Chapter 3. Absolutely Continuous Functions

§1. Absolutely Continuous Functions

A function $f : [a,b] \to \mathbb{R}$ is said to be **absolutely continuous** on $[a,b]$ if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,$$

whenever $\{[x_i, y_i] : i = 1, \ldots, n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

Clearly, an absolutely continuous function on $[a,b]$ is uniformly continuous. Moreover, a Lipschitz continuous function on $[a,b]$ is absolutely continuous. Let f and g be two absolutely continuous functions on $[a,b]$. Then $f+g$, $f-g$, and fg are absolutely continuous on $[a,b]$. If, in addition, there exists a constant $C > 0$ such that $|g(x)| \geq C$ for all $x \in [a,b]$, then f/g is absolutely continuous on $[a,b]$.

If f is integrable on $[a,b]$, then the function F defined by

$$F(x) := \int_a^x f(t)\, dt, \quad a \leq x \leq b,$$

is absolutely continuous on $[a,b]$.

Theorem 1.1. *Let f be an absolutely continuous function on $[a,b]$. Then f is of bounded variation on $[a,b]$. Consequently, $f'(x)$ exists for almost every $x \in [a,b]$.*

Proof. Since f is absolutely continuous on $[a,b]$, there exists some $\delta > 0$ such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < 1$ whenever $\{[x_i, y_i] : i = 1, \ldots, n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. Let N be the least integer such that $N > (b-a)/\delta$, and let $a_j := a + j(b-a)/N$ for $j = 0, 1, \ldots, N$. Then $a_j - a_{j-1} = (b-a)/N < \delta$. Hence, $\vee_{a_{j-1}}^{a_j} f < 1$ for $j = 0, 1, \ldots, N$. It follows that

$$\bigvee_a^b f = \sum_{j=1}^{N} \bigvee_{a_{j-1}}^{a_j} f < N.$$

This shows that f is of bounded variation on $[a,b]$. Consequently, $f'(x)$ exists for almost every $x \in [a,b]$. □

Theorem 1.2. *If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ for almost every $x \in [a,b]$, then f is constant.*

Proof. We wish to show $f(a) = f(c)$ for every $c \in [a,b]$. Let $E := \{x \in [a,c] : f'(x) = 0\}$. For given $\varepsilon > 0$, there exists some $\delta > 0$ such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$ whenever $\{[x_i, y_i] : i = 1, \dots, n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. For each $x \in E$, we have $f'(x) = 0$; hence there exists an arbitrary small interval $[a_x, c_x]$ such that $x \in [a_x, c_x] \subseteq [a,c]$ and

$$|f(c_x) - f(a_x)| < \varepsilon(c_x - a_x).$$

By the Vitali covering theorem we can find a finite collection $\{[x_k, y_k] : k = 1, \dots, n\}$ of mutually disjoint intervals of this sort such that

$$\lambda\big(E \setminus \cup_{k=1}^{n} [x_k, y_k]\big) < \delta.$$

Since $\lambda([a,c] \setminus E) = 0$, we have

$$\lambda\big([a,c] \setminus \cup_{k=1}^{n} [x_k, y_k]\big) = \lambda\big(E \setminus \cup_{k=1}^{n} [x_k, y_k]\big) < \delta.$$

Suppose $a \le x_1 < y_1 \le x_2 < \cdots < y_n \le c$. Let $y_0 := a$ and $x_{n+1} := c$. Then

$$\sum_{k=0}^{n} (x_{k+1} - y_k) = \lambda\big([a,c] \setminus \cup_{k=1}^{n} [x_k, y_k]\big) < \delta.$$

Consequently,

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon.$$

Furthermore,

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \sum_{k=1}^{n} \varepsilon(y_k - x_k) \le \varepsilon(c - a).$$

It follows from the above inequalities that

$$|f(c) - f(a)| \le \sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| + \sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon(c - a + 1).$$

This shows that $|f(c) - f(a)| \le \varepsilon(c - a + 1)$ for all $\varepsilon > 0$. Therefore, $f(c) = f(a)$. □

§2. The Fundamental Theorem of Calculus

In this section we show that absolutely continuous functions are precisely those functions for which the fundamental theorem of calculus is valid.

Theorem 2.1. *If f is integrable on $[a,b]$ and*

$$\int_a^x f(t)\,dt = 0 \quad \forall\, x \in [a,b],$$

then $f(t) = 0$ for almost every $t \in [a,b]$.

