Avec l'aide du principe de maximum nous prouvons la symétrie des solutions v du problème linéarisé. A partir de ce résultat nous déduisons plusieurs propriétés de v et u; en particulier nous montrons que si f est convexe et $N = 2$ on ne peut pas avoir deux solutions différentes qui ont le même maximum. On prouve aussi qu'il y a une seule solution si $f(u) = u^p$ et $\lambda = 0$.

Dans la dernière section nous étudions le problème

$$\begin{aligned} -\Delta u &= u^p + \mu u^q \quad \text{dans } \Omega \\ u &= 0 \quad \text{sur le bord de } \Omega \end{aligned}$$

et montrons que si $1 < p < N + 2/N - 2$, $0 < q < 1$ et μ est petite il y a exactement deux solutions positives dans quelques domaines particuliers.

1. INTRODUCTION

In this paper we are interested in studying the qualitative behaviour of the solutions of the semilinear elliptic problem

$$(1.1) \qquad \begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ and $N \geq 2$. It is clear that to understand some of the properties of a solution of (1.1) it is important to study the linearized operator at u, i.e.

$$(1.2) \qquad L = \Delta - \lambda + f'(u)$$

Here we consider the case of a bounded domain Ω symmetric with respect to the hyperplanes $\{x_i = 0\}$ and convex in any direction x_i, $i = 1, \ldots, N$ and show how a very simple application of the maximum principle gives some interesting results on u. Note that this kind of domains need not be convex.

More precisely, using some sufficient condition, described for example in [6], we show that the maximum principle holds for the operators (1.2) in certain subdomains Ω_i, $i = 1, \ldots, N$ determined by the symmetry of Ω (namely Ω_i is "half of Ω", see Section 2 and 3). This simple information is the key to get all the main results of this paper.

For example we deduce the symmetry of any solution of the problem

$$(1.3) \qquad \begin{cases} L\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

which, in other words, means the symmetry of any eigenfunction of (1.2) corresponding to the zero eigenvalue.

This result was already known for the eigenfunctions corresponding to any negative eigenvalues μ of L ([4]) and, in the case of the ball, was proved in [14] for any $\mu \leq 0$, when $\lambda = 0$, using a different argument (see Remark 2.1).

Other important consequences of the validity of the maximum principle for L in Ω_i are some properties of the nodal set of any solution of (1.3) (Theorem 3.1) as well as some properties of the coincidence set of two possible solutions of (1.1) in the case f is also convex (see Theorem 3.2). From this we deduce some results which show that the solutions of (1.1), in the symmetric domain considered, behave very much like the solutions of the same problem in a ball. For example, in Theorem 3.2 we show that if f is convex and $N = 2$ there cannot exist two solutions of (1.1) which have the same maximum; this is a generalization of the uniqueness theorem for o.d.e.'s.

Exploiting a generalization of this result (Theorem 3.3), we also show that if $f(u) = u^p$, $N = 2$ and $\lambda = 0$ then (1.1) has only one solution. The proof is based only on Theorem 3.3 and does not use the nondegeneracy of the solution of (1.1). However we also show that in this case solutions of (1.1) are nondegenerate and from this we deduce again, as done in [13] for least energy solutions, the uniqueness of the solution to (1.1).

This last result has already been proved by Dancer in [9] as a consequence of a general theorem contained also in [9] and of the known uniqueness result for the ball. However our approach is different and does not rely on the uniqueness result for the ball. Actually the same proof also applies to the case of the ball in $\mathbb{R}^N$, giving so an alternative proof.

At this point we would like to quote here that, in the case $f(u) = u^p + \lambda u$, the uniqueness result for the ball was proved by Adimurthi and Yadava ([2]), Srikanth ([17]) and Zhang ([18]) using an o.d.e. approach. Other partial uniqueness results are due to Damascelli ([8]) for star-shaped domains, Lin ([13]) and Zhang ([19]) for convex set in $\mathbb{R}^2$ and $f(u) = u^p$.

We end the paper by considering the case of $f(u) = u^p + \mu u^q$, $p > 1$, $0 < q < 1$, i.e. when f is a sum of a convex and a concave nonlinearity. This problem has been extensively studied by Ambrosetti, Brezis and Cerami ([3]) who showed, among other things, that for some values of μ and p there are at least two positive solutions. In Section 5 we show that in certain symmetric domains and for some small values of μ there are exactly two solutions. This result extends to other domain and with a different proof a previous theorem of Adimurthi, Pacella and Yadava ([1]) for the case of the ball.

