

Improved bounds for composites and rigidity of gradient fields

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We determine an improved bound for the conductivity of three-component composite materials. Our bound is strictly larger than the well-known Hashin-Shtrikman bound outside the regime where the latter is known to be optimal. The main ingredient of our result is a new quantitative rigidity estimate for gradient fields in two dimensions.

1 Introduction

The theory of homogenization permits to determine the macroscopic properties of microstructured materials. This corresponds to determining an appropriate weak limit of PDEs with rapidly oscillating coefficients [7, 15, 23, 9, 13]. In the case of periodic homogenization of scalar problems, one considers periodic microstructures which can be described by a position-dependent conductivity $\sigma \in L^\infty(Y)$, with $Y = (0, 1)^2$ the unit square. Throughout the paper, σ will be scalar so that we will be concerned with locally isotropic media. We assume the ellipticity condition $\sigma \geq \sigma_1 > 0$ a.e. The homogenized (or macroscopic) conductivity $\sigma_{\text{hom}} = \sigma_{\text{hom}}^T \in \mathbb{R}^{2 \times 2}$ can be determined by

$$\xi \cdot \sigma_{\text{hom}} \xi = \min \left\{ \int_Y \sigma(x) |\nabla u|^2(x) dx : u \in W_{\#, \xi}^{1,2}(Y) \right\}, \quad (1.1)$$

for every $\xi \in \mathbb{R}^2$. Here $W_{\#, \xi}^{1,2}(Y)$ is the set of $u \in W^{1,2}(Y)$ such that $x \mapsto u(x) - \xi x$ is Y -periodic (in the sense of traces) and with zero mean over Y . It is easy to see that σ_{hom} is positive definite.

The theory of composites is aimed at understanding which are the possible values of σ_{hom} , with σ varying in some subset of $L^\infty(Y)$ [11, 28]. In this paper, we focus on the case where the macroscopic conductivity σ_{hom} is proportional to the identity matrix Id ,

$$\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}, \quad s_{\text{hom}} > 0.$$

The methods of Hashin and Shtrikman [22], developed in 1963, prove that for every isotropic microstructure σ one has

$$s_{\text{hom}} \geq s_{\text{HS}} = \left(\int_Y (\sigma + \sigma_1)^{-1} dx \right)^{-1} - \sigma_1. \quad (1.2)$$

By duality, there is also a corresponding upper bound on s_{hom} . Since the techniques for treating the dual problem are identical, we treat only the lower bound in the present paper. It is natural to ask whether this bound can be improved, given some class of possible values of σ . The answer depends of course on the chosen class.

In problems coming from shape optimization one typically deals with mixing a finite number of components, with conductivities $\sigma_1 < \sigma_2 < \dots < \sigma_N$, and prescribed volume fractions $m_1, m_2, \dots, m_N \in [0, 1]$, with $\sum_{i=1}^N m_i = 1$ [11, 4, 28]. The microscopic conductivity σ can vary in the set

$$\Sigma = \{ \sigma \in L^\infty(Y) : |\{x \in Y : \sigma(x) = \sigma_i\}| = m_i, i = 1, \dots, N \}. \quad (1.3)$$

In the case of two materials the bound (1.2) is optimal, in the sense that for any admissible $\sigma_1, \sigma_2, m_1, m_2$ one has

$$s_{\text{HS}} = \inf \{ s_{\text{hom}} : \sigma_{\text{hom}} = s_{\text{hom}} \text{Id for some } \sigma \in \Sigma \}.$$

In fact, in this case the infimum is attained [34, 35, 5].

For the rest of this paper, we will focus on the case of three materials ($N = 3$). This case turns out to be substantially more complex, and one has to distinguish several parameter ranges. In the case

$$m_1 \geq \alpha(1 - m_2)$$

with

$$\alpha = \frac{2\sigma_1(\sigma_3 - \sigma_2)}{(\sigma_2 + \sigma_1)(\sigma_3 - \sigma_1)} \quad (1.4)$$

optimality of (1.2) was proven by Milton [27] by a construction based on “coated spheres”. Later, Lurie and Cherkaev [25] found different microstructures saturating the bound exactly in the same regime. A corresponding result for anisotropic composites was also given by Milton and Kohn [29].

In the broader range

$$m_1 \geq \alpha(\sqrt{m_2} - m_2) \quad (1.5)$$

optimality of (1.2) was proven by Gibiansky and Sigmund [21] with constructions that were inspired by numerical simulations. A simpler proof, using

finite-order laminates, which additionally permits treatment of the case of anisotropic composites, was then obtained in [1, 2]. The question of optimality of (1.2) in the case that (1.5) does not hold remained open.

The classical lower bound (1.2) was improved by Nesi in 1995 [31], who proved that

$$s_{\text{hom}} \geq s_{\text{HS}} + \Delta_N = \max_{\lambda \in [\sigma_1, \sigma_2]} \left\{ \left(\frac{m_1}{2\sigma_1} + \frac{m_2}{\sigma_2 + \lambda} + \frac{m_3}{\sigma_3 + \lambda} \right)^{-1} - \lambda \right\} \quad (1.6)$$

The function Δ_N is however strictly positive only in a subset of the range of parameters for which (1.5) does not hold.

In 2006, Albin, Cherkaev and Nesi [2] showed that, if (1.5) is not satisfied, then

$$s_{\text{hom}} > s_{\text{HS}} \quad (1.7)$$

for all $\sigma \in \Sigma$ with C^1 smooth level sets (again in the case $\sigma_{\text{hom}} = s_{\text{hom}}\text{Id}$). The question of the existence and regularity of a $\sigma \in \Sigma$ which minimizes s_{hom} in general remains open.

In this paper, we make no smoothness assumption and prove that the Hashin-Shtrikman bound (1.2) can be improved whenever (1.5) does not hold, thereby answering completely the question of optimality of (1.2) for the case of three components.

Theorem 1.1. *There is a function Δ such that for all σ_i and m_i , and all $\sigma \in \Sigma$ such that $\sigma_{\text{hom}} = s_{\text{hom}}\text{Id}$, it holds that*

$$s_{\text{hom}} \geq s_{\text{HS}} + \Delta, \quad (1.8)$$

with $\Delta > 0$ whenever (1.5) is not satisfied.

