

# Riemann surfaces, arc systems and Weil-Petersson form

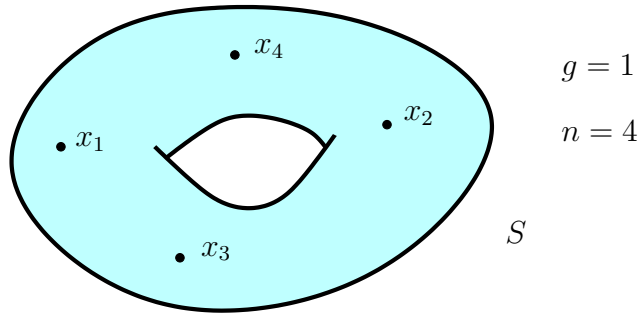
Gabriele Mondello

January 28, 2008

## 1. – Introduction

Let  $S$  be a Riemann surface of genus  $g$  together with  $n$  distinct points  $x = \{x_1, \dots, x_n\} \subset S$  and let  $\dot{S} = S \setminus x$  be the corresponding pointed surface.

Assume throughout the paper that  $\chi(\dot{S}) = 2 - 2g - n < 0$ .



The **moduli space** of  $(S, x)$  is the orbifold  $\mathcal{M}(S, x)$  that classifies  $x$ -marked Riemann surfaces diffeomorphic to  $(S, x)$  up to isomorphism.

Its universal cover (in the orbifold category) is the **Teichmüller space**  $\mathcal{T}(S, x)$  that parametrizes complex structures on  $S$  up to isotopy relative to  $x$ .

In other words,  $\mathcal{T}(S, x)$  is the space of equivalence classes of oriented diffeomorphisms  $f : S \rightarrow \Sigma$  of Riemann surfaces, where two oriented diffeomorphisms  $f : S \rightarrow \Sigma$  and  $g : S \rightarrow \Sigma'$  of Riemann surfaces are equivalent if there exists a biholomorphic map  $h : \Sigma \rightarrow \Sigma'$  such that  $h \circ f \simeq g$  are isotopic relative to  $x$ .

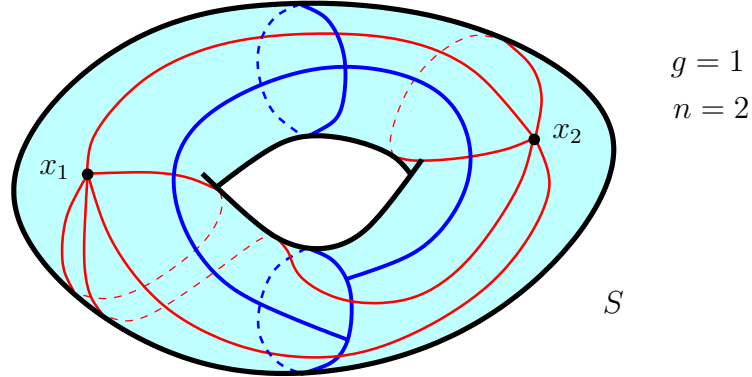
In this context, the cover  $\mathcal{T}(S, x) \rightarrow \mathcal{M}(S, x)$  is simply the forgetful map  $[f : S \rightarrow \Sigma] \mapsto [(\Sigma, f(x))]$ . It is also clear that the **mapping class group**  $\Gamma(S, x) = \pi_0 \text{Diff}_+(S, x)$  of isotopy classes of oriented diffeomorphisms of  $(S, x)$  acts on  $\mathcal{T}(S, x)$  by right composition and that  $\mathcal{M}(S, x)$  identifies to the quotient  $\mathcal{T}(S, x)/\Gamma(S, x)$ . The group  $\Gamma(S, x)$  is infinite but it acts properly and discontinuously on  $\mathcal{T}(S, x)$ ; in particular, the stabilizer of  $[f : S \rightarrow \Sigma]$  is naturally isomorphic to the automorphism group  $\text{Aut}(\Sigma, f(x))$ , which is known to be finite (because  $\chi(\dot{S}) < 0$ ).

A classical result of Teichmüller states that  $\mathcal{T}(S, x)$  is differentiably isomorphic to a Euclidean space (of dimension  $6g - 6 + 2n$ ). Thus,  $\mathcal{M}(S, x)$  locally looks like  $\mathbb{C}^{3g-3+n}/G$  around  $[(\Sigma, f(x))]$ , where  $G = \text{Aut}(\Sigma, x)$ .