2. SYMMETRY RESULT FOR THE LINEARIZED EQUATION

Let D be a bounded domain in $\mathbb{R}^N, N \geq 2$. Before proving the main result we need to recall a few facts about the maximum principle for second order elliptic operators of the form $Lu = \Delta u + c(x)u$ with $c(x) \in L^\infty(D)$, $u \in W^{2,N}_{\mathrm{loc}} \cap C(\overline{D})$.

DEFINITION 2.1. – *We say that the maximum principle holds for L in D if $Lu \leq 0$ in D and $u \geq 0$ on ∂D imply $u \geq 0$ in D.*

Two well known sufficient conditions for the maximum principle to hold are the following (see [12],[16])

$$c(x) \leq 0 \quad \text{in } D \tag{2.1}$$

(2.2)
there exists a function $g \in W^{2,N}_{\mathrm{loc}} \cap C(\overline{D}), g > 0$ in $\overline{D}$ such that $Lg \leq 0$ in D

Now we denote by $\lambda_1(L, D)$ the principal eigenvalue of L in D. The meaning and the properties of $\lambda_1(L, D)$ are, of course, well known when ∂D is smooth; however in order not to be worried in the sequel about the regularity of the domains involved we prefer to refer to the general definition of principal eigenvalue given by Berestycki, Nirenberg and Varadhan in [6]. This definition is the following

$$\lambda_1(L, D) = \sup\{\lambda : \text{ there exists } \phi > 0 \text{ in } D \text{ satisfying } (L + \lambda)\phi \leq 0\}$$

In [6] they show that even with this definition all the main properties of the "classical" principal eigenvalue continue to hold. In particular we have

PROPOSITION 2.1. – *The principal eigenvalue $\lambda_1(L, D)$ is strictly decreasing in its dependence on D and on the coefficient $c(x)$. Moreover the "refined" maximum principle holds for L in D if and only if $\lambda_1(L, D)$ is positive.*

We refer to [6] for the definition of "refined" maximum principle which is a generalized formulation of the maximum principle in the case when one cannot prescribe boundary values of the functions involved.

It is important to notice that, by using this generalized definition of the first eigenvalue, it is possible to prove that also the following condition, which is slightly different from (2.2), is sufficient for the maximum principle to hold.

(2.3)
there exists $g \in W^{2,N}_{\mathrm{loc}} \cap C(\overline{D}), g > 0$
in D such that $Lg \leq 0$ in D but $g \not\equiv 0$
on some regular part of ∂D.

We also recall the following sufficient condition for the maximum principle (see [5], [6])

PROPOSITION 2.2. – *There exists $\delta > 0$, depending only on N, $diam(D)$, $\|c\|_{L^\infty(D)}$ such that the maximum principle holds for L in any domain $D' \subset D$ with $|D'| < \delta$*

Finally we remark that regardless of the sign of c if $Lu \leq 0$ in D and $u \geq 0$ in D then $u > 0$ in D unless $u \equiv 0$ (Strong Maximum Principle).

Now we consider a solution $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function with $f(0) \geq 0$. We are interested in studying the linearized problem

$$\begin{cases} -\Delta v + \lambda v = f'(u)v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

We have

THEOREM 2.1. – *Let u be a solution of (2.4) and assume that Ω is convex in the x_1- direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$.*

Then any solution v of (2.5) is symmetric in x_1, i.e. $v(x_1, x_2, \dots, x_N) = v(-x_1, x_2, \dots, x_N)$.

Proof. – The proof is the same as the one shown in a lecture of L. Nirenberg in a slightly different case (see also the remark after the proof).

Let us denote a point x in $\mathbb{R}^N$ by $(x_1, y), y \in \mathbb{R}^{N-1}$. Applying the symmetry result of Gidas, Ni, Nirenberg ([10]) to problem (2.4) we get that u is symmetric with respect to x_1 and $\frac{\partial u}{\partial x_1} > 0$ in $\Omega_1^- = \{x = (x_1, y) \in \Omega \text{ such that } x_1 < 0\}$.

We consider the operator

$$L = \Delta - \lambda + f'(u) \tag{2.6}$$

and want to prove that the maximum principle holds for L in Ω_1^-. To do this we show that the sufficient condition (2.3) is satisfied.

If we set

$$g = \frac{\partial u}{\partial x_1} \quad \text{in } \Omega_1^- \tag{2.7}$$

we have that g satisfies (2.3) since by the Hopf Lemma $\frac{\partial u}{\partial x_1} \not\equiv 0$ on $\partial\Omega \cap \partial\Omega_1^-$ (note that we assumed $f(0) \geq 0$ in (2.4)). So the maximum principle holds for L in Ω_1^-.

