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I.7. Symmetry of Solutions – Moving Plane Method

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i. A symmetry problem in potential theory

This paper appeared in 1971, and can be considered a pioneering work on the theory of symmetry of solutions of elliptic problems. It concerns the following easily formulated question about an overdetermined problem:

Let Ω be a bounded smooth connected open domain in $\mathbb{R}^n$ and suppose that there exists a function $u = u(x)$ satisfying the Poisson equation

$$-\Delta u = 1 \quad in \quad \Omega$$

together with the boundary conditions

$$u = 0\,, \quad u_\nu = \; constant \; on \;\; \partial\Omega$$

where ν is the outer normal to $\partial\Omega$. Must then Ω be a ball?

Serrin answered this question affirmatively and also showed that the solution must be radially symmetric and equal to $\frac{b^2-r^2}{2n}$, where b is the radius of the ball and r denotes the distance from the center.

Like many other results of Serrin, this one has important consequences in applications. For example for a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω it states that "the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section". Another applications was given in the linear theory of torsion of a solid straight bar of cross section Ω in which case Serrin's theorem states that "when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section". It also applies to the study of a liquid rising in a straight capillary tube of cross section Ω.

From a mathematical point of view, in spite of its simple formulation, the question, and hence the result, is important in understanding how the boundary conditions can determine the shape of the domain and of the solution.

Equally important and innovative was Serrin's proof of this result. It is, in fact, in this paper that the, by now classical, "moving plane method" was introduced into the theory of partial differential equations. It had previously been used by A.D. Alexandroff in differential geometry.

This efficient and elegant method is based on the moving parallel planes[1] up to a critical position and showing, via the maximum principle, that the solution is symmetric about the limiting plane. Since then, this method has been used, mainly to prove monotonicity and symmetry results, in a large variety of differential equation problems. To understand its importance, it is enough to think of the famous symmetry result of Gidas, Ni and Nirenberg.

Another important tool used by Serrin in his proof is an extension of Hopf's boundary point Lemma to domains with corners, which he derived in the same paper. This result is known as Serrin's lemma.

Finally, the paper also contains extensions of the above result to more general elliptic equations or to more general boundary conditions involving the mean curvature of the boundary surface $\partial\Omega$, which lead to other consequences in applications.

ii. Symmetry of ground states of quasilinear elliptic equations

This paper addresses the problem of radial symmetry of *non-negative* solutions of elliptic equations of the general form

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0 \qquad \text{in } \mathbb{R}^n, \quad n \geq 2,$$

under the ground state condition

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

Two cases have to be considered:

a) the "regular" case when $[tA(t)]' > 0$ for all $t \geq 0$,
b) the "singular" case when either $A(t)$ is undefined at $t = 0$ or fails to satisfy a) at $t = 0$.

The first case includes, in particular, the mean curvature operator, while a relevant example for the second one is the m-Laplace operator, when $m > 1$.

On the nonlinearity $f(u)$ only the following hypotheses are assumed:

i) f is continuous for $u \geq 0$ and locally Lypschitz continuous for $u > 0$,
ii) $f(0) = 0$ and f is nonincreasing near zero.

Serrin–Zou's symmetry results are the following:

Case a) Any non-negative C^1-weak ground state solution whose (open) support is connected must be radially symmetric about some point and radially decreasing.

Case b) Any non-negative C^1-weak ground state solution with only one critical point where it is positive must be radially symmetric about that point and radially decreasing.

These results are nontrivial extensions of previous theorems, including the classical result by Gidas, Ni and Nirenberg for the Laplace operator ($A(t) \equiv 1$). Among the significant improvements of Serrin–Zou's results are weaker requirements on the nonlinearity $f(u)$ which permit the existence of non-negative solutions with zeros. This is very

[1] by moving parallel planes we mean the motion of a single plane with the property that the plane remains parallel to its initial orientation

important, since it can be proved that solutions with compact support in $\mathbb{R}^n$ do exist. Indeed, necessary and sufficient conditions on $f(u)$ can be given (as shown in a paper by Pucci and Serrin) for such phenomenon to appear. This is related to the celebrated paper of Vazquez for the m-Laplace operator. As a consequence, Serrin–Zou's theorems show that the condition of positivity used in several previous papers can be dropped without affecting the conclusion.

For Case b) some counterexamples, involving the m-Laplace operator, with $m > 2$, are presented to show that some assumption on the critical set of the solution is needed in order to get the radial symmetry. On the other hand, when the operator is singular at $t = 0$ (which, for the m-Laplace operator corresponds to the assumption $1 < m < 2$) the hypothesis on the critical set of the non-negative solution can be replaced by a connectivity assumption. It was shown by Damascelli, Pacella and Ramaswamy that if $1 < m < 2$ and the solution is positive, then no connectivity condition is needed.

Finally Serrin–Zou's approach also applies to more general nonlinearities and even allows to consider singular ground states.

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A Symmetry Problem in Potential Theory

JAMES SERRIN

The following problem has been posed by Professor R. L. FOSDICK. Let Ω be a bounded open connected domain in the Euclidean space R^n having a smooth boundary $\partial\Omega$. Suppose there exists a function $u=u(x)=u(x_1, \ldots, x_n)$ satisfying the Poisson differential equation

$$\Delta u = -1 \quad \text{in } \Omega \tag{1}$$

together with the boundary conditions

$$u=0, \qquad \frac{\partial u}{\partial n}=\text{constant} \quad \text{on } \partial\Omega. \tag{2}$$

Must Ω then be a ball? We shall show here that the answer is affirmative, and that u must have the specific form $(b^2-r^2)/2n$ where b is the radius of the ball and r denotes distance from its center. The precise result is as follows.

Theorem 1. *Let Ω be a domain whose boundary is of class C^2. Suppose there exists a function $u \in C^2(\bar{\Omega})$ satisfying conditions* (1) *and* (2). *Then Ω is a ball and u has the specific form noted above.*

The proof of this result is given in Section 1; in Section 3 we give various generalizations to elliptic differential equations other than (1). Before turning to the detailed arguments it will be of interest to discuss the physical motivation for the problem itself.

Consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω. If we fix rectangular coordinates in space with the z axis directed along the pipe, it is well known that the flow velocity u is then a function of x, y alone satisfying the Poisson differential equation (for $n=2$)

$$\Delta u = -A \quad \text{in } \Omega$$

where A is a constant related to the viscosity and density of the fluid and to the rate of change of pressure per unit length along the pipe. Supplementary to the differential equation one has the adherence condition

$$u=0 \quad \text{on } \partial\Omega.$$

Finally, the tangential stress per unit area on the pipe wall is given by the quantity $\mu\, \partial u/\partial n$, where μ is the viscosity. Our result states that *the tangential stress on the* *pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section.*

Exactly the same differential equation and boundary condition arise in the linear theory of torsion of a solid straight bar of cross section Ω; see [3] pp. 109–119. Theorem 1 then states that, *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.*

A more sophisticated example occurs in the case of a liquid rising in a straight capillary tube of cross section Ω. The function $u(x, y)$ describing the upper surface of the liquid satisfies the differential equation

$$(3) \qquad (1+u_y^2)\,u_{xx}-2u_x u_y u_{xy}+(1+u_x^2)\,u_{yy}=\kappa u(1+u_x^2+u_y^2)^{3/2}$$

where κ is a certain positive constant; also the requirement that the wetting angle γ be constant leads to the boundary condition

$$(4) \qquad \frac{\partial u}{\partial n}=-\cot\gamma=\text{constant} \quad \text{on } \partial\Omega$$

($\vec{n}$ being the inner normal direction). By the theorem of Section 3 it then follows that *provided the wetting angle γ is different from $\pi/2$, a liquid will rise to the same height at each point of the wall of a capillary tube if and only if the tube has circular cross section.* (When $\gamma=\pi/2$, the unique solution of (3), (4) is $u\equiv 0$ for *any* cross sectional form of the tube.)

In the final section of the paper we show that our results can be applied to somewhat more general boundary conditions than (2). A curious consequence of this generalization is the following result. Consider a viscous fluid flowing in straight streamlines through a straight pipe whose cross section is non-circular. Then there must be two points P and Q on the wall, such that the curvature of the wall is greater at P than at Q but the tangential stress is greater at Q than at P. A similar result also holds for the torsion problem.

It may be noted in conclusion that (2) constitutes Cauchy data on the boundary surface for the elliptic equation (1). It is of course well known that such data is generally overdetermined with regard to solving (1) in a given domain Ω; in light of this remark, our results provide a concrete example where the overdetermined nature of the condition can be rigorously analyzed.

1. Proof of Theorem 1

Let T_0 by a hyperplane in R^n not intersecting the domain Ω. We suppose this plane to be continuously moved normal to itself to new positions, until ultimately it begins to intersect Ω. From that moment onward, at each stage of the motion the resulting plane T will cut off from Ω a cap $\Sigma(T)$: that is, $\Sigma(T)$ will be that portion of Ω which lies on the same side of T as the original plane T_0.

For any cap $\Sigma(T)$ thus formed, we let $\Sigma'(T)$ be its reflection in T. Evidently $\Sigma'(T)$ will be contained in Ω at the beginning of the process; and indeed as T advances into Ω *the resulting cap $\Sigma'(T)$ will stay within Ω at least until one of the following two events occurs:*

(*i*) $\Sigma'(T)$ becomes internally tangent to the boundary of Ω at some point P not on T, or

(*ii*) T reaches a position where it is orthogonal to the boundary of Ω at some point Q.

We denote the plane T when it reaches either one of these positions by T'.

We now assert that Ω must be symmetric about T'. In fact, if this assertion is proved the theorem follows immediately. To see this, we observe that for *any given direction* in R^n there would then be a plane T' with normal in that direction such that Ω is symmetric about T'. Moreover, according to the construction Ω would have to be simply connected. But the only simply connected domains which have the symmetry property just noted are balls. Having thus proved that Ω is a ball we say that the function $(b^2-r^2)/2n$ is then the unique solution of the given boundary value problem ([2], pages 68–69).

In proving the assertion, we introduce a new function v defined in $\Sigma' \equiv \Sigma'(T')$ by the formula

(5) $$v(x)=u(x') \quad (x\in\Sigma')$$

where x' is the reflected value of x across T'. Evidently v satisfies the differential equation

$$\Delta v=-1 \quad \text{in } \Sigma'$$

and the boundary conditions

$$\begin{aligned} &v=u && \text{on } \partial\Sigma'\cap T' \\ v=0, \quad &\frac{\partial v}{\partial n}=\text{constant} && \text{on } \partial\Sigma'\cap \operatorname{Comp}(T'), \end{aligned}$$

the constant being the same as in (2).

Since Σ' is contained in Ω by construction, we may consider the function $u-v$ in Σ'. Evidently

(6) $$\Delta(u-v)=0 \quad \text{in } \Sigma'$$

and

(7) $$\begin{aligned} &u-v=0 \quad \text{on } \partial\Sigma'\cap T' \\ &u-v\geqq 0 \quad \text{on } \partial\Sigma'\cap \operatorname{Comp}(T'); \end{aligned}$$

the latter condition is a consequence of the fact that $u>0$ in Ω.* If we apply the strong version of the maximum principle ([2], page 53), it is easy to see from (6) and (7) that either

(8) $$u-v>0 \quad \text{at all interior points of } \Sigma'$$

or else $u\equiv v$ in Σ'. In the latter case it is clear that the reflected cap Σ' must coincide with that part of Ω on the same side of T' as Σ'; that is, Ω must be symmetric about T'.

* For otherwise u would have a minimum at some interior point of Ω, which is impossible since $\Delta u=-1$.

To complete the proof of the theorem it must therefore be shown that (8) is impossible. Suppose first that we are in case (*i*), that is, Σ' is internally tangent to the boundary of Ω at some point P not on T'. Then $u-v=0$ at P. Consequently, using (6), (8) and the boundary point version of the maximum principle ([2], page 65), we conclude that

$$\frac{\partial}{\partial n}(u-v)>0 \quad \text{at } P.$$

This however contradicts the fact that $\partial u/\partial n=\partial v/\partial n=$constant at P. Hence (8) is impossible in case (*i*).

In case (*ii*) the situation is more complicated, for even though $u-v=0$ at Q the boundary point version of the maximum principle does not apply (this is because Q is a right angled corner of Σ' and the requisite internally tangent ball [see [2], page 65] is not available). Consequently we must proceed in an alternate fashion. It will be shown (*a*) that $u-v$ has a zero of second order at Q and then (*b*) a contradiction will be obtained from a more delicate version of the boundary point maximum principle.

