence the original subsolution v is indeed 1–Lipschitz–continuous in Ω, as desired. □

6 Semiconvex and semiconcave functions

We proceed introducing two special classes of locally Lipschitz continuous functions for which the properties assumed in the statement of Proposition 5.4 hold true.

Definition 6.1. A function u is said *semiconvex* if one of the following equivalent conditions is valid, for some $\alpha \geq 0$

- $u(x) + \alpha|x|^2$ is a convex function;
- $u(x) + \alpha|x - x_0|^2$ is a convex function for some x_0;
- $u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y) + \alpha\lambda(1-\lambda)|x - y|^2$ for any x, y.

We refer to α as a semiconvexity constant for u.
A function u is said *semiconcave* if one of the following equivalent conditions hold for some $\alpha \geq 0$

- $u(x) - \alpha|x|^2$ is concave ;
- $u(x) - \alpha|x - x_0|^2$ is concave for some x_0;
- $u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y) - \alpha\lambda(1-\lambda)|x-y|^2$ for any x, y.

We refer to α as a semiconcavity constant for u.

See [5] for a comprehensive survey on semiconvex and semiconcave functions. They inherit the regularity properties of convex and concave functions, in particular they are locally Lipschitz continuous and twice differentiable almost everywhere, in force of Alexandrov Theorem. With the term *paraboloid*, we mean a function of the form

$$x \mapsto \beta + p \cdot (x - x_0) + \alpha\, |x - x_0|^2$$

for some α, β in $\mathbb{R}$, p, x_0 in $\mathbb{R}^N$. The constant α, which can have any sign, is called *opening* of the paraboloid

Proposition 6.2. *Let u be semiconcave (resp. semiconvex) then $\partial u(x) = D^- u(x)$ (resp. $\partial u(x) = D^+ u(x)$) for all x. Moreover, there exists a paraboloid globally supertangent (resp. subtangent) to u at any point.*

Proof. We only prove the assertion for u semiconvex, the other case being similar. Let us fix a point x_0 and denote by α a semiconvexity constant for u. Then

$$u(x) = \underbrace{(u(x) + \alpha|x - x_0|^2)}_{\varphi(x)} - \alpha|x - x_0|^2$$

with φ convex, and by (7)

$$\partial u(x) = \partial\varphi(x) - 2\alpha(x - x_0),$$

and, in particular $\partial u(x_0) = \partial\varphi(x_0)$. Take a $p \in \partial u(x_0)$, since φ is convex then by (8)

$$\begin{aligned} u(x) &= \varphi(x) - \alpha|x - x_0|^2 \geq \varphi(x_0) + p \cdot (x - x_0) - \alpha|x - x_0|^2 \\ &= u(x_0) + p \cdot (x - x_0) - \alpha|x - x_0|^2, \end{aligned}$$

for any x, which shows that $p \in D^- u(x_0)$, and that u admits a global subtangent paraboloid at x_0. $\square$

As a consequence of the above result, we can write a semiconvex function u at any point x_0 as

$$u(x_0) = \sup_{x, p \in \partial u(x)} (u(x) + p(x - x_0) - \alpha |x - x_0|^2),$$

namely as the sup envelope of a family of paraboloid with fixed opening. In the next section we will perform an inverse construction: starting from any upper semicontinuous (resp. lower semicontinuous) function we will define semiconvex (resp semiconcave) functions through sup (resp. inf) envelope of suitable classes of paraboloids.

7 Convolutions and weak semilimits

We start by the relevant definitions

Definition 7.1. Given an usc function u bounded from above in $\mathbb{R}^N$ and ε, the ε–sup of u is given by

$$u^\varepsilon(x) = \max_{y \in \mathbb{R}^N} \left(u(y) - \frac{1}{2\varepsilon} |y - x|^2 \right), \tag{16}$$

notice that the maximum in the formula does exist in force of upper semicontinuity and boundedness of u. Similarly, for a lsc function v bounded from below in $\mathbb{R}^N$ and ε, the ε–inf convolution of v is given by

$$v_\varepsilon(x) = \min_{y \in \mathbb{R}^N} \left(v(y) + \frac{1}{2\varepsilon} |y - x|^2 \right), \tag{17}$$

It is apparent that maxima and minima in the previous definitions do exist in force of boundedness assumptions on u, v. We also clearly have

- $u^\varepsilon \geq u$, $v_\varepsilon \leq v$ for any ε.
- $u^\varepsilon(x)$ is nonincreasing and $v_\varepsilon(x)$ nonincreasing with respect to ε for any fixed x.

From now on we give definitions and statement of results mainly for sup–convolutions. By slightly adapting them, we get the corresponding entities and facts for inf–convolutions.

Definition 7.2. We say that y_0 is u^ε*–optimal* for a given x_0 if

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon} |x_0 - y_0|^2$$

Proposition 7.3. *For any usc (resp. lsc) function u bounded from above (resp. below), any $\varepsilon > 0$ the sup (resp. inf)– convolution is semiconvex (resp. semiconcave) with semiconvexity (resp. semiconcavity) constant $\frac{1}{2\,\varepsilon}$.*

Proof. We just prove the part of the statement about sup convolution. We will show that $u^\varepsilon(x) + \dfrac{1}{2\varepsilon}|x|^2$ is convex. We compute

$$\begin{aligned} u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2 &= \sup_{y \in \mathbb{R}^N} \{u(x) - \frac{1}{2\varepsilon}|y - x|^2 + \frac{1}{2\varepsilon}|x|^2\} \\ &= \sup_{y \in \mathbb{R}^N} \{u(y) - \frac{1}{2\varepsilon}|y|^2 + \frac{1}{\varepsilon}(y \cdot x)\} \end{aligned}$$

Then $u^\varepsilon(x) + \dfrac{1}{2\varepsilon}|x|^2$ can be written as the supremum of linear function, and so it is convex, as claimed.

□

Proposition 7.4. *Given an usc (resp. lsc) function u bounded from above (resp. below), then the family of convolutions u^ε (resp. u_ε), for $\varepsilon > 0$, is equibounded from above. If, in addition u is bounded then u^ε (resp. u_ε) is equibounded.*

Proof. We have for any x, ε, any y u^ε–optimal for x

$$u^\varepsilon(x) = u(y) - \frac{1}{2\varepsilon}\,|x - y|^2 \leq u(y) \leq \sup u.$$

This shows the first part of the assertion. If u is bounded, we in addition have

$$\inf u \leq u(x) \leq u^\varepsilon(x) \qquad \text{for any } x.$$

□

To put it in a general way: any theory of weak solutions for a given class of partial differential equations needs to be complemented by a regularization procedure, allowing to approximate the corresponding weak solutions by more regular functions. This is, for instance, the case of mollification for solutions in the distributional sense. In our setting, sup and inf–convolutions are actually the relevant operations in this sense, and their use will be of paramount importance in what follows

It is then incumbent on us the task of making precise in which sense such convolutions constitute an approximation of the initial functions. To this aim, we introduce the weak semilimits.

Definition 7.5. Given a sequence u_n of functions locally equibounded from above (resp. below) and with $u_n(x)$ bounded from below (resp. above), for any x, we say that u is *weak upper* (resp, *lower*) *semilimit*, and write

$$u = \limsup{}^{\#} u_n \qquad (\text{resp. } u = \liminf{}_{\#} u_n)$$

if

$$\begin{array}{rrcl} & u(x) & = & \sup\{\limsup_n u_n(x_n) \mid x_n \to x\} \\ (\text{resp.} & u(x) & = & \inf\{\liminf_n u_n(x_n) \mid x_n \to x\}). \end{array}$$

Note that the boundedness conditions on u_n guarantee that both semilimits take finite values. Further, remark that, in contrast to the usual notions of pointwise lim sup and lim inf of a sequence of functions, in the previous definition the argument of u_n, when taking the limit, is not fixed but varies with n.

Proposition 7.6. *Let u_n be a sequence of functions with the same properties as in Definition 7.5. Then the upper weak semilimit u is usc (resp. the lower weak semilimit is lsc).*

Proof We prove the part of the assertion regarding the upper semilimits. Let us fix a point x_0 and a sequence x_k converging to it. It directly comes from the definition of semilimit that, for any k, we can find an optimal subsequence, not relabelled for simplicity of notations, x_n^k converging to x_k with

$$u_n(x_n^k) \to u(x_k),$$

we can therefore select indices n_k, with $n_k \to +\infty$ as $k \to +\infty$, such that

$$|x_{n_k}^k - x_k| < \frac{1}{k} \quad \text{and} \quad u(x_k) \le u_{n_k}(x_{n_k}^k) + \frac{1}{k}.$$

We get

$$\limsup_k u(x_k) \le \limsup_k u_{n_k}(x_{n_k}^k) + \frac{1}{k} \le u(x_0),$$

as desired. $\square$

The stability of maxima and maximizers (resp. minima and minimizers) is one the main properties of the above introduced weak convergences. We prove such a result on maxima and $\limsup^{\#}$, a similar statement, with obvious adaptations, holds true for minima and $\liminf_{\#}$.

Proposition 7.7. *Let u_n be a sequence of usc function, with the same properties as in Definition 7.5, in the closure of some bounded open set Ω and*

$$u := \limsup{}^{\#} u_n.$$

If x_n is a sequence of maximizers of u_n in $\overline{\Omega}$, then any of its limit points is a maximizer of u. In addition, the corresponding maximum values M_n converge to $M_0 = \max_{\overline{\Omega}} u - w$.

Proof We denote by x_n a sequence of maximizers of u_n in $\overline{\Omega}$, then it converges, up to a subsequence, to some $x_0 \in \overline{\Omega}$. We have, by the very definition of sup weak semilimit

$$\limsup M_n = \limsup u_n(x_n) - w(x_n) \leq u(x_0) - w(x_0) \leq M_0. \tag{18}$$

On the other side, let y_0 be a maximizer for u, and y_n a sequence converging to y_0 with

$$\lim_n u_n(y_n) = u(y_0).$$

We have

$$M_0 = \lim_n u_n(y_n) \leq \liminf_n M_n. \tag{19}$$

The assertion follows combining (18), (19). □

Remark 7.8. It is apparent from the definition that if $\limsup^{\#} u_n = u$ ($\liminf_{\#} u_n = u$), for some sequence u_n, limit function u then $\limsup^{\#} u_n - \varphi = u - \varphi$ ($\liminf_{\#} u_n - \varphi = u - \varphi$) for any continuous function φ. We will exploit this property in combination with stability of maxima (minima) when φ is a viscosity test function.

Proposition 7.9. *Let u be a bounded usc function, then*

$$\limsup{}^{\#}{}_{\varepsilon \to 0} u^\varepsilon = u.$$

In addition, u^ε pointwise converges to u.

We preliminarily prove a lemma.

Lemma 7.10. *Let u be a bounded usc function. Given $x \in \mathbb{R}^N$, if y_ε is u^ε–optimal for x_ε then*

$$|x_\varepsilon - y_\varepsilon| = \mathrm{O}(\sqrt{\varepsilon})$$

Proof We set $M = \sup |u|$. We have

$$u(x) \leq u^\varepsilon(x) = u(y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2.$$

We derive

$$-|x - y_\varepsilon|^2 \geq 2\,\varepsilon(u(x) - u(y_\varepsilon))$$

and

$$|x - y_\varepsilon|^2 \leq 2\,\varepsilon(u(y_\varepsilon) - u(x)) \leq 4\,\varepsilon\,M.$$

This proves the assertion. $\square$

Proof of Proposition 7.9 We fix a point x_0 and consider a (not relabelled) subsequence $x_\varepsilon \to x_0$ with $u^\varepsilon(x_\varepsilon) \to \alpha$. We will show $u(x_0) \geq \alpha$. In fact

$$u^\varepsilon(x^\varepsilon) \leq u(y^\varepsilon), \tag{20}$$

where y^ε is u^ε–optimal for x^ε and, by Lemma 7.10, $|x_\varepsilon - y_\varepsilon| \to 0$, so that

$$y_\varepsilon \to x_0 \quad \text{as } \varepsilon \to 0.$$

Hence, keeping also in mind (20) and that u is usc

$$\alpha \leq \limsup u(y^\varepsilon) \leq u(x_0),$$

as claimed. Now we take x_ε identically equal to x_0, then, by the previous point, $\limsup u^\varepsilon(x_0) \leq u(x_0)$ and, on the other side, $\liminf u^\varepsilon(x_0) \geq u(x_0)$, being $u^\varepsilon(x_0) \geq u(x_0)$, for any ε. Summing up, we get

$$\lim u^\varepsilon(x_0) = u(x_0),$$

which fully proves the assertion. $\square$

In the case where u is continuous or Lipschitz–continuous, we can improve the above convergence result and the estimate in Lemma 7.10.

Proposition 7.11. *Let u be a bounded continuous function, then u^ε locally uniformly converge to u.*

Proof. Let B be a bounded subset of $\mathbb{R}^N$. We denote by ω an uniform continuity modulus of u in B. We invoke Lemma 7.10 to get the estimate

$$u^\varepsilon(x) - u(x) \leq u(y) - u(x) \leq \omega(|x - y|) \leq \omega(2\sqrt{M\varepsilon})$$

for $x \in B$, y u^ε–optimal for x, and M an upper bound of u in $\mathbb{R}^N$. This gives the assertion. $\square$

We derive:

Corollary 7.12. *if u is bounded uniformly continuous then u^ε uniformly converge to u in $\mathbb{R}^N$.*

Proposition 7.13. *Let u, ε be a Lipschitz continuous function and a positive constant, respectively. Let x be any point and y_ε u^ε–optimal for x then*

$$|x - y_\varepsilon| = \mathrm{O}(\varepsilon).$$

Proof. Arguing as in Lemma 7.10 and exploiting Lipschitz–continuity of u, we get

$$|x - y_\varepsilon| \leq \varepsilon\, L$$

where L is a Lipschitz constant for u, which gives item the estimate in the statement. $\square$

We proceed inquiring about first order properties of sup–convolutions.

Proposition 7.14. *Fix $\varepsilon > 0$, then, for any x_0*

$$\partial u^\varepsilon(x_0) = co \left\{ \frac{y_0 - x_0}{\varepsilon} \mid y_0 \;\; u^\varepsilon\text{–}optimal\ for\ x_0 \right\},$$

consequently u^ε is (strictly) differentiable at x_0 if and only if it admits an unique u^ε–optimal point y_0, and then $Du^\varepsilon(x_0) = \frac{y_0 - x_0}{\varepsilon}$.

We preliminarily show a continuity property for u^ε–optimal points.

Lemma 7.15. *Let x_n be a sequence convergent to some x_0. If, for any n, y_n is u^ε–optimal for x_n and $y_n \to y_0$, then y_0 is u^ε–optimal for x_0 and $u(y_n) \to u(y_0)$.*

Proof We have

$$u^\varepsilon(x_n) = u(y_n) - \frac{1}{2\varepsilon} |x_n - y_n|^2,$$

for any n. Passing at the limit, and taking into account that u is usc, we obtain:

$$u^\varepsilon(x_0) \leq u(y_0) - \frac{1}{2\varepsilon} |x_0 - y_0|^2.$$

By the very definition of sup–convolution, the inequality in the previous formula must indeed be an equality, which shows that y_0 is optimal, as desired, and, in addition the claimed convergence of $u(y_n)$ to $u(y_0)$. $\square$

Proof of Proposition 7.14 Let us fix x_0 and take y_0 u^ε–optimal for it, then, by the very definition of u^ε, the quadratic function

$$x \mapsto u(y_0) - \frac{1}{2\varepsilon} |x - y_0|^2$$

is subtangent to u^ε at x_0. This shows

$$\frac{y_0 - x_0}{\varepsilon} \in D^- u(x_0) = \partial u(x_0)$$

and, since the generalized gradient is convex valued

$$\partial v^\varepsilon(x_0) \supset \text{co} \left\{ \frac{y_0 - x_0}{\varepsilon} \mid y_0 \;\; u^\varepsilon\text{–optimal for } x_0 \right\}. \tag{21}$$

Now, take a sequence x_n, where u^ε differentiable, with $x_n \to x_0$ and $Du^\varepsilon(x_n)$ convergent, then

$$Du^\varepsilon(x_n) = \frac{1}{\varepsilon}(y_n - x_n) \quad \text{for any } n, \tag{22}$$

where y_n is u^ε–optimal for x_n. By Lemma 7.15 and (22)

$$\lim_n Du^\varepsilon(x_n) = \frac{y_0 - x_0}{\varepsilon}$$

for some y_0 optimal for x_0. Keeping in mind the definition of generalized gradient, given in (7), we deduce

$$\partial v^\varepsilon(x_0) \subset \operatorname{co}\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \ \ u^\varepsilon\text{–optimal for } x_0\right\},$$

which, together with (21), yields the assertion. $\square$

We proceed showing how viscosity test information is transferred from u to the sup–convolution u^ε (resp. inf–convolutions u_ε).

Proposition 7.16. *Fix $\varepsilon > 0$ and $x \in \mathbb{R}^N$. Let y be a u^ε–optimal (resp. u_ε–optimal) point for x, then*

$$\frac{y-x}{\varepsilon} \in D^+u(y) \ \left(\textit{resp. } \frac{x-y}{\varepsilon} \in D^-u(y))\right)$$

and consequently

$$\partial u^\varepsilon(x) \cap D^+u(y) \neq \emptyset \ (\textit{resp. } \partial u_\varepsilon(x) \cap D^-u(y) \neq \emptyset.$$

Proof. We just show the part of the assertion about sup–convolutions, some argument, with obvious adaptations, works for the inf–convolutions case. We have by the very definition of sup convolution

$$u(y) - \frac{1}{2\varepsilon}|x-y|^2 = \max_z u(z) - \frac{1}{2\varepsilon}|x-z|^2$$

we derive $\frac{y-x}{\varepsilon} \in D^+u(y)$ and consequently $\partial u^\varepsilon(x) \cap D^+u(y) \neq \emptyset$ by Proposition 7.14. $\square$

We derive

Corollary 7.17. *Assume u to be Lipschitz–continuous, then u^ε (resp. u_ε) has the same Lipschitz constant of u for any ε.*

Proof. We detail the argument in the case of sup–convolutions. We fix $\varepsilon > 0$. Let L be a Lipschitz constant for u, then

$$|p| \ \leq L \qquad \text{for any } x, \text{ any } p \in \partial u(x).$$

Thanks to Proposition 7.16, we know that if y is u^ε–optimal for a certain x then

$$\frac{y-x}{\varepsilon} \in \partial u(y) \ \Rightarrow \ \frac{|y-x|}{\varepsilon} \leq L. \tag{23}$$

We further know by Proposition 7.14 that $\partial u^\varepsilon(x)$ is the convex hull of elements as in (23). We derive

$$|p| \ \leq L \qquad \text{for any } p \in \partial u^\varepsilon(x),$$

which in turn implies that L is a Lipschitz constant for u^ε.

□

Due to semiconvexity (resp. semiconcavity) properties, we know that u^ε (resp. u_ε) is differentiable, and consequently possess an unique optimal point, if and only if $D^+u^\varepsilon(x_0) \neq \emptyset$ (resp. $D^-u_\varepsilon(x) \neq \emptyset$). We derive from the previous result:

Corollary 7.18. *If D^+u^ε (resp. u_ε) is differentiable at x_0 then $Du^\varepsilon(x_0) \in D^+u^\varepsilon(y_0)$ (resp. $Du_\varepsilon(x_0) \in D^-u_\varepsilon(y_0)$, where y_0 is the unique optimal point.*

We end the section with a remark.