Now we consider the function

$$\psi(x) = v(x_1, y) - v(-x_1, y), \qquad x = (x_1, y) \in \Omega_1^- \tag{2.10}$$

where v is a solution of (2.5). By easy calculation, using that u is symmetric in x_1, we get

$$\begin{cases} L\psi = 0 & \text{in } \Omega_1^- \\ \psi = 0 & \text{in } \partial\Omega_1^- \end{cases} \tag{2.11}$$

and hence $\psi \equiv 0$ in Ω_1^- because of the maximum principle. So v is symmetric in x_1. □

Remark 2.1. – Let us consider the following eigenvalue problem

$$\begin{cases} -\Delta v + \lambda v = f'(u)v + \mu v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.12}$$

where u is a solution of (2.4).

If $\lambda = 0$ and $\mu < 0$ in [4] it is shown that v is symmetric in x_1.

Of course if Ω is a ball, the previous theorem gives the radial symmetry of v. This was already shown by Lin and Ni in [14], using a different argument, for any $\mu \leq 0$ and $\lambda = 0$.

3. SOME PROPERTIES OF THE COINCIDENCE SET OF TWO SOLUTIONS AND AN UNIQUENESS RESULT

In this section we assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ convex in the direction x_i, $i = 1, \dots, N$ and symmetric with respect to the hyperplanes $x_i = 0$, $i = 1, \dots, N$.

Let us consider a solution u of (2.4), where f is a C^1-function with $f(0) \geq 0$, and a nontrivial solution v of the corresponding linearized problem (2.5).

We make now some important remarks about the nodal set of v that will also be used in the sequel. Let us set

$$\mathcal{N} = \overline{\{x \in \Omega \text{ such that } v(x) = 0\}}$$

$$\widetilde{\Omega} = \{x \in \Omega : v(x) \neq 0\}$$

$$\Omega_i^- = \{x = (x_1, \dots, x_N) \in \Omega \text{ such that } x_i < 0\} \quad i = 1, \dots, N$$

We have

THEOREM 3.1. – *The following properties hold*
i) there cannot exist any component of $\widetilde{\Omega}$ all contained in one Ω_i^-, $i = 1, \dots, N$.
ii) if $N = 2$ then the origin $(0, \dots, 0)$ *does not belong to $\mathcal{N}$.*
iii) if $N = 2$ then $\mathcal{N} \cap \partial\Omega = \emptyset$.

Proof.

i) Suppose that there exists a component D of $\widetilde{\Omega}$ all contained in Ω_i^- and $v > 0$ in D. Then $\lambda_1(L, D) = 0$ (where L is the operator defined in (2.6)) since v is an eigenfunction of L in D corresponding to the zero eigenvalue and does not change sign in D (being $v = 0$ on $\partial\Omega$ we have $v = 0$ on ∂D). On the other hand, in the proof of Theorem 2.1 we have shown that L satisfies the maximum principle in Ω_i^- and this implies, by Proposition 2.1, that $\lambda_1(L, \Omega_i^-) > 0$. Then, by monotonicity, also $\lambda_1(L, D)$ should be positive which gives a contradiction.