One key idea, which permits in particular a simple proof of (1.2), is to “vectorialize”. This idea appeared in print for the first time in [33] and later in a more detailed version in [34] and [24]. By considering (1.1) for two vectors ξ and $\xi' \in \mathbb{R}^2$ one obtains

$$E^T \cdot \sigma_{\text{hom}} E^T = \min \left\{ \int_Y \sigma(x) |\nabla u|^2(x) dx : u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2) \right\}. \quad (1.9)$$

Here $E = e_1 \otimes \xi + e_2 \otimes \xi' \in \mathbb{R}^{2 \times 2}$, $W_{\#,E}^{1,2}$ is the vectorial counterpart of $W_{\#, \xi}^{1,2}$, and $A \cdot B = \text{Tr } A^T B = \sum A_{ij} B_{ij}$. Having transformed the problem into a vectorial one permits the use, in a much more direct way, of rigidity of gradient fields. It also displays in a very direct way the strong connection between bounds on conductivities and vectorial variational problems, including in particular the

concept of quasiconvexity. One crucial observation is that, the determinant being a divergence, one has

$$\int_Y \det \nabla u \, dx = \det E \quad \text{for all } u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2). \quad (1.10)$$

This condition permits a straightforward proof of (1.2), see Section 2; in the language of the calculus of variations it turns out that the Hashin-Shtrikman bound is the best polyconvex bound.

Gradient fields, however, satisfy many more constraints than (1.10). In particular, the improved bound obtained by Nesi and stated in (1.6) can be obtained from the theorem of Alessandrini and Nesi [3, Th 1] which, in our case, shows that for any $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ which solves

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ weakly in } Y$$

for some $\sigma \in \Sigma$, the sign of the determinant is prescribed by the boundary data in the sense that if $\det E > 0$ then $\det \nabla u > 0$ almost everywhere [3]. A precise statement is given in Theorem 3.1 below. Originally, the bound was based on a weaker result established in [6].

The key idea in the proof of Theorem 1.1 is that for all $\sigma \in \Sigma$ such that the two sides of (1.2) are close, the gradient field ∇u entering (1.9) is close (after scaling) to the set

$$Z = \{F \in \mathbb{R}^{2 \times 2} : F = F^T, \operatorname{Tr} F = 2\} \cup \{F \in \mathbb{R}^{2 \times 2} : F = \alpha \operatorname{Id}, 0 < \alpha \leq 1\}. \quad (1.11)$$

This argument is discussed in detail in Section 2. Using Alessandrini and Nesi's result, we then show that ∇u is actually close to the smaller set

$$Z^b = \{F \in Z : \det F \geq 0\}.$$

The same sets Z and Z^b were already used in proving (1.7) in [2]; one key ingredient of that proof was indeed that $\nabla u \in Z$ a.e. implies $\nabla u \in Z^b$ a.e.. In the present work, we make the estimate quantitative. Precisely, we obtain the following optimal quantitative version of Alessandrini and Nesi's estimate for this pair of sets.

Theorem 1.2. *For all $E \in \mathbb{R}^{2 \times 2}$ with $\det E > 0$, and all $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$, one has*

$$\|\operatorname{dist}(\nabla u, Z^b)\|_{L^2} \leq c \|\operatorname{dist}(\nabla u, Z)\|_{L^2}^{2/3} + c \|\operatorname{dist}(\nabla u, Z)\|_{L^2}. \quad (1.12)$$

The scaling of the estimate is optimal; the constant is universal.

A similar result can be obtained for gradient Young measures supported on the set Z . A brief explanation of the concept of gradient Young measures is given in Section 3.3.

Theorem 1.3. *Let ν be a $W^{1,p}$ -gradient Young measure for some $p > 1$ and with $\text{supp } \nu \subset Z$, $\det\langle \nu, \text{Id} \rangle > 0$. Then ν is actually a $W^{1,\infty}$ -gradient Young measure, and $\text{supp } \nu \subset Z^b$.*

The study of rigidity results for gradient fields, such as Theorem 1.2, or for Young measures, such as Theorem 1.3, has been an important theme in the vectorial calculus of variations over many years. This ranges from classical results such as Korn's inequality or Liouville's rigidity theorem, to recent breakthroughs, including in particular the optimal quantitative version of Liouville's rigidity derived by Friesecke, James and Müller [20], corresponding two-well results [10, 16], rigidity for the four gradient problem by Chlebík and Kirchheim [12], rigidity of conformal matrices [19], and the localization result for gradient Young measures by Faraco and Székelyhidi [18]. In all these cases one deals with gradient fields supported around a set with some specific structure (such as Z here), and proves that the gradient constraint makes only a part of the set efficiently usable (Z^b here).

The work by Faraco and Székelyhidi [18] contains some results on sets which are very similar to the set Z considered here. In particular, they show that if a gradient field or a $W^{1,2}$ gradient Young measure is supported on the set

$$Z_k = \left\{ F \in \mathbb{R}^{2 \times 2} : F = F^T, \frac{1}{k} \leq \frac{2 - \lambda_1}{\lambda_2}, \frac{2 - \lambda_2}{\lambda_1} \leq k \right\},$$

for some $k > 0$, then it is supported either in its bounded or in its unbounded component (by taking the closure, the same extends to the limit $k \rightarrow \infty$). Here λ_1 and λ_2 denote the singular values of F . Although the context is similar, their result and their method of proof differ significantly from the present work. In particular, if one tries to make a quantitative version like Theorem 1.2 starting from the result on Z_k one achieves only the suboptimal exponent $1/2$, and not the optimal exponent $2/3$. Theorem 1.3 would follow from their result only if higher integrability of the measure was assumed.

One common key ingredient in the proof of most of the rigidity results cited above is that one separates the gradient field in two components: one which solves some "good" equation, which has to be constructed for the purpose, and one which is small. In Friesecke, James and Müller [20] and generalizations the equation was Laplace's equation, in Faraco and Székelyhidi [18] it was a nonlinear Beltrami equation. In the present paper it is a linear elliptic equation, see (3.3) below, or equivalently, a linear Beltrami equation.

The proofs of these results are given in the following sections. In Section 2 we prove Theorem 1.1; in Section 3 we prove Theorem 1.2; and in Section 4 we prove Theorem 1.3.

2 The improved bound

Before proving the improved bound, we present a proof of the bound (1.2) using the techniques developed in [33, 24, 34].

Theorem 2.1 (The Hashin-Shtrikman Bound). *Let $0 < \sigma_1 < \sigma_2 < \sigma_3$ and $m_1, m_2, m_3 \geq 0$ be given with $m_1 + m_2 + m_3 = 1$. Let $\sigma \in \Sigma$ (see (1.3)) be given such that $\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}$. Then*

$$s_{\text{hom}} \geq \left(\int_Y (\sigma + \sigma_1)^{-1} dx \right)^{-1} - \sigma_1 = \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} - \sigma_1. \quad (2.1)$$

Proof. We begin with the characterization of σ_{hom} given in (1.9), and rewrite the integral using (1.10), as

$$\int_Y \sigma(x) |\nabla u|^2(x) = \int_Y (\sigma(x) |\nabla u|^2(x) + 2\sigma_1 \det \nabla u(x)) dx - 2\sigma_1 \det E.$$