Remark 7.19. So far we have considered sup and inf–convolutions in the whole space. It is natural to inquire whether the definition could adapted to any open subset Ω, namely we want to study

$$x \mapsto \sup_{y \in \Omega} \left(u(y) - \frac{1}{2\varepsilon}|y-x|^2 \right) \tag{24}$$

assuming u to be upper semicontinuous, and similarly the inf–convolution case. What we need to extend to this case the theory we have developed above is

– for any ε small enough there exists $\Omega_\varepsilon \subset \Omega$ such that the supremum in (24) is attained for $x \in \Omega_\varepsilon$.

– $\cup_{\varepsilon > 0} \Omega_\varepsilon = \Omega$.

These properties are fulfilled if u is bounded from below as well as from above. In this case we define

$$\Omega_\varepsilon = \{x \mid d^\#(x, \Omega) \leq -\sqrt{2\,\varepsilon\,(M - m + 1)}\},$$

where $m = \inf_\Omega u$, $M = \sup_\Omega u$. If $x \in \Omega_\varepsilon$, then there is a maximizer y of (24) in Ω. We can in fact restrict the search for a maximum to y with

$$u(x) - \frac{1}{2\varepsilon}\,|x-y|^2 > m - 1$$

so that

$$|x-y| < \sqrt{2\,\varepsilon\,(M-m+1)}.$$

Therefore any maximizing sequence for (24) when $x \in \Omega_\varepsilon$ stays away from $\partial\Omega$ and so we can pass to the limit, exploiting the upper semicontinuity of u, to get a maximum. In addition $2\,\sqrt{2\,\varepsilon\,(M-m+1)}$ goes to 0 with ε.

In this setting we can define a sup–convolution via (24) for any $x \in \Omega_\varepsilon$ recovering all the results we have proved for some sup/inf convolution in the whole space.

8 Some applications of the regularization by sup/inf convolutions

We generalize Proposition 5.5 removing the convexity condition of the domain Ω. We first get local Lipschitz continuity without assuming any condition on $\partial\Omega$.

Proposition 8.1. *Let Ω be an open subset of $\mathbb{R}^N$ and u a bounded usc function defined in Ω. Assume that*

$$|Du| \le L \quad \textit{in the viscosity sense in } \Omega$$

then u is locally Lipschitz–continuous in Ω.

Proof. We denote, for any $\varepsilon > 0$, by Ω_ε the set introduced in Remark 7.19. By Proposition 7.16

$$|Du^\varepsilon| \le L \quad \text{in the viscosity sense in } \Omega_\varepsilon,$$

for any ε, and by Proposition 5.4

$$|Du^\varepsilon(x)| \le L \quad \text{a.e. in } \Omega_\varepsilon,$$

for any ε. Consequently any u^ε satisfies the Lipschitz property in the assertion, and, since it is stable for pointwise convergence, it still holds for u in the whole Ω, thanks to Proposition 7.9.

□

Assuming some additional conditions on $\partial\Omega$, we can strengthen the previous result in the spirit of proposition 5.5, namely we prove global Lipschitz–continuity on the domain and remove boundedness from below requirement.

To explain this point, we introduce the notion of (Euclidean) *geodesic distance* on Ω, denoted by d_Ω. For any pair of points x, y in Ω it is defined as

$$d_\Omega(x,y) = \inf\left\{ \int_0^1 |\dot\xi|\, dt \mid \xi(0)=x,\, \xi(1)=y,\, \xi\subset\Omega \right\}.$$

Notice that the main point is that we are requiring the connecting curve to have support entirely contained in Ω. Also notice that to require the curves to be defined in $[0,1]$ is not restrictive because the integrand is positively homogeneous, and so invariant for change of parameter.

One natural question is to investigate about the relationship between d_Ω and the Euclidean distance. It is clear from the very definition of geodesic distance that

$$|x-y| \leq d_\Omega(x,y) \qquad \text{for any } x,\, y \text{ in } \Omega.$$

So, the problem is to establish under which conditions an opposite estimate holds true. To answer, we need a further definition.

We say that Ω has *locally Lipschitz–continuous boundary* if for any $x_0 \in \partial\Omega$ we can select $N-1$, among the N independent real variables of $\mathbb{R}^N$, say the first ones to ease notations, and a Lipschitz–continuous function Ψ

$$\Psi : (x_1, \cdots, x_{N-1}) \mapsto x_N$$

in such way that

– $\operatorname{graph} \Psi \cap B = \partial\Omega \cap B$

– $\operatorname{hypo} \Psi \cap B = \Omega \cap B$

– $\operatorname{epi} \Psi \cap B = (\mathbb{R}^N \setminus \overline{\Omega}) \cap B$

for some open neighborhood B of x_0. Where graph, hypo, epi stand for graph, hypograph, epigraph, we recall that

$$\begin{aligned} \operatorname{hypo} \Psi &= \{(x_1,\cdots,x_{N-1},z) \mid z \leq \Psi(x_1,\cdots,x_{N-1})\} \\ \operatorname{epi} \Psi &= \{(x_1,\cdots,x_{N-1},z) \mid z \geq \Psi(x_1,\cdots,x_{N-1})\} \end{aligned}$$

Proposition 8.2. *If Ω has locally lipschitz–continuous boundary then*

$$d_\Omega(x,y) \leq C\,|x-y| \quad \textit{for any } x,\, y \textit{ in } \Omega, \textit{ some } C>0.$$

Notice that this result implies in particular that any function which is Lipschitz–continuous with respect to d_Ω with Lipschitz constant L, is also Lipschitz–continuous with respect to the Euclidean distance with Lipschitz constant $C\,L$. A particular case is when Ω is convex, it can be proved that then the boundary is locally Lipschitz–continuous, but we directly see from the definition of convexity that d_Ω simply coincides with the Euclidean distance. We have

Proposition 8.3. *Let Ω be an open subset of $\mathbb{R}^N$ with locally Lipschitz–continuous boundary and v an usc function defined in $\bar{\Omega}$. Assume that*

$$|Du| \leq L \quad \textit{in the viscosity sense in } \Omega$$

then u is Lipschitz–continuous in Ω.

Proof. Given $M > 0$ large, we define

$$v = u \vee -M$$

then v is a bounded upper semicontinuous function and by Proposition 3.10 satisfies

$$|Du| \leq L \quad \text{in the viscosity sense in } \Omega$$

and consequently it is locally Lipschitz–continuous in Ω in force of Proposition 8.1. Therefore we deduce from Proposition 5.3

$$|p| \leq L \quad \text{for any } x \in \Omega, \, p \in \partial v(x). \tag{25}$$

Now fix a pair x, y of points of Ω and a curve ξ, defined in $[0, 1]$, linking them and lying in Ω. Then we get

$$|v(x) - v(y) \leq \int_0^1 \left| \frac{d}{dt} v(\xi(t)) \right| dt = \int_0^1 |p(t) \cdot \dot{\xi}(t)| \, dt,$$

where, for a.e. t, $p(t)$ is a suitable element of $\partial v(\xi(t))$. We deduce from (25)

$$|v(x) - v(y)| \leq L \int_0^1 |\dot{\xi}| \, ds$$

and, taking into account the definition of geodesic distance and that the curve ξ has been chosen arbitrarily, we conclude

$$|v(x) - v(y)| \leq L \, d_\Omega(x, y)$$

or, in other terms, that v is Lipschitz–continuous with respect to the d_Ω. But then by Proposition 8.2 v is Lipschitz–continuous in Ω with respect to the Euclidean distance, and so can be continuously extended in $\overline{\Omega}$. The related Lipschitz constant does not depend on M but just on L and d_Ω, this property, together with the fact that u is bounded from above, implies that v is bounded from below independently of M, so that we can select an M so large that $v > -M$ in Ω. For such choice $v = u$. This concludes the proof. □

The aim of next two results is to enlarge the class of viscosity test functions modifying or weakening the standard requirement of C^1–regularity, at least for the first order case.

Lemma 8.4. *Let u be an usc subsolution (resp. lsc supersolution) to equation* (3)*, and ψ a semiconcave (resp. semiconvex) supertangent (resp. subtangent) to u at some point x. Then*

$$H(x, u(x), p) \leq 0 \text{ (resp. } \geq 0) \qquad \text{for all } p \in \partial\psi(x)$$

Proof. We just prove the case subsolution with semiconcave supertangent. Since any supertangent to ψ at x is also supertangent to u at the same point, we get

$$\partial\psi(x) = D^+\psi(x) \subset D^+u(x),$$

which shows the assertion. □

Proposition 8.5. *Let u be an usc subsolution (resp. lsc supersolution) to equation* (3)*, and ψ a Lipschitz–continuous supertangent (resp. subtangent) to u at some point x. Then*

$$H(x, u(x), p) \leq 0 \text{ (resp. } \geq 0) \qquad \text{for some } p \in \partial\psi(x)$$

Proof. We treat the subsolution case. Being the argument local, we can take, without loosing generality, ψ bounded. Up to adding a quadratic term $y \to |y - x|^2$, we can assume ψ to be strict supertangent, and so x to be the unique maximizer of $u - \psi$ in a suitable closed ball B centered at x.

From this uniqueness property we deduce, taking into account Proposition 7.11, that any sequence x_ε of maximizers of $u - \psi_\varepsilon$ in B converges to x, where ψ_ε denotes the inf–convolution. Hence x_ε is in the interior of B for ε sufficiently small, and then for such ε, ψ_ε is supertangent to u at x_ε so that

$$H(x_\varepsilon, u(x_\varepsilon), p_\varepsilon) \leq 0 \qquad \text{for any } p_\varepsilon \in \partial\psi_\varepsilon(x_\varepsilon) \tag{26}$$

in force of Lemma 8.4. Further

$$u(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) = \max_B u - \psi_\varepsilon \to \max_B u - \psi = u(x) - \psi(x),$$

which, implies, bearing in mind that $\lim \psi_\varepsilon(x_\varepsilon) = \psi(x)$ by Proposition 7.13

$$\lim u(x_\varepsilon) = u(x) \tag{27}$$

We also know, by Proposition 7.16, that for any y_ε ψ_ε–optimal for x_ε

$$\partial\psi_\varepsilon(x_\varepsilon) \cap D^-\psi(y_\varepsilon) \subset \partial\psi_\varepsilon(x_\varepsilon) \cap \partial\psi(y_\varepsilon) \neq \emptyset.$$

Taking into account that $y_\varepsilon \to x$, as ε goes to 0 by Proposition 7.13, exploiting (27), plus the continuity properties of H and generalized gradients, see Proposition 4.3, we find for $q_\varepsilon \in \partial\psi_\varepsilon(x_\varepsilon) \cap \partial\psi(y_\varepsilon)$

$$\begin{aligned} q_\varepsilon &\to p \in \partial\psi(x) \\ H(x_\varepsilon, u(x_\varepsilon), q_\varepsilon) &\to H(x, u(x), p) \end{aligned}$$

then, thanks to (26)

$$H(x, u(x), p) \leq 0 \quad \text{and} \quad p \in \partial\psi(x)$$

as claimed. $\square$

9 Eikonal equation

We give a comparison result for the Eikonal equation appearing in (14). We will apply for this the material developed in the previous sections.

Theorem 9.1. *Let u, w be an usc (in $\overline{\Omega}$) subsolution and lsc (in $\overline{\Omega}$) supersolution of* (14)*, respectively, then*

$$u \leq v \qquad \text{in } \Omega.$$

Proof. By Proposition 8.1 u is locally Lipschitz–continuous in Ω. The argument is by contradiction, we assume the minimum of $v - u$ in $\overline{\Omega}$ strictly negative, which implies that all the minimizers must be in Ω, by the assumption $v \geq u$ on $\partial\Omega$.

We consider a strictly increasing sequence λ_n of positive numbers converging to 1, then the sequence $u_n := \lambda_n u$ is made up by locally Lipschitz–continuous functions uniformly converging to u in $\overline{\Omega}$. This implies that any sequence of minimizers of $v - u_n$ in $\overline{\Omega}$ converges, up to subsequences, to a minimizer of $v - u$ which we know to be in Ω. Hence $x_n \in \Omega$ for n large enough.

The function u_n is therefore a Lipschitz–continuous subtangent to v at x_n, then by Proposition 8.5

$$|p| \geq 1 \qquad \text{for some } p \in \partial u_n(x_n). \tag{28}$$

On the other side, u_n satisfies

$$|Du_n(x)| \leq \lambda_n < 1 \qquad \text{for a.e. } x \in \Omega,$$

which implies by Proposition 5.3

$$|p| \leq \lambda_n < 1 \qquad \text{for all } x \in \Omega,\ p \in \partial u_n(x). \tag{29}$$

The relations in (28), (29) are in contradiction. We therefore conclude that there is some minimizer of $u - v$ on $\partial\Omega$, which in turn implies $v \geq u$ in $\overline{\Omega}$, as desired. $\square$

We proceed providing a representation formula which generalizes that of one–dimensional equation.

Proposition 9.2. *The unique solution to* (14) *is*

$$u(x) = d(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|.$$

Proof. If there is a solution, it must be unique in force of the comparison principle established in Theorem 9.1. Given a a point $x_0 \in \Omega$ and $q \in \mathbb{R}^N$, we denote by y_λ, for λ positive suitably small, a projection of $x + \lambda\, q$ on $\partial\Omega$. If φ is a regular supertangent to u at x_0, we have, using the triangle inequality

$$\begin{aligned}
D\varphi(x_0)\cdot q &= \lim_{\lambda\to 0^+} \frac{\varphi(x_0)-\varphi(x_0-\lambda\, q)}{\lambda} \\
&\leq \lim_{\lambda\to 0^+}\inf \frac{u(x_0)-u(x_0-\lambda\, q)}{\lambda} \\
&\leq \lim_{\lambda\to 0^+}\inf \frac{|x_0-y_\lambda|-|x_0-\lambda\, q-y_\lambda|}{\lambda} \\
&\leq \lim_{\lambda\to 0^+}\inf \frac{|\lambda\, q|}{\lambda} = |q|.
\end{aligned}$$

Since q has been taken arbitrarily, this implies $|D\varphi(x_0)| \leq 1$. Now, let us denote by y_0 a projection point of x_0 on $\partial\Omega$ ad set $q = \frac{y_0 - x_0}{|y_0 - x_0|}$, therefore

$$u(x_0+\lambda q) = u(x_0) - \lambda \quad \text{for } \lambda > 0 \text{ suitably small.} \tag{30}$$

If ψ is regular supertangent to u at x_0, we have, keeping in mind (30)

$$\begin{aligned}
D\psi(x_0)\cdot q &= \lim_{\lambda\to 0^+} \frac{\psi(x_0+\lambda\, q)-\varphi(x_0)}{\lambda} \\
&\leq \lim_{\lambda\to 0^+} \frac{u(x_0+\lambda\, q)-u(x_0)}{\lambda} \\
&= \lim_{\lambda\to 0^+} \frac{-\lambda}{\lambda} = -1.
\end{aligned}$$

We deduce

$$|D\psi(x_0)| \geq |D\psi(x_0)\cdot q| \geq 1.$$

□

10 Generalized Eikonal equations

Here we generalize the material developed in the previous section to study the equation

$$H(x, Du) = 0 \tag{31}$$

possibly coupled with Dirichlet boundary conditions in some bounded open subset Ω of $\mathbb{R}^N$. We assume the Hamiltonian H to be

- continuous in $\mathbb{R}^N \times \mathbb{R}^N$.

– convex in the momentum variable for any $x \in \mathbb{R}^N$.

– coercive in p, locally uniformly in x, namely for any ball B of $\mathbb{R}^N$

$$\lim_{|p| \to +\infty} \inf_{x \in B} H(x,p) = +\infty \tag{32}$$

– there exists a strict subsolution of (31) in $\mathbb{R}^N$, namely there exists φ upper semicontinuous with $H(x, D\varphi) \leq -\delta$ in the viscosity sense in $\mathbb{R}^N$ for some positive δ.

The above hypotheses clearly hold true for the Hamiltonian $H(x,p) = |p|$ appearing in (14), which has been object of investigation in the previous section. One strict subsolution is the null function.

Proposition 10.1. *Under the above assumption of continuity and coercivity, any usc subsolution to* (31) *is locally Lipschitz–continuous.*

Proof. Let B be a ball in $\mathbb{R}^N$, then by (32)

$$\sup_{x \in B} \{|p| \mid H(x,p) \leq 0 \text{ with } x \in B\} \leq L$$

for a suitable L. This implies that if u is a subsolution to (31) then

$$|Du| \leq L \qquad \text{in the viscosity sense in } B$$

then we are in the condition to apply Proposition 8.3 to get that u is Lipschitz–continuous in B. This gives the assertion.

□

We define for any x the sublevel

$$Z(x) = \{p \mid Z(x,p) \leq 0\}$$

It is a patent consequence of the convexity and coercivity assumptions that $Z(x)$ is convex and compact for any x, it also possess nonempty interior because of the supposed existence of a strict subsolution. Moreover, directly from the continuity of the Hamiltonian we also get the following continuity properties for the set–valued function Z

$$x_n \to x, \, p_n \in Z(x_n), \, p_n \in Z(x_n) \Rightarrow p \in Z(x). \tag{33}$$

$$p \in Z(x), \, x_n \to x \Rightarrow \exists p_n \in Z(x_n) \mid p_n \to p \tag{34}$$

We proceed considering the *support function*

$$\sigma(x,q) = \max\{p \cdot q \mid p \in Z(x)\}$$

we see that it is positively homogeneous in p and, in addition convex because of the convexity of $Z(x)$. This implies subadditivity, namely

$$\sigma(x, p_1 + p_2) \leq \sigma(x, p_1) + \sigma(x, p_2) \quad \text{for any } x,\ p_1,\ p_2.$$

It moreover comes from (33), (34) that

$$x \mapsto \sigma(x, p) \qquad \text{is continuous for any fixed } p.$$

Next step is to introduce a metric on $\mathbb{R}^N$ intrinsically related to H and the 0–sublevels. Starting point is a *length functional* on the (Lipschitz–continuous) curves of $\mathbb{R}^N$ Given a curve ξ, defined in $[0, T]$ for some positive T, we define

$$\ell_H(\xi) = \int_0^T \sigma(\xi, \dot{\xi})\, dt. \tag{35}$$

Being the integrand homogeneous it is invariant by change of parameter, so it is not restrictive to just consider curves defined in $[0, 1]$. Due to classical results in one–dimensional calculus of variations, see [2], we also have that ℓ_H is lsc continuous with respect to uniform convergence of curves, namely, given a sequence of curves ξ_n defined in $[0, 1]$, if

$$\xi_n \to \xi \qquad \text{uniformly in } [0, 1]$$

then

$$\liminf \ell_H(\xi_n) \geq \ell_H(\xi).$$

A first link between intrinsic length and equation (31) is given by

Proposition 10.2. *Assume that u is a subsolution to* (31)*, then for any pair of points x, y and a curve ξ with $\xi(0) = x$, $\xi(1) = y$ we have*

$$u(y) - u(x) \leq \ell_H(\xi).$$

Proof. Since u is locally Lipschitz–continuous by Proposition 10.1 then in force of Remark 4.5

$$u(y) - u(x) = \int_0^1 \frac{d}{dt} u(\xi(t))\, dt = \int_0^1 p(t) \cdot \dot{\xi}(t)\, dt, \tag{36}$$

for a.e. $t \in [0, 1]$ and suitable $p(t) \in \partial u(\xi(t))$, by Proposition 5.3

$$H(x, p) \leq 0 \qquad \text{for any } x,\ p \in \partial u(x),$$

or, in other terms

$$p \cdot q \leq \sigma(x, q) \qquad \text{for any } x,\ q \text{ in } \mathbb{R}^N,\ p \in \partial u(x). \tag{37}$$

We get the the assertion by combining (36), (37).