ii) We will show that if $v(0) = 0$ then $v \equiv 0$. Suppose $v(0) = 0$ and $v \not\equiv 0$ and set $U_0 = \Omega$. Since $v \not\equiv 0$ and $v(0) = 0$ by the Strong Maximum Principle it cannot be $v \leq 0$ in Ω, so that $U_0^+ = \{x \in U_0 \ : \ v(x) > 0\}$ is open and nonempty. Choose a component A_1 of U_0^+. If S_i, $i = 1, 2$ is the operator that sends a point to the symmetric one with respect to the x_i-axis, we have that $S_i(A_1)$ is also a component of U_0^+ because of the symmetry of v. It cannot happen that $A_1 \cap S_1(A_1) = \emptyset$ or $A_1 \cap S_2(A_1) = \emptyset$ for otherwise A_1 or $S_1(A_1)$ would be contained in Ω_1^-, which is impossible by (i). So $A_1 = S_1(A_1) = S_2(A_1)$ is symmetric with respect to the coordinate axes and is open and connected, therefore arcwise connected. If we choose four symmetric points P_j, $j \in \{1, \dots, 4\}$ and join them with simple polygonal curves symmetric in pairs, we can costruct a simple closed polygonal curve $C_1 \subset A_1$ which is symmetric with respect to the axes. By the Jordan Curve Theorem $U_0 \setminus C_1$ has two components and, because C_1 is symmetric, the origin belongs to the component which has not ∂U_0 as part of the boundary. Let us denote by U_1 the component that contains 0 and call it the interior of C_1, while by the exterior of C_1 we mean the other component. On $\partial U_1 = C_1$ we have $v > 0$, so that $v \not\equiv 0$ in U_1 and, by the Strong Maximum Principle, it is not possible that $v \geq 0$ in U_1, since $v(0) = 0$, so that $U_1^- = \{x \in U_1 \; : \; v(x) < 0\}$ is open and nonempty. Taking a component A_2 of U_1^- we observe that $v = 0$ on ∂A_2 because $v \geq 0$ on ∂U_1 so that A_2 is also a component of $\widetilde{\Omega}$. As before we can costruct a closed symmetric simple curve $C_2 \subset A_2$ and in the interior U_2 of C_2 (the component of $U_1 \setminus C_2$ to which the origin belongs) we can choose a component A_3 of $U_2^+ = \{x \in U_2 \; : \; v(x) > 0\}$ which is also a component of $\widetilde{\Omega}$. Moreover A_3 is disjoint from A_1 because A_1 contains $C_1 = \partial\Omega_1$ which belongs to the exterior of C_2. Proceeding in this way we obtain infinitely many disjoint components $\{A_n\}_{n \geq 1}$ of $\widetilde{\Omega}$. This is not possible because by Proposition 2.2 there exists $\delta > 0$ such that $|A_n| \geq \delta$ for each n, otherwise by the Maximum Principle v would be 0 in A_n, since $v = 0$ on ∂A_n and $Lv = 0$ in A_n with $L = \Delta - \lambda + f'(v)$. Hence there are only finitely many components A_n which gives a contradiction.

iii) We will show that in a neighborhood of $\partial\Omega$ we have $v > 0$ or $v < 0$. Suppose the contrary and choose a component A_1 of $U_0^+ = \{x \in U_0 \; : \; v(x) > 0\}$. Since $v = 0$ on $\partial\Omega$ we have $v = 0$ on ∂A_1 and as in (ii) we costruct a closed simple curve $C_1 \subset A_1$ symmetric with respect to the axes. In the exterior U_1 of C_1, i.e. in the component containing $\partial\Omega$ there are points where $v < 0$ by what we assumed. So we can costruct a closed simple curve $C_2 \subset A_2$ where A_2 is a nonempty component of $U_1^- = \{x \in U_1 \; : \; v(x) < 0\}$. Proceeding as in the proof of (ii) we obtain infinitely many components of $\widetilde{\Omega}$ which is not possible by Proposition 2.2, as we remarked before.

Remark 3.1. – If Ω is a ball in $\mathbb{R}^N$, the properties i) - iii) are easy consequences of the radial symmetry of v.

Now we consider two solutions u_1 and u_2 of the problem (2.4) and set

$$\mathcal{M} = \overline{\{x \in \Omega \text{ such that } u_1(x) = u_2(x)\}}, \quad \widehat{\Omega} = \{x \in \Omega \text{ such that } u_1 \neq u_2\}$$

The next theorem contains some information on $\mathcal{M}$ and a partial uniqueness result.

THEOREM 3.2. – *Suppose that f is convex. Then we have*

(3.1) *there cannot exist any component D space of $\widehat{\Omega}$ all contained in one Ω_i^-, $i = 1, \dots, N$.*

$$\text{if } N = 2 \quad \text{then} \quad \mathcal{M} \cap \partial\Omega = \emptyset \tag{3.2}$$

$$\text{if } N = 2 \quad \text{and} \quad \max_{x \in \overline{\Omega}} u_1(x) = \max_{x \in \overline{\Omega}} u_2(x) \quad \text{then} \quad u_1 \equiv u_2 \tag{3.3}$$

Proof. – Set $w(x) = u_1(x) - u_2(x)$, $x \in \Omega$. Since f is convex w satisfies

$$\begin{cases} \Delta w - \lambda w + f'(u_2)w \leq 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

and

$$\begin{cases} \Delta w - \lambda w + f'(u_1)w \geq 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

First we notice that if $w \gneq 0$ by (3.4) and the strong maximum principle $w > 0$ in Ω so that $\Omega = \widehat{\Omega}$. Thus we assume that w changes sign in Ω. To prove (3.1) let us argue by contradiction supposing that there exists a component D of $\widehat{\Omega}$ all contained in Ω_i^- for some $i \in \{1, \dots, N\}$ and $w > 0$ in D.