The integrand on the right-hand side can be seen as a quadratic form in ∇u . We denote it more compactly by

$$\sigma(x) |F|^2 + 2\sigma_1 \det F = F \cdot L(x)F$$

where $L(x) : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ is the position-dependent linear operator defined as

$$L(x)F = \sigma(x)F + \sigma_1 \text{cof } F. \quad (2.2)$$

It is straightforward to check that this operator is self-adjoint and positive semi-definite almost everywhere. Furthermore, L has a nontrivial null-space only for those x for which $\sigma(x) = \sigma_1$ (see Remark 2.2). This notation gives an alternate form for the characterization of σ_{hom} . Namely,

$$E^T \cdot \sigma_{\text{hom}} E^T = \min \left\{ \int_Y \nabla u \cdot L \nabla u dx - 2\sigma_1 \det E : u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2) \right\}. \quad (2.3)$$

The bound (2.1) arises by removing the constraint that the field ∇u is a gradient. More precisely,

$$E^T \cdot \sigma_{\text{hom}} E^T \geq \min \left\{ \int_Y F \cdot LF dx - 2\sigma_1 \det E : \right. \\ \left. F \in L^2(Y; \mathbb{R}^{2 \times 2}), \int_Y F dx = E \right\}. \quad (2.4)$$

The inequality follows from the observation that for any $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$, ∇u is an admissible F in the latter minimization problem.

This minimization problem is straightforward to address. We seek to minimize a quadratic functional subject to a linear constraint. Fix

$$E = \frac{1}{\sqrt{2}} \text{Id}. \quad (2.5)$$

From the Euler-Lagrange equations of (2.4), it is easy to see that a minimizer F of the problem satisfies $LF = G$ a.e. in Y for some constant $G \in \mathbb{R}^{2 \times 2}$. In particular, if we take

$$F(x) = \frac{1}{\sqrt{2}(\sigma(x) + \sigma_1)} \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} \text{Id}, \quad (2.6)$$

then F has the correct average and

$$LF = \frac{1}{\sqrt{2}} \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} \text{Id} \quad \text{a.e. in } Y. \quad (2.7)$$

It follows easily that F is a minimizer for (2.4). Substituting (2.5), (2.6) and (2.7) into (2.4) gives (2.1). \square

Remark 2.2. *To understand L , it is useful to study the operator independently in the sets*

$$Y_i = \{x \in Y : \sigma(x) = \sigma_i\},$$

and to recall that for all $F \in \mathbb{R}^{2 \times 2}$ one has

$$\text{cof} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix}, \quad \text{cof}(\text{cof}(F)) = F.$$

The operator L has eigenvalues $\sigma_i + \sigma_1$ and $\sigma_i - \sigma_1$, with associated eigenspaces

$$\mathbb{R}_+^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \quad \text{and} \quad \mathbb{R}_-^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

respectively. In particular, L is invertible in Y_i for $i \neq 1$. In Y_1 , L has the non-trivial nullspace $\mathbb{R}_-^{2 \times 2}$. Its eigenvalues are $2\sigma_1$ and 0 with the same eigenspaces as above. Thus, in Y_1 , (2.7) uniquely determines only the projection of F onto $\mathbb{R}_+^{2 \times 2}$.

We now turn our attention to the proof of Theorem 1.1. The idea is to get a quantitative estimate on the error in passing from (2.3) to (2.4).

Proof of Theorem 1.1. In order to simplify some computations, we shall rescale the problem so that

$$E = \sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \text{Id}. \quad (2.8)$$

Then the optimality condition (2.7) becomes instead

$$LF = \text{Id} \quad \text{a.e. in } Y. \quad (2.9)$$

We fix the volume fractions m_1, m_2, m_3 and let $\sigma \in \Sigma$ be an admissible material layout with $\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}$. We now split any $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ into a solution to (2.9) and an error. Specifically, we define $F_u \in L^2(Y; \mathbb{R}^{2 \times 2})$ by

$$F_u(x) = \begin{cases} \frac{1}{\sigma_i + \sigma_1} \text{Id} & \text{if } x \in Y_i, i \neq 1 \\ \frac{1}{2\sigma_1} \text{Id} + \frac{1}{2}(\nabla u(x) - \text{cof}(\nabla u(x))) & \text{if } x \in Y_1. \end{cases} \quad (2.10)$$

The subscript u makes explicit the dependence of F_u on u . Again we define $Y_i = \{x \in Y : \sigma(x) = \sigma_i\}$.

Since, by construction, F_u satisfies (2.9) almost everywhere, we have

$$\int_Y (\nabla u - F_u) \cdot LF_u \, dx = \int_Y \text{Tr}(\nabla u - F_u) \, dx = 0. \quad (2.11)$$

The last equality holds because

$$\int_Y \text{Tr} F_u \, dx = 2 \sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} = \text{Tr} E = \int_Y \text{Tr} \nabla u \, dx.$$

From (2.9), (2.10) and (2.11), we find

$$\begin{aligned} \int_Y \nabla u \cdot L \nabla u \, dx &= \int_Y (\nabla u - F_u + F_u) \cdot L(\nabla u - F_u + F_u) \, dx \\ &= \int_Y F_u \cdot LF_u + \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx \\ &= 2 \sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} + \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx. \end{aligned}$$

This, together with the characterization (2.3) of $\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}$ shows

$$s_{\text{hom}} = s_{\text{HS}} + \mathcal{E}$$

where

$$\mathcal{E} = \frac{1}{2} \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min \left\{ \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx : u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2) \right\}.$$

The improved bound arises by proving a lower bound on \mathcal{E} .

In Y_i for $i \neq 1$, we have (see Remark 2.2)

$$(\nabla u - F_u) \cdot L(\nabla u - F_u) \geq (\sigma_i - \sigma_1) |\nabla u - F_u|^2.$$

Furthermore, since $\nabla u - F_u \in \mathbb{R}_+^{2 \times 2}$, from Remark 2.2 we see that in Y_1

$$L(\nabla u - F_u) = 2\sigma_1(\nabla u - F_u).$$

From these two observations, we find that

$$\begin{aligned} \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) &\geq 2\sigma_1 \int_{Y_1} |\nabla u - F_u|^2 \\ &\quad + \sum_{i=2}^3 (\sigma_i - \sigma_1) \int_{Y_i} |\nabla u - F_u|^2. \end{aligned} \quad (2.12)$$

Thus, we have

$$\mathcal{E} \geq C \inf \left\{ \|\nabla u - F_u\|_{L^2}^2 : u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2) \right\} \quad (2.13)$$

where

$$C = 2\sigma_1^2 \min\{2\sigma_1, \sigma_2 - \sigma_1\} \leq \frac{1}{2} \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min\{2\sigma_1, \sigma_2 - \sigma_1\}.$$

Indeed, (2.13) holds if C is replaced by the larger term in the above inequality. C has been chosen to be independent of the volume fractions m_i .