. $\square$

We derive:

Corollary 10.3. *If ξ is a cycle, i.e. a closed curve, not reduced to a point, then $\ell_H(\xi) > 0$.*

Proof. Set $x = \xi(0)$, being φ a strict subsolution, we have

$$p \cdot q < \sigma(x, q) \qquad \text{for any } x, q, p \in \partial\varphi(x), \tag{38}$$

bearing in mind that

$$\frac{d}{dt}\varphi(t) = p(t) \cdot \dot{\xi}(t) \qquad \text{for a.e. } t, \text{ suitable } p(t) \in \partial\varphi(\xi(t))$$

we derive from (38)

$$0 = \varphi(x) - \varphi(x) = \int_0^1 \frac{d}{dt}\varphi(t)\, dt < \ell_H(\xi),$$

as claimed

□

We proceed defining the *intrinsic distance*, for any pair of points x, y

$$S_H(x, y) = \inf\{\ell_H(\xi) \mid \xi(0) = x, \, \xi(0) = y\}. \tag{39}$$

We have

– $S_H(x, x) = 0$ for any x.

– (triangle inequality) $S_H(x, y) \leq S_H(x, z) + S_H(z, y)$ for any x, y, z.

The first property comes from Corollary 10.3, while the second is consequence of the very definition of S_H. The function S_H is not a distance for it does not enjoy any sign or symmetry property, but more precisely a *semidistance*; triangle inequality will be specially useful in what follows.

The next proposition relates the intrinsic distance to equation (31) and shows, among other things, that it is finite for any pair of points.

Theorem 10.4. *An usc function u is subsolution to* (31) *in $\mathbb{R}^N$ if and only if*

$$u(y) - u(x) \leq S_H(x, y) \qquad \textit{for any } x, \, y \textit{ in } \mathbb{R}^N.$$

Proof. If u is subsolution then the inequality in the statement directly comes from Proposition 10.2. Conversely, let ψ, q be a C^1 supertangent to u at a point x and an unit vector, respectively, we have

$$\begin{aligned} \frac{\psi(x) - \psi(x - t\,q)}{t} &\leq \frac{u(x) - u(x - t\,q)}{t} \leq \frac{1}{t}\, S_H(x + t\,q, x) \qquad (40) \\ &\leq \frac{1}{t} \int_0^t \sigma(x + (t+s)\,q, q)\, ds \qquad (41) \end{aligned}$$

for $t < 0$ sufficiently small. Passing at the limit for $t \to 0$ we get

$$D\psi(x) \cdot q \leq \sigma(x, q).$$

Since q has been arbitrarily chosen among the unit vector of $\mathbb{R}^N$, we conclude, thanks to Hahn–Banach Theorem

$$D\psi(x) \in Z(x)$$

or, in other terms

$$H(x, D\psi(x)) \leq 0$$

as desired. □

We couple equation (31) with Dirichlet boundary condition

$$u(x) = g(x) \qquad \text{on } \partial\Omega$$

where Ω is the usual bounded open set and g a given function on $\partial\Omega$. We derive a representation formula.

Theorem 10.5. *Assume the boundary datum to satisfy*

$$g(y) - g(x) \leq S_H(x, y) \qquad \textit{for any } x,\ y \textit{ in } \partial\Omega$$

then the function

$$v(x) = \inf\{S_H(y, x) + g(y) \mid y \in \partial\Omega\} \qquad (42)$$

is subsolution to (14) *in* $\mathbb{R}^N$*, solution in* $\mathbb{R}^N \setminus \partial\Omega$ *and agrees with* g *on* $\partial\Omega$.

Proof. Fix $\varepsilon > 0$. Let x_1, x_2 be in $\mathbb{R}^N$, denote by y an ε–optimal point for $v(x_2)$, then we get applying the triangle inequality to S_H

$$v(x_1) - v(x_2) \leq g(y) + S_H(y, x_1) - g(y) - S_H(y, x_2) + \varepsilon \leq S_H(x_2, x_1) + \varepsilon.$$

Being ε arbitrary, this gives

$$v(x_1) - v(x_2) \leq S_H(x_2, x_1)$$

and consequently implies v being subsolution in the light of Theorem 10.4. It comes from the definition of v that $v \leq g$ on $\partial\Omega$; assume for purposes of contradiction that there is $y \in \partial\Omega$ with $v(x) < g(x)$, consequently there must be $z \in \partial\Omega$ satisfying

$$g(z) + S(z, y) < g(y)$$

but this is in contrast with (42), showing the end the equality $v = g$.

We proceed proving that v is the maximal subsolution agreeing with g on $\partial\Omega$. This is in fact a consequence of Theorem 10.4. If u is another subsolution equal to g on $\partial\Omega$ then, given a fixed x we have

$$u(x) - g(y) = u(x) - u(y) \leq S_H(y, x)$$

for any $y \in \partial\Omega$, then

$$u(x) \leq v(x)$$

as claimed

It is left to prove v being supersolution in $\mathbb{R}^N \setminus \partial\Omega$, for this we exploit the previously proved maximality property via a pull–up argument which actually has an interest and utility going beyond the present frame. Assume the supersolution property being violated at some x, then there is a C^1 subtangent ψ to v at x with

$$H(x, D\psi(x)) < 0$$

we can assume, without loosing generality ψ to be strict subtangent, to fix ideas we denote by K a compact neighborhood of x such that

$$\psi(y) > v(y) \qquad \text{for } y \in K \setminus \{x\}. \tag{43}$$

We claim that consequently for any open neighborhood of x $B \subset K$ there is $\varepsilon > 0$ with

$$\{y \mid \psi(y) + \varepsilon > v(y)\} \cap K \subset B. \tag{44}$$

In fact, if it were not the case, there should be sequence ε_n converging to 0 and x_n in $K \setminus B$ with

$$\psi(x_n) + \varepsilon > v(x_n),$$

any limit point $\bar{x}$ of x_n is $K \setminus \{x\}$ and

$$\psi(\bar{x}) \geq v(\bar{x}),$$

in contrast with (43). Exploiting the continuity of the Hamiltonian and of $D\psi$ we can choose B such that

$$H(y, D\psi(y)) < 0 \qquad \text{for any } y \in B. \tag{45}$$

We take in correspondence an ε satisfying (44), and define

$$w = \begin{cases} \max\{u, \psi + \varepsilon\} & \text{in } B \\ u & \text{outside } B. \end{cases}$$

then w is subsolution in the whole of $\mathbb{R}^N$ because it is equal to u in a neighborhood of $\mathbb{R}^N \setminus B$ and is the maximum of two subsolutions in B. In this way we get a contradiction with the maximality property of v, being $w > v$ at x. This ends the proof.

□

We get a comparison result for (31) following the same lines of Theorem 9.1 with some adjustments. We compare functions u, v being sub/supersolutions of the equation just in the interior of Ω. This in particular implies that the subsolution is in general not more that locally Lipschitz–continuous in Ω.

Theorem 10.6. *Let u, v be an usc (in $\overline{\Omega}$) subsolution and lsc (in $\overline{\Omega}$) supersolution of* (14)*, respectively, then*

$$u \leq v \qquad \text{in } \Omega.$$

Proof. Arguing by contradiction, we get that all the minimizers of $w - u$ must be in Ω.

The new point is how to construct a sequence of strict subsolutions, say u_n, uniformly converging to u in $\overline{\Omega}$. For this we essentially exploit the existence of a strict global subsolution φ. Given a strictly increasing sequence λ_n of positive numbers converging to 1, we define

$$u_n = \lambda_n \, u + (1 - \lambda_n) \, \varphi.$$

It is easy to see, exploiting the convex character of the Hamiltonian, that the u_n are indeed strict subsolutions. From this point the assertion can be obtained as in Theorem 9.1.

□

From this we derive:

Corollary 10.7. *The function v, defined in* (42)*, is the unique solution to* (31) *in Ω agreeing with g on $\partial\Omega$.*

We end the section exhibiting an example which shows that if the condition on the existence of a strict subsolution is removed, then equation (31) is not any more uniquely solved. We consider the problem

$$\begin{cases} |Du(x)| = f(x) & \text{in } \Omega \\ u \equiv 0 & \text{on } \partial\Omega \end{cases} \tag{46}$$

in this case the intrinsic length is given for any curve ξ defined in $[0, 1]$ by

$$\ell_H(\xi) = \int_0^1 f(\xi) \, |\dot{\xi}(t)| \, dt.$$

If the potential f is strictly positive then we are in the setting previously investigated, being the null function a strict subsolution. We therefore assume $f(x_0) = 0$, for some $x_0 \in \Omega$, and define

$$w(x) = \min\{S_H(x, y) : y \in \partial\Omega \cup \{x_0\}, S(x, y)\}$$

The functions w is apparently different from

$$v(x) = \min\{S_H(y, x) \mid y \in \partial\Omega\}$$

since $w(x_0) = 0$ and $v(x_0) > 0$.

The function w is subsolution in Ω in force of Theorem 10.4. We aim at showing that it is in addition solution, not only in $\Omega \setminus \{x_0\}$, but in the whole Ω. For this, it is enough to check the supersolution property at x_0. We have

$$|p| \leq f(x_0) \quad \text{for any } p \in \partial w(x_0),$$

but then, being $f(x_0) = 0$, $\partial w(x_0) = \{0\}$, which implies that w is strictly differentiable at x_0 with $Dw(x_0) = 0$. Therefore w satisfies the equation at x_0 and so the claim is proved.

11 Discounted equation

We assume strengthened conditions of continuity on the Hamiltonian H. We consider the following conditions

$$|H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)) \tag{47}$$
$$|H(x, p) - H(x, q)| \leq L\,|p - q|\,(1 + |x|) \tag{48}$$

for x, y, p, q varying in some subset of $\mathbb{R}^N$ to be specified , where ω is a modulus, namely a nondecreasing function from $[0, +\infty)$ to $[0, +\infty)$ vanishing and continuous at 0.

We consider the discounted equation

$$\lambda u + H(x, Du) = 0, \tag{49}$$

for a positive λ. The Hamiltonian is now strictly increasing with respect to u, but in general not any more coercive in p. Coercivity will be added later.

We need some preliminary results.

Lemma 11.1. *Let u, w be bounded usc and lsc function, respectively. Then* $\limsup^{\#} u^\varepsilon - w = u - w$.

Proof. We set $v = \limsup^{\#} u^\varepsilon - w$, then for any given x,

$$v(x) = \lim_n u_n^\varepsilon(x_n) - w(x_n)$$

for some sequence x_n converging to x. We deduce from $\limsup^{\#} u^\varepsilon = u$ and $-w$ upper semi continuous

$$v(x) \leq u(x) - w(x).$$

On the other side,taking into account that u^ε pointwise converges to u, we have

$$u(x) - w(x) = \lim_\varepsilon u^\varepsilon(x) - w(x) \leq v(x).$$

□

Proposition 11.2. *Let u, w be usc and bounded, and lsc functions, respectively, in the closure of a bounded open domain Ω. Given $\varepsilon > 0$, let x_ε be a maximizer of $u^\varepsilon - w$ in $\overline{\Omega}$, and y_ε an u^ε–optimal point for x_ε, then*

$$|x_\varepsilon - y_\varepsilon| = \mathrm{o}(\sqrt{\varepsilon}).$$

Proof. We denote by M_ε, M_0 the maximum values of $u^\varepsilon - w$, $u - w$ on $\overline{\Omega}$, respectively. We have

$$M_\varepsilon = u^\varepsilon(x_\varepsilon) - w(x_\varepsilon) = u(y_\varepsilon) - \frac{1}{2\,\varepsilon}|x_\varepsilon - y_\varepsilon|^2 - w(x_\varepsilon)$$

and

$$\limsup_\varepsilon \frac{1}{2\,\varepsilon}|x_\varepsilon - y_\varepsilon|^2 = \limsup_\varepsilon u(y_\varepsilon) - w(x_\varepsilon) - M_\varepsilon. \tag{50}$$

Now, we know by Lemma 7.10 that x_ε and y_ε have the same limit points, and, in addition, that $u - w$ is usc, therefore

$$\limsup u(y_\varepsilon) - w(x_\varepsilon) \leq M_0. \tag{51}$$

We also know from Lemma 11.1 that $M_\varepsilon \to M_0$. Putting together this last information with (50), (51), we get the assertion. □

Theorem 11.3. *Let us consider the equation* (49) *assuming* (47) *in $\mathbb{R}^N \times \mathbb{R}^N$ plus continuity of H in p. If u is an usc (in $\overline{\Omega}$) subsolution and v a lsc (in $\overline{\Omega}$)supersolution to* (49)*, with $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

Proof.
Step 0:*Preliminary.*

We can assume, without loosing generality that u is bounded in $\bar{\Omega}$ in order to apply sup–convolutions. In fact $x \mapsto H(x,0)$ is continuous in $\bar{\Omega}$ and so bounded, consequently we can take $\alpha \in \mathbb{R}$ with

$$\begin{aligned} \lambda\,\alpha + H(x,0) &\leq 0 \qquad \text{in } \Omega \\ \alpha &\leq \min_{\partial\Omega} v \end{aligned}$$

the function constantly equal to α is therefore subsolution to (49) in Ω, thanks to the first relation and, in addition, $\alpha \leq v$ on $\partial\Omega$. We in turn deduce by Proposition 3.10 that $u \vee \alpha$ is subsolution in Ω and $u \vee \alpha \leq v$ on $\partial\Omega$. If we show $v \geq u \vee \alpha$ in Ω we then get the assertion, so we can replace u by $u \vee \alpha$ in the following, or in other terms we can assume u bounded as claimed.

Step 1:*We start arguing by contradiction.*

Here we exploit the limit relation $\limsup^{\#} u^\varepsilon = u$, proved in Proposition 7.9, and the stability property of maximizers under sup semilimits.

We assume, by contradiction

$$\operatorname{argmax} u - v \subset \Omega \quad \text{and} \quad \max_{\overline{\Omega}} u - v > 0.$$

Given $\varepsilon > 0$ small, we deduce

$$\operatorname{argmax} u^\varepsilon - v \subset \Omega \quad \text{and} \max_{\overline{\Omega}} u^\varepsilon - v > \theta > 0 \quad \text{for a } \theta \text{ independent of } \varepsilon. \tag{52}$$

Step 2:*We fix an ε such that (52) holds and x_ε maximizer of $u^\varepsilon - v$. We use u^ε to test v from below at x^ε.*

We will exploit that $D^- u^\varepsilon(x^\varepsilon) = \partial u^\varepsilon(x^\varepsilon) \neq \emptyset$, being u^ε is semiconvex. Owing to $x^\varepsilon \in \Omega$, u^ε is subtangent to v at x^ε, and consequently $D^- u^\varepsilon(x_\varepsilon) \subset D^- v(x_\varepsilon)$. Recalling that v supersolution, we thus get

$$\lambda v(x_\varepsilon) + H(x_\varepsilon, p) \geq 0 \quad \text{for } p \in \partial u^\varepsilon(x_\varepsilon) = D^- u^\varepsilon(x_\varepsilon). \tag{53}$$

Step 3:*We approximate the same x^ε as in the previous step by a sequence x_n of points where u^ε is differentiable.*

We denote by y_n, for any n, the u^ε-optimal points for x_n (univocally determined since x^ε is a differentiability point). We use the relations $Du^\varepsilon(x_n) \in D^+ u(y_n)$, see Proposition 7.16, $Du^\varepsilon(x_n) = \frac{y_n - x_n}{\varepsilon}$, see Proposition 7.14, and that u is subsolution to get

$$\lambda u(y_n) + H\left(y_n, \frac{y_n - x_n}{\varepsilon}\right) \leq 0. \tag{54}$$

We pass to the limit over n in (54). The sequence y_n, being bounded, converges, up to a subsequence, to some y_ε, which is u^ε–optimal for x_ε, thanks to Lemma 7.15; in addition, the same result tells us that $u(y_n) \to u(y_\varepsilon)$. We therefore get

$$\lambda u(y_\varepsilon) + H\left(y_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right) \leq 0. \tag{55}$$

Step 4: *We reach a contradiction by putting together* (53) *and* (55)*, and letting* ε *go to* 0.

Since $\frac{y_\varepsilon - x_\varepsilon}{\varepsilon} \in \partial u^\varepsilon(x_\varepsilon)$, we derive from (53)

$$\lambda v(x_\varepsilon) + H\left(x_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right) \geq 0. \tag{56}$$

By subtracting (56) to (55) and exploiting condition (47), we get

$$\begin{aligned}
\lambda(u(y_\varepsilon) - v(x_\varepsilon)) &\leq H\left(x_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right) - H\left(y_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right) \\
&\leq \omega\left(|x_\varepsilon - y_\varepsilon|\left(1 + \left|\frac{y_\varepsilon - x_\varepsilon}{\varepsilon}\right|\right)\right) \\
&= \omega\left(|x_\varepsilon - y_\varepsilon| + \frac{|y_\varepsilon - x_\varepsilon|^2}{\varepsilon}\right).
\end{aligned}$$

Regarding the left hand–side of the previous formula, we have

$$u(y_\varepsilon) - v(x_\varepsilon) \geq u(x_\varepsilon) - v(x_\varepsilon) > \theta > 0, \tag{57}$$

for ε small. Regarding the right hand–side, we recall that $\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0$, as $\varepsilon \to 0$, thanks to Proposition 11.2, so that

$$\omega\left(|x_\varepsilon - y_\varepsilon| + \frac{|y_\varepsilon - x_\varepsilon|^2}{\varepsilon}\right) \to 0. \tag{58}$$

Formulae (57) and (58) are not compatible with the inequality

$$\lambda(u(y_\varepsilon) - v(x_\varepsilon)) \leq \omega\left(|x_\varepsilon - y_\varepsilon| + \frac{|y_\varepsilon - x_\varepsilon|^2}{\varepsilon}\right)$$

This ends the proof. □

If we replace condition (47) by coercivity of the Hamiltonian H, the above result still holds but the proof can be greatly simplified. We give two results in this vein, in the first one we also suppose H being convex in p

Theorem 11.4. *Let us consider the equation* (49) *assuming the Hamiltonian* H *to be continuous in both variable plus convex and coercive in* p*. If* u *is an usc (in* $\overline{\Omega}$*) subsolution and* v *a lsc (in* $\overline{\Omega}$*) supersolution to* (49)*, with* $u \leq v$ *on* $\partial\Omega$ *then* $u \leq v$ *in* Ω*.*

Proof. Arguing as in Step 0 of the proof of Theorem 11.3, we see tat u can be assumed bounded in $\bar{\Omega}$ without loosing generality. We have

$$H(x, Du) \leq -\lambda \inf_{\bar{\Omega}} u$$

and so, by the coercivity assumption, u is locally Lipschitz–continuous in Ω in view of Proposition 8.1. Owing to Proposition 5.3, we derive that

$$\lambda\, u(x) + H(x,p) \leq 0 \quad \text{for any } x \in \Omega,\, p \in \partial u(x) \tag{59}$$

The argument proceeds as usual by contradiction: we assume

$$\max_{\bar{\Omega}} u - v > 0 \tag{60}$$

and denote by $x_0 \in \Omega$ a corresponding maximizer. The function u being subtangent to v at x_0, we deduce from Proposition 8.5

$$\lambda\, v(x_0) + H(x_0, p_0) \geq 0 \quad \text{for some } p_0 \in \partial u(x_0).$$

By combining this information with (59) (60), we obtain

$$0 \geq \lambda\, u(x_0) + H(x_0, p_0) > \lambda\, v(x_0) + H(x_0, p_0) \geq 0,$$

which is impossible. This concludes the proof. □

We remove the convexity assumption and prove, preliminarily to comparison result, a lemma.