Proof. By our assumption,

$$\int_c^d f(t)\,dt = 0$$

for all c, d with $a \le c < d \le b$. If O is an open subset of $[a,b]$, then O is a countable union of mutually disjoint open intervals (c_n, d_n) $(n = 1, 2, \ldots)$; hence,

$$\int_O f(t)\,dt = \sum_{n=1}^{\infty} \int_{c_n}^{d_n} f(t)\,dt = 0.$$

It follows that for any closed subset K of $[a,b]$,

$$\int_K f(t)\,dt = \int_{[a,b]} f(t)\,dt - \int_{[a,b]\setminus K} f(t)\,dt = 0.$$

Let $E_+ := \{x \in [a,b] : f(x) > 0\}$ and $E_- := \{x \in [a,b] : f(x) < 0\}$. We wish to show that $\lambda(E_+) = 0$ and $\lambda(E_-) = 0$. If $\lambda(E_+) > 0$, then there exists some closed set $K \subseteq E_+$ such that $\lambda(K) > 0$. But $\int_K f(t)\,dt = 0$. It follows that $f = 0$ almost everywhere on K. This contradiction shows that $\lambda(E_+) = 0$. Similarly, $\lambda(E_-) = 0$. Therefore, $f(t) = 0$ for almost every $t \in [a,b]$. □

Theorem 2.2. *If f is integrable on $[a,b]$, and if F is defined by*

$$F(x) := \int_a^x f(t)\,dt, \quad a \le x \le b,$$

then $F'(x) = f(x)$ for almost every x in $[a,b]$.

Proof. First, we assume that f is bounded and measurable on $[a,b]$. For $n = 1, 2, \ldots$, let

$$g_n(x) := \frac{F(x+1/n) - F(x)}{1/n}, \quad x \in [a,b].$$

It follows that

$$g_n(x) = n \int_x^{x+1/n} f(t)\,dt, \quad x \in [a,b].$$

Suppose $|f(x)| \le K$ for all $x \in [a,b]$. Then $|g_n(x)| \le K$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} g_n(x) = F'(x)$ for almost every $x \in [a,b]$, by the Lebesgue dominated convergence theorem, we see that for each $c \in [a,b]$,

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c g_n(x)\,dx.$$

But F is continuous; hence,

$$\lim_{n\to\infty} \int_a^c g_n(x)\,dx = \lim_{n\to\infty} n\left[\int_c^{c+1/n} F(x)\,dx - \int_a^{a+1/n} F(x)\,dx\right] = F(c) - F(a).$$

Consequently,

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c g_n(x)\,dx = F(c) - F(a) = \int_a^c f(x)\,dx.$$

It follows that

$$\int_a^c [F'(x) - f(x)]\,dx = 0$$

for every $c \in [a,b]$. By Theorem 2.1, $F'(x) = f(x)$ for almost every x in $[a,b]$.

Now let us assume that f is integrable on $[a,b]$. Without loss of any generality, we may assume that $f \ge 0$. For $n = 1, 2, \ldots$, let f_n be the function defined by

$$f_n(x) := \begin{cases} f(x) & \text{if } 0 \le f(x) \le n, \\ 0 & \text{if } f(x) > n. \end{cases}$$

It is easily seen that $F = F_n + G_n$, where

$$F_n(x) := \int_a^x f_n(t)\,dt \quad \text{and} \quad G_n(x) := \int_a^x [f(t) - f_n(t)]\,dt, \quad a \le x \le b.$$

Since $f(t) - f_n(t) \ge 0$ for all $t \in [a,b]$, G_n is an increasing function on $[a,b]$. Moreover, by what has been proved, $F_n'(x) = f_n(x)$ for almost every $x \in [a,b]$. Thus, we have

$$F'(x) = F_n'(x) + G_n'(x) \ge F_n'(x) = f_n(x) \quad \text{for almost every } x \in [a,b].$$

Letting $n \to \infty$ in the above inequality, we obtain $F'(x) \ge f(x)$ for almost every $x \in [a,b]$. It follows that

$$\int_a^b F'(x)\,dx \ge \int_a^b f(x)\,dx = F(b) - F(a).$$

On the other hand,

$$\int_a^b F'(x)\,dx \le F(b) - F(a).$$

Consequently,

$$\int_a^b [F'(x) - f(x)]\,dx = 0.$$

But $F'(x) \ge f(x)$ for almost every $x \in [a,b]$. Therefore, $F'(x) = f(x)$ for almost every x in $[a,b]$. □

Theorem 2.3. *A function F on $[a,b]$ is absolutely continuous if and only if*

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

for some integrable function f on $[a,b]$.