Since in Theorem 2.1 we proved that in Ω_i^- the maximum principle holds for the operators $L_i = \Delta - \lambda + f'(u_i)$ $i = 1, 2$, by Proposition 2.1 we have that $\lambda_1(L_1, \Omega_i^-) > 0$, for $i = 1, 2$. Hence also $\lambda_1(L_1, D) > 0$ and, again by Proposition 2.1, the "refined" maximum principle holds for L_1 in D. This last fact together with (3.5) would imply that $w \leq 0$ in D against what we assumed. If instead we suppose $w \leq 0$ in D then we argue in the same way using the operator L_2 and (3.4).

To prove (3.2) it is enough to observe that, by the Gidas, Ni and Nirenberg symmetry result, u_1 and u_2 are symmetric in any x_i and hence so is w. Thus arguing as in iii) of the previous theorem the assumption $\mathcal{M} \cap \partial\Omega \neq \emptyset$ would bring a contradiction.

Finally, to prove (3.3), we notice that, again by the Gidas, Ni and Nirenberg result, $\max_{x \in \overline{\Omega}} u_i(x) = u_i(0)$, $i = 1, 2$; therefore if the two maxima coincide the origin belongs to $\mathcal{M}$. As in ii) of Theorem 3.1 this gives a contradiction. □

Now we prove a generalization of (3.3) of Theorem 3.2 that will be used in the proof of Theorem 4.1.

Let Ω be as before and $N = 2$. Let us call a function $u \in C^1(\overline{\Omega})$ symmetric and monotone if u is symmetric in x_1, x_2 and $\frac{\partial u}{\partial x_i} > 0$ in Ω_i^-, $i = 1, 2$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function.

THEOREM 3.3. – *Suppose that $N = 2$, f is convex and $u_1, u_2 \in C^3(\Omega) \cap C^1(\overline{\Omega})$ are symmetric and monotone functions that satisfy the equation*

$$-\Delta u + \lambda u = f(u) \quad \text{in } \Omega \tag{3.6}$$

If $u_1(0) = u_2(0)$ and $u_1 \le u_2$ on $\partial\Omega$ then u_1 and u_2 coincide.

Proof. – As in the proof of Theorem 2.1 we deduce that the operators $L = \Delta - \lambda + f'(u_i)$, $i = 1, 2$ satisfy the maximum principle in Ω_j^-, $j = 1, 2$.

Since the difference $w = u_1 - u_2$ satisfies a linear equation $\Delta w - \lambda w + c(x)w = 0$ with $c \in L^\infty(\Omega)$ and $f \in C^1$ we have that Proposition 2.2 and the strong maximum principle apply to w. Arguing as in Theorem 3.1 we first deduce that cannot exist any component D of $\widehat{\Omega} = \{x \in \Omega : u_1 \ne u_2\}$ such that $u_1 = u_2$ on ∂D and contained in Ω_j^-, $j = 1, 2$.

Then we can follow exactly the proof of Theorem 3.1 with the only remark that in the first step we choose a component A_1 of $\Omega_0^+ = \{x \in \Omega : w(x) > 0\}$ and we have $w = 0$ on ∂A_1, because of the hypothesis $w(x) \le 0$ on $\partial\Omega$. So A_1 is also a component of $\widehat{\Omega}$ with $u_1 = u_2$ on ∂A_1. The same property holds, by construction, also for the other components A_2, A_3; therefore we conclude as in Theorem 3.1.

Remark 3.2. – If Ω is a ball then any solution u of (2.4) is radial and hence the claim (3.3) follows immediately from the theory of ordinary differential equation. Therefore this result can be seen as a generalization of the uniqueness theorem for an o.d.e.

Nevertheless it is instructive to see how we can get very easily this result in a ball without using the underlaying ordinary equation but exploiting only maximum principles. Therefore suppose $\Omega = B_R(0) \subset \mathbb{R}^N$ and $u_i \in C^2(\overline{\Omega})$, $i = 1, 2$, satisfying $-\Delta u = f(u)$ in Ω. Let us prove that if $u_1(0) = u_2(0)$ then $u_1 \equiv u_2$. In fact the difference $w = u_1 - u_2$ satisfies a linear equation $\Delta w + c(x)w = 0$. By Proposition 2.2 there exists $\delta > 0$ such that if $0 \le r_1 < r_2 < R$ and $r_2 - r_1 < \delta$ then the Maximum Principle holds for $\Delta + c$ in $B_{r_2} \setminus B_{r_1}$. We claim that u_1 and u_2 coincide on ∂B_r for any $r < \delta$. In fact it cannot be $u_1 > u_2$ on ∂B_r because by Proposition 2.2 and the strong maximum principle it would be $u_1 > u_2$ on B_r, against the assumption $u_1(0) = u_2(0)$. In the same way it is not possible that $u_1 < u_2$