Lemma 11.5. *Assume* Ω *to have locally Lipschitz–continuous boundary. Assume* u *to be usc bounded subsolution of* (49) *in* Ω *and continuous up to the boundary. For any* $\delta > 0$ *there is* ε *such the sup–convolution* u^ε *satisfies*

$$\lambda\, u^\varepsilon + H(x, Du^\varepsilon(x)) \leq \delta \qquad \textit{a.e. in } \Omega^\delta,$$

where

$$\Omega^\delta = \{x \mid d^\#(x, \Omega) \leq -\delta\}$$

Proof. Using assumptions on the domain, Proposition 8.3, boundedness of u and coercivity of H, we deduce that u is Lipschitz–continuous in Ω. We denote by ℓ the corresponding Lipschitz constant. By Proposition 7.13 and Remark 7.19, we have that

$$\frac{y-x}{\varepsilon} \in \partial u^\varepsilon(x) \cap D^+ u(y)$$

for any $x \in \Omega^{\rho(\varepsilon)}$, y u^ε–optimal for x, where

$$\Omega^{\rho(\varepsilon)} = \{x \mid d^{\#}(x,\Omega) \leq -\rho(\varepsilon)\}$$

and $\rho(\varepsilon) \longrightarrow 0$ as ε goes to 0. Consequently by the subsolution property of u

$$\lambda\, u(y) + H\left(y, \frac{y-x}{\varepsilon}\right) \leq 0 \qquad \text{in } \Omega^{\rho(\varepsilon)}.$$

We denote by ω a continuity modulus of H in $\overline{\Omega} \times B(0,\ell)$, and from the estimate

$$u(y) \geq u^\varepsilon(x), \ \ |x-y| \leq \ell\, \varepsilon$$

which holds for any $x \in \Omega^{\rho(\varepsilon)}$, any y u^ε–optimal for x, we derive

$$\lambda\, u_\varepsilon(x) + H\left(x, \frac{y-x}{\varepsilon}\right) \leq \lambda\, u(y) + H\left(y, \frac{y-x}{\varepsilon}\right) + \omega(\ell\varepsilon) \leq \omega(\ell\varepsilon) \qquad \text{in } \Omega^{\rho(\varepsilon)}.$$

The assertion is then proved taking ε sufficiently small.

□

Note carefully that the argument of the next result is based on the possibility of uniformly approximating a subsolution to (49) by strict subsolutions plus a regularization with sup–convolutions.

Theorem 11.6. *Let us consider the equation* (49) *in* Ω *assuming the Hamiltonian* H *to be continuous in both variables plus coercive in* p*, and* Ω *with locally Lipschitz–continuous boundary. If* u *is an usc (in* $\overline{\Omega}$*) subsolution and* v *a lsc (in* $\overline{\Omega}$*) supersolution to* (49)*, with* $u \leq v$ *on* $\partial\Omega$ *then* $u \leq v$ *in* Ω*.*

Proof. Arguing as in Step 0 of the proof of Theorem 11.3, we see tat u can be assumed bounded in $\bar{\Omega}$ without loosing generality. By the coercivity assumption, and assumption on $\partial\Omega$, u is Lipschitz–continuous in Ω in view of Proposition 8.3. The argument proceeds as usual by contradiction: we assume that

$$\max_{\bar{\Omega}} u - v > 0,$$

and define $w = u - \delta$ for some $\delta > 0$. Such a w satisfies

$$\lambda\, w + H(x, Dw) \leq -\lambda\, \delta \qquad \text{in the viscosity sense in } \Omega.$$

The constant δ can be chosen so small that

$$\max_{\bar{\Omega}} w - v > 0,$$

and the maximizers are in Ω. Taking into account that w^ε uniformly converge to w in Ω, we consequently find a maximizer $x_\varepsilon \in \Omega$ of $w^\varepsilon - v$, for ε small, with

$$w^\varepsilon(x_\varepsilon) - v(x_\varepsilon) > 0. \tag{61}$$

Since w^ε is a semiconvex subtangent to v at x_ε and v is supersolution, we get

$$\lambda\, v(x_\varepsilon) + H(x_\varepsilon, p) \geq 0 \qquad \text{for any } p \in \partial w^\varepsilon(x_\varepsilon).$$

On the other side by the previous proposition we know that w^ε is a.e. approximate subsolution to $\lambda\, u + H(x, Du) = -\lambda\, \delta$, we therefore have for a suitable choice of ε

$$\lambda\, w^\varepsilon(x_\varepsilon) + H(x_\varepsilon, p_0) \leq 0 \qquad \text{for some } p_0 \in \partial w^\varepsilon(x_\varepsilon).$$

By combining the last two inequalities, we get

$$\lambda\, v(x_\varepsilon) + H(x_\varepsilon, p_0) \geq \lambda\, w^\varepsilon(x_\varepsilon) + H(x_\varepsilon, p_0)$$

and so

$$\lambda\, v(x_\varepsilon) \geq \lambda\, w^\varepsilon(x_\varepsilon),$$

which is in contrast with (61). □

By assuming (48), in addition to (47), we are going to establish a comparison result in the whole space.

Theorem 11.7. *Let H be an Hamiltonian satisfying (47) and (48). Let u be a bounded upper semicontinuous subsolution to (49) in $\mathbb{R}^N$ and w a bounded lower semicontinuous equation of the same equation. Then $u \leq w$ in $\mathbb{R}^N$.*

Proof. To simplify, we put $\lambda = 1$. The idea is to apply Theorem 11.3. To this aim, we will preliminarily reduce the comparison to a suitable bounded domain of $\mathbb{R}^N$ through a perturbation of u. Condition (48) will be essentially exploited at this point.

Step 1:*By contradiction.*

We assume

$$\sup_{\mathbb{R}^N} u - w > 0,$$

and fix x_0 with

$$\theta := u(x_0) - w(x_0) > \sup_{\mathbb{R}^N} u - w - \varepsilon > 0 \tag{62}$$

and

$$2\,\varepsilon < \frac{\theta}{2L}, \tag{63}$$

where L is the constant appearing in (48). We can assume without loss of generality that $x_0 = 0$.

Step 2:*Perturbation of u.*

We set for $R > 0$

$$\alpha_R = \frac{2\varepsilon}{R}. \tag{64}$$

We introduce a differentiable function φ, which is equal to $|x|$, apart a slight adjustment in a small neighborhood of 0 to make it C^1 in the whole space. We in addition assume

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi \geq 0, \ |D\varphi| \leq 2 \ \text{ in } \mathbb{R}^N \tag{65}$$

We then define

$$\bar{u}_R(x) = u(x) - \alpha_R \, \varphi(x) - \theta.$$

We proceed writing down the inequality satisfied, in the viscosity sense, by $\bar{u}_R$ in $\mathbb{R}^N$. By Remark 3.3 any element of $D^+\bar{u}_R(x)$, for all x, is of the form $p - \alpha \, D\varphi(x)$, with $p \in D^+u(x)$, so that we have

$$\begin{aligned} \bar{u}_R(x) + H(x, p - \alpha_R \, D\varphi(x)) &= (u(x) - \alpha_R \, \varphi(x) - \theta) + H(x,p) + L \, \alpha_R \, |D\varphi(x)| \, (1 + |x|) \\ &\leq 2 \, \alpha_R \, L \, (1 + |x|) - \theta. \end{aligned}$$

We deduce that $\bar{u}_R(x)$ satisfies in the ball $B(0,R)$

$$\bar{u}_R + H(x, D\bar{u}_R) \leq -\theta + 2 \, \alpha_R \, L \, (1 + R) \tag{66}$$

in the viscosity sense. We moreover have for $x \in \partial B(0,R)$

$$\begin{aligned} \bar{u}_R(x) - w(x) &= u(x) - \alpha_R \, R - \theta - w(x) \\ &\leq \varepsilon - \alpha_R \, R = \varepsilon - 2\varepsilon < 0. \end{aligned} \tag{67}$$

Step 3: *Reaching a Contradiction.*

Letting R go to infinity and taking into account (63), (64), we get

$$\lim_{R \to +\infty} 2 \, \alpha_R \, L \, (1 + R) = 4 \, L \, \varepsilon < \theta.$$

Owing to (66), we conclude that in a ball $B(0,R)$, with R suitably large, the function $\bar{u}_R$ is strict subsolution, $\bar{u} - w$ is strictly negative on $\partial B(0,R)$ by (67), and

$$\bar{u}_R(0) - w(0) = u(0) - \theta - w(0) = 0. \tag{68}$$

Since $\bar{u}_R$ is strict subsolution, we can add to it a small positive constant, say δ, to get that $\bar{u}_R + \delta$ is subsolution in $B(0,R)$, $\tilde{u}_R + \delta - w$ is strictly negative on $\partial B(0,R)$, and, clearly, $\bar{u}_R(0) + \delta - w(0) > 0$. We then reach a contradiction with the comparison principle given in Theorem 11.3 applied to $\Omega = B(0,R)$. □

12 The Hamilton–Jacobi–Bellman equation

A relevant example of Hamiltonian satisfying (47), (48) is given by the so–called Bellman Hamiltonian

$$H(x,p) = \max_{a \in A} \{-p \cdot f(x,a) - \ell(x,a)\} \tag{69}$$

where A is a compact subset of some $\mathbb{R}^M$, with M possibly different from N, $f : \mathbb{R}^N \times A :\to \mathbb{R}^N$ is continuous in both arguments, and, in addition, Lipschitz-continuous in x, uniformly with respect to a, namely

$$|f(x,a) - f(y,a)| \leq L_f \, |x - y| \quad \text{for any } x,\, y,\, a, \text{ some } L_f > 0.$$

Finally, the function $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ is bounded and uniformly continuous in $\mathbb{R}^N \times A$. Notice that H is convex in p, being the maximum of linear functions.

Lemma 12.1. *The Hamiltonian H, defined in* (69)*, satisfies* (47) *in $\mathbb{R}^N \times \mathbb{R}^N$ and* (48)*.*

Proof. We start by proving (48). Given x, p, q in $\mathbb{R}^N$, we first select $a_0 \in A$ with

$$-p \cdot f(x,a_0) - \ell(x,a_0) = \max_{a \in A} \{-p \cdot f(x,a) - \ell(x,a)\}$$

By possibly interchanging p and q, we have

$$\begin{aligned} |H(x,p) - H(x,q)| &= H(x,p) - H(x,q) \\ &\leq |p - q| \big(|f(x,a_0) - f(0,a_0)| + |f(0,a_0)| \big) \\ &\leq |p - q| \, (L_f \, |x| + |f(0,a_0)|) \\ &\leq |p - q| \, (L_f \, |x| + \max_{a \in A} |f(0,a)|) \end{aligned}$$

This shows property (48). We denote by ν an uniform continuity modulus of ℓ in $\mathbb{R}^N \times A$. Let x, y, p be in $\mathbb{R}^N$. We select a_0 with

$$-p \cdot f(x,a_0) - \ell(x,a_0) = \max_{a \in A} \{-p \cdot f(x,a) - \ell(x,a)\}$$

By possibly interchanging x and y, we have

$$\begin{aligned} |H(x,p) - H(y,p)| &= H(x,p) - H(y,p) \\ &\leq -p \cdot (f(x,a_0) - f(y,a_0)) + (\ell(y,a_0) - \ell(x,a_0)) \qquad (70) \\ &\leq |p| \, L_f \, |x - y| + \nu(|x - y|). \end{aligned}$$

This shows (47) and ends the proof. □

Equation (49) with H of Bellman type is called Hamilton–Jacobi–Bellman equation. We mention, without entering in many details, that it is related to the following infinite horizon control problem

$$\inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda t} \ell(y(t, x_0, \alpha), \alpha) \, dt, \tag{71}$$

where $\mathcal{A} = L^\infty(0, +\infty; A)$, i.e. the space of measurable functions defined in $[0, +\infty)$ and taking values in A, the boundedness condition being just a consequence of the fact that A has been assumed to be compact. We consider the trajectory solution of the Cauchy problem

$$\begin{cases} y'(t) = f(y(t), \alpha(t)) \\ y(0) = x_0 \end{cases} \tag{72}$$

Notice that under our assumptions such a problem actually admits unique solution defined in the whole $[0, +\infty)$, that will be denoted in what follows by

$$t \mapsto y(t, x, \alpha).$$

Moreover, using Gronwall's inequality and the global Lipschitz–continuity of f, we get:

Lemma 12.2.

(i) Given two initial data x, z and $\alpha \in \mathcal{A}$, we have

$$|y(t, x, \alpha) - y(t, z, \alpha)| \le e^{L_f t} |x - z|.$$

(ii) Given a time $T > 0$ and a bounded subset K, there exists a bounded set $B \supset K$ such that all the solutions to (72)*, for any $\alpha \in \mathcal{A}$ starting in K, lie in B for $t \in [0, T]$.*

In the optimal control jargon the set A is the *control set* and the entities making up $\mathcal{A}$, are called *controls* tout court. The previously introduced dynamics is qualified as *controlled*, while ℓ is the *running cost* and the constant λ plays the role of a *discount factor*. Finally, the functional appearing in (71) takes the name of *payoff*. See [1] for a general treatment of this topics using the *Dynamic Programming Principle* . The Hamilton–Jacobi–Bellman equation can be viewed as an infinitesimal version of this principle.

The relationship of the above model with (49) is given by the *value function* v associating to any initial point x the infimum of the payoff, namely

$$v(x) = \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda t} \ell(y(t, x, \alpha), \alpha)\, dt.$$

Proposition 12.3. *The value function v is bounded and continuous in $\mathbb{R}^N$.*

Proof. We directly get from the formula defining v

$$|v| \le \frac{R}{\lambda},$$

where $R = \sup_{\mathbb{R}^N \times A} |\ell|$. Fix ε, which will play the role of infinitesimal quantity, take T such that

$$R \int_T^{+\infty} e^{-\lambda s}\, ds < \varepsilon \tag{73}$$

and consider a sequence x_n converging to some x. Select a bounded subset B such that all the trajectories of the controlled dynamics starting at x_n, x lie in B for $s \in [0, T]$, this is possible thanks to Lemma 12.2 (ii). Finally denote by ν an uniform continuity modulus of ℓ in $\mathbb{R}^N \times A$ and by α_n an ε–optimal control for $v(x_n)$. We have by (73) and Lemma 12.2 (i)

$$\begin{aligned}
v(x) - v(x_n) &\leq \int_0^{+\infty} e^{-\lambda s}\, |\ell(y(s,x,\alpha_n),\alpha_n) - \ell(y(s,x_n,\alpha_n),\alpha_n)|\, ds + \varepsilon \\
&\leq \int_0^T e^{-\lambda s}\, |\ell(y(s,x,\alpha_n),\alpha_n) - \ell(y(s,x_n,\alpha_n),\alpha_n)|\, ds + 2\,\varepsilon + \varepsilon \\
&\leq \int_0^T e^{-\lambda s}\, \nu(|y(s,x,\alpha_n) - y(s,x_n,\alpha_n)|)\, ds + 3\,\varepsilon \\
&\leq \int_0^T e^{-\lambda s}\, \nu(e^{L_f s}\, |x - x_n|)\, ds + 3\,\varepsilon \\
&\leq \nu(e^{L_f T}\, |x - x_n|) \int_0^T e^{-\lambda s}\, ds + 3\,\varepsilon.
\end{aligned}$$

and a similar estimate can be obtained for $v(x_n) - v(x)$. This implies

$$\limsup_{n \to +\infty} |v(x_n) - v(x)| \leq 3\,\varepsilon$$

and we get the asserted continuity of v being ε arbitrary. □

We proceed proving that the value function is solution of the Hamilton–Jacobi–Bellman equation, and so it is the unique solution in $\mathbb{R}^N$ thanks to Theorem 11.7.

An historical remark in in order: the connection between the value function and (49) was clear since, more or less, the end of the sixties, but it took a long time and endless investigations to find out the right notion of weak solution capable to uniquely select v. This has been, without any doubt, the important contribution of Crandall–Lions and of viscosity solutions theory.

We start by a preliminary lemma.

Lemma 12.4. *Assume u to be bounded and continuous in $\mathbb{R}^N$. Let B be a bounded subset, then there exists a modulus μ such that*

$$\frac{|x-y|^2}{\varepsilon} < \mu(\varepsilon) \qquad \text{for } x \in B,\ y\ u^\varepsilon\text{–optimal for } x,\ \varepsilon \text{ small.}$$

Proof. Let x, y be as in the statement. By Lemma 7.10

$$|x - y| \le 2\sqrt{R\,\varepsilon}. \tag{74}$$

where $R = \sup |u|$. This implies that y belongs to some bounded set $B' \supset B$ for ε small. We denote by ω an uniform continuity modulus in B', then taking into account (74)

$$\frac{|x-y|^2}{2\,\varepsilon} = u(y) - u(x) \le \omega(|x-y|) \le \omega(2\sqrt{R\,\varepsilon}).$$

The assertion is then obtained setting

$$\mu(t) = 2\,\omega(2\sqrt{R\,t}).$$

□

Proposition 12.5. *The value function is viscosity subsolution to* (49) *with Hamiltonian of the Bellman form given in* (69)*.*

Proof. We consider the class of all controls which are constantly equal to a in $[0,t]$ and free to assume any value in A in $(t,+\infty)$. If α is such a control we get for any x

$$\begin{aligned} v(x) &\le \int_0^{+\infty} e^{-\lambda s}\ell(y(s,x,\alpha),\alpha)\,ds \\ &= \int_0^t e^{-\lambda s}\ell(y(s,x,a),a)\,ds + \int_t^{+\infty} e^{-\lambda s}\ell(y(s,x,\alpha),\alpha)\,ds \end{aligned}$$

We set $\bar\alpha(s) = \alpha(s+t)$, then $\bar\alpha$ is an admissible control and we derive from the previous inequality

$$v(x) \le \int_0^t e^{-\lambda s}\ell(y(s,x,a),a)\,ds + e^{-\lambda t}\int_0^{+\infty} e^{-\lambda s}\ell(y(s,y(t,x,a),\bar\alpha))\,ds$$

Since $\bar u$ can be any possible control in $\mathcal{A}$ we further deduce the formula in the statement.

$$v(x) \le \int_0^t e^{-\lambda s}\ell(y(s,x,a),a)\,ds + e^{-\lambda t}\,v(y(t,x,a)). \tag{75}$$

The trajectory $s \mapsto y(s,x,a)$ is of class C^1 in $[0,t]$ since it corresponds to a constant control, and we have

$$\frac{d}{ds}y(s,x,a) = f(y(s,x,a),a) \qquad \text{for } s \in [0,t]$$

We consider a C^1 supertangent ψ to v at x, we get from (75)

$$\begin{aligned}\lim_{t\to 0}\frac{\psi(x)-e^{-\lambda t}\,\psi(v(y(t,x,a)))}{t} &\leq \lim_{t\to 0}\frac{v(x)-e^{-\lambda t}v(v(y(t,x,a)))}{t}\\ &\leq \lim_{t\to 0}\frac{1}{t}\int_0^t e^{-\lambda s}\,\ell(y(s,x,a),a)\,ds\end{aligned}$$

and consequently

$$\frac{d}{dt}(-e^{-\lambda t}\,\psi(y(t,x,a)))_{|t=0}\leq \ell(x,a).$$

Using that $y(0,x,a)=x$, we further derive

$$v(x)-D\psi(x)\cdot f(x,a)\leq \ell(x,a).$$

Since control a has been arbitrarily chosen, we get in the end

$$v(x)+\max_{a\in A}\{-D\psi(x)\cdot f(x,a)-\ell(x,a)\}$$

which proves v being subsolution, as claimed. □

Proposition 12.6. *The value function is the maximal bounded subsolution to* (49).

Proof. We consider a bounded subsolution u, $x_0\in\mathbb{R}^N$, $\delta>0$ small and large time T such that

$$e^{-\lambda T}\,u(x) \;<\; \delta \qquad \text{for any } x\in\mathbb{R}^N. \tag{76}$$

$$\left(\sup_{\mathbb{R}^N\times A}\ell\right)\int_T^{+\infty} e^{-\lambda s}\,ds \;<\; \delta. \tag{77}$$

Owing to Lemma 12.2 (ii), there exists a ball B centered at x_0 such that any trajectory of the controlled dynamics starting at x_0 is contained in B for $s\in[0,T]$.