(*a*) By hypothesis the boundary of Ω is of class C^2. Consider a rectangular coordinate frame with origin at Q, the x_n axis being directed along the inward normal to $\partial\Omega$ at Q, and the x_1 axis being normal to T'. In this frame we can represent the boundary of Ω locally by the equation

$$x_n=\phi(x_1,\dots,x_{n-1}), \qquad \phi\in C^2.$$

Since u is in $C^2(\bar{\Omega})$ the condition $u=0$ on $\partial\Omega$ can then be expressed as a twice differentiable identity

$$u(x_1,\dots,x_{n-1},\phi)\equiv 0. \tag{9}$$

Similarly, the boundary condition $\partial u/\partial n=$constant$=c$ on $\partial\Omega$ can be written as an identity,

$$\frac{\partial u}{\partial x_n}-\sum_1^{n-1}\frac{\partial u}{\partial x_k}\frac{\partial\phi}{\partial x_k}=c\left\{1+\sum_1^{n-1}\left(\frac{\partial\phi}{\partial x_k}\right)^2\right\}^{1/2} \tag{10}$$

where x_n is to be replaced throughout by $\phi(x_1,\dots,x_{n-1})$.

At this stage some simple notation will be convenient; thus

$$u_i=\frac{\partial u}{\partial x_i} \quad (i=1,\dots,n-1); \qquad u_n=\frac{\partial u}{\partial x_n}.$$

Differentiating (9) with respect to x_i, $i=1,\dots,n-1$, we now obtain

$$u_i+u_n\phi_i=0. \tag{11}$$

Evaluating this at Q, where $\phi_i=0$, we find

$$u_i=0, \qquad u_n=c \quad (\text{at } Q).$$

Next differentiating (11) with respect to x_j, $j=1,\dots,n-1$, and evaluating at Q yields

$$u_{ij}+c\,\phi_{ij}=0 \quad (\text{at } Q)$$

where $u_{ij}=\partial^2 u/\partial x_i\,\partial x_j$. Lastly, differentiating (10) with x_l and evaluating at Q gives

$$u_{nl}=0 \quad (\text{at } Q).$$

Since also $u_{nn}=-\sum_1^{n-1} u_{ii}-1=c\Delta\varphi-1$ at Q, we have accordingly determined all the first and second derivatives of u at Q.

Since the reflected cap Σ' lies inside Ω, it is not hard to see that the second derivatives of ϕ must also satisfy

$$\phi_{1l}=0 \quad \text{at } Q, \qquad l=2,\dots,n-1.$$

Taking these relations into account, and observing that (5) implies

$$v(x_1,x_2,\dots,x_n)=u(-x_1,x_2,\dots,x_n),$$

we find that the first and second derivatives of u and v agree at Q. This completes the proof of (a).

Turning now to (b), we require the following preliminary result.

Lemma 1. *Let D^* be a domain with C^2 boundary and let T be a plane containing the normal to ∂D^* at some point Q. Let D then denote the portion of D^* lying on some particular side of T.*

Suppose that w is of class C^2 in the closure of D and satisfies

$$\Delta w \leqq 0 \quad in\ D,$$

while also $w\geqq 0$ in D and $w=0$ at Q. Let $\vec{s}$ be any direction at Q which enters D non-tangentially. Then either

$$\frac{\partial w}{\partial s}>0 \quad or \quad \frac{\partial^2 w}{\partial s^2}>0 \quad at\ Q$$

unless $w\equiv 0$.

The proof of this result will be given in Section 2. Assuming that the lemma holds, we may apply it to the function $w=u-v$ in Σ'. Since $w>0$ there, and $w=0$ at Q, this yields

$$\frac{\partial(u-v)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial s^2}>0 \quad \text{at } Q$$

contradicting the fact that both u and v have the same first and second partial derivatives at Q. This completes the proof of the theorem.

2. Proof of Lemma 1

Let K_1 be a ball which is internally tangent to D^* at Q, and which touches the boundary of D^* only at Q. Such a ball exists by virtue of the fact that the boundary of D^* is of class C^2.

Construct a ball K_2 with center at Q and radius $\frac{1}{2}r_1$ where r_1 is the radius of K_1. Finally let $K'=K_1\cap K_2\cap D$. Now define the auxiliary function

$$z=z(x)=x_1\left(e^{-\alpha r^2}-e^{-\alpha r_1^2}\right)$$

where α is a positive constant to be determined (we have chosen coordinates with origin at the center of K_1 and with T being the plane $x_1=0$; moreover it can be assumed that D is on the side of T where $x_1>0$). It is clear that

(12) $$z>0 \quad \text{in } K', \qquad z=0 \quad \text{on } \partial K_1 \text{ and on } T$$

(at this stage we mean by K' the *interior* of the region described, i.e., K_1, K_2 and D are taken to be open sets). We compute

$$\Delta z=2\alpha x_1 e^{-\alpha r^2}\{2\alpha r^2-(n+2)\}.$$

Now $r \geqq \frac{1}{2} r_1$ in K' so that by choosing α suitably large, say $\alpha=(n+2) r_1^{-2}$, we obtain $\Delta z>0$ in K'.

Now suppose w is not identically zero in D. Then by the strong maximum principle we have $w>0$ in D. We consider the part of the boundary of K' lying on ∂K_2. This set intersects the boundary of D only on the plane T. Moreover the intersection set lies at a finite distance from the corners of D. By virtue of the boundary point lemma ([2], page 65), therefore, it is not hard to see that there exists a constant $\varepsilon>0$ such that

$$w \geqq \varepsilon x_1 \quad \text{on } \partial K' \cap \partial K_2 .$$

Moreover

$$w \geqq 0 \quad \text{on } \partial K' \cap \partial K_1 \text{ and } \partial K' \cap T.$$

On the other hand, it is clear that

(13) $$z \leqq x_1 \quad \text{on } \partial K' \cap \partial K_2 .$$

Consequently, by use of (12), the function $w-\varepsilon z$ is non-negative on the entire boundary of K', and is zero at Q. Moreover

$$\Delta(w-\varepsilon z)=\Delta w-\varepsilon \Delta z<0$$

in K'. By the maximum principle, therefore, $w-\varepsilon z>0$ in K'. Hence at Q, where $w-\varepsilon z=0$, we have either

$$\frac{\partial(w-\varepsilon z)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(w-\varepsilon z)}{\partial s^2} \geqq 0 .$$

Since by direct calculation

$$\frac{\partial z}{\partial s}=0, \quad \frac{\partial^2 z}{\partial s^2}>0 \quad \text{at } Q,$$

this completes the proof of the lemma.

3. More General Elliptic Equations

If the proof of Theorem 1 is re-examined, one finds that the properties of the Poisson equation which were applied are the following.

(A) It is invariant to the symmetric substitution $x \to x'$.

(B) The second partial derivatives u_{nn} in an arbitrary rectangular coordinate frame can be determined in terms of the remaining second partial derivatives.

(C) The difference of two solutions obeys the strong maximum principle, and

(D) The difference of two solutions obeys both boundary point versions of the maximum principle.

Now conditions (A) and (B) are satisfied by any nonlinear *elliptic* differential equation of the form

$$a\Delta u + h u_i u_j u_{ij} = f \tag{14}$$

provided that a, f and h are functions only of u and $|p|$, where $p=(u_1, \dots, u_n)$ denotes the gradient vector of the solution.* To see this, note that (14) is invariant under coordinate rotations as well as under substitutions $(x_1, \dots, x_n) \to (-x_1, \dots, x_n)$ which change the sign of a single coordinate.

On the other hand, the strong maximum principle does *not* hold for differences of solutions of (14) in the form which was applied in the proof of Theorem 1. Nevertheless, a somewhat restricted version does remain valid. Suppose in fact that the functions a, f and $h p_i p_j$ are continuously differentiable in u and p, and that u and v are two solutions in a domain D such that $u-v \geqq 0$ in D and $u-v=0$ at some interior point of D. Then according to a theorem of E. HOPF ([1], pages 149–150) we have $u \equiv v$.

In the proof of Theorem 1 the maximum principle was used only to show that either (8) holds or that $u \equiv v$ in Σ'. This conclusion can be reached by an alternate argument, however, using only the restricted version of the maximum principle.

Deferring this proof for a moment, it remains only to consider the two boundary point versions of the maximum principle. In this regard, the standard boundary point lemma ([2], page 65) applies immediately to the difference of solutions of (14) since this difference is zero at the point P. (The argument is the same as for the restricted version of the maximum principle above.) Finally, the required genralization of Lemma 1 needed for application to the difference of solutions of (14) will be proved in the following section.

We may thus turn our attention to showing that either (8) holds or else $u \equiv v$ in Σ'. For this purpose we shall make an additional assumption about the behavior of the solution, namely**

$$u > 0 \quad \text{in } \Omega. \tag{15}$$

(We note that this assumption applies automatically in the case of Theorem 1, as a consequence of (1), (2).) Letting c denote the constant which occurs in the boundary condition (2), we assert to begin with that either $c>0$ or else $\partial^2 u/\partial n^2 > 0$ at each point of the boundary. Thus suppose $c=0$. Then, according to the formulas given in part (a) of the previous proof, if we introduce at any fixed point P of the boundary a special coordinate frame with x_n axis directed along the inner normal to the boundary, all the first and second derivatives of u vanish at P except $\partial^2 u/\partial n^2 = u_{nn}$, and this derivative has the value

$$\frac{f(0,0)}{a(0,0)} = d$$

* From here onward we adopt the standard convention that repeated indices are to be summed from 1 to n.

** Recall that Ω is assumed to be open so that (15) is not in conflict with the given boundary condition (2).

according to (14). (Note that $a>0$ for all values of u and p because (14) is elliptic.)

Since $c=0$ by supposition, the inequality $d<0$ is obviously contradictory to (15); it remains to show that $d=0$ also cannot occur. In such a case, however, we would have $f(0,0)=0$. Then the function $\bar{u}\equiv 0$ would be a solution of (14), and correspondingly the solution u under consideration could be regarded as the *difference* of two solutions. Then in view of (15) we could apply the standard boundary point lemma to infer that $\partial u/\partial n>0$ at every point of the boundary, contradicting the original supposition that $c=0$. Hence $d>0$.

From what has just been proved it is now clear that u increases monotonically as one enters Ω along any non-tangential direction $\vec{s}$, for some positive distance so into the domain. Moreover, s_0 can be chosen independent of position on $\partial\Omega$, depending only on a bound for $\vec{s}$ away from the tangent direction.

Let us now return to the opening stages of the proof of Theorem 1. By what has been shown it is clear that immediately after T has penetrated Ω not only will $\Sigma'(T)$ be contained in Ω but also

$$(16) \qquad u>v \quad \text{at interior points of } \Sigma'(T).$$

Here we construct v in $\Sigma'(T)$ in exactly the same way as the previous construction of v in $\Sigma'(T')$.

We assert that (16) persists as T advances into Ω, for all positions of T prior to T'. Suppose in fact that there is some instant where (16) fails prior to T reaching T', Then there would be a position T'' of T such that either

$$(17) \qquad u\geqq v \quad \text{in } \Sigma'(T''), \qquad u=v \quad \text{at some interior point of } \Sigma'(T'')$$

or

$$(18) \qquad \begin{array}{ll} u>v & \text{at interior points of } \Sigma'(T'') \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n} & \text{at some interior point of } T''\cap\Omega. \end{array}$$

If (17) holds we reach a contradiction with the restricted version of the maximum principle, and if (18) holds, with the boundary point principle. Thus the assertion is proved.

It follows by continuity that when T reaches T' we have

$$u\geqq v \quad \text{at interior points of } \Sigma'.$$

By applying the restricted maximum principle once more we see that either the strict inequality holds or else $u\equiv v$. But this is what we originally set out to demonstrate. Summing up, we have proved the following result.

Theorem 2. *Let Ω be a domain whose boundary is of class C^2. Suppose there exists a function $u\in C^2(\bar{\Omega})$ satisfying the elliptic differential equation*

$$(19) \qquad a(u,|p|)\Delta u+h(u,|p|)u_i u_j u_{ij}=f(u,|p|) \quad \text{in } \Omega$$

where a, f and $h\,p_i\,p_j$ are continuously differentiable functions of u and p (here $p=(u_1,\ldots,u_n)$ denotes the gradient vector of u). Suppose also that

$$(20) \qquad u>0 \quad \text{in } \Omega$$

and that u satisfies the boundary conditions

$$u=0, \quad \frac{\partial u}{\partial n}=\text{constant} \quad on\ \partial\Omega. \tag{2}$$

Then Ω must be a ball and u is radially symmetric.

