FISICA MATEMATICA
Prova scritta del 20 febbraio 2018

Cognome, nome, matricola ....

Esercizio 1. Determinare la soluzione del problema di Cauchy

$$\begin{cases} \partial_t u(x,v,t) + 2\,\partial_x u(x,v,t) - e^t\,\partial_v[v\,u(x,v,t)] = 0, \\ u(x,v,1) = x. \end{cases}$$

Sia A_t un insieme tale che $|A_1| = 2$. Determinare $|A_0|$.

$$\begin{cases} \dot{x} = 2 \\ \dot{v} = -v e^t \end{cases} \quad \begin{matrix} x(t_0) = x_0 \\ v(t_0) = v_0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} x(t) = x_0 + 2(t-t_0) \\ v(t) = v_0\, e^{-[e^t - e^{t_0}]} \end{matrix} \quad \varphi^{t,t_0}_{(x,v)} = \begin{pmatrix} x + 2(t-t_0) \\ v e^{-(e^t - e^{t_0})} \end{pmatrix}$$

$$\operatorname{div} F = -e^t$$

- $u(x,v,t) = [x + 2(1-t)]\, e^{\int_1^t ds\, e^s} = (x - 2t + 2)\, e^{(e^t - e)}$

- $\frac{d}{dt}|A_t| = -|A_t|\, e^t \qquad |A_t| = |A_{t_0}|\, e^{-(e^t - e^{t_0})}$

$$|A_t| = 2 e^{-(e^t - e)} \qquad |A_0| = 2 e^{-(1-e)}$$

Esercizio 2. Sia $\Omega = (0,1)\times(0,1)$. Determinare la soluzione del problema di Cauchy-Dirichlet omogeneo

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) = 2\Delta u(x,t) & (x,t)\in\Omega\times\mathbb{R}^+ \\ u(x,0) = \sin(3\pi x_1)\sin(3\pi x_2), \quad \partial_t u(x,0) = h(x) & \forall x\in\Omega \\ u(x,t) = 0 \quad \forall x\in\partial\Omega,\ \forall t\geq 0, \end{cases}$$

per (i) $h=0$; (ii) $h(x) = \sin^2(\pi x_1)\sin(2\pi x_2)$.

(i) $h=0$

$u(x,t) = \sin(3\pi x_1)\sin(3\pi x_2)\cos(\omega_{3,3}t)$

dove $\omega_{3,3} = \sqrt{2}\,\pi\sqrt{9+9} = \sqrt{2}\,\pi\sqrt{18} = \sqrt{2}\,\pi\cdot 3\sqrt{2} = 6\pi$

$18 = 9\cdot 2$

$u(x,t) = \sin(3\pi x_1)\sin(3\pi x_2)\cos(6\pi t)$

(ii) $h(x) = \sin^2(\pi x_1)\sin(2\pi x_2)$

$u(x,t) = \sin(3\pi x_1)\sin(3\pi x_2)\cos(6\pi t) + \sum_{n\geq 1}\sin(n\pi x_1)\sin(2\pi x_2)\,B_n \sin\left(\sqrt{2}\,\pi\sqrt{4+n^2}\,t\right)$

$\sqrt{2}\,\pi\sqrt{4+n^2}\,B_n = 2\int_0^1 dy\,\sin^2(\pi y)\sin(n\pi y)$

$= -\frac{1}{2}\frac{1}{\pi n}\left[(-1)^n - 1\right] + \frac{1}{4}\frac{1}{(n-2)\pi}\left[(-1)^n - 1\right] + \frac{1}{4}\frac{1}{(n+2)\pi}\left[(-1)^n-1\right]$

$\frac{1}{\pi n} - \frac{1}{2}\frac{1}{\pi}\frac{1}{(n-2)} - \frac{1}{2}\frac{1}{\pi}\frac{1}{(n+2)} = \frac{1}{\pi n} - \frac{1}{2}\frac{1}{\pi}\frac{2n}{(n^2-4)} = \frac{1}{\pi}\frac{[n^2-4-n^2]}{n(n^2-4)} = -\frac{4}{\pi}\frac{1}{n(n^2-4)}$

$B_n = \begin{cases} 0 & \text{se } n \text{ è pari} \\ -\frac{2}{\sqrt{2}\,\pi}\frac{1}{\sqrt{n^2+4}}\frac{4}{\pi n(n^2-4)} & \text{se } n \text{ è dispari} \end{cases}$

$u(x,t) = \sin(3\pi x_1)\sin(3\pi x_2)\cos(6\pi t) - \sum_{n\geq 1,\ n\ \text{dispari}} \frac{8}{n\pi(n^2-4)}\frac{1}{\sqrt{2}\,\pi\sqrt{n^2+4}}\sin(n\pi x_1)\sin(2\pi x_2)\cdot\sin\left(\sqrt{2}\sqrt{n^2+4}\,\pi t\right)$