Proof. The sufficiency part has been established. To prove the necessity part, let F be an absolutely continuous function on $[a,b]$. Then F is differentiable almost everywhere and F' is integrable on $[a,b]$. Let

$$G(x) := F(a) + \int_a^x F'(t)\,dt, \quad x \in [a,b].$$

By Theorem 2.2, $G'(x) = F'(x)$ for almost every $x \in [a,b]$. It follows that $(F-G)'(x) = 0$ for almost every $x \in [a,b]$. By Theorem 1.2, $F - G$ is constant. But $F(a) = G(a)$. Therefore, $F(x) = G(x)$ for all $x \in [a,b]$. □

§3. Change of Variables for the Lebesgue Integral

Let f be an absolutely continuous function on $[c,d]$, and let u be an absolutely continuous function on $[a,b]$ such that $u([a,b]) \subseteq [c,d]$. Then the composition $f \circ u$ is not necessarily absolutely continuous. However, we have the following result.

Theorem 3.1. *Let f be a Lipschitz continuous function on $[c,d]$, and let u be an absolutely continuous function on $[a,b]$ such that $u([a,b]) \subseteq [c,d]$. Then $f \circ u$ is absolutely continuous. Moreover,*

$$(f \circ u)'(t) = f'(u(t))u'(t) \quad \text{for almost every } t \in [a,b],$$

where $f'(u(t))u'(t)$ is interpreted to be zero whenever $u'(t) = 0$ (even if f is not differentiable at $u(t)$).

Proof. Since f is a Lipschitz continuous function on $[c,d]$, there exists some $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$ whenever $x, y \in [c,d]$. Let $\varepsilon > 0$ be given. Since u is absolutely continuous on $[a,b]$, there exists some $\delta > 0$ such that $\sum_{i=1}^{n} |u(t_i)-u(s_i)| < \varepsilon/M$, whenever $\{[s_i,t_i] : i = 1,\ldots,n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n}(t_i - s_i) < \delta$. Consequently,

$$\sum_{i=1}^{n} |(f\circ u)(t_i) - (f\circ u)(s_i)| = \sum_{i=1}^{n} |f(u(t_i)) - f(u(s_i))| \le \sum_{i=1}^{n} M|u(t_i) - u(s_i)| < \varepsilon.$$

This shows that $f\circ u$ is absolutely continuous on $[a,b]$.

Since both u and $f\circ u$ are absolutely continuous on $[a,b]$, there exists a measurable subset E of $[a,b]$ such that $\lambda(E) = 0$ and both $u'(t)$ and $(f\circ u)'(t)$ exist for all $t \in [a,b]\setminus E$. Suppose $t_0 \in [a,b]\setminus E$. If $u'(t_0) = 0$, then for given $\varepsilon > 0$, there exists some $h > 0$ such that $|u(t) - u(t_0)| \le \varepsilon|t - t_0|$ whenever $t \in (t_0 - h, t_0 + h)\cap[a,b]$. It follows that

$$|f\circ u(t) - f\circ u(t_0)| \le M|u(t) - u(t_0)| \le M\varepsilon|t - t_0|$$

for all $t \in (t_0 - h, t_0 + h)\cap[a,b]$. This shows that

$$(f\circ u)'(t_0) = 0 = f'(u(t_0))u'(t_0).$$

Now suppose $t_0 \in [a,b]\setminus E$ and $u'(t_0) \ne 0$. Suppose $u(t) \ne u(t_0)$. Then we have

$$\frac{(f\circ u)(t) - (f\circ u)(t_0)}{t - t_0} = \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)}\,\frac{u(t) - u(t_0)}{t - t_0}.$$

Since $u'(t_0)$ and $(f\circ u)'(t_0)$ exist, we obtain

$$\lim_{t\to t_0} \frac{(f\circ u)(t) - (f\circ u)(t_0)}{t - t_0} = (f\circ u)'(t_0) \quad \text{and} \quad \lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0} = u'(t_0) \ne 0.$$

Consequently,

$$\lim_{t\to t_0} \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} = \frac{(f\circ u)'(t_0)}{u'(t_0)}.$$

Let $r := (f\circ u)'(t_0)/u'(t_0)$. For given $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$r - \varepsilon < \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} < r + \varepsilon \quad \forall\, t \in (t_0 - \delta, t_0 + \delta)\cap[a,b].$$