Remarks 1. It is clear that condition (20) can be replaced by the alternate assumption that $u<0$ in Ω. Also (20) can be deleted from the statement of the theorem if it is assumed that f is never zero. To see this, note that if $f<0$, say, then u cannot take on an interior minimum value, so that the boundary condition $u=0$ on $\partial\Omega$ implies $u>0$ at interior points of Ω. In particular, one has $f=-1$ for equation (1), explaining why it was unnecessary to make the explicit assumption $u>0$ in Theorem 1.

2. Equation (3) describing the upper surface of a liquid in a capillary tube (i.e. a liquid under the combined influence of gravity and surface tension) is a special case of (19). The result stated at the end of the introduction is therefore a consequence of the following remark. *Let u be a solution of* (3), (4) *in Ω, such that $u=\alpha=$constant on $\partial\Omega$. Suppose that $\gamma\neq\pi/2$. Then $u\neq\alpha$ in Ω.* Proof. Assume first $0<\gamma<\pi/2$. Then by (4), since solutions of (3) can take on neither a positive maximum nor a negative minimum in Ω, we have $\alpha>0$ and $0\leqq u<\alpha$ in Ω. Similarly if $\gamma>\pi/2$ then $\alpha<0$ and $\alpha<u\leqq 0$ in Ω.

A further class of equations to which Theorem 2 is applicable can be obtained from regular variational problems of the form

$$\delta\int F(u,|p|)\,dx=0. \tag{21}$$

Indeed if we assume that $F(u,t)$ is three times differentiable, and that $F'(u,0)=F'''(u,0)=0$ for consistency, then the Euler-Lagrange equation for (21) can be written in the form (19) with

$$a=1, \quad h=|p|^{-2}\left(\frac{|p|F''}{F'}-1\right), \quad f=\frac{|p|}{F'}(F_u-|p|F'_u)$$

where primes here denote differentiation with respect to the second argument of F.

3. Under the hypotheses of the theorem we cannot conclude any more about the form of u then that it must be a function of the radial distance r from the center of Ω. If we write $u=u(r)$ and use primes to denote differentiation with respect to r, then one obtains the differential equation

$$a(u,|u'|)\left(u''+\frac{n-1}{r}u'\right)+h(u,|u'|)u'^2u''=f(u,|u'|), \tag{22}$$

for $0\leqq r\leqq b$ with the end conditions

$$u'(0)=0, \quad u(b)=0. \tag{23}$$

The ellipticity of (20) implies

$$a(u,|u'|)+h(u,|u'|)|u'|^2>0$$

so that in any case (22) is non-singular and can be written in the form

$$u''=F(r,u,u'). \tag{22'}$$

To make further progress it may be assumed that a and h are independent of u and that $\partial f/\partial u \geqq 0$, as is the case for the equation describing a capillary surface. Then standard arguments show that there cannot be more than one solution h. There would be some interest in pursuing the discussion of equation (22) further, as JOHNSON and PERKO have done in the case of the capillary equation (Arch. Rational Mech. Analysis, vol. 29).

4. It is of interest to observe that without some assumption of the type described in the preceeding remark, equation (22) may have more than one solution (or even infinitely many solutions) satisfying the end conditions (23). In consequence, the property of having unique solutions corresponding to given boundary data is *not* a prerequisite for Theorem 2 to hold for a given equation.

4. The Boundary Point Lemma at a Corner

Here we shall prove a generalization of Lemma 1 suitable for application to non-linear elliptic equations. We then use this result to obtain the conclusion

$$\frac{\partial(u-v)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial s^2}>0 \quad \text{at } Q \tag{24}$$

which is needed, just as in the case of Theorem 1, for completing the proof of Theorem 2.

Lemma 2. *Let D be a domain of the type described in Lemma 1. Suppose that w is of class C^2 in the closure of D and satisfies the elliptic differential inequality*

$$Lw=a_{ij}(x)\,w_{ij}+b_i(x)\,w_i\leqq 0 \quad \textit{in } D$$

where the coefficients are uniformly bounded. We assume that the matrix a_{ij} is uniformly definite

$$a_{ij}(x)\,\xi_i\,\xi_j\geqq\kappa\,|\xi|^2 \quad (\kappa=\text{constant}>0), \tag{25}$$

and that

$$|a_{ij}\,\xi_i\,\eta_j|\leqq K(|\xi\cdot\eta|+|\xi|\cdot|d|) \quad (K=\text{constant}>0) \tag{26}$$

where $\xi=(\xi_1,\ldots,\xi_n)$ is an arbitrary real vector, $\eta=(\eta_1,\ldots,\eta_n)$ is the unit normal to the plane T, and d is the distance from T.

Suppose also that $w\geqq 0$ in D and $w=0$ at Q. Then either

$$\frac{\partial w}{\partial s}>0 \quad \textit{or} \quad \frac{\partial^2 w}{\partial s^2}>0 \quad \textit{at } Q$$

unless $w\equiv 0$, where $\vec{s}$ is any direction at Q which enters D non-tangentially.

Proof. We proceed in the same way as in the proof of Lemma 1, except that the comparison function z and the ball K_2 must be chosen with greater care.

As in the proof of Lemma 1, we introduce the ball K_1, and then construct K_2 to have center at Q and radius θr_1, where $\theta\leqq 1/2$ is a constant to be determined.

Then in $K'=K_1\cap K_2\cap D$ we define the auxiliary function

$$z=z(x)=\{e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2}\}\cdot\{e^{-\alpha r^2}-e^{-\alpha r_1^2}\}$$

where α is also to be determined. With the choice of coordinates as before, it is clear that

$$z>0 \quad \text{in } K', \qquad z=0 \quad \text{on } \partial K_1 \text{ and on } T.$$

We compute

$$\begin{aligned} Lz=&e^{-\alpha r^2}(e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2})\cdot\{4\alpha^2 a_{ij}x_i x_j-2\alpha[a_{ii}+b_i x_i]\}\\ &+e^{-\alpha(x_1-r_1)^2}(e^{-\alpha r^2}-e^{-\alpha r_1^2})\cdot\{4\alpha^2 a_{11}(x_1-r_1)^2-2\alpha[a_{11}+b_1(x_1-r_1)]\}\\ &+8\alpha^2 e^{-\alpha r^2}e^{-\alpha(x_1-r_1)^2}\cdot(x_1-r_1)\,a_{1j}x_j. \end{aligned}$$

Because of the ellipticity condition (25)

$$a_{ij}x_i x_j \geqq \kappa r^2 \geqq \tfrac{1}{4}\kappa r_1^2 \quad \text{in } K'$$

and, for the same reason, $a_{11}(x_1-r_1)^2\geqq\frac{1}{4}\kappa r_1^2$ in K'. Moreover by (26),

$$|a_{1j}x_j|=|a_{ij}\eta_i x_j|\leqq K(|x_1|+|x_1|)$$

since in the present case $\eta=(1,0,\ldots,0)$. Thus

$$|(x_1-r_1)\,a_{ij}x_j|\leqq 2x_1 r_1 K \quad \text{in } K'.$$

Finally we observe that by the mean value theorem

$$e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2}\geqq 2\alpha(1-\theta)\,r_1 e^{-\alpha r_1^2}x_1\geqq\alpha x_1 r_1 e^{-2\alpha\theta r_1^2-\alpha(x_1-r_1)^2}.$$

Inserting these inequalities into the earlier expression for Lz, and using the fact that the terms $[a_{ii}+b_i x_i]$ and $[a_{11}+b_1(x_1-r_1)]$ are bounded, we find for large α

$$\begin{aligned} Lz\geqq&\alpha^2 x_1 r_1 e^{-\alpha(r^2+(x_1-r_1)^2)}\cdot\{(\alpha\kappa r_1^2-B)\,e^{-2\alpha\theta r_1^2}-16K\}\\ &+\alpha e^{-\alpha(x_1-r_1)^2}(e^{-\alpha r^2}-e^{-\alpha r_1^2})\{\alpha\kappa r_1^2-B\} \end{aligned}$$

where B is an appropriate constant. By choosing $\theta=1/\alpha$ and then α suitably large it is clear that we can make the quantities in braces positive. That is, we have now constructed a function z such that $Lz>0$ in K'.

The remaining part of the proof is the same as for Lemma 1, since both the maximum principle and the boundary point lemma apply to the elliptic operator L in the same way as to the Laplace operator ([2], pages 61 and 65). The only slight difference to be noted is that (13) must now be replaced by

$$z\leqq 2\alpha r_1 x_1, \tag{13'}$$

which follows easily from the mean value theorem and the definition of z. Consequently the function $w-\varepsilon z$ of Lemma 1 must be replaced by $w-(\varepsilon/2\alpha r_1)\,z$ but otherwise the argument is left unchanged. This completes the demonstration of Lemma 2.

To complete the proof of Theorem 2 we must show that the final argument in the proof of Theorem 1 can be carried over to the more general case of equation (19). Thus let u and v be, respectively, the original solution and the reflected solution in the region Σ'. It has already been established that either $u-v>0$ or $u\equiv v$ in Σ'. We must show in the former case that (24) holds (the plane $T=T'$ being in the position indicated by case (*ii*) of the proof of Theorem 1). We begin by obtaining an appropriate second order differential equation for the difference function $u-v$.

Since both u and v satisfy (19) we have

$$a[u]\Delta u+h[u]u_i u_j u_{ij}=f[u]$$

$$a[v]\Delta v+h[v]v_i v_j v_{ij}=f[v]$$

where $a[u]\equiv a(u,|p|)$ and similarly for the other square brackets. Differencing these equations yields easily

$$\begin{aligned}&\{a[u]+a[v]\}\Delta(u-v)+\{h[u]u_i u_j+h[v]v_i v_j\}(u-v)_{ij}\\ &\qquad+\{a[u]-a[v]\}\Delta(u+v)+\{h[u]u_i u_j-h[v]v_i v_j\}(u+v)_{ij}=2\{f[u]-f[v]\}.\end{aligned}$$

Now by the mean value theorem of multidimensional calculus

$$a[u]-a[v]=\left(\frac{\partial a}{\partial u}\right)_0(u-v)+\left(\frac{\partial a}{\partial p_i}\right)_0(u-v)_i$$

with similar expressions for $h[u]u_i u_j-h[v]v_i v_j$ and $f[u]-f[v]$. Thus if we define

$$a_{ij}(x)=\{a[u]+a[v]\}\delta_{ij}+\{h[u]u_i u_j+h[v]v_i v_j\}$$

it follows from the above identity that

$$a_{ij}(x)(u-v)_{ij}+b_i(x)(u-v)_i+c(x)(u-v)=0 \tag{27}$$

where $b_i(x)$ and $c(x)$ are certain bounded functions.

Here the matrix a_{ij} is uniformly definite:

$$a_{ij}\xi_i\xi_j\geqq\kappa|\xi|^2$$

since both expressions $a[u]\delta_{ij}+h[u]u_i u_j$ and $a[v]\delta_{ij}+h[v]v_i v_j$ have this property (recall that equation (19) is elliptic). Consider next the expression $a_{ij}\xi_i\eta_j$ for which the estimate (26) must be established. We have by computation

$$a_{ij}\xi_i\eta_j=\{a[u]+a[v]\}\xi\cdot\eta+\{h[u](p\cdot\xi)(p\cdot\eta)+h[v](q\cdot\xi)(q\cdot\eta)\} \tag{28}$$

where $p=(u_1,\dots,u_n)$ and $q=(v_1,\dots,v_n)$. Now v is the reflection of u in the plane $T=T'$. Hence $u=v$ on T, and moreover (in the usual rectangular coordinate frame centered at Q with x_1 axis normal to T)

$$p=(u_1,u_2,\dots,u_n),\qquad q=(-u_1,u_2,\dots,u_n)\quad\text{on } T.$$

Thus on T there holds

$$|p|=|q|,\qquad a[u]=a[v],\qquad h[u]=h[v];$$

and, since $\eta=(1, 0, \ldots, 0)$,

$$I \equiv h[u](p\cdot\xi)(p\cdot\eta)+h[v](q\cdot\xi)(q\cdot\eta)$$
$$=h[u]\,u_1(p-q)\cdot\xi=2h[u]\,u_1^2\xi_1=2h[u]\,u_1^2\,\xi\cdot\eta.$$

By continuity it then follows that off T

$$|I|\leqq M(|\xi\cdot\eta|+|\xi|\cdot|x_1|) \tag{29}$$

for some constant M. Noting that also $\{a[u]+a[v]\}\leqq M'$ for some M', we have finally from (28) and (29)

$$|a_{ij}\xi_i\eta_j|\leqq K(|\xi\cdot\eta|+|\xi|\cdot|x_1|) \qquad (K=M+M')$$

as required.