on ∂B_r. So $u_1 \equiv u_2$ in $\overline{B}_\delta$. Making the same reasoning in $B_{\frac{3}{2}\delta} \setminus B_{\frac{1}{2}\delta}$ (that has ∂B_δ in the interior) we get $u_1 \equiv u_2$ in $B_{\frac{3}{2}\delta}$ and after a finite number of steps we get $u_1 \equiv u_2$ in B_R. □

4. THE CASE OF $f(u) = u^p$

Here we assume $\Omega \subset \mathbb{R}^N$ as in the previous section and consider the case of $f(u) = u^p$, $p > 1$, so that (2.4) and (2.5) become, respectively

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\Delta v + \lambda v = pu^{p-1}v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

We recall that u is said to be a nondegenerate solution of (2.4) if (2.5) admits only the trivial solution $v \equiv 0$, i.e. if zero is not an eigenvalue for the operator $L = -\Delta + \lambda - pu^{p-1}$.

We have

THEOREM 2.1. – *Let $\lambda = 0$. If $N = 2$ or Ω is a ball in $\mathbb{R}^N$ then problem (4.1) has only one solution.*

Proof. – Let u, v be solutions of the problem (4.1) with $\lambda = 0$ and suppose that $u(0) \le v(0)$. For each k, $0 < k \le 1$ the function $v_k(x) = k^{\frac{2}{p-1}} v(kx)$ satisfies the same equation $-\Delta v_k = v_k^p$ in $\frac{\Omega}{k}$. Moreover since u and v are symmetric and monotone functions (in the sense of section 3) so is v_k. If we choose $\overline{k} = \left(\frac{u(0)}{v(0)}\right)^{\frac{p-1}{2}} \in]0, 1]$ we have that $u(0) = v_{\overline{k}}(0)$, $u = 0 \le v_{\overline{k}}$ on $\partial\Omega$ and $u, v_{\overline{k}}$ are symmetric and monotone solutions to the equation $-\Delta u = u^p$ in Ω. Therefore, by Theorem 3.3 u and $v_{\overline{k}}$ must coincide in Ω. If $\overline{k} < 1$ then $0 = u < v_{\overline{k}}$ on $\partial\Omega$ so that it must be $\overline{k} = 1$ which means $u \equiv v_1 \equiv v$ in Ω. □

Now we state a nondegeneracy result

THEOREM 4.2. – *Let $\lambda = 0$. If $N = 2$ or Ω is a ball in $\mathbb{R}^N$ then any solution of (4.1) is nondegenerate.*

Proof. – As in [13] we deduce a useful integral identity.
Multiplying (4.1) by v and (4.2) by u and integrating we get

$$\int_\Omega u^p v dx = 0 \tag{4.3}$$

Now let us consider the function $\zeta(x) = x \cdot \nabla u(x)$. Easy calculations show that ζ solves

$$-\Delta\zeta = pu^{p-1}\zeta + 2u^p \tag{4.4}$$

and from (4.1)-(4.4) we get

$$\int_{\partial\Omega} (x\cdot\nu)\frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} d\sigma = \int_{\partial\Omega} \zeta \frac{\partial v}{\partial \nu} d\sigma = 2\int_\Omega u^p v dx = 0 \tag{4.5}$$

where ν is the outer normal to $\partial\Omega$.

On the other hand since we are in dimension two by iii) of Theorem 3.1 we know that the nodal set of v does not intersect $\partial\Omega$; hence near the boundary of Ω v has always the same sign, say $v > 0$. Hence by the Hopf boundary lemma $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$ unless $v \equiv 0$. Also $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ for the same reason while $(x\cdot\nu) \geq 0$ and $(x\cdot\nu) \not\equiv 0$ on $\partial\Omega$ by the geometric assumption on Ω. This makes the identity (4.5) impossible unless $v \equiv 0$ in Ω as we wanted to prove. The same argument applies to the case of a ball Ω in $\mathbb{R}^N$, using the radial symmetry of v. □

Next theorem gives an uniqueness result for p near 1; it was already proved by Lin [13] in the case $\lambda = 0$, assuming Ω convex but not necessarily symmetric. For sake of completeness we state the proof here for $\lambda > -\lambda_1(\Delta, \Omega)$ and our domain Ω.