If it happens that $\nabla u = F_u$ a.e. in Y , then translating u and applying Theorem 1.2 shows that

$$\det \left(\nabla u - \frac{1}{\sigma_3 + \sigma_1} \text{Id} \right) > 0 \text{ a.e.}$$

For this reason, we now make the following definitions

$$\begin{aligned} v(x) &= \frac{2\sigma_1(\sigma_3 + \sigma_1)}{\sigma_3 - \sigma_1} \left(u(x) - \frac{1}{\sigma_3 + \sigma_1} x \right), \\ G(x) &= \frac{2\sigma_1(\sigma_3 + \sigma_1)}{\sigma_3 - \sigma_1} \left(F_u(x) - \frac{1}{\sigma_3 + \sigma_1} \text{Id} \right). \end{aligned}$$

The leading constant is chosen so that $G \in Z$ almost everywhere with Z as in (1.11). From (2.10), we find

$$G(x) = \begin{cases} \alpha_i \text{Id} & \text{if } x \in Y_i, i \neq 1 \\ \alpha_1 \text{Id} + \frac{1}{2}(\nabla v(x) - \text{cof}(\nabla v(x))) & \text{if } x \in Y_1. \end{cases} \quad (2.14)$$

where

$$\alpha_i = \frac{2\sigma_1(\sigma_3 - \sigma_i)}{(\sigma_i + \sigma_1)(\sigma_3 - \sigma_1)} \quad \text{for } i = 1, 2, 3.$$

That is, $\alpha_1 = 1$, $\alpha_3 = 0$, and $\alpha_2 = \alpha$ from (1.4). It is straightforward to check that

$$\begin{aligned} \int_Y \det \nabla v \, dx &= \det \int_Y \nabla v \, dx \\ &= \det \left(\sum_{i=1}^3 m_i \alpha_i \text{Id} \right) = (m_1 + m_2 \alpha)^2. \end{aligned} \quad (2.15)$$

For $i \neq 1$, we find

$$\begin{aligned} \int_{Y_i} \det \nabla v \, dx &= \int_{Y_i} \det(\nabla v - G + G) \, dx \\ &= \int_{Y_i} [\det(\nabla v - G) + (\nabla v - G) \cdot \text{cof } G + \det G] \, dx. \end{aligned}$$

Using that $G = \alpha_i \text{Id}$ in Y_i and applying Hölder's inequality to the second term, we find

$$\int_{Y_i} \det \nabla v \, dx \geq -\frac{1}{2} \int_{Y_i} |\nabla v - G|^2 \, dx - \sqrt{2} \alpha_i m_i^{\frac{1}{2}} \left(\int_{Y_i} |\nabla v - G|^2 \, dx \right)^{\frac{1}{2}} + m_i \alpha_i^2.$$

Equivalently,

$$\frac{\gamma^2}{2} \|\nabla u - F_u\|_{L^2(Y_i)}^2 + \gamma \sqrt{2} \alpha_i m_i^{\frac{1}{2}} \|\nabla u - F_u\|_{L^2(Y_i)} \geq m_i \alpha_i^2 - \int_{Y_i} \det \nabla v \, dx, \quad (2.16)$$

where

$$\gamma = 2\sigma_1(\sigma_3 + \sigma_1)/(\sigma_3 - \sigma_1). \quad (2.17)$$

For Y_1 we proceed differently. Let H be an arbitrary mapping $H \in L^\infty(Y; Z^b)$. Then we estimate as before with H in place of G , noting that $|H| \leq 2$ and $\det H \geq 0$ almost everywhere.

$$\begin{aligned} \int_{Y_1} \det \nabla v \, dx &= \int_{Y_1} [\det(\nabla v - H) + (\nabla v - H) \cdot \text{cof } H + \det H] \, dx \\ &\geq -\frac{1}{2} \int_{Y_1} |\nabla v - H|^2 \, dx - 2m_1^{\frac{1}{2}} \left(\int_{Y_1} |\nabla v - H|^2 \, dx \right)^{\frac{1}{2}} \\ &\geq -\frac{1}{2} \int_Y |\nabla v - H|^2 \, dx - 2m_1^{\frac{1}{2}} \left(\int_Y |\nabla v - H|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Taking the supremum in H of the right-hand side, we obtain

$$\int_{Y_1} \det \nabla v \, dx \geq -\frac{1}{2} \|\text{dist}(\nabla v, Z^b)\|_{L^2}^2 - 2m_1^{\frac{1}{2}} \|\text{dist}(\nabla v, Z^b)\|_{L^2}.$$

Clearly if $\|\nabla u - F_u\|_{L^2} \geq \gamma^{-1}$ for every $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$, then (2.13) proves that we may choose $\Delta = \gamma^{-2}C > 0$ in (1.8). From now on, we assume that u is chosen such that $\|\nabla u - F_u\|_{L^2} < \gamma^{-1}$, which corresponds to $\|\text{dist}(\nabla v, Z)\|_{L^2} < 1$. After applying Theorem 1.2, we have

$$\int_{Y_1} \det \nabla v \, dx \geq -\frac{1}{2}c \|\text{dist}(\nabla v, Z)\|_{L^2}^{\frac{4}{3}} - 2cm_1^{\frac{1}{2}} \|\text{dist}(\nabla v, Z)\|_{L^2}^{\frac{2}{3}}$$

which translates into

$$\frac{1}{2}c\gamma^{\frac{4}{3}} \|\nabla u - F_u\|_{L^2(Y)}^{\frac{4}{3}} + 2cm_1^{\frac{1}{2}}\gamma^{\frac{2}{3}} \|\nabla u - F_u\|_{L^2(Y)}^{\frac{2}{3}} \geq -\int_{Y_1} \det \nabla v \, dx, \quad (2.18)$$

for a universal constant $c > 0$.

Combining (2.15), (2.16) and (2.18), we find

$$\begin{aligned} & \frac{\gamma^2}{2} \|\nabla u - F_u\|_{L^2(Y \setminus Y_1)}^2 + \sqrt{2}\gamma\alpha m_2^{\frac{1}{2}} \|\nabla u - F_u\|_{L^2(Y_2)} \\ & + \frac{1}{2}c\gamma^{\frac{4}{3}} \|\nabla u - F_u\|_{L^2(Y)}^{\frac{4}{3}} + 2cm_1^{\frac{1}{2}}\gamma^{\frac{2}{3}} \|\nabla u - F_u\|_{L^2(Y)}^{\frac{2}{3}} \geq \kappa \end{aligned} \quad (2.19)$$

where

$$\kappa = \max \{0, m_2\alpha^2 - (m_1 + m_2\alpha)^2\}. \quad (2.20)$$

Notice that two terms in (2.16) disappear for $i = 3$ since $\alpha_3 = 0$. We may take the maximum of the second term above with 0 since the left-hand side of (2.19) is clearly nonnegative.