The differential equation (27) for $u-v$ is not quite in the form to which Lemma 2 applies. Hence as in the proof of the restricted maximum principle we make HOPF's substitution $w=(u-v)\,e^{\beta x_1}$. Then obviously

$$a_{ij}(x)\,w_{ij}+\tilde{b}_i(x)\,w_i+\tilde{c}(x)\,w=0$$

where $\tilde{b}_i=b_i-2\beta a_{1i}$, $\tilde{c}=a_{11}\beta^2-b_1\beta+c$. Since $u-v\geqq 0$ by assumption it follows that for β sufficiently large

$$a_{ij}(x)\,w_{ij}+\tilde{b}_i(x)\,w_i\leqq 0.$$

Consequently Lemma 2 may be applied directly to w (note that w satisfies the hypotheses of Lemma 2 since $u-v=0$ at Q); we find therefore

$$\frac{\partial w}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2}>0 \quad \text{at } Q$$

since $w\not\equiv 0$. But at Q

$$\frac{\partial(u-v)}{\partial s}=\frac{\partial w}{\partial s}, \qquad \frac{\partial^2(u-v)}{\partial s^2}=\frac{\partial^2 w}{\partial s^2}-2\beta s_1\frac{\partial w}{\partial s},$$

so that also

$$\frac{\partial(u-v)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial s^2}>0 \quad \text{at } Q.$$

Thus the difference $u-v$ satisfies (24), and the proof of Theorem 2 is complete.

5. A Different Boundary Condition

The previous results can be extended to more general boundary conditions than (2) without changing the basic method. We let H denote the *mean curvature* of the boundary surface $\partial\Omega$, chosen so that H is positive when the surface is "convex." Analytically, if a portion of the boundary is locally represented by the equation

$$x_n=\phi(x_1, \ldots, x_{n-1})$$

with the x_n axis directed into Ω, then on this part

$$H=\frac{(1+\phi_k\phi_k)\,\Delta\phi-\phi_i\phi_j\phi_{ij}}{(n-1)(1+\phi_k\phi_k)^{3/2}}. \tag{30}$$

We now replace (2) by the more general condition

$$u=0, \quad \frac{\partial u}{\partial n}=c(H) \quad \text{on } \partial\Omega \tag{31}$$

where c is a continuously differentiable non-decreasing function of H. Then the following result holds

Theorem 3. *The conclusions of Theorems* 1 *and* 2 *remain valid under the more general boundary condition* (31).

Proof. It will be enough to show how the proof of Theorem 1 needs to be modified to cover the more general boundary condition (31), and then to make several comments concerning the further modification required for the case of equation (19).

For the generalization of Theorem 1, then, we proceed as in the initial steps of the proof given in Section 1 to show that either (8) holds or $u\equiv v$ in Σ'. To prove then that (8) is impossible, suppose first that case (*i*) holds in the definition of Σ'. Then $u-v=0$ at P, and by the boundary point version of the maximum principle we conclude as before that

$$\frac{\partial}{\partial n}(u-v)>0 \quad \text{at } P. \tag{32}$$

On the other hand, if H is the mean curvature of $\partial\Omega$ at P and H' is the mean curvature of $\partial\Sigma'$ at P, then $H\leqq H'$ since $\Sigma'\subset\Omega$. Consequently

$$\frac{\partial u}{\partial n}=c(H)\leqq c(H')=\frac{\partial v}{\partial n}$$

which contradicts (32). Hence (8) is impossible in case (*i*).

In case (*ii*) we must further analyze the second partial derivatives of u and v at Q. For this purpose we assume for simplicity that the boundary of Ω is of class C^3, though by taking more care one can weaken this assumption. Fixing coordinates as before, we obtain then

$$u_i=0, \quad u_n=c, \quad u_{ij}+c\phi_{ij} \quad (\text{at } Q)$$

where i, j range from 1 to $n-1$. Moreover, differentiating (10) and evaluating at Q gives

$$u_{ni}=\frac{dc}{dH}H_i \quad (\text{at } Q)$$

since c is no longer constant. Also from (30),

$$H_i=(\Delta\varphi)_i/(n-1) \quad (\text{at } Q).$$

Since v is the reflection of u, it now follows that all the first and second partial derivatives of these functions agree at Q, with the possible exception of u_{1n} and v_{1n}, which have the values

$$u_{1n}=-v_{1n}=\frac{dc}{dH}H_1 \quad (\text{at } Q). \tag{33}$$

Now Σ' is contained in Ω; continuing the analysis of this situation (which has already yielded the relations $\phi_{1l}=0$ at Q), we find that the third partial derivatives of ϕ at Q must satisfy the condition

$$\varepsilon\{\phi_{lm1}\zeta_l\zeta_m+\phi_{111}\}\leqq 0 \quad (\varepsilon=\operatorname{sign} x_1 \text{ in } \Sigma')$$

where $\zeta=(\zeta_2, \ldots, \zeta_{n-1})$ is an arbitrary real vector and the indices l and m are to be summed from 2 to $n-1$. It follows that

$$\phi_{111}\leqq 0, \quad \varphi_{ll1}\leqq 0 \quad (\text{at } Q) \tag{34}$$

since Σ' may be assumed to lie in the space $x_1>0$. Clearly (34) implies

$$H_1=(\Delta\varphi)_1/(n-1)\leqq 0.$$

Thus by (33), recalling that $dc/dH\geqq 0$, we find

$$u_{1n}\leqq 0, \quad v_{1n}\geqq 0 \quad (\text{at } Q).$$

But these are the only (possibly) unequal partial derivatives of u, v at Q. Since $u>v$ in Σ', it follows that in fact $u_{1n}=v_{1n}=0$, at Q. Hence all the first and second partial derivatives of u and v agree at Q. The remainder of the proof is the same as before.

The argument for the elliptic equation (19) proceeds almost exactly as it did earlier in Section 2, though of course taking into account the modifications in the proof of Theorem 1 which were described above.

The only slight change required involves the demonstration that u increases monotonically as one enters Ω along any non-tangential direction $\vec{s}$. The difficulty is that one might have $c(H)=0$ at some points of the boundary and $c(H)>0$ at others (of course $c(H)$ can never be negative at any boundary point because $u>0$ in Ω). In any case, just as in Section 3 one can show that at any point where $\partial u/\partial n=0$ all the first and second partial derivatives of u vanish except for $\partial^2 u/\partial n^2$, and this is positive. A not too difficult compactness argument then yields the required monotonicity property of u near the boundary. This completes the proof of Theorem 3.

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Symmetry of Ground States of Quasilinear Elliptic Equations

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Abstract

We consider the problem of radial symmetry for *non-negative* continuously differentiable weak solutions of elliptic equations of the form

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 2, \tag{1}$$

under the ground state condition

$$u(x) \to 0 \text{ as } |x| \to \infty. \tag{2}$$

Using the well-known moving plane method of Alexandrov and Serrin, we show, under suitable conditions on A and f, that all ground states of (1) are radially symmetric about some origin O in $\boldsymbol{R}^n$. In particular, we obtain radial symmetry for compactly supported ground states and give sufficient conditions for the positivity of ground states in terms of the given operator A and the nonlinearity f.

1. Introduction

We consider the problem of radial symmetry for non-negative solutions of elliptic equations of the form

$$\operatorname{div}(A(|Du|)Du) + f(u) = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 2, \tag{1.1}$$

under the ground state condition

$$u(x) \to 0 \text{ as } |x| \to \infty. \tag{1.2}$$

This problem was first considered by Gidas, Ni & Nirenberg [10] for the case of the Laplace operator ($A(t) \equiv 1$), that is, for the equation

$$\Delta u + f(u) = 0, \quad x \in \boldsymbol{R}^n,$$

and for C^1 functions $f(u)$ which are (essentially) of class $C^{1+\gamma}$ near $u = 0$ and satisfy $f(0) = 0$, $f'(0) < 0$. Under the additional restriction that solutions are *positive* in $\boldsymbol{R}^n$, they showed that corresponding ground states are radially symmetric about some origin O in $\boldsymbol{R}^n$. Their work depends crucially on the moving plane method of ALEXANDROV and SERRIN (as also does that in the present paper), as well as upon appropriate asymptotic estimates for the behavior of ground states as $|x| \to \infty$.

Here we extend their result in three main and comprehensive directions — first, by asking of the function $f(u)$ only the following properties,

(i) f is continuous for $u \geqq 0$ and locally Lipschitz continuous for $u > 0$, and
(ii) $f(0) = 0$ and f is non-increasing on some interval $0 \leqq u < \delta$;

second, by replacing the Laplace operator with appropriate quasilinear operators for which (1.1) is locally uniformly elliptic for any vector $Du \in \boldsymbol{R}^n$; and third by treating degenerate operators.

In particular, our conclusions cover as special cases the Laplace operator, the mean curvature operator

$$A(t) = \frac{1}{\sqrt{1+t^2}} \qquad \text{(see Theorem 1}'\text{)},$$

and the degenerate Laplace operator

$$A(t) = |t|^{m-2}, \ m > 1 \qquad \text{(Theorems 2, 3)}.$$

The importance of (1.1) in applications is discussed in [3], including the case when f is not Lipschitz continuous at $u = 0$.

In view of the generality of the hypotheses, it is necessary to deal carefully with the meaning of solutions of (1.1). To be specific, we understand a *ground state* of (1.1) to be a non-negative, non-trivial C^1 distribution (weak) solution of (1.1) on $\boldsymbol{R}^n$, which satisfies condition (1.2).

We can now describe our main results, first considering the case of *regular operators* A.

Theorem 1. *Let* $f(u)$ *satisfy conditions* (i) *and* (ii) *above. Suppose* $A(t)$ *is of class* $C^{1,1}([0,\infty))$ *and has the property*

$$\Omega'(t) > 0 \quad \textit{for all } t \geqq 0, \tag{1.3}$$

where $\Omega(t) = tA(t)$. *Then any ground state of* (1.1) *whose* (*open*) *support is connected must be radially symmetric about some origin* O *in* $\boldsymbol{R}^n$, *and the corresponding radial function* $u(r)$ *obeys* $u'(r) < 0$ *for all* $r > 0$ *such that* $u(r) > 0$.

Condition (1.3) guarantees the ellipticity of equation (1.1); see Section 2. It is immediate that both the Laplacian and the mean curvature operator satisfy (1.3), as well as operators arising from regular variational problems.

In Theorem 1 it is explicitly understood that ground states need not be everywhere positive, this being reflected in the final statement of the theorem, that $u'(r) < 0$ *only* on the open support of u.

Under appropriate conditions on the nonlinearity $f(u)$ the ground state is necessarily positive and the connectivity condition in Theorem 1 can be dropped, but in other cases this condition is necessary for the validity of the theorem. The precise behavior to be expected is described in the next results.

Proposition 1. *Let the hypotheses of Theorem 1 be satisfied, and let* $F(t) = \int_0^t f(s)ds$. *Suppose that u is a ground state of* (1.1). *Then u is positive on* $\boldsymbol{R}^n$ *if and only if* [1]

$$\int_0 \frac{dt}{|F(t)|^{1/2}} = \infty, \tag{1.4}$$

while conversely u has compact support in R^n *if and only if*

$$\int_0 \frac{dt}{|F(t)|^{1/2}} < \infty. \tag{1.5}$$

This result, which combines ideas due to VAZQUEZ [20] and CORTÁZAR, ELGUETA & FELMER [3], is proved in [17]. In view of Theorem 1, the first part of the proposition has the following corollary.

Corollary. *Suppose the hypotheses of the first paragraph of Theorem 1 hold, and let f be locally Lipschitz continuous for* $u \geqq 0$. *Then ground states of* (1.1) *are necessarily positive, radially symmetric about some origin O, and satisfy* $u'(r) < 0$ *for all* $r > 0$.

Under the a priori assumption that the ground states in question are *everywhere positive* this result is due to LI & NI [14] (in fact their work also covers regular fully nonlinear equations). The conclusion of course also generalizes the Gidas-Ni-Nirenberg theorem, and shows even more that *the condition of positivity in that result can be dropped without affecting the conclusion*. See also [8] and [13], and the remarks at the end of Section 3.