Step 1:*sup–convolution as approximated subsolution.*

We fix a bounded subset B of $\mathbb{R}^N$. We consider the ε–sup–convolution of u, denoted by u^ε, for an ε small to be determined later on. Given a differentiability point x of u^ε and y u^ε–optimal, we have because of Proposition 7.16 and the fact that u is subsolution

$$\lambda u(y)+H(y,Du^\varepsilon(x))\leq 0$$

which implies, being $u(y)\geq u^\varepsilon(x)$

$$\lambda u^\varepsilon(x)+H(y,Du^\varepsilon(x))\leq 0. \tag{78}$$

Using estimate (70) in the proof of Lemma 12.1, Lemma 7.10, Proposition 7.14 and Lemma 12.4 we get

$$\begin{aligned} |H(y, Du^\varepsilon(x)) - H(x, Du^\varepsilon(x))| &\leq |Du^\varepsilon(x)|\, L\, |x-y| + \nu(|x-y|) \\ &\leq L\, \frac{|x-y|^2}{\varepsilon} + \nu(2\sqrt{M\varepsilon}) \\ &\leq \mu(|x-y|) + \nu(2\sqrt{M\varepsilon}), \end{aligned}$$

where ν is an uniform continuity modulus for ℓ in $\mathbb{R}^N \times A$. We can take ε so small that

$$\mu(|x-y|) + \nu(2\sqrt{M\varepsilon}) \leq \delta$$

and

$$\lambda u^\varepsilon(x) + H(x, Du^\varepsilon(x)) \leq \delta \quad \text{a.e. in } B.$$

Owing to the convex character of H this implies

$$\lambda u^\varepsilon(x) + \max_{a \in A}\{-p \cdot f(x,a) - \ell(x,a)\} - \delta \leq 0 \tag{79}$$

for any $x \in B$, any $p \in \partial u^\varepsilon(x)$.

Step 2:*Computing on controlled trajectories.*

Let $z(t)$ be any trajectory of the controlled dynamics starting at x_0, exploiting that u^ε is Lipschitz–continuous we get

$$u^\varepsilon(x_0) - e^{-\lambda T}\, u^\varepsilon(z(T)) = \int_0^T \frac{d}{ds}\left(-e^{-\lambda s}\, u^\varepsilon(z(s))\right) ds \tag{80}$$

We have from Remark 4.5 and the fact that z is an admissible trajectory

$$\begin{aligned} \frac{d}{ds}\left(-e^{-\lambda s}\, u^\varepsilon(z(s))\right) &= \lambda\, e^{-\lambda s}\, u^\varepsilon(z(s)) - e^{-\lambda s}\, p \cdot \dot z(s) \\ &= \lambda\, e^{-\lambda s}\, u^\varepsilon(z(s)) - e^{-\lambda s}\, p(s) \cdot f(z(s), \alpha(s)) \end{aligned}$$

for a.e. s, a suitable $p(s) \in \partial u^\varepsilon(z(s))$ and control $\alpha \in \mathcal{A}$. Taking into account (79) we deduce

$$\frac{d}{ds}\left(-e^{-\lambda s}\, u^\varepsilon(z(s))\right) \leq e^{-\lambda s}\, \ell(z(s), \alpha(s)) + \delta \quad \text{for a.e. } s.$$

Combining this inequality with (80), (76), (77) we reach

$$u(x_0) \leq \int_0^T \ell(z(s), u(s))\, ds + \frac{\delta}{\lambda} + \delta < v(x_0) + \frac{\delta}{\lambda} + 2\,\delta$$

Being δ arbitrary, this implies that $u \leq v$ in $\mathbb{R}^N$, as desired. □

Proposition 12.7. *The value function is solution to* (49) *in* $\mathbb{R}^N$*.*

Proof. We use the pull–up argument introduced in the proof of Theorem 10.5, via the maximality property given in Proposition 12.6. □

13 Jensen Lemma and related comparison results

We start by the statement of Alexandrov Theorem:

Theorem 13.1. *Any semiconvex or semiconcave function from $\mathbb{R}^N$ to $\mathbb{R}$ is twice differentiable, up to a set of vanishing Lebesgue measure.*

Previous result can be viewed as the second order counterpart of Rademacher Theorem in the case where not just Lipschitz continuity is assumed but, in addition, some kind of convexity as well.

There is however a relevant difference which makes by far more involved the corresponding second order analysis. Namely, by exploiting the local boundedness of differentials of Lipschitz–continuous functions, we can pass to the limit for any sequence of differentials, up to subsequences, constructing in this way sets of weak first order objects at any point.

This is instead not possible for second order differentials, since the semiconvexity or semiconcavity just gives an unilateral estimate for them, which is not enough to pass to the limit and does not allow in general to get second order entities at any point.

However the fundamental Jensen Lemma, that we are going to prove in the section, allows in a sense to perform such asymptotic construction at any strict local maximizers of semiconvex functions and strict local minimizers of semiconcave functions.

We need some preliminary material.

Given a locally Lipschitz–continuous map $T : \mathbb{R}^N \to \mathbb{R}^N$, we recall that by Rademacher Theorem the differential of T, denoted by DT, does exist, up to a set of vanishing N–dimensional Lebesgue measure. We will use the following version of Area Formula:

Theorem 13.2. *Let T, A be a locally Lipschitz–continuous map from $\mathbb{R}^N$ to $\mathbb{R}^N$, and a measurable subset of $\mathbb{R}^N$, respectively, then*

$$\int_A |DT|\, dx \geq |T(A)|.$$

In the previous statement, and in what follows, $|A|$ stands for the Lebesgue measure of the set A.

We recall the regularization procedure by mollification. The standard mollifier is defined as

$$\rho(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

where the constant c is taken in such a way that

$$\int_{\mathbb{R}^N} \rho\, dx = 1$$

or, in other terms, $\rho\, dx$ is a probability measure in $\mathbb{R}^N$. We set

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\, \rho\left(\frac{x}{\varepsilon}\right) \qquad \text{for any } \varepsilon > 0,$$

we still have

$$\int \rho_\varepsilon\, dx = 1$$

and ρ_ε is supported in the ball centered at 0 with radius ε

Given a continuous function u, the mollification of u, with kernel ρ_ε, at x is given by the equivalent formulas

$$\int_{\mathbb{R}^N} u(y)\, \rho_\varepsilon(|x-y|)\, dy = \int_{\mathbb{R}^N} u(x+h)\, \rho_\varepsilon(|h|)\, dh.$$

Loosely speaking, the regularization procedure replaces the value of u at x by an average of u with respect to the probability measure $\rho_\varepsilon\, dy$.

The mollification of u is a C^∞ function, in addition, by setting $\varepsilon = \frac{1}{n}$, we produce a sequence of mollified functions locally uniformly converging to u. If the initial function u is semiconvex/concave, the regularization procedure does not affect semiconvexity or semiconcavity constant, as made precise in the following

Proposition 13.3. *The ρ_ε–mollification of a semiconvex (semiconcave) function u is semiconvex (semiconcave) with the same semiconvexity (semiconcavity) constants, for any $\varepsilon > 0$.*

Proof. Assume that μ is a semiconvexity constant for u, then

$$u(\lambda\, x + (1-\lambda)\, z) \leq \lambda\, u(x) + (1-\lambda)\, u(z) + \mu\, \lambda\, (1-\lambda)\, |x-z|^2$$

for any x, z, $\lambda \in [0,1]$. We have to show that the same inequality holds true for the ρ_ε– mollification denoted by v. We in fact have

$$\begin{gathered} v(\lambda\, x + (1-\lambda)\, z) = \int u(\lambda\, (x+h) + (1-\lambda)\, (z+h))\, \rho(|h|)\, dh \\ \leq \int \big(\lambda\, u(x+h) + (1-\lambda)\, u(z+h) + \mu\, \lambda\, (1-\lambda)\, |x-z|^2\big)\, \rho(|h|)\, dh \\ = \lambda \int u(x+h)\, \rho(|h|)\, dh + (1-\lambda) \int u(z+h)\, \rho(|h|)\, dh + \mu\, \lambda\, (1-\lambda)\, |x-z|^2 \\ = \lambda\, v(x) + (1-\lambda)\, v(z) + \mu\, \lambda\, (1-\lambda)\, |x-z|^2. \end{gathered}$$

This shows the assertion. □

We will also use the following algebraic lemma:

Lemma 13.4. *Let X a matrix of $\mathbb{S}^N$ with*

$$0 \geq X \geq -\mu\, I \qquad \textit{for some } \mu > 0,$$

then

$$|\det X| \leq \mu^N.$$

For proving it, we first establish a couple of elementary facts.

Lemma 13.5. *Let λ_i, $i = 1, \cdots N$ be N nonnegative constants then*

$$\prod_{i=1}^{N} \lambda_i \leq \left(\frac{\sum_i \lambda_i}{N} \right)^N .$$

Proof. We consider the function

$$h : \Lambda = (\lambda_1, \cdots, \lambda_N) \to \prod_{i=1}^{N} \lambda_i$$

on the compact constraint

$$E = \left\{ \Lambda \geq 0 \mid \sum_{i=1}^{N} \lambda_i = c \right\}$$

where c is a given positive constant. We denote by $\Lambda_0 = (\lambda_1^0, \cdots \lambda_N^0)$ a maximizer of h in E, then the first order condition

$$Dh(\Lambda_0) = \rho\, \mathbf{1}$$

holds true, where $\mathbf{1}$ is the vector with all components equal to 1 and ρ is a positive constant. We deduce

$$\prod_{i \neq j} \lambda_i^0 = \rho \qquad \text{for any } j,$$

which in turn implies

$$\Lambda_0 = \frac{c}{N}\, \mathbf{1}.$$

We finally get for any $(\lambda_1, \cdots, \lambda_N) \in E$

$$\prod_{i=1}^{N} \lambda_i \leq \left(\frac{c}{N} \right)^N = \left(\frac{\sum_i \lambda_i}{N} \right)^N$$

which gives the assertion since c has been arbitrarily taken.

□

Lemma 13.6. *Let $X \in \mathbb{S}^N$ with $X \leq 0$ then*

$$|\det X| \leq \left(\frac{-\operatorname{tr} X}{N}\right)^N .$$

Proof. Since the trace is invariant and X is symmetric, we can assume that X is diagonal. We denote by λ_i, $i = 1, \cdots, N$ the entries in the main diagonal. Since all the λ_i are nonpositive by the sign condition on X, then $|\det X| = \prod_i -\lambda_i$ and by the previous lemma

$$|\det X| \leq \left(\frac{-\sum_i \lambda_i}{N}\right)^N = \left(\frac{-\operatorname{tr} X}{N}\right)^N .$$

□

Proof. (of Lemma 13.4) Since $X \geq -\mu I$ all the entries in the main diagonal are greater than or equal to $-\mu$, which implies $\operatorname{tr} X \geq -N\mu$. By exploiting Lemma 13.6, we get

$$|\det X| \leq \left(\frac{-\operatorname{tr} X}{N}\right)^N \leq \left(\frac{N\mu}{N}\right)^N .$$

□

Next theorem will be crucial for the proof of comparison result in Theorem 13.11. As already said, it highlights a key property of local maximizer of semi-convex functions or, equivalently, local minimizer of semiconcave functions, which clearly does not by any means extend to minimizers of semiconvex or maximizers of semiconcave functions, as the function modulus or minus modulus shows.

The argument exploits properties of semiconvex/semiconcave functions, stability of maximizers/minimizers and measure–theoretic facts via Area Formula.

Theorem 13.7. *(Jensen) Given a semiconvex function u, we set for any $\delta > 0$*

$$A_\delta = \{x \mid \text{ local maximizer of } y \mapsto u(y) + p \cdot (y - x) \text{ for some } p \text{ with } |p| \leq \delta\}.$$

Then for any strict local maximizer x_0 of u

$$|A_\delta \cap B(x_0, r)| > C \qquad \text{for } \delta,\ r \text{ small enough,}$$

where C is a positive constant solely depending on r, δ, the dimension N of the ambient space and the semiconvexity constant of u.

The same statement holds true by replacing semiconcavity and maximizers with semiconcavity and minimizers, respectively.

Proof. We take r so small that x_0 is the unique maximizer of u in the closed ball $B(x_0, 2r)$. We locally uniformly approximate u by a sequence u_n of smooth functions obtained by $\frac{1}{n}$–mollification, thanks to Proposition 13.3 the u_n have the same semiconvexity constant of u, say μ. By exploiting stability property of maximizers under uniform convergence, we derive

$$\operatorname{argmax}_{B(x_0,2r)} \big(u_n + p \cdot (x - x_0)\big) \subset B(x_0, r)$$

whenever $|p| \leq \delta$, and δ is sufficiently small, n sufficiently large. In other terms, for such n, δ

$$Du_n(A^n_\delta) \supset B(0, \delta) \tag{81}$$

where

$$A^n_\delta := \{x \in B(x_0, r) \mid \text{max. of } \big(u_n + p \cdot (x - x_0)\big) \text{ in } B(x_0, 2r) \text{ for some } |p| \leq \delta\}.$$

Since A^n_δ is closed, and so measurable, and Du_n locally Lipschitz–continuous, we can apply Area Formula to obtain

$$\int_{A^n_\delta} |\det D^2 u_n| \, dx \geq |Du_n(A^n_\delta)| \geq \omega_N \, \delta^N,$$

where $\omega_N = \frac{|B(0,\rho)|}{\rho^N}$, for any $\rho > 0$. By the very definition of A^n_δ and semiconvexity properties of u_n

$$-\mu \, I \leq D^2 u_n \leq 0 \qquad \text{in } A^n_\delta$$

so that we derive from (81) via the previous Algebraic Lemma 13.4

$$|A^n_\delta| \geq \frac{\omega_N \, \delta^N}{\mu^N}, \tag{82}$$

with estimate independent of n. We set

$$B_\delta = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} A^n_\delta,$$

it is apparent that

$$\limsup_n \mathbb{I}(A^n_\delta) = \mathbb{I}(B_\delta),$$

where $\mathbb{I}$ stands for the indicator function, which is equal 1 in the set at the argument and vanishing outside. By reverse Fatou Lemma we get

$$\limsup_n |A^n_\delta| = \int \mathbb{I}(A^n_\delta) \, dx \leq \int \limsup \mathbb{I}(A^n_\delta) \, dx = |B_\delta|,$$

which in turn implies by (82)

$$|B_\delta| \geq \frac{\omega_N \, \delta^N}{\mu^N}.$$

We claim that

$$B_\delta \subset A_\delta \cap B(x_0, r).$$

In fact, if $x \in B_\delta$ then $x \in A_\delta^{j_n}$ for all the indices of some subsequence j_n, this means that there are p_{j_n} with modulus less than or equal to δ such that x is maximizer in $B(x_0, 2r)$ of

$$y \mapsto u_n(y) + p_{j_n} \cdot (y - x).$$

Passing at the limit for $j_n \to +\infty$, we have that p_{j_n} converges, up to subsequences, to some p with $|p| \leq \delta$ and, also recalling that u_n uniformly converges to u, we get that x is also maximizer in $B(x_0, 2r)$ of

$$y \mapsto u(y) + p \cdot (y - x).$$

This concludes the proof.

□

We deduce from the previous theorem a crucial second order information at any local maximizer, not necessarily strict, of a semiconvex function (analogous statement holds true at minimizers of semiconcave functions).

Theorem 13.8. *Let u be a semiconvex function with semiconvexity constant μ, and x_0 a local maximizer of u. There exists a symmetric matrix $-\mu\, I \leq X \leq 0$ and a sequence y_m of points where u is twice differentiable converging to x_0 such that*

$$X = \lim_m D^2 u(y_m).$$

Proof. The point x_0 is a local strict maximizer of $u - \frac{1}{m}\,|x - x_0|^2$, for any $m \in \mathbb{N}$. Given a sequence $\delta_n \to 0$, we find by Alexandroff Theorem and Theorem 13.7 a sequence $x_n^m \in A_{\delta_n}$, where A_{δ_n} is defined as in the statement of Theorem 13.7 for the function $u - \frac{1}{m}\,|x - x_0|^2$, such that $u - \frac{1}{m}\,|x - x_0|^2$, an so u, is differentiable twice in x_n^m and $x_n^m \to x_0$ as $n \to +\infty$.

By the very definition of A_{δ_n}, we have $D^2 u(x_n^m) \leq 2\,\frac{1}{m}\, I$ and by semiconvexity $D^2 u(x_n^m) \geq -\mu\, I$. We find through a diagonal extraction a sequence $y_m = x_{n_m}^m$ such that

$$2\,\frac{1}{m}\, I \geq D^2 u(y_m) \geq -\mu\, I.$$

Therefore the sequence $D^2 u(y_m)$ is bounded in $\mathbb{S}^N$, and any limit matrix X satisfies the assertion. □

The previous result can be rephrased as follows:

Theorem 13.9. *Assume that u is a semiconvex function and ψ is a C^2 supertangent to u at a point x_0, then there exists $X \in \mathbb{S}^N$ and a sequence of points y_m where u is twice differentiable, converging to x_0 such that*

– $D^2u(y_m) \to X$;

– $X \leq D^2\psi(x_0)$.

We get as a consequence:

Lemma 13.10. *A semiconvex (semiconcave) function u is viscosity subsolution (supersolution) of some elliptic continuous equation if and only if it is a.e. subsolution (supersolution).*

Proof. We just treat the semiconvex case. If u is viscosity subsolution and x_0 is a point where it is twice differentiable, then

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + \frac{1}{2}\,(D^2u(x_0)\,(x - x_0)) \cdot (x - x_0) + \mathrm{o}(|x - x_0|^2))$$

and so the function

$$\psi_\delta(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + \frac{1}{2}\,((D^2u(x_0) + \delta\, I)\,(x - x_0)) \cdot (x - x_0)$$

is a smooth supertangent to u at x_0 for any $\delta > 0$, and consequently the supersolution test holds true for ψ_δ. Letting δ go to 0, we get that u is a classical pointwise subsolution of the equation at x_0. Then u is a.e. subsolution according to Alexandroff Theorem.

Conversely, let us assume u to be a.e. subsolution, and consider a C^2 supertangent ψ to u at a point x_0. Then u is strictly differentiable at x_0 and $Du(x_0) = D\psi(x_0)$. We apply Jensen Lemma and Theorem 13.9 to find that the subsolution inequality holds taking arguments $(x_0, u(x_0), Du(x_0), X)$ with $X \leq D^2\psi(x_0)$. By exploiting ellipticity of the equation, we finally find that ψ satisfies the supersolution test at x_0. This shows that u is actually viscosity subsolution, as desired.

□

We consider the equation

$$\lambda\, u + F(Du, D^2u) = f(x) \qquad \text{in } \Omega \tag{83}$$

assuming $\lambda > 0$, Ω to be a bounded open set of $\mathbb{R}^N$, $F : \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \to \mathbb{R}$ to be continuous and F in addition elliptic. We prove:

Theorem 13.11. *Let u, v be a subsolution and supersolution of* (83). *Assume, in addition, that u is upper semicontinuous and v lower semicontinuous in $\overline{\Omega}$. If $u \leq v$ in $\partial\Omega$ then $u \leq v$ in Ω.*

Proof. We can assume without loosing generality that both u and v are bounded in $\overline{\Omega}$ in order to apply sup/inf convolutions techniques. If, for instance, this were not the case for u, we could choose c such that

$$\lambda\, c + F(0,0) \leq \min_{\overline{\Omega}} f, \quad c \leq \min_{\overline{\Omega}} v$$

then

$$u_0 = u \vee c$$

could replace u in the statement without modifying the assertion.