Using straightforward estimates on the left-hand side of (2.19), we find

$$\|\nabla u - F_u\|_{L^2}^2 + \|\nabla u - F_u\|_{L^2} + \|\nabla u - F_u\|_{L^2}^{\frac{4}{3}} + \|\nabla u - F_u\|_{L^2}^{\frac{2}{3}} \geq c^* \kappa,$$

where c^* is a constant that depends on c and the σ_i . Thus we have (see the comments before (2.18)) either \mathcal{E} is sufficiently large already, or else the above inequality holds, which implies that at least one of the terms on the left-hand side is larger than $\kappa/4$. Thus, by choosing a constant $C^* > 0$ sufficiently small and recalling (2.13), we have

$$\mathcal{E} \geq C^* \min \{1, \kappa^3\}$$

and the right-hand side can be taken as Δ for the theorem. We observe that C depends only on c and the σ_i , but not on the m_i . That $\kappa > 0$ when (1.5) is violated is straightforward to check. \square

By more carefully estimating (2.12) and (2.19), we obtain the following refinement of the theorem.

Proposition 2.3. *Theorem 1.1 holds with*

$$\begin{aligned} \Delta = & \frac{1}{2\gamma^2} \left(\sum_{i=1}^3 \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min\{2\sigma_1, \sigma_2 - \sigma_1\} \\ & \times \min \left\{ 1, \frac{\kappa}{2}, \left(\frac{\kappa}{4\alpha} \right)^2 \frac{1}{2m_2}, \left(\frac{\kappa}{2c} \right)^{\frac{3}{2}}, \left(\frac{\kappa}{8c} \right)^3 \frac{1}{m_1^{\frac{3}{2}}} \right\} \end{aligned}$$

where α is defined in (1.4), γ is defined in (2.17), κ is defined in (2.20), and c is universal.

3 Rigidity of the set Z

3.1 The underlying PDE

In order to illustrate the main ideas in the proof we first present a special case. Assume that $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ satisfies the differential inclusion

$$\nabla u \in Z_0 \text{ a.e.}, \quad (3.1)$$

where

$$Z_0 = \{F \in \mathbb{R}^{2 \times 2} : F = F^T, \text{Tr } F = 2\} \cup \left\{ \frac{1}{2} \text{Id} \right\},$$

and assume that $\det E > 0$. The key property of Z_0 is that for any $F \in Z_0$ there is a number $\sigma_F > 0$ such that

$$\sigma_F F + \text{cof } F = 2\text{Id}.$$

Indeed, it suffices to take $\sigma_F = 3$ if $F = \text{Id}/2$, and $\sigma_F = 1$ else. Performing this basic construction at each point $x \in Y$ we obtain a function $\sigma \in L^\infty(Y; \mathbb{R})$ such that

$$\sigma \nabla u + \text{cof } \nabla u = 2\text{Id} \quad \text{a.e. in } Y. \quad (3.2)$$

Precisely, σ is defined by

$$\sigma(x) = \begin{cases} 3 & \text{if } \nabla u(x) = \frac{1}{2}\text{Id} \\ 1 & \text{otherwise.} \end{cases}$$

Since the cofactor of a gradient field is divergence-free, taking the divergence (in a distributional sense) of (3.2) shows that u solves the elliptic equation

$$\operatorname{div}(\sigma \nabla u) = 0. \quad (3.3)$$

Since $1 \leq \sigma \leq 3$ this equation has a unique solution in $W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$, hence it uniquely identifies u . This permits us to use the following injectivity result by Alessandrini and Nesi.

Theorem 3.1 (From Alessandrini and Nesi [3], Thm. 1). *Let $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ be the weak solution of*

$$\operatorname{div}(\sigma \nabla u) = 0,$$

where $\sigma \in L^\infty(Y; \mathbb{R}^{2 \times 2})$, $\sigma = \sigma^T$, $1/k \leq \sigma \leq k$ a.e., for some $k \geq 1$, and $E \in \mathbb{R}^{2 \times 2}$. If $\det E > 0$ then

$$\det \nabla u > 0 \text{ a.e. in } Y.$$

By (3.3), we can apply this result to our u . Then, (3.1) gives the improved differential inclusion

$$\nabla u \in Z_0 \cap \{F : \det F > 0\} \text{ a.e.} \quad (3.4)$$

In particular, we have shown that any $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ which solves the differential inclusion $\nabla u \in Z_0$, where Z_0 is the unbounded set defined above, is actually Lipschitz, i.e., that only a bounded part of Z_0 is actually used. The key argument used to derive (3.4) has been to construct an elliptic PDE which has the given u as a solution, and then to use properties of that PDE. This basic argument will be refined below in several respects. In Section 3.2 we shall consider maps $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ which are only approximate solutions of the differential inclusion, and derive an optimal quantitative rigidity for them. In Section 4 we shall instead consider maps with weaker integrability, and in particular $W^{1,p}$ -Gradient Young Measures, and show that they exhibit the same rigidity. In Section 3.4 we show that our rigidity result cannot follow from polyconvexity alone.

3.2 Quantitative estimate

Proof of Theorem 1.2. Let $F : Y \rightarrow Z$ be a measurable map such that

$$\operatorname{dist}(\nabla u, Z) = |\nabla u - F|.$$

Pick some $\delta \in (0, 1/2)$. We define

$$F_\delta = (1 - \delta)F + \delta \operatorname{Id}, \quad u_\delta(x) = (1 - \delta)u(x) + \delta x.$$

Clearly $F_\delta \in Z$; if F_δ is of the form αId , then $\delta \leq \alpha \leq 1$. At the same time $|F_\delta - \nabla u_\delta| = (1 - \delta)|F - \nabla u|$.

We define $\sigma_\delta : Y \rightarrow \mathbb{R}$ by setting

$$\sigma_\delta(x) = \begin{cases} (2 - \alpha)/\alpha & \text{if } F_\delta(x) = \alpha \text{Id} \\ 1 & \text{otherwise.} \end{cases}$$

As above, this choice ensures that

$$\sigma_\delta F_\delta + \text{cof } F_\delta = 2\text{Id} \text{ a.e. in } Y. \quad (3.5)$$

Now let

$$f_\delta = \sigma_\delta \nabla u_\delta + \text{cof } \nabla u_\delta - 2\text{Id}. \quad (3.6)$$

It is clear that $f_\delta \in L^2(Y; \mathbb{R}^{2 \times 2})$, and that

$$\text{div}(\sigma_\delta \nabla u_\delta) = \text{div } f_\delta. \quad (3.7)$$

Given f_δ , the elliptic equation (3.7) has a unique solution in $W_{\#, E_\delta}^{1,2}(Y; \mathbb{R}^2)$, and this is u_δ . Here and below $E_\delta = (1 - \delta)E + \delta \text{Id}$.