A second corollary of the proposition concerns the case when (1.5) applies:

Corollary. *Suppose that* (1.5) *holds, and let u be a ground state of* (1.1) *with connected open support. Then the support of u is a ball and u is radially symmetric about the center.*

The assumption that the open support of *u* be connected *cannot be omitted when* (1.5) *holds, or even weakened to the requirement that the* (*closed*) *support of u be connected*, since this would allow the possibility of multibump, non-radial ground states (even though each bump, in isolation from the others, would be radially symmetric about some origin).

Theorem 1′. *Let the hypotheses of the first paragraph of Theorem 1 hold, and let f be locally Hölder continuous for* $u \geqq 0$. *Assume also that* (1.5) *is satisfied. Then any ground state of* (1.1) *has* (*open*) *support on a finite number of open balls in* $\boldsymbol{R}^n$, *on each of which it is radially symmetric about the center of the ball.*

[1] If $f(u) \equiv 0$ for all sufficiently small u, then the integral (1.4) is understood to be divergent. The same convention is used later for the integral in (1.8).

Theorem 1′ generalizes Theorem 1.2 of [3], which deals in particular with the Laplace operator Δu and the special nonlinearity $f(u) = -u^q + u^p,\ 0 < q < 1 < p$; see also [4].

For the degenerate Laplace operator, condition (1.3) fails at $t = 0$ when $m \neq 2$, and Theorem 1 must accordingly be modified. In the next result we turn our attention to such exceptional operators, namely those which either are undefined at $t = 0$ or fail to satisfy (1.3) at $t = 0$. Here it is necessary to add further conditions on the critical set of u before symmetry can be expected.

Theorem 2. *Suppose that $f(u)$ satisfies conditions* (i) *and* (ii)*, and assume $A(t)$ is of class $C^{1,1}((0,\infty))$ such that*

$$\Omega'(t) > 0 \ \ for\ t > 0, \qquad \Omega(t) \to 0 \ \ as\ t \to 0. \tag{1.6}$$

Let u be a ground state of (1) *with the property that the set*

$$\{x \in \boldsymbol{R}^n \mid u(x) > 0,\ Du(x) = 0\} \tag{1.7}$$

contains exactly one point O. Then u is radially symmetric about O, and the corresponding function $u(r)$ satisfies $u'(r) < 0$ for all $r > 0$ such that $u(r) > 0$.

Critical point conditions, as in Theorem 2, are well-known for the symmetry problem for the degenerate Laplace operator; see [1, 2, 5]. In the generality of Theorem 2 they are moreover *necessary*, as shown by the counter-examples in Section 6.

We also have the following complementary result to Theorem 2, in which the function

$$H(t) = t\Omega(t) - \int_0^t \Omega(s)ds, \quad t > 0,$$

plays a vital part. Note that $H'(t) > 0$ for all $t > 0$, in view of (1.6), and that $H(0) \to 0$ as $t \to 0$. Clearly then $H(t) > 0$ for all $t > 0$.

Proposition 2. *Let the hypotheses of the first sentence of Theorem 2 be satisfied, and let u be a ground state of* (1.1). *Then u is positive on $\boldsymbol{R}^n$ if*

$$\inf_{t\to 0} \frac{H(t)}{t\Omega(t)} > 0, \qquad \int_0 \frac{dt}{H^{-1}(|F(t)|)} = \infty, \tag{1.8}$$

while conversely it has compact support in R^n if

$$\int_0 \frac{dt}{H^{-1}(|F(t)|)} < \infty. \tag{1.9}$$

Proposition 2 is proved in [17] under the additional restriction that $\inf_{t\to 0} H(t)/t\Omega(t) > 0$. That this condition is not in fact necessary has been shown in recent work of PUCCI & SERRIN. We note also that Proposition 1 is a special case of Proposition 2; that is, when $\Omega'(t) > 0$ for $t \geqq 0$, we get $\Omega(t) = \Omega'(0)t + o(t)$ for small t, so also $H(t) = \{\frac{1}{2}\Omega'(0)\}t^2 + o(t^2)$. Thus the integral conditions (1.8) and (1.9) reduce to (1.4) and (1.5).

For the degenerate Laplace operator $A(t) = t^{m-2}$, $m > 1$, we have $t\Omega(t) = t^m$ and $H(t) = \frac{m-1}{m}t^m$, so (1.8) reduces to the condition

$$\int_0 \frac{dt}{|F(t)|^{1/m}} = \infty. \tag{1.10}$$

For functions $f(u)$ which are locally Lipschitz continuous on $u \geqq 0$ this condition is satisfied for all $m \leqq 2$, and hence corresponding ground states are positive. When $m > 2$, there is a converse result:

Corollary. *Suppose* $\sup_{u\to 0} f(u)/u < 0$ (*for example,* $f'(0) < 0$). *Let u be a corresponding ground state for the degenerate Laplace operator with $m > 2$, such that the set* (1.7) *contains exactly one point. Then* (1.9) *holds and u has support in an open ball in* $\boldsymbol{R}^n$.

When the operator A is *singular* at $t = 0$ the result of Theorem 2 can be improved.

Theorem 3. *Let the hypotheses of the first sentence of Theorem 2 be satisfied. Suppose also that $\Omega'(t) \to \infty$ as $t \to 0$. Let u be a ground state of* (1.1) *with the property that the* (*open*) *support of Du is connected. Then u must be radially symmetric about some origin O in $\boldsymbol{R}^n$, and the corresponding function $u(r)$ obeys $u'(r) < 0$ for all $r > 0$ such that* $u(r) > 0$.

An important example for Theorem 3 is the degenerate Laplace operator $A(t) = t^{m-2}$ with $1 < m < 2$, a case also considered in the following paper of Damascelli, Pacella & Ramaswamy [6]. In particular, using special properties of the degenerate Laplace operator, but without requiring the connectivity condition of Theorem 3, these authors show that corresponding *positive* ground states are radially symmetric about some center in $\boldsymbol{R}^n$.

Remark. Once one is able to show that ground states are radially symmetric, a great deal is known about their uniqueness as functions of r and their asymptotic behavior as $r \to \infty$. For these issues, we refer the reader to [3, 9, 16] and the references noted therein.

In conclusion, it is worth emphasizing that by allowing *non-negative* ground states into consideration we obtain an altogether richer theory than that for which ground states are a priori assumed to be positive, for besides the virtue of additional generality one moreover avoids undue restrictions on the behavior of the nonlinearity $f(u)$ near $u = 0$; see Propositions 1 and 2. Moreover, compactly supported ground states, which arise in this context, can even be considered as solutions of the *free boundary problem* to find solutions of (1.1) which are *positive* on a domain B and have the property that $u(x),\ Du(x) \to 0$ as $x \to \partial B$.

Theorems 1, 1′ and 2 will be proved in Sections 2 and 3. Theorem 3 will be proved in Sections 4 and 5. In Section 6, we present two counter-examples of *non-symmetric* ground states.

Finally, we remark that our approach applies to more general nonlinearities $f(r, u, |Du|)$ and that we also can treat singular ground states. With appropriately corresponding assumptions, results similar to those above can be obtained in both these cases. For singular ground states, in particular, we give an outline of the required details in Section 7.

2. Preliminaries

Consider the quasilinear elliptic equation

$$\operatorname{div}(A(|\nabla u|)\nabla u) + f(u) = 0, \qquad x \in \boldsymbol{R}^n. \tag{2.1}$$

We are interested in C^1 non-negative weak solutions u in $\boldsymbol{R}^n$ such that

$$\lim_{|x|\to\infty} u(x) = 0, \tag{2.2}$$

the so-called *ground state* condition.

Throughout the paper we treat the following two separate cases for the operator A, that is, *regular* operators and *degenerate* operators. Our assumptions are as follows.

1. *Regular case.* A is in $C^{1,1}([0,\infty))$. We let

$$\Omega(t) = tA(t), \quad t \geqq 0,$$

and suppose that

$$\Omega'(t) > 0 \quad \text{for } t \geqq 0. \tag{2.3}$$

2. *Degenerate case.* A is of class $C^{1,1}((0,\infty))$. Here $\Omega(t)$ is only defined for $t > 0$, and correspondingly we may suppose only that

$$\Omega'(t) > 0 \quad \text{for } t > 0.$$

Additionally, it is then assumed that

$$\Omega(t) \to 0 \quad \text{as } t \to 0,$$

a condition which is of course automatic in the regular case.

Finally, the function f is assumed to satisfy the conditions (i), (ii) in the introduction.

In the regular case, it is evident from (2.3) that both $A(t)$ and $\Omega(t)$ are *positive* for $t > 0$. Clearly also, by L'Hôpital's rule,

$$A(0) = \lim_{t\to 0} A(t) = \Omega'(0) > 0.$$

When A is degenerate we obviously continue to have $A(t)$ and $\Omega(t)$ positive for $t > 0$, though A itself remains undefined at 0. Note moreover that

$$\lim_{p\to 0} A(|p|)p = 0,$$

where $p = (p_1, p_2, ..., p_n)$, so that with the obvious meaning that $A(|Du|)Du = 0$ when $Du = 0$ the weak form of (2.1) remains valid even for degenerate operators A.

Let $p = (p_1, \ldots, p_n)$. Then for regular operators A we have

$$\frac{\partial}{\partial p_j}(A(|p|)p_i) = A(|p|)\delta_{ij} + |p|A'(|p|)\frac{p_i p_j}{|p|^2} \tag{2.4}$$

for all $p \in \boldsymbol{R}^n$, while for degenerate A the same calculation applies provided $p \neq 0$.

Let the right-hand side of (2.4) be denoted by $a_{ij}(p)$.

Lemma 2.1. *For regular operators A the matrix $a_{ij}(p)$ is Lipschitz continuous and positive-definite for all $p \in \boldsymbol{R}^n$. When A is degenerate, the same result holds provided $p \neq 0$.*

Proof. By linear algebra, for $p = (p_1, p_2, ..., p_n) \neq 0$ the only non-zero eigenvalue of the matrix $\{p_i p_j |p|^{-2}\}$ is 1, with eigenvector p; in fact we have

$$\lambda_1 = 1, \qquad \lambda_2 = \cdots = \lambda_n = 0.$$

In turn, $a_{ij}(p)$ has the n eigenvalues

$$\lambda_1 = A(|p|) + |p|A'(|p|) = \Omega'(|p|), \qquad \lambda_2 = \cdots = \lambda_n = A(|p|),$$

and so is positive-definite. When $p = 0$ and A is regular, we have $a_{ij}(0) = A(0)\delta_{ij}$, so again a_{ij} is positive-definite.

Lipschitz continuity is obvious from the basic hypotheses for both regular and degenerate operators. □

Lemma 2.2. *If A is regular, then any C^1 non-negative weak solution u of* (2.1) *is of class C^2 in the neighborhood of points y where $u(y) > 0$. For degenerate operators, the same result is valid provided that $Du(y) \neq 0$.*

In view of Lemma 2.1 and condition (i) for the nonlinearity f, this is an immediate consequence of [12, Theorems 6.2, 6.3, pp. 282–283].

In any domain in which u is of class C^2 we can clearly write (2.1) in the classical form

$$a_{ij}(Du)u_{ij} + f(u) = 0, \tag{2.5}$$

where $u_{ij} = \partial^2 u/\partial x_i \partial x_j$. By Lemma 2.1 one sees that (2.5) is locally uniformly elliptic.

In what follows it will be convenient to introduce the function

$$\lambda(s) \stackrel{\text{def}}{=} \inf_{0<t\leqq s}\{A(t), \Omega'(t)\}, \qquad s > 0.$$

Clearly λ is well-defined for both regular and degenerate operators, and is positive and non-increasing. We have the following monotonicity theorem.

Lemma 2.3. *Let A be either regular or degenerate, and suppose that $p \neq q$. Then*

$$\{A(|p|)p - A(|q|)q\} \cdot \{p - q\} > 0$$

and, more specifically,

$$\{A(|p|)p - A(|q|)q\} \cdot \{p - q\} \geqq \lambda(\sigma)|p - q|^2,$$

where $\sigma = \max(|p|, |q|)$.