The argument is by contradiction. We assume

$$\max_{\overline{\Omega}} u - v =: 2\,\theta > 0, \quad \operatorname{argmax}_{\overline{\Omega}} u - v \subset \Omega.$$

In contrast to what done for first order approximation results, we perform sup/inf convolutions for both u and v. We consider the sequence $u^\varepsilon - v_\varepsilon$ which approximates $u - v$ as $\varepsilon \to 0$ in the sense of upper weak semilimit, and, thanks to the corresponding stability properties of maxima and maximizers, we get

$$\max_{\overline{\Omega}} u^\varepsilon - v_\varepsilon > \theta > 0, \quad \operatorname{argmax}_{\overline{\Omega}} u^\varepsilon - v_\varepsilon \subset \Omega$$

for ε small. For such an ε, we denote by x_ε a maximizer in $\overline{\Omega}$ of $u_\varepsilon - v_\varepsilon$. Being $u^\varepsilon - v_\varepsilon$ semiconvex, we can apply Jensen Lemma and get a sequence $x_n \to x_\varepsilon$ where $u^\varepsilon - v_\varepsilon$, u^ε, v_ε are twice differentiable and a symmetric matrix $Z \leq 0$ with

$$D^2(u^\varepsilon - v_\varepsilon)(x_n) = D^2 u^\varepsilon(x_n) - D^2 v_\varepsilon(x_n) \to Z.$$

We deduce the estimate

$$-\frac{1}{2\varepsilon}\, I \leq D^2 u^\varepsilon(x_n) \leq D^2 v_\varepsilon(x_n) \leq \frac{1}{2\varepsilon}\, I$$

for n large, which imply that $D^2 u^\varepsilon(x_n)$, and consequently $D^2 v_\varepsilon(x_n)$ are convergent, up to a subsequence, to matrices X and Y, respectively, with

$$X - Y = Z \leq 0.$$

We in addition have that both u_ε, v^ε are differentiable at x_ε and

$$D u^\varepsilon(x_\varepsilon) = \frac{y_\varepsilon - x_\varepsilon}{\varepsilon} = \frac{x_\varepsilon - z_\varepsilon}{\varepsilon} = D v_\varepsilon(x_\varepsilon), \tag{84}$$

where y_ε, z_ε are the u^ε–optimal and the v_ε–optimal point at x_ε, univocally determined by the differentiability of both functions.

By the transfer principle of test functions from a function to its sup/inf convolution, we have

$$\begin{aligned} \lambda\, u(y_n) + F\left(\tfrac{y_n - x_n}{\varepsilon}, D^2 u^\varepsilon(x_n)\right) &\leq f(y_n) \\ \lambda\, v(z_n) + F\left(\tfrac{x_n - z_n}{\varepsilon}, D^2 v_\varepsilon(x_n)\right) &\geq f(z_n) \end{aligned}$$

where y_n, z_n are the u^ε–optimal and the v_ε–optimal points at x_n, again univocally determined by the differentiability of both functions at x_n. We exploit continuity properties of optimal points, Lemma 7.15 and convergence properties of $D^2u^\varepsilon(x_n)$, $D^2v_\varepsilon(x_n)$, outlined above, to get at the limit for $n \to +\infty$

$$\begin{array}{l} \lambda\, u(y_\varepsilon) + F\left(\frac{y_\varepsilon - x_\varepsilon}{\varepsilon}, X\right) \leq f(y_\varepsilon) \\ \lambda\, v(z_\varepsilon) + F\left(\frac{x_\varepsilon - z_\varepsilon}{\varepsilon}, Y\right) \geq f(z_\varepsilon). \end{array}$$

By (84), $X \leq Y$, $u(y_\varepsilon) \geq u(x_\varepsilon)$, $v(z_\varepsilon) \leq v(x_\varepsilon)$ and ellipticity of F, we further derive, subtracting side by side the above inequalities

$$\lambda\, \theta \leq \lambda\, (u(x_\varepsilon) - v(x_\varepsilon)) \leq f(y_\varepsilon) - f(z_\varepsilon).$$

Since this estimate can be obtained for ε sufficiently small and $|y_\varepsilon - z_\varepsilon| = O(\sqrt{\varepsilon})$, we can exploit that f is continuous and eventually take ε so small to reach a contradiction. □

14 Perron Method

Perron–Ishii method is the adaptation of the classical Perron method to viscosity solution theory. As a first step we deal with the pointwise supremum (infimum) of possibly infinite families of sub(super)solutions. Here we face a basic difficulty due the limited information available on the continuity properties for functions so obtained. We in fact have:

Proposition 14.1. *Let u_α be a family of locally equibounded lsc (usc) functions, for $\alpha \in \mathcal{A}$, then*

$$u = \sup_\alpha u_\alpha \quad (\inf_\alpha u_\alpha)$$

is lsc (usc).

Proof. We fix x_0 and consider $x_n \to x_0$. Given $\varepsilon > 0$, let $\beta \in \mathcal{A}$ with $u(x_0) < u_\beta(x_0) + \varepsilon$, then

$$u(x_0) \leq u_\beta(x_0) + \varepsilon \leq \liminf u_\beta(x_n) + \varepsilon \leq \liminf u(x_n) + \varepsilon$$

which proves the assertion being ε arbitrary. □

However, in general, pointwise infima of families of lsc functions are not lsc and pointwise suprema of families of usc functions are not usc. This setup requires a generalization of the notion of viscosity sub/supersolutions to functions which are merely locally bounded.

To do that, we introduce the usc and lsc envelope of a locally bounded function u, denoted by $u^\#$, $u_\#$, respectively. More precisely, we need local boundedness from above to perform the usc envelope and from below for the lsc envelope, to ease presentation, we will not specify this in what follows. Roughly speaking these envelopes are obtained by performing an upper or lower semilimit of a sequence constantly equal to u. We have in formulas for any x:

$$\begin{aligned} u^{\#}(x) &= \sup\{\limsup u(x_n) \mid x_n \to x\} \\ u_{\#}(x) &= \inf\{\liminf u(x_n) \mid x_n \to x\} \end{aligned}$$

By invoking corresponding properties holding for weak semilimits, we have that there exists sequences y_n, z_n converging to x such that

$$\lim u(y_n) = u^{\#}(x) \quad \text{and} \quad \lim u(z_n) = u_{\#}(x). \tag{85}$$

This in turn implies, taking also into account Proposition 14.1, the following characterization:

Proposition 14.2.

$$\begin{aligned} u^{\#} &= \inf\{v \textit{ usc} \mid v \geq u\} \\ u_{\#} &= \sup\{w \textit{ lsc} \mid w \leq u\}. \end{aligned}$$

We say that a locally bounded function u is viscosity subsolution (supersolution) of a certain first or second order equation if $u^{\#}$ ($u_{\#}$) is viscosity subsolution (supersolution) of the same equation in the usual sense. We say that u is solution if it is super and subsolution at the same time. All the construction so far performed is justified by the following:

Theorem 14.3. *Let u_α, $\alpha \in \mathcal{A}$, a locally equibounded family of subsolutions (supersolutions) to some continuous equation, then*

$$u = \sup_\alpha u_\alpha (\inf_\alpha u_\alpha)$$

is a subsolution (supersolution) of the same equation.

Proof. We just prove the subsolution part of the statement. Let ψ be a supertangent, we can assume strict without loss of generality, to $u^{\#}$ at some point x_0. We know from (85) that $u^{\#}(x_0) = \lim_n u(y_n)$, for a suitable sequence y_n converging to x_0, and, taking into account that u is a pointwise supremum, we can select a sequence α_n in $\mathcal{A}$ with

$$u^{\#}(x_0) = \lim_n u_{\alpha_n}(y_n). \tag{86}$$

By the definition of $u^{\#}$, $u^{\#}_{\alpha_n} \leq u^{\#}$ for any n, and consequently

$$v := \limsup{}^{\#} u^{\#}_{\alpha_n} \leq u^{\#},$$

on the other side, by (86)

$$u^{\#}(x_0) = \lim_n u_{\alpha_n}(y_n) \leq \limsup u^{\#}_{\alpha_n}(y_n) \leq v(x_0)$$

which yields

$$v(x_0) = u^{\#}(x_0).$$

Therefore x_0 is a local strict maximizer of $v - \psi$, and by exploiting standard stability properties of maxima under upper semilimit, we deduce the existence of a sequence of points x_n converging to x_0 where ψ is supertangent to $u^{\#}_{\alpha_n}$.

Being the $u^{\#}_{\alpha_n}$ subsolutions, the subsolution tests for the equation in object holds true for ψ at x_n, so that exploiting the continuity of the equation we get the desired subsolution inequality for ψ at x_0. This shows that u is indeed subsolution, and concludes the proof.

□

Theorem 14.4. *Assume that there is a lsc supersolution v to the equation in an open set Ω possibly unbounded and possibly equal to the whole $\mathbb{R}^N$, such that the family*

$$S = \{w \ \ \textit{subsolution} \mid w \leq v\}$$

is nonempty then

$$w_0 = \sup\{w(x) \ \mid w \in S\} \tag{87}$$

is a solution. If there is an usc subsolution u such that

$$T = \{w \ \ \textit{supersolution} \mid w \geq u\}$$

is nonempty, then

$$w^0 = \inf\{w \ \mid w \in T\} \tag{88}$$

is a solution.

Proof. We prove that (87) yields a solution, the other formula can be handled similarly. We already know, from Propositions 14.3, that w_0 is indeed a subsolution, it is then left to show the supersolution property.

For this purpose we consider a C^2 subtangent ψ to $(w_0)_{\#}$ at a point x_0 that we can assume, without loosing generality, strict, we stress that we are taking $\psi = (w_0)_{\#}$ at x_0. We discuss two cases according on whether $(w_0)_{\#} = v$ or $(w_0)_{\#} < v$ at x_0.

In the first instance, $(w_0)_{\#}$ is subtangent to v at x_0 and, by the transitivity property of subtangency, we deduce that ψ is subtangent to v as well, so that

$$F(x, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0$$

as desired. In the second case, we argue by contradiction and assume instead

$$F(x, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) < 0,$$

then, by continuity of F and C^2 regularity of ψ, we deduce that ψ is a strict subsolution of the equation in some neighborhood U_0 of x_0.

Being $(w_0)_\#$ lsc, strictly less v at x_0 and equal to ψ at the same point, we can assume without loosing generality that

$$v(x) > \psi(x) + \delta_0 \qquad \text{for any } x \in U_0, \text{ a suitable } \delta_0 > 0, \tag{89}$$

and in addition that x_0 is a strict minimizer of $(w_0)_\# - \psi$ in the closure of U_0. This last property implies that for any given open neighborhood of x_0 compactly contained in U_0, denoted by U, we can find $0 < \delta < \delta_0$ with

$$\{x \in U_0 \mid (w_0)^\#(x) \le \psi(x) + \delta\} \subset \overline{U}. \tag{90}$$

If not, there should be a sequence $\delta_n \to 0$ and x_n with

$$(w_0)_\#(x_n) - \psi(x_n) \le (w_0)^\#(x_n) - \psi(x_n) \le \delta_n \text{ and } x_n \in U_0 \setminus \overline{U}, \tag{91}$$

and for any limit point y_0 of x_n, by lower semicontinuity of $(w_0)_\#$, we should get

$$0 = \lim\,(w_0)_\#(x_n) - \psi(x_n) \ge (w_0)_\#(y_0) - \psi(y_0).$$

This is in contrast with x_0 being strict minimizer of $(w_0)_\# - \psi$ in the closure of U_0, since, clearly $y_0 \in \overline{U}$, $y_0 \neq x_0$ by (91).

We fix $\delta < \delta_0$ satisfying (90) and define

$$\overline{w} = \begin{cases} \max\{(w_0)^\#, \psi + \delta\} & \text{in } U_0 \\ (w_0)^\# & \text{outside } U_0 \end{cases}$$

Taking into account that $\overline{w} = (w_0)^\#$ in a neighborhood of ∂U_0, by (90), we see that $\overline{w}$ is usc and subsolution on the whole of $\mathbb{R}^N$.

We proceed showing that $\overline{w} = (\overline{u})^\#$, where

$$\overline{u} = \begin{cases} \max\{w_0, \psi + \delta\} & \text{in } U_0 \\ w_0 & \text{outside } U_0 \end{cases}$$

In fact, the function $(\overline{u})^\#$ is greater than or equal to w_0 in $\mathbb{R}^N$, which implies $(\overline{u})^\# \ge (w_0)^\#$ in $\mathbb{R}^N$, in addition $(\overline{u})^\# \ge \overline{u} \ge \psi + \delta$ in U_0. This gives $(\overline{u})^\# \ge (w_0)^\#$ in $\mathbb{R}^N$. On the other side, $\overline{w}$ is upper semicontinuous and dominates w_0 in $\mathbb{R}^N$ and $\psi + \delta$ in U_0, then it must dominates $(\overline{u})^\#$ in the whole space. This proves the claim. We have by (89)

$$\begin{aligned} \overline{u} &= \max\{w_0, \psi + \delta\} < \psi + \delta_0 < v \qquad \text{in } U_0 \\ \overline{u} &= w_0 \le v \qquad \text{outside } U_0. \end{aligned}$$

Finally we observe that the relation $(w_0)_\#(x_0) < \psi(x_0, \delta)$ implies, by the very definition of $(w_0)_\#$, $w_0(\overline{x}) < \psi(\overline{x}) + \delta$ for some $\overline{x} \in U_0$, and so

$$\overline{u}(\overline{x}) = \psi(\overline{x}) + \delta > w_0(\overline{x}).$$

This is impossible because $\overline{u}$ belongs to the family of functions S of which w_0 is the pointwise supremum. The proof is then concluded. $\square$

We say that an equation satisfies the comparison principle on a bounded open set Ω if, given a supersolution v_0 lsc in $\overline{\Omega}$, and a subsolution u_0 usc in $\overline{\Omega}$,

$$u_0 \leq v_0 \text{ in } \partial\Omega \;\Rightarrow\; u_0 \leq v_0 \text{ in } \Omega.$$

We deduce from Theorem 14.4:

Theorem 14.5. *Consider an equation satisfying the comparison principle in Ω and let g be a continuous function on $\partial\Omega$. Assume that there is a lsc supersolution v_0 and an usc subsolution u_0 in Ω which can be extended continuously to g in $\partial\Omega$, namely with*

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u_0(x) = \lim_{\substack{x \to x_0 \\ x \in \Omega}} v_0(x) = g(x_0) \qquad \text{for any } x_0 \in \partial\Omega. \tag{92}$$

Then there exists one and only one solution w, sandwiched in between v_0 and u_0, which is continuous in $\overline{\Omega}$ and takes the value g on the boundary.

Proof. We adopt the same notations as in the statement of Theorem 14.4. Let x_n be a sequence in Ω converging to $x_0 \in \partial\Omega$, assume that $(w_0)^\#(x_n)$ converges to some y_0, then by the very definition of upper semicontinuous envelope, there is a sequence z_n in Ω with $z_n \to x_0$ and $w_0(z_n) \to y_0$. We then have by (92)

$$y_0 = \lim_n (w_0)^\#(x_n) = \lim_n w_0(z_n) \leq v_0(z_n) = g(x_0).$$

This implies that $(w_0)^\#$ can be extended as g on $\partial\Omega$ keeping its upper semicontinuous character. Arguing similarly we find that if we extend $(w^0)_\#$ putting g on the boundary we get a lower semicontinuous function in $\overline{\Omega}$. By the comparison principle we deduce that

$$(w_0)^\# \leq (w^0)_\# \qquad \text{in } \Omega. \tag{93}$$

On the other side we have by the very definition of w_0, w^0

$$(w^0)_\# \leq w^0 \leq w_0 \leq (w_0)^\#. \tag{94}$$

By combining (93), (94) we obtain

$$(w^0)_\# = (w^0)^\# = (w_0)^\# = (w_0)_\#.$$

This shows that $w_0 = w^0 =: w$ is a continuous solution to the equation with

$$v_0 \geq w \geq u_0$$

taking the value g on $\partial\Omega$. It is the unique solution of the corresponding Dirichlet because the comparison principle holds in Ω. $\square$

15 Applications

Here we apply the previous results to two model equations:

$$\begin{aligned} \lambda\, u + \max\{< -A_1, D^2 u >, < -A_2, D^2 u >\} + b(x) \cdot Du - f(x) &= 0 \quad (95) \\ - \triangle\, u + |Du| - f(x) &= 0 \quad (96) \end{aligned}$$

set in a bounded open set Ω with C^2 boundary, and both coupled with homogeneous boundary conditions. We explain the notations and assumptions in (95), (96). We set

$$< X, Y >= \mathrm{tr}(X\,Y) \qquad \text{for any } X,\, Y \text{ in } \mathbb{S}^N.$$

It is apparently a bilinear form in $\mathbb{S}^N$. We also set

$$\triangle \psi =< I, D^2 \psi > \qquad \text{for any } C^2 \text{ function } \psi,$$

where I stands for the identity matrix. We further assume $\lambda > 0$, $A_1,\, A_2 \in \mathbb{S}^N$, $A_1 > 0$, $A_2 > 0$, $b : \overline{\Omega} \to \mathbb{R}^N$ Lipschitz–continuous, f continuous and nonnegative.

Lemma 15.1.

$$< X, Y > \geq 0 \qquad \textit{whenever } X \geq 0,\, Y \geq 0.$$

Proof. It is apparent that

$$< X, Y >=< X', Y' >$$

whenever $X' = A\,X\,A^{-1}$, $Y' = A\,Y\,A^{-1}$ for some invertible matrix A. We can therefore assume, without loosing generality, that X is of diagonal form, in this case

$$< X, Y >= \sum_i X_{ii}\, Y_{ii}$$

and the assertion follows since any nonnegative matrix has nonnegative entries in the main diagonal. □

The above result immediately implies

Corollary 15.2. *Equations* (95), (96) *are elliptic.*

Proposition 15.3. *There are a supersolution v and a subsolution u of* (95) *in Ω with $v \geq u$ in $\overline{\Omega}$, both continuous in $\overline{\Omega}$ and vanishing on $\partial\Omega$.*

Proof. To ease notations, we set

$$F(x,p,X) = \max\{< -A_1, X >, < -A_2, X >\} + b(x)\cdot p - f(x).$$

It comes from $f \geq 0$ that the null function is subsolution to the equation. To prove the assertion, it is therefore enough to construct a nonnegative supersolution vanishing on $\partial\Omega$. We select a constant $R > 0$ with

$$\lambda R + F(x,0,0) \geq 0 \qquad \text{on } \Omega.$$

The function constantly equal to R is therefore positive supersolution of the equation but fails vanishing on the boundary. The idea is to modify it in a neighborhood of $\partial\Omega$ to obtain a continuous supersolution vanishing on the boundary. We consider the function

$$\psi(x) := M(1 - e^{-\mu d(x)}), \tag{97}$$

with M and μ positive constants to be determined, and $d(\cdot)$ denoting the distance from the boundary. Due to the C^2 regularity assumed on $\partial\Omega$, the function is C^2 as well in a suitable neighborhood of $\partial\Omega$ denoted by U. We find, by direct calculation in $U \cap \Omega$

$$\begin{aligned} Dw(x) &= M\,\mu\, e^{-\mu d(x)}\, Dd(x) \\ D^2 w(x) &= M\,\mu\, e^{-\mu d(x)}\, D^2 d(x) - M\,\mu^2\, e^{-\mu d(x)}\, Dd(x)\otimes Dd(x), \end{aligned}$$

and in addition

$$|Dd(x)| = 1.$$

Given two vectors a, b in $\mathbb{R}^N$ the tensor product $a\otimes b$ denotes the matrix with entries $(a_i\, b_j)$ for i, $j = 1,\cdots,N$. We compute for $x \in U\cap\Omega$

$$F(x, D\psi(x), D^2\psi(x)) = M\,\mu\, e^{-\mu d(x)}\, B - f(x),$$

where

$$B = \max_{i=1,2} < -A_i, D^2 d(x) - \mu\, Dd(x)\otimes Dd(x) > + b(x)\cdot Dd(x).$$

We can therefore choose μ so large that B is positive, and then M so large to satisfy

$$\lambda\,\psi(x) + F(x, D\psi(x), D^2\psi(x)) > 0 \quad \text{in } U\cap\Omega, \quad \psi(x) > R \quad \text{in } \Omega\setminus U.$$

Consequently the function

$$w(x) = \min\{R,\, \psi(x)\}$$

is equal to R in $\Omega\setminus U$ and to ψ in a neighborhood of $\partial\Omega$. It is in conclusion a continuous supersolution of the equation vanishing on the boundary. This concludes the proof.