We define $v \in W_{\#, E_\delta}^{1,2}(Y; \mathbb{R}^{2 \times 2})$ as the solution of

$$\text{div}(\sigma_\delta \nabla v_\delta) = 0.$$

By Theorem 3.1 we have

$$\det \nabla v_\delta > 0 \text{ a.e. in } Y,$$

which implies, by a geometric estimate illustrated in Figure 1,

$$\text{dist}(\nabla v_\delta, Z^b) \leq \sqrt{2} \text{dist}(\nabla v_\delta, Z). \quad (3.8)$$

It remains to estimate the distance between ∇u_δ and ∇v_δ . To do this, we consider the difference $w_\delta = u_\delta - v_\delta$. By the linearity of the equation, w_δ is the unique solution in $W_{\#, 0}^{1,2}(Y; \mathbb{R}^2)$ of

$$\text{div}(\sigma_\delta \nabla w_\delta) = \text{div } f_\delta.$$

Testing this equation with w_δ gives

$$\int_Y \sigma_\delta |\nabla w_\delta|^2 dx = \int_Y f_\delta \cdot \nabla w_\delta dx.$$

Let $p_\delta = \sigma_\delta^{1/2} |\nabla w_\delta|$. Then $\|p_\delta\|_{L^2}^2 \leq \|p_\delta\|_{L^2} \|\sigma_\delta^{-1/2} f_\delta\|_{L^2}$, and thus

$$\|p_\delta\|_{L^2(Y)} \leq \|\sigma_\delta^{-1/2} f_\delta\|_{L^2(Y; \mathbb{R}^2)}. \quad (3.9)$$

In order to estimate the norm of $\sigma_\delta^{-1/2} f_\delta$, we combine (3.5) and (3.6) to obtain

$$f_\delta = \sigma_\delta(\nabla u_\delta - F_\delta) + (\text{cof } \nabla u_\delta - \text{cof } F_\delta)$$

which implies

$$\sigma_\delta^{-1/2} |f_\delta| \leq (\sigma_\delta^{1/2} + \sigma_\delta^{-1/2}) |\nabla u_\delta - F_\delta| \leq (\sigma_\delta^{1/2} + \sigma_\delta^{-1/2}) \text{dist}(\nabla u, Z), \quad (3.10)$$

pointwise. Therefore, recalling (3.9) and the fact that by definition $1 \leq \sigma_\delta \leq 2/\delta$, we obtain

$$\|\nabla w_\delta\|_{L^2} \leq \|\sigma_\delta^{-1/2}\|_{L^\infty} \|p_\delta\|_{L^2} \leq \|p_\delta\|_{L^2} \leq \frac{c}{\delta^{1/2}} \|\text{dist}(\nabla u, Z)\|_{L^2}.$$

Finally, recalling (3.8),

$$\begin{aligned} \|\text{dist}(\nabla u_\delta, Z^b)\|_{L^2} &\leq \|\text{dist}(\nabla v_\delta, Z^b)\|_{L^2} + \|\nabla u_\delta - \nabla v_\delta\|_{L^2} \\ &\leq \sqrt{2} \|\text{dist}(\nabla v_\delta, Z)\|_{L^2} + \|\nabla w_\delta\|_{L^2} \\ &\leq \sqrt{2} \|\text{dist}(\nabla u_\delta, Z)\|_{L^2} + (1 + \sqrt{2}) \|\nabla w_\delta\|_{L^2} \\ &\leq \sqrt{2} \|\text{dist}(\nabla u, Z)\|_{L^2} + (1 + \sqrt{2}) \|\nabla w_\delta\|_{L^2}, \end{aligned}$$

and

$$\begin{aligned} \|\text{dist}(\nabla u, Z^b)\|_{L^2} &\leq \frac{\sqrt{2}\delta}{1-\delta} + \frac{1}{1-\delta} \|\text{dist}(\nabla u_\delta, Z^b)\|_{L^2} \\ &\leq c\delta + c \|\text{dist}(\nabla u, Z)\|_{L^2} + \frac{c}{\delta^{1/2}} \|\text{dist}(\nabla u, Z)\|_{L^2}. \end{aligned}$$

Optimizing δ we conclude

$$\|\text{dist}(\nabla u, Z^b)\|_{L^2} \leq c \|\text{dist}(\nabla u, Z)\|_{L^2}^{2/3} + c \|\text{dist}(\nabla u, Z)\|_{L^2}.$$

Optimality of the scaling follows from Lemma 3.2 below. \square

3.3 Optimality: gradient Young measures and laminates

We now turn to the question of proving optimality of the 2/3 scaling in Theorem 1.2. In order to do this, we need to construct a gradient field such that its L^2 distance from Z is of order ε , and its L^2 distance from Z^b is of order $\varepsilon^{2/3}$. In principle one could do this by writing an explicit test function. It is, however, much simpler to first perform a construction in matrix space, and then to use general tools, in particular the concept of a laminate, to

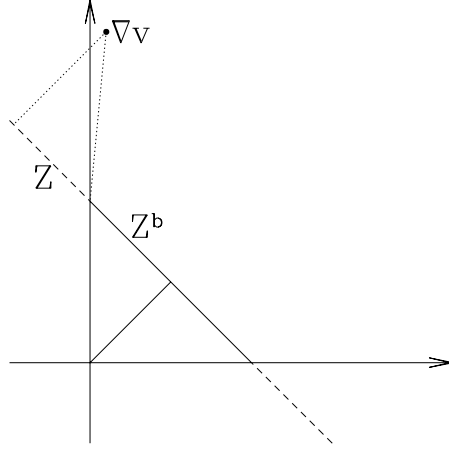


Figure 1: Estimating $\|\text{dist}(\nabla v_\delta, Z^b)\|_{L^2}$ by $\|\text{dist}(\nabla v_\delta, Z)\|_{L^2}$.

obtain existence of a test function. We therefore start by giving a brief sketch of the necessary background material and notation.

Given a sequence u_k converging weakly in $W_{\#,E}^{1,p}(Y; \mathbb{R}^2)$ to the affine function $x \mapsto Ex$, one says that u_k generate the $W^{1,p}$ (homogeneous) gradient Young measure $\nu \in \mathcal{M}(\mathbb{R}^{2 \times 2})$ if

$$\lim_{k \rightarrow \infty} \int_Y f(\nabla u_k) dx = \int_{\mathbb{R}^{2 \times 2}} f(F) d\nu(F) \quad (3.11)$$

for all $f \in C_c^0(\mathbb{R}^{2 \times 2})$. By simple truncation arguments the same will automatically hold for all continuous f satisfying the growth condition

$$|f(F)| \leq C(1 + |F|^q)$$

for some $q < p$. Taking f to be the identity mapping one sees that the average of ν is E . It can be shown that every such sequence generates such a measure. The Young measure gives the *volume distribution* of the values of the gradient, see, e.g., [32, 30] for details.