Proof. By an obvious interpolation identity we have

$$(A(|p|)p - A(|q|)q) \cdot (p - q) = \int_0^1 a_{ij}(p_\theta)(p_i - q_i)(p_j - q_j)\, d\theta$$

where $p_\theta = \theta p + (1 - \theta)q$. (*Note.* The integrand may have a singularity when $\theta p + (1 - \theta)q = 0$, which however can occur at most at a single value θ_0 of θ. On the other hand, since $a_{ij}(p)$ is positive-definite for $p \neq 0$, it is not hard to see that the identity is valid even in this exceptional case — e.g., delete an interval $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ from the integration domain, calculate the resulting integral and then let $\varepsilon \to 0$.) The first conclusion now follows from the fact that the integrand is continuous and positive (in θ) except for at most one point.

To obtain the second conclusion, it is enough to note that, with the possible exception of a single value θ,

$$a_{ij}(p_\theta)\xi_i\xi_j \geqq \min\{A(|p_\theta|),\ \Omega'(|p_{\theta|})\}|\xi|^2 \geqq \lambda(|p_\theta|)|\xi|^2$$

(since the eigenvalues of $a_{ij}(p)$ are $A(|p|)$ and $\Omega(|p|)$; see the proof of Lemma 2.1), and

$$|p_\theta| \leqq \sigma = \max(|p|, |q|),$$

where of course $\sigma > 0$.

Now let u, v be two non-negative weak solutions of (2.1) defined in a half space Σ in $\boldsymbol{R}^n$. Put $w = u - v$. Then by differencing the classical equations (2.5) satisfied by u and v, we obtain the principal differential equation

$$a_{ij}(p)w_{ij} + (a_{ij}(p) - a_{ij}(q))v_{ij} + c(x)w = 0 \tag{2.6}$$

in Σ, where p stands for Du and q for Dv, and

$$c(x) = \begin{cases} \frac{f(u(x)) - f(v(x))}{u(x) - v(x)}, & u(x) \neq v(x), \\ 0, & u(x) = v(x) \end{cases}.$$

By Lemma 2.2 and the remarks immediately afterward, it is clear for regular operators A that equation (2.6) is well-defined and locally uniformly elliptic in any open subset of Σ where $u,\ v \neq 0$, while for degenerate operators it is necessary also to have $Du,\ Dv \neq 0$.

Because a_{ij} is Lipschitz continuous, one also has

$$(a_{ij}(p) - a_{ij}(q))v_{ij} = b_i(x)w_i,$$

where $b_i(x)$ is a locally bounded function in the domains of validity indicated above, while of course $c(x)$ is locally bounded in any open set in which either u or v is nonzero. Thus (2.6) can finally be written in the standard quasilinear elliptic form

$$a_{ij}(p)w_{ij} + b_i(x)w_i + c(x)w = 0. \qquad \square \tag{2.7}$$

3. Symmetry: Proof of Theorems 1, 2

Let u be a C^1 non-negative weak solution of (2.1), and write

$$\operatorname{supp}_+ u = \{x \in \boldsymbol{R}^n \mid u(x) > 0\},$$

the *open support* of u, and

$$Z = \{x \in \boldsymbol{R}^n \mid u(x) > 0,\ Du(x) = 0\},$$

the set of critical points of u on its open support.

For $\gamma \in \boldsymbol{R}$ and $x = (x_1, \ldots, x_n)$, we define

$$\Sigma = \Sigma_\gamma = \{x \in \boldsymbol{R}^n \,|\, x_1 < \gamma\}$$

and denote by Γ the hyperplane

$$\Gamma = \Gamma_\gamma = \{x \in \boldsymbol{R}^n \,|\, x_1 = \gamma\}.$$

Let x^γ be the *reflection* in Γ of a point x in $\boldsymbol{R}^n$, that is,

$$x^\gamma = (2\gamma - x_1, x_2, \ldots, x_n),$$

and similarly let Ω^γ be the reflection in Γ of a set Ω in $\boldsymbol{R}^n$,

$$\Omega^\gamma = \{x^\gamma \mid x \in \Omega\}.$$

For $x \in \overline{\Sigma}$, put

$$v(x) = v_\gamma(x) = u(x^\gamma), \qquad w(x) = u(x) - v(x).$$

Then by the reflection invariance of equation (2.1), it follows that v as well as u is a solution of (2.1) in Σ, of class C^1 in $\overline{\Sigma}$. We observe also that $w = 0$ on Γ, and that, furthermore, if $Du(x) = 0$ at some point x in Γ, then also $Dv(x) = Dw(x) = 0$.

We next obtain a strong comparison lemma and a Hopf boundary lemma for the function w, which will be crucial in our moving plane procedure.

Lemma 3.1. *Let u be a C^1 non-negative weak solution of* (2.1) *and let $A(t)$ be regular* (*that is, of class $C^{1,1}([0, \infty))$ such that* (1.3) *is satisfied*)*. Assume that* $\operatorname{supp}_+ u$ *is connected and suppose that*

$$w \geqq 0, \qquad w \not\equiv 0 \qquad \text{in } \Sigma.$$

Then

$$w > 0 \quad \text{in } \Sigma \cap \operatorname{supp}_+ u, \qquad \frac{\partial w}{\partial x_1} < 0 \quad \text{on } \Gamma \cap \operatorname{supp}_+ u. \tag{3.1}$$

Proof. Suppose for contradiction that $(3.1)_1$ fails, that is, there exists $y \in \Sigma \cap \operatorname{supp}_+ u$ with $w(y) = 0$. By (2.7) we can write

$$a_{ij}(x)w_{ij} + b_i(x)w_i + c(x)w = 0 \tag{3.2}$$

as in Section 2. Clearly (3.2) is uniformly elliptic in a neighborhood of y since $v(y) = u(y) > 0$. For the same reason, the coefficients b_i are uniformly bounded near y, and finally $c(x)$ is also uniformly bounded near y.

Put $c(x) = c_+(x) + c_-(x)$, where $c_+ \geqq 0$ and $c_- \leqq 0$ are respectively the positive and negative parts of c. Then (3.2) can be rewritten as a uniformly elliptic differential inequality

$$a_{ij}(x)w_{ij} + b_i(x)w_i + c_-(x)w = -c_+(x)w \leqq 0 \tag{3.3}$$

for x near y. The strong maximum principle [11, Theorem 3.5, p. 35], then implies that

$$w \equiv w(y) = 0$$

in a neighborhood of y.

Let Y be the (open) component of $\Sigma \cap \operatorname{supp}_+ u$ which contains y. Then by a chaining argument one immediately concludes that w vanishes in Y. We claim in fact that Y coincides with $\Sigma \cap \operatorname{supp}_+ u$.

To see this, we note first the relations

$$u = v > 0 \text{ in } Y, \qquad u > 0 \text{ in } Y^\gamma$$

(where the second inequality follows by reflection), and

$$u = v = 0 \text{ on } \partial Y \backslash \Gamma, \qquad u = 0 \text{ on } \partial Y^\gamma \backslash \Gamma.$$

There are now two cases to consider.

First, suppose that $u = v = 0$ at all points of ∂Y. It is clear that Y is a component of $\operatorname{supp}_+ u$. Then, by reflection, Y^γ is a second component of $\operatorname{supp}_+ u$. This is impossible, however, since $\operatorname{supp}_+ u$ is assumed to be connected.

In the remaining case, the set $Y_\Gamma = \partial Y \cap \operatorname{supp}_+ u$ is non-empty, in which case we must have $Y_\Gamma \subset \Gamma$. Consider the set

$$Y^* = Y \cup Y^\gamma \cup Y_\Gamma.$$

It is not difficult to see that each point of Y_Γ has a neighborhood consisting entirely of points in Y^*. Hence Y^* is open, and $u > 0$ in Y^*. Let $x \in \partial Y^*$. Obviously either $x \in \partial Y \backslash \Gamma$, or $x \in \partial Y^\gamma \backslash \Gamma$ or $x \in \Gamma$. In the first two cases we have $u(x) = 0$. In the last, we see easily that x is also in ∂Y. On the other hand $x \notin \operatorname{supp}_+ u$, for otherwise $x \in Y_\Gamma$, and x is an interior point of Y^*. Hence $u(x) = 0$. We have thus shown that $u = 0$ on ∂Y^*.

It is easy to see that Y^* is connected. It follows therefore that Y^* is itself a component of $\operatorname{supp}_+ u$. Since $\operatorname{supp}_+ u$ is connected by hypothesis, this means that Y^* *coincides* with $\operatorname{supp}_+ u$. Consequently,

$$Y = \Sigma \cap Y^* = \Sigma \cap \operatorname{supp}_+ u,$$

as claimed.[2]

This being shown, we conclude that $w \equiv 0$ on $\Sigma \cap \mathrm{supp}_+ u$. But also (since $w \geqq 0$, $v \geqq 0$ and $u = 0$ on $\Sigma \backslash \mathrm{supp}_+ u$) we have $w = u = v = 0$ on $\Sigma \backslash \mathrm{supp}_+ u$. Hence $w \equiv 0$ in Σ, contradicting the principal assumption of the lemma. This proves $(3.1)_1$.

Next, let $y \in \Gamma \cap \mathrm{supp}_+ u$. Then $w(y) = 0$, $v(y) = u(y) > 0$ and so w again satisfies the uniformly elliptic differential inequality (3.3) for $x \in \Sigma$ near y. Moreover $w > 0$ in $\Sigma \cap \mathrm{supp}_+ u$ by $(3.1)_1$. Thus the boundary point lemma [11, Lemma 3.4, p. 34] yields

$$\frac{\partial w}{\partial x_1}(y) < 0,$$

which completes the proof. □

Lemma 3.2. *Let u be a C^1 non-negative weak solution of* (2.1) *and let $A(t)$ be degenerate. Assume that* $\mathrm{supp}_+ u$ *is connected, and moreover let*

$$Z \subset \Sigma. \tag{3.4}$$

Then the conclusion of Lemma 3.1 continues to hold.

Proof. This is essentially the same as for Lemma 3.1, requiring only slight modifications due to the degeneracy of A at the origin. Suppose for contradiction that $(3.1)_1$ fails, that is, there exists $y \in \Sigma \cap \mathrm{supp}_+ u$ with $w(y) = 0$. Obviously $Dw(y) = 0$ because $w(y) = 0$ is a minimum on the open set Σ. Moreover $v(y) = u(y) > 0$. Hence by (3.4) we have $Dv(y) \neq 0$, and in turn

$$Du(y) = Dv(y) \neq 0.$$

Consequently w satisfies the uniformly elliptic differential inequality (3.3) near y, and the first part of the proof of Lemma 3.1 thus carries over immediately.

Next we prove $(3.1)_2$. Suppose for contradiction that it fails. Then there exists $y \in \Gamma \cap \mathrm{supp}_+ u$ such that

$$\frac{\partial w}{\partial x_1}(y) = 0.$$

Therefore, since $w = 0$ on Γ, we have $Dw(y) = 0$ and $v(y) = u(y) > 0$. Hence by (3.4) again, there results $Du(y) = Dv(y) \neq 0$. Therefore the proof of Lemma 3.1 carries over, yielding

$$\frac{\partial w}{\partial x_1}(y) < 0,$$

an immediate contradiction. This completes the proof. □

Lemma 3.3. *Let u be a C^1 non-negative weak solution of* (2.1) *and let $A(t)$ be either regular or degenerate. Suppose there exists $y \in \Sigma$ such that $w(y) < 0$. Then there exists $\xi \in \Sigma$ with the properties*

$$w(\xi) < 0, \qquad v(\xi) > \delta,$$

where $\delta > 0$ is defined in condition (ii).

[2] The same point is discussed in [3, Lemma 3.2] with a more involved proof.

Proof. Choose $\varepsilon > 0$ so small that $w(y)+\varepsilon < 0$. Then, since w vanishes at infinity and on Γ, clearly

$$w_\varepsilon = \min\{w+\varepsilon, 0\}$$

is non-positive and has compact support in Σ. By the distribution meaning of solutions, taking the Lipschitzian function w_ε as test function one gets

$$\int_\Sigma [A(|Du|)Du - A(|Dv|)Dv]Dw_\varepsilon = \int_\Sigma [f(u) - f(v)]w_\varepsilon. \tag{3.5}$$

Clearly the left-hand side of (3.5) is positive by the first part of Lemma 2.3, since $Dw_\varepsilon \equiv Dw$ and $Dw \not\equiv 0$ in the set $\{w+\varepsilon < 0\}$, while otherwise $Dw_\varepsilon = 0$ (a.e.).