□

Proposition 15.4. *The comparison principle holds for* (95)*.*

Proof. As already pointed out, the equation is elliptic. We denote by u, v an usc supersolution and a lsc supersolution with $v \geq u$ on $\partial\Omega$. We argue as in Theorem 13.11 and, using the same notation, we get

$$\begin{array}{l} \lambda\, u(y_\varepsilon) + \max\{< A_1, X >, < A_2, X >\} + b(y_\varepsilon) \cdot \frac{y_\varepsilon - x_\varepsilon}{\varepsilon} \leq f(y_\varepsilon) \\ \lambda\, u(z_\varepsilon) + \max\{< A_1, Y >, < A_2, Y >\} + b(z_\varepsilon) \cdot \frac{y_\varepsilon - x_\varepsilon}{\varepsilon} \geq f(z_\varepsilon) \end{array}$$

where y_ε, z_ε are u^ε and v_ε–optimal points for x_ε, respectively, $X \leq Y$

$$u^\varepsilon(x_\varepsilon) - v_\varepsilon(x_\varepsilon) = \max_{\overline{\Omega}} u^\varepsilon - v_\varepsilon =: \theta > 0.$$

We subtract side by side and exploit ellipticity to obtain

$$\lambda\, \theta \leq \big(b(z_\varepsilon) - b(y_\varepsilon)\big) \cdot \frac{y_\varepsilon - x_\varepsilon}{\varepsilon} + f(y_\varepsilon) - f(z_\varepsilon) \tag{98}$$

By slightly adapting Proposition 11.2, we get $|x_\varepsilon - y_\varepsilon| = \mathrm{o}(\sqrt{\varepsilon})$ and

$$|y_\varepsilon - z_\varepsilon| \leq |y_\varepsilon - x_\varepsilon| + |x_\varepsilon - z_\varepsilon| = \mathrm{o}(\sqrt{\varepsilon})$$

and then deduce from (98)

$$\lambda\, \theta \leq L\, |y_\varepsilon - x_\varepsilon|\, \frac{|x_\varepsilon - z_\varepsilon|}{\varepsilon} + \nu(|z_\varepsilon - x_\varepsilon|) \leq \frac{\mathrm{o}(\varepsilon)}{\varepsilon} + \nu(|z_\varepsilon - x_\varepsilon|),$$

where L is a Lipschitz constant for $b(\cdot)$, ν an uniform continuity modulus for f in $\overline{\Omega}$. Sending ε to 0, we get a contradiction and conclude the proof. □

Thanks to Propositions 15.3, 15.4, we are in position to apply Theorem 14.5 and get

Proposition 15.5. *The homogeneous Dirichlet problem for* (95) *admits one and only one continuous solution.*

We proceed studying the viscous Eikonal equation (96). In contrast to what happens in the first order Eikonal equations, we do not need existence of a strict subsolution, or in other terms the strict positivity of f, to establish a comparison principle. Loosely speaking, the presence of the Laplacian allows constructing strict supersolutions, which make the argument by contradiction working in the comparison result.

Proposition 15.6. *There are a subsolution u and a supersolution v to* (96)*, both continuous in $\overline{\Omega}$, vanishing on $\partial\Omega$ and with $v \geq u$ in $\overline{\Omega}$.*

Proof. The null function is a subsolution to (96). We proceed constructing a continuous nonnegative supersolution vanishing on the boundary.

We indicate with Θ a neighborhood of $\partial\Omega$ where the distance from $\partial\Omega$, denoted by $d(\cdot)$, is of class C^2. We define

$$\begin{aligned} \varphi(x) &= (1 - e^{-\mu\, d(x)}) && (99)\\ \psi(x) &= (1 - e^{-\mu\,|x-x_0|^2}) && (100) \end{aligned}$$

where x_0 is a point in the exterior of Ω, μ, positive constants to be determined later on. We set

$$k = \frac{\min_{\Omega\setminus\Theta} v}{\max_{\Omega\setminus\Theta} w}$$

Given $x \in \Omega \setminus \Theta$, we have

$$k\,\varphi(x) \leq \frac{\varphi(x)}{\max_{\Omega\setminus\Theta} w}\,\psi(x) \leq \psi(x)$$

while $k\,\varphi > \psi$ in a suitable neighborhood of $\partial\Omega$ because φ vanishes on $\partial\Omega$ while ψ stays strictly positive. We compute (in Θ for φ and in Ω for v)

$$\begin{aligned} D\varphi(x) &= \mu\, e^{-\mu d(x)}\, Dd(x)\\ D^2\varphi(x) &= \mu\, e^{-\mu d(x)}\, D^2 d(x) - \mu^2\, e^{-\mu d(x)}\, Dd(x)\otimes Dd(x)\\ D\psi(x) &= +2\,\mu\, e^{-\mu\,|x-x_0|^2}\,(x-x_0)\\ D^2\psi(x) &= +2\,\mu\, e^{-\mu\,|x-x_0|^2}\, I - 4\,\mu^2\, e^{-\mu\,|x-x_0|^2}\,(x-x_0)\otimes(x-x_0) \end{aligned}$$

We set to ease notations

$$\begin{aligned} A &= -\triangle\, d + \mu + 1\\ B &= -N + 2\,\mu\,|x-x_0|^2 + |x-x_0| \end{aligned}$$

Given a constant M devoted to become large, we have

$$-\triangle\,(kM\varphi) + |D(kM\varphi)| - f(x) = k\,M\,\mu\, e^{-\mu\, d(x)}\, A - f(x)$$

in Ω and

$$-\triangle\,(M\psi) + |D(M\psi)| - f(x) = 2\,M\,\mu\, e^{-\mu\,|x-x_0|^2}\, B - f(x)$$

in Θ. We can therefore select μ, M in such a way that $M\psi$ is a strict supersolution of the equation in Ω and $Mk\varphi$ supersolution in Θ. Taking into account the order relation between $k\varphi$ and ψ, we deduce that the function

$$u(x) = \min\{Mk\varphi, M\psi\}$$

is a supersolution of the equation vanishing on the boundary. □

Theorem 15.7. *The equation* (96) *satisfies the comparison principle for bounded usc subsolution and lsc supersolution on any bounded open set* Ω.

Proof. We denote by u, v bounded usc subsolution and lsc supersolution, respectively, with $u \leq v$ on the boundary. We argue by contradiction as in Theorem 13.11 and we find, for ε small, a sequence of maximizers of $u^\varepsilon - v_\varepsilon$ contained in the interior of Ω and converging to a maximizer $x_0 \in \Omega$ of $u - v$. We define ψ as in (99) for some x_0 exterior to Ω, arguing as Proposition 15.6 we see that for a suitable choice of μ, we have

$$- \triangle \psi(x) - |D\psi(x)| \geq \delta > 0 \qquad \text{for any } x \in \Omega, \text{ some } \delta > 0, \tag{101}$$

We have that, for $\theta > 0$, ε suitably small, there exists a sequence of maximizers x_ε of

$$u^\varepsilon - v_\varepsilon - \theta\, \psi(x)$$

on $\overline{\Omega}$ belonging to Ω. We can also assume, up to extracting a subsequence, that

$$x_\varepsilon \to \bar{x} \qquad \text{as } \varepsilon \to 0, \text{ for some } \bar{x} \in \Omega.$$

Notice that both u^ε, v_ε are differentiable at x_ε with

$$Du^\varepsilon(x_\varepsilon) = Dv_\varepsilon(x_\varepsilon) + \theta\, D\psi(x_\varepsilon). \tag{102}$$

We freeze ε, and, by applying Jensen Lemma, we find a sequence x_n converging to x_ε where both u^ε and v_ε are twice differentiable with

$$D^2u(x_n) \to X, \quad D^2v(x_n) \to Y$$

and in addition

$$X - Y \leq \theta\, D^2\psi(x_\varepsilon). \tag{103}$$

sending n to infinity, we get

$$- \operatorname{tr}(X - Y) + |Du^\varepsilon(x_\varepsilon)| - f(y_\varepsilon) - |Dv_\varepsilon(x_\varepsilon)| + f(z_\varepsilon) \leq 0,$$

where y_ε, z_ε are u^ε and v_ε–optimal points for x_ε, respectively. We deduce from (102), (103)

$$-\theta \triangle \psi(x_\varepsilon) + |Du^\varepsilon(x_\varepsilon)| - |Du^\varepsilon(x_\varepsilon) - \theta D\psi(x_\varepsilon)| + \big(f(z_\varepsilon) - f(y_\varepsilon)\big) \leq 0$$

and consequently

$$-\theta \triangle \psi(x_\varepsilon) - \theta\, |D\psi(x_\varepsilon)| + \big(f(z_\varepsilon) - f(y_\varepsilon)\big) \leq 0. \tag{104}$$

We reach a contradiction sending ε to 0, because of (102) and the fact that both y_ε and z_ε converge to $\bar{x}$ so that the term $f(z_\varepsilon) - f(y_\varepsilon)$ becomes infinitesimal. $\square$

Thanks to the previous two results, we can conclude by Theorem 14.5

Theorem 15.8. *There exists one and only one continuous solution to* (96) *plus homogeneous boundary conditions on* $\partial\Omega$.

16 Homogenization

We consider a $\mathbb{Z}^N$–periodic Hamiltonian denoted by $H(x,p)$, the periodicity condition means that

$$H(x,p) = H(x+z,p) \qquad \text{for any } x,\, p \text{ in } \mathbb{R}^N,\ z \in \mathbb{Z}^N.$$

The Hamiltonian is consequently defined for x varying in the flat torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, p varying in $\mathbb{R}^N$. We in addition assume H to be

- continuous in both variables;
- convex in the momentum variable;
- superlinear in the momentum variable uniformly with respect to x, namely

$$\lim_{|p|\to+\infty} \min\left\{\frac{H(x,p)}{|p|} \mid x \in \mathbb{T}^N\right\} = +\infty \qquad \text{for any } x. \tag{105}$$

The superlinearity condition allows defining the Lagrangian L via

$$L(x,q) = \max_p p \cdot q - H(x,p) \tag{106}$$

It is straightforward to show that L is convex in q and superlinear.

We consider the equation

$$u_t + H(x,Du) = 0 \qquad \text{in } \mathbb{R}^N \times (0,+\infty) \tag{107}$$

coupled with the initial condition

$$u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^N \tag{108}$$

where u_0 is a given Lipschitz continuous initial datum. We will say that a function u, usc (resp. lsc) in $\mathbb{R}^N \times [0,+\infty)$ is subsolution (resp. supersolution) to (107), (108) if it is subsolution to the equation and satisfies $u(\cdot,0) \leq u_0$ (resp. $u(\cdot,0) \geq u_0$).

We introduce a parameter ε devoted to become infinitesimal and introduce the family of time–dependent problems

$$\begin{cases} \frac{\partial}{\partial t}u_\varepsilon + H(x/\varepsilon, Du_\varepsilon) = 0 & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u_\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{HJε}$$

We aim at proving that the solutions u_ε to (HJε) uniformly converges, as $\varepsilon \to 0$, to the solution of a limit problem of the form

$$\begin{cases} \frac{\partial}{\partial t}u + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{$\overline{\text{HJ}}$}$$

with $\overline{H}$, called the effective Hamiltonian, must be determined. As $\varepsilon \to 0$, the size of the cells of periodicity of H become infinitesimal, and H is therefore highly oscillating. This shows the difficulty in studying this kind of asymptotic problems.

16.1 Time dependent equations and Lax–Oleinik formula

We start by the following comparison result

Proposition 16.1. *Let u, v be subsolution and supersolution to* (107) *Assume u to be usc and v lsc in $\mathbb{R}^N \times [0,+\infty)$, assume in addition that either u or v are Lipschitz continuous. If $u(\cdot,0) \leq v(\cdot,0)$ then $u \leq v$ in $\mathbb{R}^N \times [0,+\infty)$.*

A lemma is preliminary, it shows that the class of viscosity test function can be enlarged for time–dependent equations.

Lemma 16.2. *Given a continuous subsolution u resp. supersolution) to* (107)*, let ψ be a C^1 function and $(x_0,t_0) \in \mathbb{R}^N \times (0,+\infty)$ a local maximizer (resp. minimizer) of $u - \psi$ in $\mathbb{R}^N \times [0,t_0]$, then*

$$\psi_t(x_0,t_0) + H(x, D\psi(x_0,t_0)) \leq 0 \quad \text{(resp. } \geq 0\text{).}$$

Proof. We just treat the subsolution case. Let ψ be as in the statement. We can assume $u - \psi$ to be a strict maximizer on $B := \overline{B((x_0,t_0),r)} \cap (0,t_0]$, for a suitable r. Let (x_n,t_n) be the maximizer on $\overline{B((x,t_0),r)} \cap (0,t_0)$ of

$$u - \psi - \frac{1}{n}\,\frac{1}{t_0 - t}.$$

Let $(y_0,s_0) \in B$ be a limit point of (x_n,t_n), we claim that $s_0 = t_0$, assume on the contrary that $s_0 < t_0$, then

$$u(y_0,s_0) - \psi(y_0,s_0) < u(x_0,t_1) - \psi(x_0,t_1) \tag{109}$$

for a $t_1 < t_0$ suitably close to t_0. We have

$$\lim_n u(x_n,t_n) - \psi(x_n,t_n) - \frac{1}{n}\,\frac{1}{t_0 - t_n} + \frac{1}{n}\,\frac{1}{t_0 - t_1} = u(y_0,s_0) - \psi(y_0,s_0)$$

and we derive from (109)

$$u(x_n,t_n) - \psi(x_n,t_n) - \frac{1}{n}\,\frac{1}{t_0 - t_n} < u(x_0,t_1) - \psi(x_0,t_1) - \frac{1}{n}\,\frac{1}{t_0 - t_1}$$

which is impossible. The claim is therefore proved. We deduce from maximality properties of (x_n,t_n)

$$u(x_n,t_n) - \psi(x_n,t_n) \geq u(x_0,t_n) - \psi(x_0,t_n)$$

and at the limit

$$u(y,t_0) - \psi(y_0,t_0) \geq u(x_0,t_0) - \psi(x_0,t_0)$$

Since (x_0, t_0) is strict maximizer, we deduce that

$$(x_n, t_n) \to (x_0, t_0)$$

from the relations

$$\psi_t(x_n, t_n) + H(x_n, D\psi(x_n, t_n)) \le 0$$

which holds because u is subsolution and ψ supertangent to u at (x_n, t_n), we deduce passing at the limit

$$\psi_t(x_0, t_0) + H(x_0, D\psi(x_0, t_0)) \le 0$$

which concludes the proof. □

Proof. (*of Theorem 16.1*) Comparison results for time dependent equations can be proved as for stationary discounted equations. The key point is that any subsolution u (resp. supersolution) to (107), (108) can be uniformly approximated by strict subsolution (resp. supersolution) of the form $-\delta\, t + u$ (resp. $\delta\, t + u$), for $\delta > 0$.

We first show the comparison in sets of the form $\Omega \times [0, +\infty)$, where Ω is a bounded open subset of $\mathbb{R}^N$. We assume $u \le v$ on the parabolic boundary

$$\Omega \times \{0\} \cup \partial\Omega \times [0, +\infty),$$

the difficulty of operating in a noncompact set can be overcome thanks to Lemma 16.2.

If the subsolution u is Lipschitz continuous, we can adapt the argument of Theorem 11.4 to get the comparison in $\Omega \times [0, +\infty)$. If instead the supersolution v is Lipschitz continuous, the proof is slightly more complicated, and we need resorting to some regularization of v. In this case, we follow the argument of Theorem 11.6.

After having obtained in this way a local comparison result, we follow the argument of Theorem 11.7 to get the comparison in the whole of $\mathbb{R}^N \times (0, +\infty)$ □

We define

$$w(x) = \sup\{u(x) \mid u \text{ Lip. subsol. in } \mathbb{R}^N \times (0, +\infty) \text{ to (107), (108)}\}, \tag{110}$$

we further define

$$M = \max\{|H(x,p)| \mid x \in \mathbb{R}^N,\ |p| \le |Du_0|_\infty\} \tag{111}$$

$$R = \min\left\{T \mid \min_{|p| \ge T} H(x,p) \ge M + 1\right\} \tag{112}$$

We show:

Proposition 16.3. *The definition of w is well posed. The function w is in addition Lipschitz continuous with*

$$|w_t|_\infty \leq M \qquad \text{and} \qquad |Dw|_\infty \leq R, \tag{113}$$

where M, R are defined as in (111), (112)*, and is the unique solution to* (107), (108).

Proof. The function $\underline{u} = u_0 - Mt$ satisfies

$$-M + H(x,p) \leq 0$$

for any point x, $p \in \partial u_0(x)$, which shows that $\underline{u}$ is a Lipschitz continuous subsolution to (107). The set appearing in the definition of w is therefore nonempty. Arguing along the same lines, we also show that $\overline{u} = u_0 + Mt$ is supersolution to (107), and deduce, in force of Proposition 16.1

$$u \leq \overline{u} \qquad \text{for any subsolution to (107).} \tag{114}$$

We conclude that the supremum in the definition of w must be finite, showing that the definition of w is well posed. By (114) and the very definition of w, we further get

$$\underline{u} \leq w \leq \overline{u} \qquad \text{in } \mathbb{R}^N \times [0,+\infty),$$

which in turn implies $w(\cdot,0) = u_0$ and

$$-M\,t \leq w(x,t) - u_0(x) \leq M\,t \qquad \text{for any } x \in \mathbb{R}^N\ t \in [0,+\infty).$$

We fix a time $h > 0$ and set

$$C = |w(\cdot,h) - u_0|_\infty \leq M\,h \tag{115}$$

so that

$$u_0 - C \leq w(\cdot,h) \leq u_0 + C. \tag{116}$$

We see that $w \pm C$, $w(\cdot,\cdot+h)$ are defined via formula (110) with $u_0 \pm C$, $w(\cdot,h)$ in place of u_0. This implies by (116)

$$w(x,t) - C \leq w(x,t+h) \leq w(x,t) + C \quad \text{for any } x,\ t,$$

and consequently, according to (115)

$$|w(x,t+h) - w(x,t)| \leq C \leq M\,h \quad \text{for any } x,\ t,\ h. \tag{117}$$

We claim that this estimate is inherited by the usc envelope $w^\#$. To show this, we fix x, t, h, and consider a sequence (x_n,t_n) with $\lim_n w(x_n,t_n) = w^\#(x,t)$. We then have in the light of(117), up to extracting subsequences

$$w^\#(x,t) - w^\#(x,t+h) \leq \lim_n w(x_n,t_n) - w(x_n,t_n+h) \leq Mh$$

and arguing similarly we also get

$$w^{\#}(x,t+h) - w^{\#}(x,t) \leq Mh,$$

showing the claim. Taking into account that by Theorem 14.3 $w^{\#}$ is subsolution to (107), we deduce

$$-M + H(x, Dw^{\#}) \leq w_t^{\#} + H(x, Dw^{\#}) \leq 0.$$

This in turn implies that the functions $x \mapsto w^{\#}(x,t)$ are equiLipschitz continuous with $|Dw^{\#}(x,t)| \leq R$ for t varying in $(0,+\infty]$ which together with (117) gives that $w^{\#}$ is Lipschitz continuous in both variables and satisfies the estimates in the statement.