It is a very difficult question to decide which measures on $\mathbb{R}^{2 \times 2}$ can be generated this way. There is, however, a large class of measures that can be easily generated, the class of so-called laminates. A laminate of zeroth order with average F is a Dirac delta, i.e., it has the form $\nu = \delta_F$, and is generated by the (constant) sequence $u_k(x) = Fx$. A laminate of n -th order is defined inductively from a laminate of order $n - 1$ by replacing each of the terms $c\delta_F$ by a sum $c(\lambda\delta_{F_1} + (1 - \lambda)\delta_{F_2})$, where $\lambda F_1 + (1 - \lambda)F_2 = F$, $\lambda \in [0, 1]$, and

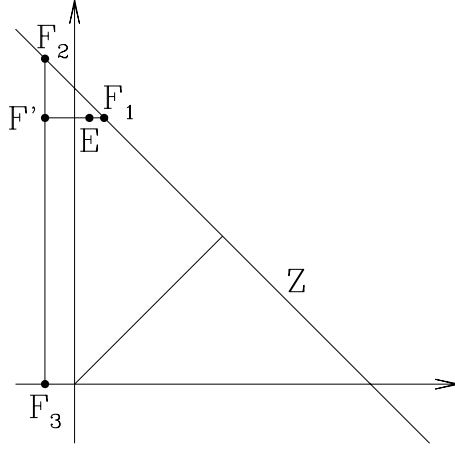


Figure 2: Sketch of the laminate used in the proof of Lemma 3.2.

$\text{rank}(F_1 - F_2) \leq 1$. One says that the matrix F has been split into F_1 and F_2 . For example, first-order laminates have the form $\nu = \lambda\delta_{F_1} + (1 - \lambda)\delta_{F_2}$, with $F_1 - F_2 = a \otimes n$, and are the limits of the gradient distributions of the maps

$$u_k(x) = F_2 x + \frac{1}{k} a \chi(kx \cdot n) .$$

The Lipschitz function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\chi(0) = 0$, $\chi'(t) = 1$ if $t \in (z, z + \lambda)$ and $\chi'(t) = 0$ if $t \in (z + \lambda, z + 1)$, for $z \in \mathbb{Z}$. For large k , the gradients ∇u_k oscillate on a fine scale between the values F_1 and F_2 , with average $F = \lambda F_1 + (1 - \lambda)F_2$. As $k \rightarrow \infty$, the sequence u_k converges weakly-* in $W^{1,\infty}$ to the affine function $x \mapsto Fx$; the function $f(\nabla u_k)$ converges weakly-* in L^∞ to $\int V(F) d\nu(F) = \lambda V(F_1) + (1 - \lambda)V(F_2)$. Refining this argument one can show that mixtures are always possible between rank-one connected matrices, hence that all laminates as defined above are attainable as weak limits of gradients. For details see, e.g., [14, 30, 17].

We are now in a position to present the construction proving optimality of Theorem 1.2.

Lemma 3.2. *For any $\varepsilon > 0$ there exists $E \in \mathbb{R}^{2 \times 2}$ with $\det E > 0$ and $u \in W_{\#,E}^{1,2}(Y; \mathbb{R}^2)$ such that*

$$\|\text{dist}(\nabla u, Z)\|_{L^2} \leq \varepsilon$$

and

$$\|\text{dist}(\nabla u, Z^b)\|_{L^2} \geq \frac{1}{4} \varepsilon^{2/3} .$$

Proof. If $\varepsilon \geq 1$ it suffices to take $u(x) = Ex = (1 + \varepsilon/2)x$, therefore we concentrate on the case $\varepsilon < 1$.

We shall first construct a laminate which obeys the mentioned inequalities, and then a test function u .

The laminate is supported on diagonal matrices. Let $0 < \delta < 1$. We consider $F_1 = \text{diag}(\delta, 2 - \delta) \in Z^b$; $E = \text{diag}(\delta/2, 2 - \delta)$; $F' = (-\delta, 2 - \delta)$. Then

$$\nu' = \frac{3}{4}\delta_{F_1} + \frac{1}{4}\delta_{F'}$$

is a laminate with average E . We now split F' into the two matrices $F_2 = \text{diag}(-\delta, 2 + \delta)$ and $F_3 = \text{diag}(-\delta, 0)$, to obtain the laminate

$$\nu = \frac{3}{4}\delta_{F_1} + \frac{1}{4}\frac{2 - \delta}{2 + \delta}\delta_{F_2} + \frac{1}{4}\frac{2\delta}{2 + \delta}\delta_{F_3}$$

(see Figure 2). Since $F_1, F_2 \in Z$, we have

$$\int_{\mathbb{R}^{2 \times 2}} \text{dist}^2(F, Z) d\nu(F) = \frac{1}{4}\frac{2\delta}{2 + \delta} \text{dist}^2(F_3, Z) = \frac{1}{4}\frac{2\delta}{2 + \delta}\delta^2 \leq \frac{1}{4}\delta^3.$$

However, $F_2 \notin Z^b$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^{2 \times 2}} \text{dist}^2(F, Z^b) d\nu(F) &= \frac{1}{4}\frac{2 - \delta}{2 + \delta} \text{dist}^2(F_2, Z^b) + \frac{1}{4}\frac{2\delta}{2 + \delta} \text{dist}^2(F_3, Z^b) \\ &= \frac{1}{4}\frac{2 - \delta}{2 + \delta} 2\delta^2 + \frac{1}{4}\frac{2\delta}{2 + \delta}\delta^2 \geq \frac{1}{4}\delta^2. \end{aligned}$$

Let now $\delta = \varepsilon^{2/3}$, and pick a sequence u_k which generates the laminate ν . Then

$$\lim_{k \rightarrow \infty} \int_Y \text{dist}^2(\nabla u_k, Z^b) dx \geq \frac{1}{4}\varepsilon^{4/3}, \quad \lim_{k \rightarrow \infty} \int_Y \text{dist}^2(\nabla u_k, Z) dx \leq \frac{1}{4}\varepsilon^2.$$

Therefore taking k sufficiently large the Lemma is proven. \square

3.4 Rigidity does not hold for polyconvex measures

We finally show that our rigidity result uses in a substantial way the fact that we are dealing with a gradient field. In particular, the same cannot be proven just arguing with null Lagrangians, i.e., on the basis of polyconvex bounds. To prove this, we exhibit a polyconvex measure which violates the statement. By polyconvex measure we mean a measure ν on $\mathbb{R}^{2 \times 2}$ which obeys the equivalent for measures of (1.10), i.e., such that the determinant of the average is the average of the determinant. From the discussion above

it is clear that all laminates and all gradient Young measures are polyconvex measures, but the converse is false. In particular, let $E = \text{diag}(\frac{1}{7}, \frac{1}{3})$, and

$$\nu = \frac{16}{21}\delta_{\text{diag}(0,0)} + \frac{4}{21}\delta_{\text{diag}(1,1)} + \frac{1}{21}\delta_{\text{diag}(-1,3)}.$$

It is clear that $\text{supp } \nu \subset Z$, and that $\det E > 0$. Easy computations prove that

$$\langle \nu; \text{Id} \rangle = E, \quad \langle \nu; \det \rangle = \det E,$$

hence ν is a polyconvex measure with average E . However, from the definition it is apparent that the support of ν contains matrices with negative determinant, therefore $\text{supp } \nu \not\subset Z^b$.