We claim that there exists $\xi \in \operatorname{supp}_+(-w_\varepsilon)$ such that $v(\xi) > \delta$. Otherwise, when $w+\varepsilon < 0$, there holds

$$0 \leqq u < v - \varepsilon < \delta,$$

so

$$f(u) - f(v) \geqq 0$$

since $f(u)$ is non-increasing for $u < \delta$. It follows that the right-hand side of (3.5) is non-positive, a contradiction. □

Now we are ready to prove Theorems 1 and 2.

Proof of Theorem 1. *Step 1.* There exists $\gamma_0 > 0$ such that, for $\gamma \geqq \gamma_0$,

$$w \geqq 0 \qquad \text{in } \Sigma_\gamma. \tag{3.6}$$

Indeed, choose γ_0 so that $u(x) \leqq \delta$ for $x_1 \geqq \gamma_0$, which can be done in view of the ground state condition (1.2). Then if (3.6) fails, there would exist y in Σ_γ with $w(y) < 0$. In this case, by Lemma 3.3 there must also be some ξ in Σ_γ with $v(\xi) > \delta$, and in turn $u(\xi^\gamma) > \delta$, contradicting the choice of γ_0.

Step 2. There exists $\gamma \in \boldsymbol{R}$ such that $w \ngeqq 0$ in Σ_γ. Indeed, let $y_0 \in \boldsymbol{R}^n$ be fixed, and choose ν (suitably large negatively) so that

$$u(x) < u(y_0) \tag{3.7}$$

whenever $x \in \Sigma_\nu$, which can be done in view of the ground state condition (1.2). Clearly $y_0 \notin \Sigma_\nu$. Thus there is $y \in \Sigma_\nu$ such that $y^\nu = y_0$. In turn

$$w(y) = u(y) - v_\nu(y) = u(y) - u(y_0) < 0$$

by (3.7). Hence $w \ngeqq 0$ in Σ_γ when $\gamma = \nu$, as required.

Step 3. Let $[\mu, \infty)$ be the maximal (obviously closed) interval for which (3.6) holds for all γ in $[\mu, \infty)$. That such an interval exists follows at once from the result of Step 2.

We show that $w \equiv 0$ in Σ_μ. Suppose the contrary. By Lemma 3.1, one then has

$$w > 0 \text{ in } \Sigma_\mu \cap \operatorname{supp}_+ u, \qquad \frac{\partial w}{\partial x_1} < 0 \text{ on } \Gamma_\mu \cap \operatorname{supp}_+ u. \tag{3.8}$$

We now derive a contradiction. Since $[\mu, \infty)$ is maximal, there exist sequences $\{\gamma_i\}$ and $\{x^i\}$ such that (in an obvious notation)

$$\gamma_i \in (-\infty, \mu), \quad x^i \in \Sigma^i = \Sigma_{\gamma_i}, \quad w^i(x^i) < 0, \qquad i = 1, 2, \ldots, \tag{3.9}$$

where $w^i = u - v^i$ and $\lim_{i\to\infty} \gamma_i = \mu$.

By Lemma 3.3 we can assume without loss of generality that $v^i(x^i) > \delta$. In turn, it is not hard to see that the sequence $\{x^i\}$ must be bounded. Moreover, there can be no subsequence which approaches any point ξ in $\overline{\Sigma}_\mu$ where $u(\xi) = 0$. In fact, otherwise, by (3.9) and the assumption that $w \geqq 0$ in Σ_μ, we would have $w(\xi) = 0$. But then $v(\xi) = 0$, an impossibility since $v^i(x^i) > \delta$.

Consequently there is a subsequence of $\{x^i\}$ tending either to a point y in $\Sigma_\mu \cap \operatorname{supp}_+ u$, or to a point z in $\Gamma_\mu \cap \operatorname{supp}_+ u$.

The first case cannot occur, for then $w(y) = 0$ as above, contradicting $(3.8)_1$. For the remaining case, since $w^i = 0$ on Γ_{γ_i}, there must exist for each i a point z^i on the line segment joining x^i to the point $(\gamma_i, x_2^i, \ldots, x_n^i)$ on Γ_{γ_i}, such that

$$\frac{\partial w^i}{\partial x_1}(z^i) \geqq 0.$$

Therefore

$$\frac{\partial w}{\partial x_1}(z) \geqq 0$$

since $z^i \to z$, and $w^i \to w$ in C^1 as $i \to \infty$. This contradicts $(3.8)_2$, so also this case is impossible.

It follows that (3.8) cannot occur and u must be symmetric with respect to the hyperplane $x_1 = \mu$. Moreover, since $\partial w/\partial x_1 = 2\partial u/\partial x_1$ on any hyperplane Γ, the proof shows clearly that

$$\frac{\partial u}{\partial x_1}(x) < 0 \tag{3.10}$$

in $\{x_1 > \mu\} \cap \operatorname{supp}_+ u$. In view of the rotational invariance of (2.1), the same argument of course applies for any assigned direction of the x_1-axis in $\boldsymbol{R}^n$.

The proof is now concluded with the help of

Lemma 3.4. *Let u be a nonnegative C^1 function on $\boldsymbol{R}^n$ satisfying the ground state condition* (1.2). *Suppose that corresponding to each direction $\boldsymbol{e}$ in $SO(n-1)$ there exists a hyperplane $\Gamma_{\boldsymbol{e}}$, with normal $\boldsymbol{e}$, having the properties that $u(x)$ is symmetric across $\Gamma_{\boldsymbol{e}}$ and*

$$Du(x) \cdot \boldsymbol{e} \leqq 0 \qquad \text{whenever } (x - x_0) \cdot \boldsymbol{e} > 0 \text{ for some } x_0 \in \Gamma_{\boldsymbol{e}}. \tag{3.11}$$

Then u is radially symmetric about some origin O in $\boldsymbol{R}^n$, and the corresponding radial function $u(r)$ satisfies $u'(r) \leqq 0$.

Proof. We determine an origin O by the intersection of n orthogonal hyperplanes

$$\Gamma_{e_1}, \Gamma_{e_2}, \dots, \Gamma_{e_n}.$$

By (3.11) it is evident that O is a maximum point of u. Now let $\boldsymbol{e} \neq \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$. We assert that $\Gamma_{\boldsymbol{e}}$ contains O.

If not, then by (3.11) it is easy to see that $u \equiv$ constant $= u(O)$ on the line segment from O to the nearest point of $\Gamma_{\boldsymbol{e}}$, say $\bar{O}$. By successive reflections in the planes Γ_{e_i} and $\Gamma_{\boldsymbol{e}}$ one sees easily that $u \equiv u(O)$ on a line segment of length $4|O\bar{O}|$ parallel to $O\bar{O}$ and with center at O. By further similar reflections, we get $u \equiv u(O)$ on segments through O of lengths $8|O\bar{O}|$, $16|O\bar{O}|$, etc. But then $u \nrightarrow 0$ as $|x| \to \infty$, which is impossible.

This being shown, to complete the proof of the lemma it is enough to verify that if $x, y \in \boldsymbol{R}^n$ with $|x| = |y| > 0$, then $u(x) = u(y)$. But in this case, let $\boldsymbol{k}$ be the unit vector with direction $\boldsymbol{x} - \boldsymbol{y}$ (we assume $x \neq y$) and $\Gamma_{\boldsymbol{k}}$ the corresponding hyperplane. Since $|x| = |y|$, it is clear that $\Gamma_{\boldsymbol{k}}$ bisects the line segment joining x and y. Hence $u(x) = u(y)$ by the reflection property of $\Gamma_{\boldsymbol{k}}$, as required. □

To finish the proof of Theorem 1, it still must be shown that $u'(r) < 0$ for all $r > 0$ such that $u(r) > 0$. This is obvious however by applying (3.10) to the radial function $u(r)$. Theorem 1 is thus complete.

Note. Lemma 3.4 is given under slightly weaker hypotheses than are actually needed for Theorems 1 and 2, but the general form will be useful later for the proof of Theorem 3.

Proof of Theorem 2. This is the same as for Theorem 1, except that we can use Lemma 3.2 instead of Lemma 3.1. In this case, moreover, the center point O must be the unique positive critical point of u, which will be taken as the origin of coordinates. We note additionally that $\mathrm{supp}_+ u$ can have at most one component (i.e., $\mathrm{supp}_+ u$ is connected) since otherwise Z would contain maximum points in each component, contradicting the hypothesis that it contains exactly one critical point.

Step 1. Exactly the same as that of Theorem 1.

Step 2. Since u has only one maximum point, the origin O, it is clear that $w \ngeqq 0$ in Σ_γ whenever $\gamma < 0$.

Step 3. Let $[\mu, \infty)$ be the maximal (obviously closed) interval for which (3.6) holds for all γ in $[\mu, \infty)$. By Step 2, we have $\mu \geqq 0$. We want to show that $\mu = 0$.

If $\mu > 0$, then $Z \subset \Sigma_\mu$, so by Lemma 3.2 either (3.8) holds or $w \equiv 0$ in Σ_μ. Obviously the latter cannot occur since then $u(x) \equiv u(x^\mu)$, from which we would infer at once that there are critical points of u in the set $\{x_1 > 0\} \cap \mathrm{supp}_+ u$, for example at O^μ, contrary to assumption. The former case also cannot occur (hence $\mu = 0$), by a virtually unchanged argument from that used in proving Theorem 1, but applying Lemma 3.2 instead of Lemma 3.1 since $Z \subset \Sigma_\mu$.

Since we have shown that $\mu = 0$, it now follows that $w \geqq 0$ in Σ_0. Moreover, the proof shows that $\partial u/\partial x_1 < 0$ in $\{x_1 > 0\} \cap \mathrm{supp}_+ u$. Reversing the x_1-axis, one sees that $w \geqq 0$ for the correspondingly reversed half-plane. By reflection this yields $w \leqq 0$ in the original half-space Σ_0, and so finally $w = 0$ in Σ_0. That is, u must be symmetric with respect to the hyperplane $x_1 = 0$.

Since the direction of the x_1-axis is arbitrary, as remarked at the end of the proof of Theorem 1, the last part of the symmetry Lemma 3.4 shows that u must be radially symmetric, of course now with respect to the origin O.

That $u'(r) < 0$ when $r > 0$ and $u(r) > 0$ also follows exactly as before. This completes the proof of Theorem 2. □

Proof of Theorem 1′. We first observe that, for functions f which are Hölder continuous at $u = 0$ and Lipschitz continuous for $u > 0$, any corresponding non-negative weak solution of (2.1) is of class $C^2(\boldsymbol{R}^n)$. (One first shows that $u \in C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$; see Theorem 1 on page 127 and Proposition 1 on page 129 of [19], or Theorem 2 on page 829 of [7], and then applies Theorem 6.3 on page 283 of [12].)

Now let u be a ground state for (1.1), and let Ω be any component of $\mathrm{supp}_+ \, u$. We define

$$u^* = \begin{cases} u(x), & x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly u^* is of class C^2 in $\overline{\Omega}$ with $u = Du = 0$ on $\partial\Omega$; hence it is of class $C^2(\boldsymbol{R}^n)$ provided also $D^2u = 0$ on $\partial\Omega$.

But if $y \in \partial\Omega$, then at this point we have $a_{ij}u_{ij} = -f(u(y)) = -f(0) = 0$ by condition (ii). On the other hand, since y is a minimum of u, either $D^2u(y) = 0$ or there are directions $\boldsymbol{n}$ at y such that $\partial^2 u/\partial n^2 > 0$. The second alternative obviously stands in contradiction with $a_{ij}u_{ij} = 0$. Thus $D^2u(y) = 0$ as required.

The function u^* is obviously a ground state of (1.1), while also $\mathrm{supp}_+ \, u^* = \Omega$ is connected. Hence, recalling that condition (1.5) is assumed to hold, by Theorem 1′ we see that u^* is radially symmetric and its support is an open ball. It follows now that the support of u consists of a set of open balls on each of which u is radially symmetric about the center.

Finally, there can be at most a *finite* number of such balls, since at any center O' (where u takes a local maximum value) we would have $a_{ij}u_{ij} \leqq 0$, so $f(u(O')) \geqq 0$. However, the case $f(u(O')) = 0$ cannot occur because of uniqueness of the radial initial-value problem. Thus $f(u(O')) > 0$ and $u(O') > \delta$ by (ii). Accordingly, if $\mathrm{supp}_+ \, u$ had an infinite number of components, then either $u \nrightarrow 0$ as $|x| \to \infty$, and u would *not* be a ground state, or there would be an accumulation point of centers, equally impossible. □

Remark. Theorem 1 can be given a simpler proof when (1.4) holds. Indeed, once it is known that u is positive (Proposition 1), it is no longer necessary to avoid points where $u = 0$ in carrying out the argument of Lemma 3.1. That is, the conclusion (3.1) can then be obtained much more immediately, and even in the stronger form $w > 0$ in Σ, $\partial w/\partial x_1 < 0$ on Γ.