By the pull–up method and the comparison result, we easily show that w is the unique solution solution to (107), (108). This completes the proof. □

We write down the Lax–Oleinik formula

$$v(x,t) = \inf \left\{ u_0(\xi(0)) + \int_0^t L(\xi, \dot{\xi})\, ds \mid \text{ curves } \xi \text{ with } \xi(t) = x \right\}, \tag{118}$$

where L is the Lagrangian defined in (106).

Proposition 16.4. *The function v given by the Lax–Oleinik formula* (118) *is a subsolution to* (107), (108) *in the discontinuous sense.*

Proof. We fix (x_0, t_0) and assume that there is a C^1 supertangent ψ to u at (x_0, t_0). We consider a sequence (x_n, t_n) converging to (x_0, t_0) with $\lim_n u(x_n, t_n) = u^{\#}(x_0, t_0)$. The argument being local, we can assume ψ uniformly continuous with continuity modulus ω. We have

$$u^{\#}(x_0, t_0) = \psi(x_0, t_0) \leq u(x_n, t_n) + |u^{\#}(x_0, t_0) - u(x_n, t_n)|$$

and for any $q \in \mathbb{R}^N$, any positive sequence h_n

$$\psi(x_n + h_n q, t_n - h_n) \leq \psi(x_0 + h_n q, t_0 - h_n) + \omega(|x_n - x_0| + |t_n - t_0|).$$

We deduce

$$\begin{aligned} \psi(x_0, t_0) - \psi(x_0 + h_n q, t_0 - h_n) &\leq u(x_n, t_n) - \psi(x_n + h_n q, t_n - h_n) + \varepsilon_n \\ &\leq u(x_n, t_n) - u(x_n + h_n q, t_n - h_n) + \varepsilon_n. \end{aligned}$$

where

$$\varepsilon_n = \omega(|x_n - x_0| + |t_n - t_0|) + |u^{\#}(x_0, t_0) - u(x_n, t_n)|$$

We assume h_n to satisfy $\lim_n \frac{\varepsilon_n}{h_n} = 0$. We compute

$$\begin{aligned}
\frac{\psi(x_0,t_0) - \psi(x_0 + h_n q, t_0 - h_n)}{h_n} &\le \frac{u(x_n,t_n) - u(x_n + h_n q, t_n - h_n)}{h_n} + \frac{\varepsilon_n}{h_n} \\
&\le \int_{t_n - h_n}^{t_n} L(x_n + (t_n - s)q, -q)\, ds + \frac{\varepsilon_n}{h_n}
\end{aligned}$$

Sending n to infinity, we get

$$\psi_t(x_0,t_0) - D\psi(x_0,t_0) \cdot q \le -L(x_0,q),$$

and letting q vary in $\mathbb{R}^N$

$$\psi_t(x_0,t_0) + H(x_0, D\psi(x_0,t_0)) \le 0.$$

This concludes the proof □

Theorem 16.5. *The Lax–Oleinik formula* (118) *provides the solution v to* (107), (108)*. The function v therefore coincides with w, see Proposition 16.3. It is Lipschitz continuous and satisfies the estimates* (113)*.*

Proof. By Proposition 16.1

$$w \ge v. \tag{119}$$

Let u be any Lipschitz continuous subsolution to (107), (108). Given $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, $x_0 \in \mathbb{R}^N$, we consider the curve $(r, \xi(r))$ with $\xi(0) = x_0$, $\xi(t) = x$, we have

$$w(x,t) - u_0(x_0) = \int_0^t s(r) + p(r) \cdot \dot{\xi}(r)\, dr$$

for suitable $(s(r), p(r)) \in \partial u(\xi(r), r)$. We further have

$$p(r) \cdot \dot{\xi}(r) \le L(\xi(r), \dot{\xi}(r)) + H(\xi(r), \dot{\xi}(r))$$

so that we get, exploiting the subsolution property of u

$$\begin{aligned}
[w(x,t) - u_0(x_0) &\le \int_0^t s(r) + L(\xi(r), r) + H(\xi(r), \dot{}xi(r)) + H(\xi(r), \dot{\xi}(r))\, dr \\
&\le \int_0^t L(\xi(r), r)\ dr
\end{aligned}$$

Due to the arbitrary choice of the curve ξ, we derive

$$u \le v \qquad \text{for any Lipschitz continuous subsolution } u \text{ to (107).(108)} \tag{120}$$

The inequalities (119), (120) yield $w = v$, which proves the assertion □

Remark 16.6. To get comparison result for (107) plus Lax-Oleinik representation formula for the solution, it is enough to assume the initial datum u_0 bounded uniformly continuous in $\mathbb{R}^N$.

If u_0 satisfies this condition then it is uniformly approximated in $\mathbb{R}^N$ by a sequence of Lipschitz continuous functions, say u_n^0, thanks to Corollary 7.12. It is immediate to check that the functions v_n defined as in (118),with u_n^0 in place of u_0, uniformly converge to v. By stability properties of viscosity solutions, the Lax–Oleinik formula yields a solution to (107), (108) also for u_0 bounded uniformly continuous.

To show the comparison result, we can assume the sub/supersolution of (107), (108) to be compared, say u, v, satisfy

$$v(\cdot,0) - u(\cdot,0) > \varepsilon \qquad \text{for some } \varepsilon > 0.$$

We can therefore find a Lipschitz continuous function w_0 with

$$v(\cdot,0) > w_0 > u(\cdot,0) \qquad \text{in } \mathbb{R}^N.$$

The solution w of (107) with initial datum w_0 is Lipschitz continuous, and we get by Proposition 16.1

$$v \geq w \geq u \qquad \text{in } \mathbb{R}^N \times [0,+\infty).$$

We conclude by sending ε to 0.

16.2 Effective Hamiltonian

To define the Hamiltonian appearing in the limit problem ($\overline{\text{HJ}}$), we start introducing the one–parameter family of Eikonal equations

$$H(x,Du) = a \qquad a \in \mathbb{R}. \tag{121}$$

We look for $\mathbb{Z}^N$–periodic sub/solutions of it, or equivalently, we set the equation in T^N. It is convenient to define an intrinsic distance in $\mathbb{T}^N$ related to H and to the constant a. This is done as in Section 10 starting from a length functional, denoted by ℓ_a defined as in (35) using σ_a, the support function of a–sublevels

$$Z_a(x) = \{p \in \mathbb{R}^N \mid H(x,p) \leq a\}.$$

In order this definition to make sense, the $Z_a(x)$ must be nonempty for any x, this amounts requiring

$$a \geq \max_x \min_p H(x,p). \tag{122}$$

We define the (semi)distance S_a as in (39), with ℓ_a in place of ℓ_H. Even if (122) holds true, it is possible that $S_a \equiv -\infty$. Arguing as in Theorem 10.4, we get

Proposition 16.7. *$S_a > -\infty$ if and only if the equation* (121) *admits subsolutions in $\mathbb{T}^N$, or equivalently periodic subsolution.*

Due to convexity and coercivity assumption on H, any subsolution is Lipschitz–continuous. Slightly adapting the argument of Theorem 10.5, we also get

Proposition 16.8. *Let a such that $S_a > -\infty$, then for any y the function*

$$x \mapsto S_a(y, x)$$

is subsolution to (121) *in $\mathbb{T}^N$ and solution in $\mathbb{T}^N \setminus \{y\}$.*

We define

$$c = \inf\{a \mid H = a \text{ admits periodic subsolutions}\}.$$

The definition is well posed because the set of the a appearing in the formula is nonempty. In fact if $a \geq \max_{x \in \mathbb{T}^N} H(x, 0)$, then any constant function is subsolution to (121). In addition c is finite because it must be greater than or equal to the value in (122). It is finally a minimum thanks to the stability property of viscosity subsolutions. We have

Proposition 16.9. *The equation* (121) *cannot have solution in $\mathbb{T}^N$, or equivalently periodic solution, if $a > c$.*

Proof. We argue by contradiction. If v is a solution in $\mathbb{T}^N$ of $H = a$ for $a > c$ and u a subsolution of $H = c$ then u is subtangent to v at any minimizer x_0 in $\mathbb{T}^N$ of $v - u$. By Proposition 8.5

$$H(x, q) \geq a > c \qquad \text{for some } q \in \partial u(x_0),$$

on the other side by the convex character of H, we also have

$$H(x, p) \leq c \qquad \text{for any } p \in \partial u(x_0).$$

This is impossible. □

Theorem 16.10. *There are solutions in $\mathbb{T}^N$, equivalently periodic solutions, to $H = c$.*

Proof. The argument is by contradiction. We set for any $y \in \mathbb{T}^N$

$$u_y = S_c(y, \cdot).$$

If there are no solutions to $H = c$, then, by Proposition 16.8 the functions u_y do not satisfy the supersolution condition at y, for every y. Therefore there exists, for every y, a strict subtangent ψ_y to u_y at y with $\psi_y(y) = u_y(y) = 0$ such that

$$H(y, D\psi_y(y)) < c.$$

By using the usual pull-up method, we can define a subsolution w_y of $H = c$ which is C^1 in some ball B_y centered at y and satisfies

$$H(x, Dw_y(x)) < c - \delta_y \qquad \text{in } B_y, \text{ for some } \delta_y > 0$$

Due to $\mathbb{T}^N$ being compact, we can extract a finite subcover $\{B_{y_i}\} =: \{B_i\}$, for $i = 1, \cdots, M$, of $\{B_i\}$, we set $w_i = w_{y_i}$, $\delta_i = \delta_{y_i}$. We select positive constants $\lambda_1, \cdots \lambda_M$, satisfying $\sum_i \lambda_i = 1$, and define

$$w := \sum_i \lambda_i \, w_i.$$

We further set

$$\delta = \min_i \lambda_i \, \delta_i.$$

Let x be a differentiability point of all w_i, and consequently of w, we know that $x \in B_j$, for some $j \in \{1, \cdots, M\}$, then we have

$$\begin{aligned} H(x, Dw(x)) &\leq \sum_i \lambda_i \, H(x, Dw_i(x)) \\ &= \sum_{i \neq j} \lambda_i \, H(x, Dw_i(x)) + \lambda_j \, H(x, Dw_j(x)) \\ &\leq c - \lambda_j \delta_j \leq c - \delta. \end{aligned}$$

This shows that w is a.e subsolution, or equivalently viscosity subsolution, of $H = c - \delta$ in $\mathbb{T}^N$, which is in contrast with the very definition of c. We have therefore reached a contradiction. □

For any $q \in \mathbb{R}^N$, we consider the Hamiltonian

$$(x, p) \mapsto H(x, q + p)$$

and the family of equations

$$H(x, Du + q) = a \qquad \text{for } a \in \mathbb{R}.$$

We define

$$\overline{H}(q) = \min\{a \mid H(x, Du + q) = a \text{ admits period. subsolutions}\}$$

Accordingly, the value c, we have previously considered, is nothing but $\overline{H}(0)$. Owing to Theorem 16.10, the equation

$$H(x, Du + q) = \overline{H}(q) \tag{123}$$

admits solutions in $\mathbb{T}^N$, or equivalently periodic solutions, for any $q \in \mathbb{R}^N$. These functions will play the role of correctors in the homogenization procedure we will describe below.

Proposition 16.11. *the function $q \mapsto \overline{H}(q)$ is convex and coercive.*

Proof. Given q_1, q_2 in $\mathbb{R}^N$, we denote by v_1, v_2 solutions to (123) with q_1, q_2 in place of q, respectively. Given $\lambda \in [0,1]$ and a differentiability point x for both v_1, v_2, we have

$$\begin{aligned} & H(x,(1-\lambda)Dv_1(x)+\lambda Dv_2(x)+(1-\lambda)q_1+\lambda q_2) \\ \leq \; & (1-\lambda)\, H(x,Dv_1(x)+q_1)+\lambda\, H(x,Dv_2(x)+q_2) \\ \leq \; & (1-\lambda)\, \overline{H}(q_1)+\lambda\, \overline{H}(q_2). \end{aligned}$$

This shows that $(1-\lambda)v_1+\lambda\, v_2$ is subsolution to

$$H(x,Du++(1-\lambda)q_1+\lambda q_2)=(1-\lambda)\, \overline{H}(q_1)+\lambda\, \overline{H}(q_2),$$

which in turn implies by the very definition of $\overline{H}$

$$(1-\lambda)\, \overline{H}(q_1)+\lambda\, \overline{H}(q_2) \geq \overline{H}((1-\lambda)q_1 \lambda\, q_2)$$

showing the claimed convex character of $\overline{H}$.

Given $q \in \mathbb{R}^N$, if u is a subsolution of $H(x,Du+q)=\overline{H}(q)$ in $\mathbb{T}^N$, the null function is supertangent at any of its maximizers, and consequently

$$\min_x H(x,q) \leq \overline{H}(q).$$

This shows that $\overline{H}$ inherits the coercivity of H, see (105).

□

16.3 Asymptotic

Theorem 16.12. *The sequence u_ε of solutions to the approximated problems* (HJε) *locally uniformly converges to the solution u of* ($\overline{\mathrm{HJ}}$).

Proof. Thanks to Proposition 16.3, the u_ε are equiLipschitz continuous and locally equibounded, they consequently locally uniformly converge to some Lipschitz continuous function u, up to subsequences. We still indicate by u_ε a converging subsequence.

Our task is to show that u is subsolution to ($\overline{\mathrm{HJ}}$). The proof for the supersolution property goes along the same lines. We preliminarily consider a solution w to the cell problem

$$H(x,Dv+D(\psi(x_0,t_0)))=\overline{H}(D(\psi(x_0,t_0))),$$

this implies that $\varepsilon\, w(x/\varepsilon)$ solves

$$H(x/\varepsilon,Dv+D(\psi(x_0,t_0)))=\overline{H}(D(\psi(x_0,t_0))). \tag{124}$$

Let $(x_0, t_0) \in \mathbb{R}^N \times (0, +\infty)$ and ψ a C^1 strict supertangent to u at (x_0, t_0). The point (x_0, t_0) is the unique maximizer of $u - \psi$ in $\overline{U}$, for a suitable choice of an open neighborhood U of (x_0, t_0) in $\mathbb{R}^N \times (0, +\infty)$. We proceed considering the functions

$$u_\varepsilon(x,t) - \psi(x,t) - \varepsilon\, w_{\varepsilon^2}(x/\varepsilon). \tag{125}$$

Here u_ε indicates the solution to (HJε), while w_ε stands for the ε^2 inf–convolution of w. Since the inf–convolution w_{ε^2} is equibounded with respect to ε by Proposition 7.4, the quantity $\varepsilon\, w_{\varepsilon^2}(x/\varepsilon)$ converges uniformly to 0 as $\varepsilon \to 0$. Owing to the stability properties of maximizers under uniform convergence, we deduce that a sequence of maximizers of (125) in $\overline{\Theta}$, denoted by $(x_\varepsilon, t_\varepsilon)$, belongs to Θ for ε small enough and

$$(x_\varepsilon, t_\varepsilon) \to (x_0, t_0) \qquad \text{as } \varepsilon \to 0. \tag{126}$$

Since

$$(\psi_t(x_\varepsilon, t_\varepsilon), D\psi(x_\varepsilon, t_\varepsilon) + p_\varepsilon) \in D^+ u(x_\varepsilon, t_\varepsilon)$$

for any $p_\varepsilon \in \partial w_{\varepsilon^2}(x_\varepsilon) = D^+ w_{\varepsilon^2}(x_\varepsilon)$, and u_ε is subsolution to (HJε), we derive

$$\psi_t(x_\varepsilon, t_\varepsilon) + H(x/\varepsilon, D\psi(x_\varepsilon, t_\varepsilon) \le 0 \quad \text{for any } p_\varepsilon \in \partial w_{\varepsilon^2}(x_\varepsilon). \tag{127}$$

Since w_{ε^2} is inf–convolution of w, and w solves the cell problem (124), we further have by Corollary 7.18

$$H(y/\varepsilon, Dw_{\varepsilon^2}(x) + D\psi(x_0, t_0)) \ge \overline{H}(D\psi(x_0, t_0))$$

at any point x where w_{ε^2} is differentiable, with y denoting the w_{ε^2}–optimal point for x. Consequently

$$H(y_\varepsilon/\varepsilon, q_\varepsilon + D\psi(x_0, t_0)) \ge \overline{H}(D\psi(x_0, t_0)) \quad \text{for some } q_\varepsilon \in \partial w_{\varepsilon^2}(x_\varepsilon), \tag{128}$$

where y_ε is u_{ε^2}–optimal for x_ε. In view of combining (127), (128), we need to estimate

$$|H(x/\varepsilon, D\psi(x_\varepsilon, t_\varepsilon) + q_\varepsilon) - H(y/\varepsilon, D\psi(x_0, t_0) + q_\varepsilon)|. \tag{129}$$

The state variable varies in a compact space because of the periodic character of H, $D\psi(x,t)$ is bounded in U because ψ is of class C^1, and finally $|q_\varepsilon|$ is bounded by the Lipschitz constant of w_{ε^2} which is equal to that of w in force of Corollary 7.17. Consequently there exist a modulus ω such that the quantity in (129) is estimated from above by

$$\begin{aligned} & \omega\left(\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} + |D\psi(x_0, t_0) - D\psi(x_\varepsilon, t_\varepsilon)|\right) \\ \le\ & \omega\left(\frac{\mathrm{O}(\varepsilon^2)}{\varepsilon} + |D\psi(x_0, t_0) - D\psi(x_\varepsilon, t_\varepsilon)|\right) =: \eta_\varepsilon \end{aligned}$$

with

$$\eta_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0. \tag{130}$$

Note that, to estimate $|x_\varepsilon - y_\varepsilon|$ we have employed Proposition 7.13. Using (127) and (128) plus the above inequality, we therefore obtain

$$\begin{aligned} & \psi_t(x_0,t_0) + \overline{H}(D\psi(x_0,t_0)) \le \psi_t(x_0,t_0) + H(y_\varepsilon/\varepsilon, q_\varepsilon + D\psi(x_0,t_0)) \\ \le \;& \psi_t(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon/\varepsilon, q_\varepsilon + D\psi(x_\varepsilon,t_\varepsilon)) + \eta_\varepsilon + |\psi_t(x_0,t_0) - \psi_t(x_\varepsilon,t_\varepsilon)| \\ \le \;& \eta_\varepsilon + |\psi_t(x_0,t_0) - \psi_t(x_\varepsilon,t_\varepsilon)|. \end{aligned}$$

Sending ε to 0, using (126), (130) and ψ being of class C^1, we show the claimed subsolution property for u.

We have therefore proved that all limits of subsequences of u_ε solve ($\overline{\text{HJ}}$). The solution to ($\overline{\text{HJ}}$) being unique, we deduce that the whole sequence u_ε converge to the solution of ($\overline{\text{HJ}}$), as was asserted. □

Remark 16.13. The homogenization result holds true for bounded uniformly continuous initial data. In this case we do not have any more that the solutions u_ε are equiLipschitz continuous, not even Lipschitz continuous in $\mathbb{R}^N \times [0, +\infty)$.

By taking $M = \max_x H(x, 0)$, and using the comparison result, we nevertheless get

$$\inf u_0 - M\, t \le u_\varepsilon \le \sup u_0 + M\, t$$

and we deduce that the u_ε are locally equibounded. We show, arguing as in Theorem 16.12, that the upper semilimit of u_ε is a subsolution to ($\overline{\text{HJ}}$), and the lower semilimit a supersolution. We get local uniform convergence of u_ε to the solution of ($\overline{\text{HJ}}$) exploiting the comparison result for ($\overline{\text{HJ}}$), and the fact that the upper semilimit is greater than or equal to the lower semilimit.

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