4 Gradient Young measures supported on Z

The key point in the proof of Theorem 1.3 is that a $W^{1,p}$ -gradient Young measure supported on Z necessarily has much better integrability. In fact, one can show that such a measure is necessarily a $W^{1,q}$ -gradient Young measure for any $q \geq 1$. Let us suppose we have shown that ν is a $W^{1,q}$ -gradient Young measure for some $q > 2$ and which is generated by a sequence $\{u_j\}$. Taking $f(F) = \text{dist}^2(F, Z)$ in (3.11) gives

$$\|\text{dist}(\nabla u^j, Z)\|_{L^2} \rightarrow 0.$$

Theorem 1.2 implies that

$$\|\text{dist}(\nabla u^j, Z^b)\|_{L^2} \rightarrow 0$$

as well. Finally, applying (3.11) now for $f(F) = \text{dist}^2(F, Z^b)$ shows that $\text{supp } \nu \subset Z^b$. Moreover, the boundedness of $\text{supp } \nu$ then implies by a truncation argument that ν is in fact a $W^{1,\infty}$ -gradient Young measure.

The proof of sufficient integrability relies on the following theorem due to Boyarskiĭ [8] (see also [26] for the generalization in n dimensions).

Theorem 4.1 (From [8, 26]). *Let $k \geq 1$ be given. There exist constants $p_c(k) < 2 < q_c(k)$ such that if $p_c(k) < p < q_c(k)$, if $\sigma = \sigma^T \in L^\infty(Y; \mathbb{R}^{2 \times 2})$ satisfies*

$$\forall \xi \in \mathbb{R}^2 \quad k^{-1}|\xi|^2 \leq \xi \cdot \sigma \xi \leq k|\xi|^2 \text{ a.e.} \quad (4.1)$$

and if $f \in L^p(Y; \mathbb{R}^{2 \times 2})$, then a unique solution to the equation

$$\begin{cases} \text{div}(\sigma \nabla u) = \text{div } f \\ u \in W_0^{1,p}(Y; \mathbb{R}^2) \end{cases}$$

exists and

$$\|u\|_{W^{1,p}} \leq C(k,p)\|f\|_{L^p}.$$

The exponents $p_c(k)$ and $q_c(k)$ obey

$$\lim_{k \rightarrow 1^+} p_c(k) = 1, \quad \lim_{k \rightarrow 1^+} q_c(k) = +\infty.$$

Theorem 1.3 now follows from the following lemma.

Lemma 4.2. *Let ν be a homogeneous $W^{1,p}$ -gradient Young measure with $\text{supp } \nu \subset Z$ and $p > 1$. Then ν is also a $W^{1,3}$ -gradient Young measure.*

Remark 4.3. *The exponent 3 can be replaced by any exponent greater than 2. We choose 3 for notational convenience only.*

Proof. By our initial discussion, we may assume $p \leq 2$. We begin by choosing $k > 1$ such that $p_c(k) < p < 3 < q_c(k)$ where $p_c(k)$ and $q_c(k)$ are defined in Theorem 4.1. Let $\{u^j\} \subset W^{1,p}$ be a generating sequence for ν . By a standard localization argument, we may assume

$$x \mapsto u^j(x) - Ex \in W_0^{1,p}(Y; \mathbb{R}^2) \quad (4.2)$$

where $E = \langle \nu; \text{Id} \rangle$ is the average of ν .

Now we proceed analogously to the proof of Theorem 1.2, with more care to the growth. Let q be such that $p_c(k) < q < p$, and let δ be such that

$$\frac{2}{k+1} < \delta < \frac{1}{2}.$$

This ensures that the σ_δ we shall construct will satisfy (4.1).

For each j we define u_δ^j , F_δ^j , σ_δ^j and f_δ^j as in the proof of Theorem 1.2. Then (3.10) gives

$$|f_\delta^j| = |\sigma_\delta^j \nabla u_\delta^j + \text{cof } \nabla u_\delta^j - 2 \text{Id}| \leq C_\delta \text{dist}(\nabla u^j, Z) \text{ a.e. in } Y,$$

and since $q < p$, (3.11) shows

$$\lim_{j \rightarrow \infty} \|f_\delta^j\|_{L^q} \leq C_\delta \lim_{j \rightarrow \infty} \int \text{dist}^q(\nabla u^j, Z) dx = C_\delta \int_{\mathbb{R}^{2 \times 2}} \text{dist}^q(F, Z) d\nu(F) = 0.$$

Now consider the two PDEs

$$\begin{cases} \text{div}(\sigma_\delta^j \nabla(u_\delta^j - v_\delta^j)) = \text{div } f_\delta^j \text{ in } Y, \\ u_\delta^j - v_\delta^j \in W_0^{1,q}(Y; \mathbb{R}^2) \end{cases} \quad (4.3)$$

and

$$\begin{cases} \operatorname{div}(\sigma_\delta^j \nabla v_\delta^j) = 0 \text{ in } Y, \\ x \mapsto v_\delta^j(x) - Ex \in W_0^{1,3}(Y; \mathbb{R}^2). \end{cases} \quad (4.4)$$

Since, by construction, $p_c(k) < q < 3 < q_c(k)$, Theorem 4.1 guarantees the existence of a unique solution to both equations. Furthermore, since $\operatorname{div}(\sigma_\delta^j \nabla u_\delta^j) = \operatorname{div} f_\delta^j$ and recalling (4.2), the solutions to (4.3) and (4.4) necessarily coincide. The estimate in Theorem 4.1 applied to (4.4) implies that $\|v_\delta^j\|_{W^{1,3}} \leq C\|\sigma_\delta^j E\|_{L^3} \leq Ck|E|$, hence the sequence $\{v_\delta^j(x) - \delta x\}$ is bounded in $W^{1,3}$ and it generates a $W^{1,3}$ -gradient Young measure. Moreover, by the estimate in Theorem 4.1 applied to (4.3), we have

$$\|u_\delta^j - v_\delta^j\|_{W^{1,q}} \leq C(k)\|f_\delta^j\|_{L^q} \rightarrow 0.$$

Combining this with (3.11) shows that the gradient Young measure generated by $\{v_\delta^j(x) - \delta x\}$ coincides with ν . \square

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