Esercizio 3. Dato il problema di Cauchy

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) = 2\Delta u(x,t) & (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \\ u(x,0) = (1 - x_1|x|^2)e^{-|x|^2}, \partial_t u(x,0) = 0 & \forall x \in \mathbb{R}^3, \end{cases}$$

determinare la soluzione per $x = 0, t > 0$.

$$u(0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(0)} d\sigma(y)\,(1 - y_1|y|^2)e^{-|y|^2}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{(ct)^2}{4\pi c^2 t}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \,\sin\vartheta\,\left(1 - (ct)^3 \cos\varphi \sin\vartheta\right)e^{-(ct)^2}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{(ct)^2}{\cancel{4\pi c^2} t}(\cancel{4\pi})\,e^{-(ct)^2}\right] = \frac{\partial}{\partial t}\left[t\,e^{-(ct)^2}\right]$$

$$= (1 - 2(ct)^2)\,e^{-(ct)^2}$$

$$\cancel{u(t,0)} \quad u(0,t) = (1 - 4t^2)\,e^{-2t^2}$$

Esercizio 4. Sia Ω la corona circolare $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$. Risolvere l'equazione di Poisson

$$\begin{cases} \Delta u + (1 - |x|^2) = 0 & \forall x \in \Omega \\ u(x) = x_1 & \forall x \in \partial\Omega. \end{cases}$$

$$u(x) = r\cos\vartheta + \tilde{u}(x)$$

$\frac{1}{r}\partial_r(r\,u'(r)) = r^2 - 1$ $\qquad r\,u'(r) = v(r)$

$v'(r) = r^3 - r$ $\qquad v(r) = \frac{r^4}{4} - \frac{r^2}{2} + C$ $\qquad u'(r) = \frac{r^3}{4} - \frac{r}{2} + \frac{C}{r}$

$u(r) = \frac{r^4}{16} - \frac{r^2}{4} + C\ln r + C_1$

$u(r) = \frac{1}{16}(r^4 - 1) - \frac{1}{4}(r^2 - 1) + C\ln 2$

$\frac{1}{16}[$ $\qquad \frac{15}{16} - \frac{3}{4} + C\ln 2 = 0$ $\qquad \frac{15-12}{16} = \frac{3}{16}$ $\qquad C\ln 2 = -\frac{3}{16}$

$C_0 = -\frac{3}{16}\frac{1}{\ln 2}$

$u(r) = \frac{1}{16}(r^4 - 1) - \frac{1}{4}(r^2 - 1) - \frac{3}{16}\frac{\ln r}{\ln 2}$

$u(2) = \frac{1}{16}[16 - 1 - 16 + 4 - 3] = 0$ ✓

$$u(r) = r\cos\vartheta + \frac{1}{16}(r^4 - 1) - \frac{1}{4}(r^2 - 1) - \frac{3}{16}\frac{\ln r}{\ln 2}$$

$$u(x) = x_1 + \frac{1}{16}(r^4 - 1) - \frac{1}{4}(r^2 - 1) - \frac{3}{16}\frac{\ln r}{\ln 2}$$

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Esercizio 1. Determinare la soluzione del problema di Cauchy

$$\begin{cases} \partial_t u(x,v,t) + vt\,\partial_x u(x,v,t) - 2t\,\partial_v[v\,u(x,v,t)] = 0, \\ u(x,v,1) = 2x + v. \end{cases}$$

Sia A_t un insieme tale che $|A_1| = 2$. Determinare $|A_0|$.

$$\begin{cases} \dot{x} = vt & x(t_0) = x_0 \\ \dot{v} = -2vt & v(t_0) = v_0 \end{cases} \qquad \begin{aligned} x(t) &= x_0 - \frac{v_0}{2}\left[e^{-(t^2-t_0^2)} - 1\right] \\ v(t) &= v_0\, e^{-(t^2-t_0^2)} \end{aligned}$$

$$\varphi^{t,t_0}(x,v) = \begin{pmatrix} x - \frac{1}{2} v\left[e^{-(t^2-t_0^2)} - 1\right] \\ v e^{-(t^2-t_0^2)} \end{pmatrix}$$

$$\operatorname{div} F = -2t$$

- $u(x,v,t) = (2x+v)\, e^{t^2-1}$

- $\frac{d}{dt}|A_t| = -2t\,|A_t| \qquad |A_t| = |A_{t_0}|\, e^{-(t^2-t_0^2)}$

$$|A_t| = 2e^{-t^2+1} = 2e^{1-t^2} \qquad |A_0| = 2e$$

Esercizio 2. Sia $\Omega = (0,1)\times(0,1)$. Determinare la soluzione del problema di Cauchy-Dirichlet omogeneo

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) = 3\Delta u(x,t) \quad (x,t)\in\Omega\times\mathbb{R}^+ \\ u(x,0) = \sin(2\pi x_1)\sin(3\pi x_2), \quad \partial_t u(x,0) = h(x) \quad \forall x\in\Omega \\ u(x,t) = 0 \quad \forall x \in \partial\Omega,\ \forall t\geq 0, \end{cases}$$

per (i) $h = 0$; (ii) $h(x) = (1-x_1)\sin(\pi x_1)\sin(2\pi x_2)$.