THEOREM 4.3. – *There exists $p_0 > 1$, $p_0 < \frac{N+2}{N-2}$ if $N \geq 3$ such that the problem (4.1) has only one solution for any $p \in]1, p_0[$ and $\lambda > -\lambda_1$.*

Proof. – If u_1 and u_2 are two distinct solutions of (4.1) then $w = u_1 - u_2$ must change sign otherwise the identity

$$0 = \int_\Omega u_1(-\Delta u_2 + \lambda u_2) - u_2(-\Delta u_1 + \lambda u_1) = \int_\Omega u_1 u_2 (u_2^{p-1} - u_1^{p-1}) \tag{4.6}$$

deduced from (4.1), would imply $u_1 \equiv u_2$.

Now let u_n be a solution of (4.1) with $p = p_n$, $p_n \searrow 1$. As already recalled, by the theorem of Gidas, Ni and Nirenberg (see [10])

$$M_n = \max_{x\in\overline{\Omega}} u_n(x) = u_n(0)$$

We claim that

$$M_n^{p_n-1} \longrightarrow \lambda_1 + \lambda \quad \text{as } n \longrightarrow \infty \tag{4.7}$$

First of all we show that $M_n^{p_n-1}$ is bounded. Suppose that $M_n^{p_n-1} \longrightarrow +\infty$ and set

$$\widetilde{u}_n(x) = \frac{1}{M_n} u_n\left(\frac{x}{M_n^{\frac{p_n-1}{2}}}\right) \tag{4.8}$$

By standard elliptic estimates $\widetilde{u}_n$ converges uniformly to a function $\widetilde{u} \in C^2(K)$, for any compact set K in $\mathbb{R}^N$ and $\widetilde{u}$ satisfies

$$\begin{cases} -\Delta\widetilde{u} = \widetilde{u} & \text{in } \mathbb{R}^N \\ \widetilde{u} > 0 & \text{in } \mathbb{R}^N \end{cases} \tag{4.9}$$

Let λ_R and ϕ_R be respectively the first eigenvalue and the relative eigenfunction of $-\Delta$ in $B_R(0)$ with respect to the zero Dirichlet boundary condition.

For R large we have

$$0 > \int_{\partial B_R(0)} \widetilde{u}\frac{\partial \phi_R}{\partial \nu} d\sigma = (1-\lambda_R)\int_{B_R(0)} \widetilde{u}\phi_R dx > 0$$

a contradiction which shows that $M_n^{p_n-1}$ is bounded. Thus, up to a subsequence, $M_n^{p_n-1} \longrightarrow \mu$. Let $\overline{u}_n = \frac{u_n}{M_n}$, which is a solution of the problem

$$\begin{cases} -\Delta\overline{u}_n + \lambda\overline{u}_n = M_n^{p_n-1}\overline{u}_n^{p_n} & \text{in } \Omega \\ \overline{u}_n = 0 & \text{on } \partial\Omega \end{cases} \tag{4.10}$$

By elliptic estimates $\overline{u}_n$ converges to $\overline{u}$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ and $\overline{u}$ satisfies

$$\begin{cases} -\Delta\overline{u} + \lambda\overline{u} = \mu\overline{u} & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.11}$$

Hence $\mu = \lambda_1 + \lambda$ and $\overline{u} = \phi_1$ the first eigenfunction of $-\Delta$. So the claim (4.7) is proved.

Now suppose that the assertion of the theorem is false, i.e. let us assume that u_n and v_n are two distinct solutions of (4.1) with $p = p_n$, $p_n \searrow 1$. From (4.7), since $\overline{u}_n \longrightarrow \phi_1$ uniformly we get $\overline{u}_n^{p_n-1} \longrightarrow 1$ and hence

$$u_n^{p_n-1} \longrightarrow \lambda_1 + \lambda \quad \text{uniformly in any compact set of } \Omega \tag{4.12}$$

Obviously the same happens to the sequence $\overline{v}_n^{p_n-1}$ where $\overline{v}_n = \frac{v_n}{\|v_n\|_\infty}$. The functions $w_n = \frac{u_n - v_n}{\|u_n - v_n\|_{L^\infty(\Omega)}}$ satisfy

$$(4.13) \qquad \begin{cases} -\Delta w_n + \lambda w_n = g_n w_n & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega \end{cases}$$

where $g_n = \frac{u_n^{p_n} - v_n^{p_n}}{u_n - v_n} \longrightarrow \lambda_1 + \lambda$. Since w_n is uniformly bounded and $\|w_n\|_{L^\infty(\Omega)} = 1$, from (4.13) and standard elliptic estimates we deduce that $w_n \longrightarrow \phi_1$ uniformly. This is not possible since ϕ_1 does not change sign while we showed at the beginnning of the proof that w_n must change sign. □

From the nondegeneracy of the solutions of (4.1) it also follows the uniqueness of the solution.