In addition, the integrand in the proof of Lemma 2.3 is never singular, further simplifying the argument, while, finally, the concluding proof of radial symmetry can be shortened (as indicated above) since (3.11) can be used with the strict inequality sign.

$u(r_0) > 0$, then the set $\{Du = 0\}$ contains the sphere $\{|x| = r_0\}$. Therefore we must have $Du(x) \equiv 0$ for $|x| \leqq r_0$, for otherwise, the (open) support of Du would be disconnected, contradicting the main assumption of the theorem. In turn $u \equiv u(r_0)$ for $r \leqq r_0$, and $f(u(r_0)) = 0$ from the equation. Recalling that f is Lipschitz continuous at the value $u(r_0) > 0$, and applying Proposition A2 in the Appendix of [9] to the singular operator A, now shows that $u \equiv u(r_0)$ for all $r \geqq 0$, which is absurd. This completes the proof of Theorem 3.

6. Counter-Examples

In this section, we give two counter-examples showing that the hypothesis of Theorem 2, namely that the set (1.7) contains only a single point, cannot be removed or even significantly weakened.

Example 1. We construct a non-symmetric ground state with compact support. Let $m > 1$ and $s \geqq 0$ be real. For $a \in (0, \min\{1/2,\ (m-1)/m\})$ to be determined later, we put

$$k = m - 1 - am > 0.$$

Let $r = |x|$ be the radial distance, and define

$$u(x) = u(r) = \begin{cases} 1, & r \leqq s, \\ [1 - (r-s)^{1/a}]^{1/a}, & s \leqq r < s+1, \\ 0, & r \geqq s+1, \end{cases}$$

Since $1/a > 2$, it is clear that $u \in C^{2,\alpha}(\boldsymbol{R}^n)$ for some $\alpha > 0$, and

$$u(s) = 1, \quad u'(s) = u''(s) = 0, \quad u(s+1) = u'(s+1) = u''(s+1) = 0,$$

where $'$ is the derivative with respect to the radius r. Moreover,

$$r = (1 - u^a)^a + s, \quad r \in [s, s+1].$$

By direct calculation, it follows that for $r \in [s,\ s+1]$

$$u'(r) = -a^{-2}[1 - (r-s)^{1/a}]^{(1-a)/a}(r-s)^{(1-a)/a} = -a^{-2}[u(1-u^a)]^{1-a} < 0.$$

A further calculation then gives

$$(|u'(r)|^{m-1})' = -(k+a)a^{-2m}[u(1-u^a)]^k[1 - (a+1)u^a].$$

Hence, for $r \in [s,\ s+1]$,

$$\begin{aligned} - \div (|Du|^{m-2}Du) &= (|u'(r)|^{m-1})' + \frac{n-1}{r}|u'(r)|^{m-1} \\ &= -(k+a)a^{-2m}[u(1-u^a)]^k[1-(a+1)u^a] \\ &\quad q + (n-1)a^{2(1-m)}[u(1-u^a)]^{k+a}[(1-u^a)^a + s]^{-1}. \end{aligned}$$

Therefore u satisfies

$$-\mathrm{div}(|Du|^{m-2}\nabla u) = f(u), \quad x \in \boldsymbol{R}^n, \tag{6.1}$$

where

$$\begin{aligned} f(u) = &-(k+a)a^{-2m}[u(1-u^a)]^k[1-(a+1)u^a] \\ &+(n-1)a^{2(1-m)}[u(1-u^a)]^{k+a}[(1-u^a)^a + s]^{-1} \end{aligned}$$

for $u \in [0, 1]$ and is identically zero elsewhere. In particular, if $m \in (1, 2]$, it is easy to see that f is locally Hölder continuous for $u \geqq 0$, with exponent $\alpha = k > 0$, while if $m > 2$, then by taking a sufficiently small one finds that f is at least of class $C^{1,\alpha}$ for some $\alpha > 0$. Finally

$$f'(u) \approx -ka^{-2m}(1-a)(m-1)u^{k-1} < 0 \quad \text{for small } u > 0.$$

Thus we have constructed a (classical) $C^{2,\alpha}$ solution $u(r)$ of the degenerate Laplace equation (6.1), where f is Hölder continuous and, if $m > 2$ and a is suitably small, at least of class $C^{1,\alpha}$. Moreover, $u(x)$ has compact support, f is decreasing for small $u > 0$ and

$$F(u(0)) > 0, \quad f(u(0)) = 0, \quad u(0) = 1.$$

Now we are going to construct the counter-example. For $s \geqq 0$, let $f_s(u)$ be the function and u_s the solution of (6.1) constructed above. Fix $s > 3$. Choose

$$x_0 \neq 0, \qquad |x_0| < 1.$$

and define

$$\hat{u}(x) = \begin{cases} u_0(|x-x_0|) + 1, & |x - x_0| < 1, \\ u_s(|x|), & \text{elsewhere.} \end{cases}$$

Clearly $\hat{u}$ is of class $C^{2,\alpha}$ by our construction. Moreover, by direct verification, $\hat{u}$ is a classical non-symmetric solution of the degenerate Laplace equation (6.1), with

$$f(u) = \begin{cases} f_s(u), & u \in [0, 1], \\ f_0(u-1), & u \in [1, 2], \\ 0, & u > 2. \end{cases}$$

Plainly f is of class C^{α} if $m \in (1, 2)$ and at least of class $C^{1,\alpha}$ if $m > 2$ and a is suitably small. It is evident finally that $\hat{u}$ has compact support.

When $m > 2$, this construction provides a counterexample to Theorem 2 when the set (1.7) contains more than one point. Moreover, condition (1.7) cannot be replaced by the simple hypothesis that $\operatorname{supp}_+ u$ is connected (as in Theorem 1), since this condition is satisfied by $\hat{u}$.

Conversely, when $m = 2$ (Laplace operator), the example shows that *for the validity of Theorem 1 it is* **not** *enough simply to have f locally Hölder continuous for $u \geqq 0$;* see condition (i) in the Introduction.

Example 2. This example is a non-symmetric *positive* ground state. In analogy to the earlier construction, for $a > 0$, $s \geqq 0$ and $m > 1$, and also with

$$\frac{1}{a} > 2, \ \frac{m}{m-1}, \ \frac{n-m}{m-1},$$

we define

$$u(x) = u(r) = \begin{cases} 1, & r \leqq s, \\ [1 + (r-s)^{1/a}]^{-1}, & r \geqq s, \end{cases}$$

where r is the radius. Clearly $u \in C^{2,\alpha}(\boldsymbol{R}^n)$ for some $\alpha > 0$, with

$$u(s) = 1, \quad u'(s) = u''(s) = 0,$$

and

$$u(r) > 0; \quad r = u^{-a}(1-u)^a + s \ \text{ for } r > s.$$

By direct calculation, it follows that for $r > s$

$$u'(r) = -a^{-1}[1 + (r-s)^{1/a}]^{-2}(r-s)^{(1-a)/a} = -a^{-1}u^{a+1}(1-u)^{1-a} < 0,$$

$$(|u'(r)|^{m-1})' = (m-1)a^{-m}u^l(1-u)^k[2u - (a+1)],$$

where

$$k = m - 1 - am > 0, \quad l = m - 1 + am > 0.$$

As before, u is a classical solution of the equation

$$-\operatorname{div}(|Du|^{m-2}Du) = f(u) \tag{6.2}$$

for $r > 0$, where

$$f(u) = (|u'(r)|^{m-1})' = (m-1)a^{-m}u^l(1-u)^k[2u - (a+1)] \\ + (n-1)a^{1-m}u^l(1-u)^{k+a}[(1-u)^a + su^a]^{-1}$$

for $u \in [0, 1]$, and is identically zero elsewhere. Moreover $u(x)$ is strictly positive on $\boldsymbol{R}^n$ and

$$f'(u) \approx -la^{-m}[(m-1) - (n-m)a]u^{l-1} < 0 \quad \text{for small } u > 0.$$

Thus as before, f is Hölder continuous for $u \geqq 0$ and decreasing for small $u > 0$; furthermore, a proper choice of a obviously allows higher differentiability when $m > 2$.

Finally, a construction similar to that above yields a strictly positive non-symmetric solution $\hat{u}$ of (6.2). Of course, the remarks following the first counter-example continue to apply.

7. Singular Ground States

Let O be an arbitrary point in $\boldsymbol{R}^n$, which can be taken as the origin of coordinates. For convenience we put $\boldsymbol{S} = \boldsymbol{R}^n \backslash \{O\}$. A non-negative function u which is of class C^1 on $\boldsymbol{S}$ is called a *singular ground state* of (1.1) if

(1) u is a weak solution of (1.1) on $\boldsymbol{S}$,
(2) u satisfies (1.2), and
(3) $\lim_{x \to 0} u(x) = \infty$.

The previous symmetry results continue to be valid for singular ground states, provided their formulations are appropriately modified. To this end, we make the following further definitions (these are slightly different from those given above, but no confusion should result),

$$\operatorname{supp}_+ u = \{x \in \boldsymbol{S} \mid u(x) > 0\},$$

$$Z = \{x \in S \mid u(x) > 0,\ Du(x) = 0\}.$$

The following results then hold.

Theorem 1 (Singular case). *Let the hypotheses of Theorem 1 be satisfied. Then any singular ground state of* (1.1) *such that* $\mathrm{supp}_+ u$ *is connected must be radially symmetric about* O*. The corresponding radial function* $u(r)$ *obeys* $u'(r) < 0$ *for all* $r > 0$ *such that* $u(r) > 0$.

Theorem 2 (Singular case). *Suppose that the hypotheses of the first sentence of Theorem 2 are satisfied, and let u be a singular ground state of* (1.1) *such that the set* Z *is empty. Then the conclusion of Theorem 2 holds.*

Theorem 3 (Singular case). *Theorem 3 holds for singular ground states of* (1.1).

We prove only the result of Theorem 1, the other theorems being treated analogously. The notation of Section 3 will be carried over, with the additional definition

$$\Sigma' = \Sigma'_\gamma = \Sigma_\gamma \backslash \{O\}.$$

Lemma 3.1 is then easily seen to hold for singular ground states, provided that in the statement Σ is replaced by Σ' and $\gamma > 0$ (recall that O is taken as origin). Similarly, Lemma 3.3 remains valid when $\gamma > 0$ and Σ is replaced by Σ', the proof being unchanged since w_ε has compact support in Σ'.

Turning to the main proof of Theorem 1, we see that Step 1 clearly carries over unchanged, while obviously Step 2 applies for any $\gamma < 0$.

We assert that $\mu = 0$ in Step 3. Otherwise, if $\mu > 0$, we reach a contradiction exactly as before. (Note that the sequence $\{x^i\}$ necessarily has the property that $x^i \in \Sigma'_{\gamma_i}$, since w^i is undefined at O.) It follows that $w \geqq 0$ in Σ_0. Then by the reflection procedure at the end of the proof of Theorem 2 we obtain $w \equiv 0$ in Σ_0. Theorem 1 now follows at once, where of course $u(r) \to \infty$ as $r \to 0$.

Results corresponding to the auxilliary conclusions in the introduction can be left to the interested reader.

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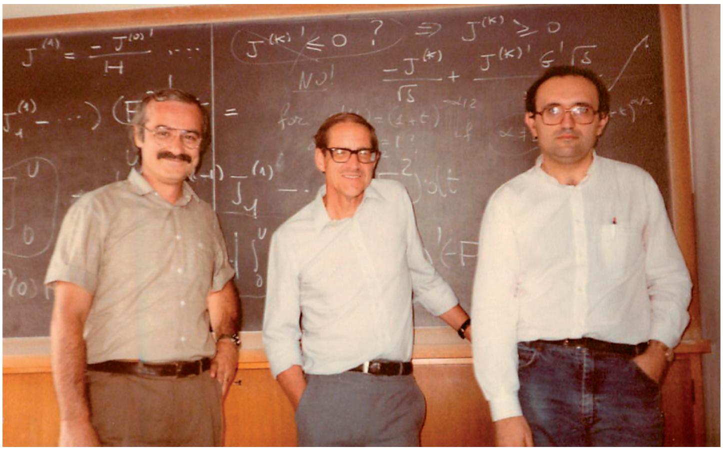

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