$$u(x,t) = \sum_{n,m\geq 1} \sin(n\pi x_1)\sin(m\pi x_2)\, W_{n,m}(t)$$

(i) $u(x,t) = \sin(2\pi x_1)\sin(3\pi x_2)\cos(\sqrt{3}\sqrt{13}\,\pi t)$

(ii) $u(x,t) = \sin(2\pi x_1)\sin(3\pi x_2)\cos(\sqrt{3}\sqrt{13}\,\pi t)$

$$+ \sum_{n\geq 1} \sin(n\pi x_1)\sin(2\pi x_2)\, \cancel{\sqrt{3}\pi\sqrt{4+n^2}}\, B_n \sin(\sqrt{3}\sqrt{4+n^2}\,\pi t)$$

con $B_n = \frac{1}{\pi\sqrt{3}}\frac{1}{\sqrt{4+n^2}}\, 2\int_0^1 dy\,(1-y)\sin(\pi y)\sin(n\pi y)$

$$\int_0^1 dy\, y\sin(k\pi y)\sin(m\pi y) = \int_0^1 dy\, y\,\frac{1}{2}\left[\cos((m-1)\pi y) - \cos((m+1)\pi y)\right]$$

$$= \frac{1}{2}\left\{-\frac{1}{(m-1)}\frac{1}{\pi}\int_0^1 dy\,\sin((m-1)\pi y) + \frac{1}{(m+1)}\frac{1}{\pi}\int_0^1 dy\,\sin((m+1)\pi y)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\pi^2}\frac{1}{(m-1)^2}\left[\cos((m-1)\pi)-1\right] - \frac{1}{\pi^2}\frac{1}{(m+1)^2}\left[\cos((m+1)\pi)-1\right]\right\} \quad m\neq 1$$

$$= \begin{cases} 0 & \text{per } m \text{ dispari},\ m\neq 1 \\ \frac{1}{2}\frac{1}{\pi^2}\left\{-\frac{2}{(m-1)^2}+\frac{2}{(m+1)^2}\right\} = -\frac{1}{\pi^2}\left\{\frac{1}{(m-1)^2}-\frac{1}{(m+1)^2}\right\} = -\frac{1}{\pi^2}\frac{4m}{(m^2-1)^2} \end{cases}$$

$B_m = 0$ per m dispari $\neq 1$

$B_m = +\frac{1}{\pi^2}\frac{8m}{(m^2-1)^2}\frac{1}{\pi\sqrt{3}}\frac{1}{\sqrt{4+m^2}}$ per m pari

$B_1 = \frac{1}{\pi}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{5}}\, 2\cdot\left\{\frac{1}{2}-\frac{1}{4}\right\} = \frac{1}{\pi}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{5}}\cdot\frac{1}{2}$

Esercizio 3. Dato il problema di Cauchy

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) = 3\Delta u(x,t) & (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \\ u(x,0) = (1 + x_3|x|^2)e^{-|x|^2}, \partial_t u(x,0) = 0 & \forall x \in \mathbb{R}^3, \end{cases}$$

determinare la soluzione per $x = 0, t > 0$.

$$u(0,t) = \frac{\partial}{\partial t}\left\{ \frac{1}{4\pi t c^2} \int_{B_{ct}(0)} d\sigma(y) \left(1 + y_3 |y|^2\right) e^{-|y|^2} \right\}$$

$$= \frac{\partial}{\partial t}\left\{ \frac{1}{4\pi t c^2} (ct)^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \left(1 + (ct)^3 \cancel{\cos\vartheta}\right) e^{-(ct)^2} \right\}$$

$$= \frac{\partial}{\partial t}\left\{ t\, e^{-(ct)^2} \right\} = e^{-(ct)^2} - 2c^2t^2 e^{-(ct)^2} = \left(1 - 2(ct)^2\right) e^{-c^2t^2}$$

$$= (1 - 6t^2)\, e^{-3t^2}$$

Esercizio 4. Sia Ω la corona circolare $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$. Risolvere l'equazione di Poisson

$$\begin{cases} \Delta u + (|x|^2 + 1) = 0 & \forall x \in \Omega \\ u(x) = x_2 & \forall x \in \partial\Omega. \end{cases}$$

$\tilde{u}_0 (r, \vartheta) = r \sin \vartheta$

$u(r) = -\frac{1}{4}(r^2 - 1) - \frac{1}{16}(r^4 - 1) + \frac{27}{16} \frac{\ln r}{\ln 2}$

$u(x) = x_2 - \frac{1}{4}(|x|^2 - 1) - \frac{1}{16}(|x|^4 - 1) + \frac{27}{16} \frac{\ln |x|}{\ln 2}$