THEOREM 4.4. – *Suppose that for any $p \in]1, \frac{N+2}{N-2}[$ if $N \geq 3$, or for any $p > 1$, if $N = 2$, any solution of (4.1) is nondegenerate. Then for any such exponent p, (4.1) has only one solution.*

Proof. – Let us consider the case $N \geq 3$, for $N = 2$ the argument is the same. From the previous theorem we know that there exists $p_0 > 1$ such that (4.1) has an unique solution for $p \in]1, p_0[$. Let $]1, \overline{p}[$ be the maximal interval with this uniqueness property. If $\overline{p} = \frac{N+2}{N-2}$ the assertion is proved otherwise, since all solutions are nondegenerate, using the implicit function theorem we deduce that there is only one solution of (4.1) also for $p = \overline{p}$.

Arguing by contradiction let us assume that there exists a sequence $p_n \searrow \overline{p}$, $p_n < \frac{N+2}{N-2}$ and two distinct solutions u_n, v_n of (4.1) with $p = p_n$. By elliptic estimates (see [11] or also Remark 5.1 of next section) we have that u_n, v_n both converge in $C^2(\Omega)$ to the unique solution $\overline{u}$ of (4.1) for $p = \overline{p}$. Set

$$(4.14) \qquad w_n = u_n - v_n \quad \text{and} \quad \overline{w}_n = \frac{w_n}{\|w_n\|_{H_0^1(\Omega)}}$$

Then w_n satisfies

$$(4.15) \qquad \begin{cases} -\Delta \overline{w}_n = \alpha_n \overline{w}_n & \text{in } \Omega \\ \overline{w}_n = 0 & \text{on } \partial\Omega \end{cases}$$

where $\alpha_n(x) = \int_0^1 p_n(tu_n(x) + (1-t)v_n(x))^{p_n - 1} dt$.
Moreover $\overline{w}_n \longrightarrow \overline{w}$ weakly in $H_0^1(\Omega)$ and $\overline{w} \not\equiv 0$. In fact, by (4.15) we have

$$(4.16) \qquad 1 = \int_\Omega |\nabla \overline{w}_n|^2 dx = \int_\Omega \alpha_n \overline{w}_n^2 dx = \overline{p} \int_\Omega \overline{u}^{\overline{p}-1} \overline{w}^2 dx + o(1)$$

which implies $\overline{w} \not\equiv 0$. Passing to the limit in (4.15) we get

$$(4.17) \qquad \begin{cases} -\Delta \overline{w} = \overline{p}\overline{u}^{\overline{p}-1} \overline{w} & \text{in } \Omega \\ \overline{w} \not\equiv 0 & \text{in } \Omega \\ \overline{w} = 0 & \text{on } \partial\Omega \end{cases}$$

which is a contradiction since we assumed that $\overline{u}$ was nondegenerate. □

COROLLARY 4.1. – *If $N = 2$ and $\lambda = 0$ then problem (4.1) has only one solution*

Proof. – The assertion follows from Theorem 4.2 and 4.4 providing so a proof different from that of Theorem 4.1. □

COROLLARY 4.2. – *If $N = 2$ there exists an interval $]\lambda', \lambda''[$ with $-\lambda_1 < \lambda' < 0 < \lambda''$ such that (4.1) has only one solution for any $\lambda \in]\lambda', \lambda''[$*

Proof. – It is a consequence of the nondegeneracy of the only solution in correspondence of $\lambda = 0$. □

Remark 4.1. – Of course the statement of the corollaries above apply also to the ball Ω in $\mathbb{R}^N$ giving in this way an alternative proof of well known results.

5. THE CASE OF $f(u) = u^p + \mu u^q$

Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 2$ and let us consider the problem

$$(5.1) \qquad \begin{cases} -\Delta u = u^p + \mu u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where μ is a real parameter, $q \in]0, 1[$ and $p > 1$ if $N = 2$ or $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$.

Problem (5.1) has been extensively studied in [3] and, among other results, they obtained the following theorem

THEOREM 5.1 [3] – *For all q, p, in the range indicated above there exists $\Lambda > 0$ such that for any $\mu \in]0, \Lambda[$ problem (5.1) has two solutions*