Since $u'(t_0) \ne 0$, there exists some $\eta > 0$ such that any $x \in (u(t_0) - \eta, u(t_0) + \eta)\cap[c,d]$ can be expressed as $x = u(t)$ for some $t \in (t_0 - \delta, t_0 + \delta)\cap[a,b]$. Therefore,

$$r - \varepsilon < \frac{f(x) - f(u(t_0))}{x - u(t_0)} < r + \varepsilon \quad \forall\, x \in (u(t_0) - \eta, u(t_0) + \eta)\cap[c,d].$$

This shows that $f'(u(t_0))$ exists and $f'(u(t_0)) = r = (f\circ u)'(t_0)/u'(t_0)$, as desired. $\square$

Theorem 3.2. *Let g be a bounded and measurable function on $[c,d]$, and let u be an absolutely continuous function on $[a,b]$ such that $u([a,b]) \subseteq [c,d]$. Then $(g \circ u)u'$ is integrable on $[a,b]$. Moreover, for any $\alpha, \beta \in [a,b]$,*

$$\int_{u(\alpha)}^{u(\beta)} g(x)\,dx = \int_{\alpha}^{\beta} g(u(t))u'(t)\,dt.$$

Proof. Let

$$F(x) := \int_{c}^{x} g(t)\,dt, \quad x \in [c,d].$$

Since g is bounded, F is Lipschitz continuous. Moreover, $F'(x) = g(x)$ for almost every $x \in [a,b]$. By Theorem 3.1, $F \circ u$ is absolutely continuous on $[a,b]$ and, for almost every $t \in [a,b]$, $(F \circ u)'(t) = g(u(t))u'(t)$. Suppose $\alpha, \beta \in [a,b]$ and $\alpha < \beta$. By Theorem 2.3, we have

$$(F \circ u)(\beta) - (F \circ u)(\alpha) = F(u(\beta)) - F(u(\alpha)) = \int_{u(\alpha)}^{u(\beta)} F'(x)\,dx = \int_{u(\alpha)}^{u(\beta)} g(x)\,dx.$$

On the other hand,

$$(F \circ u)(\beta) - (F \circ u)(\alpha) = \int_{\alpha}^{\beta} (F \circ u)'(t)\,dt = \int_{\alpha}^{\beta} g(u(t))u'(t)\,dt.$$

This proves the desired formula for change of variables. □

Theorem 3.3. *Let g be an integrable function on $[c,d]$, and let u be an absolutely continuous function on $[a,b]$ such that $u([a,b]) \subseteq [c,d]$. If $(g \circ u)u'$ is integrable on $[a,b]$, then*

$$\int_{u(\alpha)}^{u(\beta)} g(x)\,dx = \int_{\alpha}^{\beta} g(u(t))u'(t)\,dt, \quad \alpha, \beta \in [a,b].$$

Moreover, $(g \circ u)u'$ is integrable if, in addition, u is monotone.

Proof. Suppose that g is integrable on $[a,b]$. Without loss of any generality, we may assume $g \geq 0$. For $n = 1, 2, \ldots$, let g_n be the function defined by

$$g_n(x) := \begin{cases} g(x) & \text{if } 0 \leq g(x) \leq n, \\ 0 & \text{if } g(x) > n. \end{cases}$$

Then $g_n \leq g_{n+1}$ for all $n \in \mathbb{N}$. Suppose $\alpha, \beta \in [a,b]$ and $\alpha < \beta$. By Theorem 3.2 we have

$$\int_{u(\alpha)}^{u(\beta)} g_n(x)\,dx = \int_{\alpha}^{\beta} g_n(u(t))u'(t)\,dt.$$

If u is monotone, then $u'(t) \geq 0$ for almost every $t \in [a, b]$. Letting $n \to \infty$ in the above equation, by the monotone convergence theorem we obtain

$$\int_{u(\alpha)}^{u(\beta)} g(x)\, dx = \int_{\alpha}^{\beta} g(u(t))u'(t)\, dt.$$

Since g is integrable on $[c, d]$, it follows from the above equation that $(g \circ u)u'$ is integrable on $[a, b]$. More generally, we assume that $(g \circ u)u'$ is integrable on $[a, b]$ but u is not necessarily monotone. Then $|g_n(u(t))u'(t)| \leq g(u(t))|u'(t)|$ for all $n \in \mathbb{N}$ and almost every $t \in [a, b]$. Thus, an application of the Lebesgue dominated convergence theorem gives the desired formula for change of variables. $\square$