From the orbifold point of view, which we will privilege,  $\mathcal{T}(S, x) \rightarrow \mathcal{M}(S, x)$  is a  $\Gamma(S, x)$ -principal bundle; whereas it is certainly not so at the level of underlying topological spaces, whenever some Riemann surface in  $\mathcal{M}(S, x)$  has nontrivial automorphisms.

This way, to give a  $\Gamma(S, x)$ -equivariant cellular model for  $\mathcal{T}(S, x)$  is the same as giving an orbi-cellular structure to  $\mathcal{M}(S, x)$ : even though we will not speak of orbi-cells, this is our basic purpose.

Two constructions are known in literature: starting from a Riemann surface  $(S, x)$  and a collection of numbers  $p_1, \dots, p_n > 0$ , both of them produce (in different ways) a metric graph  $G$  together with an isotopy class of homotopy equivalences  $G \hookrightarrow \dot{S}$ , where “metric” means that each side of  $G$  is associated a positive weight.



Hence, both constructions induce a map

$$\mathcal{T}(S, x) \times \Delta^{n-1} \times \mathbb{R}_+ \longrightarrow \left( \begin{array}{c} \text{metric graphs } G \subset \dot{S} \\ \text{up to isotopy} \end{array} \right)$$

where  $\Delta^{n-1} \times \mathbb{R}_+$  must be understood as  $\mathbb{R}_{\geq 0}^n \setminus \{0\}$ .

Our aim is to relate these two maps.

## 2. – The arc complex

Let  $S$  be a Riemann surface with marked points  $x = \{x_i\}$  and/or boundary.

An **arc** on  $(S, x)$  is a nontrivial isotopy class (relative to  $x \cup \partial S$ ) of simple paths that intersect  $x \cup \partial S$  at the extremal points and nowhere else.

A  **$k$ -system of arcs** is a set  $\underline{\alpha} = \{\alpha_1, \dots, \alpha_k\}$  of  $k$  distinct arcs that admit disjoint representatives, up to isotopies of systems of arcs.

The **arc complex**  $\mathfrak{A}(S, x)$  (see [3]) is the simplicial complex whose  $m$ -simplices are  $(m + 1)$ -systems of arcs in  $(S, x)$ .

We say that  $\underline{\alpha} = \{\alpha_0, \dots, \alpha_k\}$  is **proper** if

$$\dot{S} \setminus \underline{\alpha} := \dot{S} \setminus \bigcup_{i=0}^k \alpha_i$$

is a disjoint union of discs and pointed discs. The subset of  $\mathfrak{A}(S, x)$  consisting of proper simplices is denoted by  $\mathfrak{A}^\circ(S, x)$ .

Lefschetz duality on  $(S, x)$  will induce the correspondences shown below, which will allow us to identify the space of embedded graphs  $G \subset \dot{S}$  to the topological realization of  $\mathfrak{A}^\circ(S, x)$ .

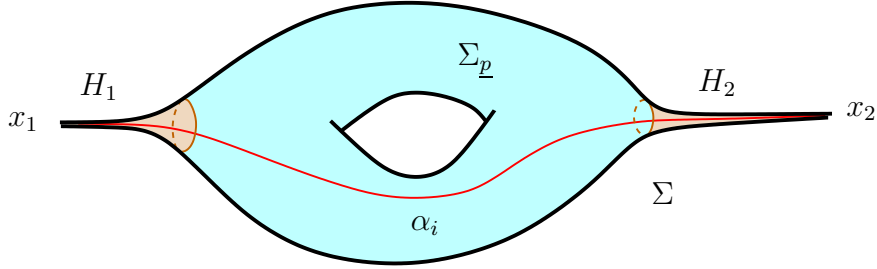
$$\begin{array}{ll}
e_i \text{ edge of } G \subset \dot{S} & \longleftrightarrow \text{ dual arc } \alpha_i \subset S \\
\text{weight of } e_i & \longleftrightarrow \text{ weight } w_i \text{ of } \alpha_i \\
G \subset \dot{S} \text{ homotopy equiv.} & \longleftrightarrow \text{ proper system of arcs } \underline{\alpha} \\
\text{metric on } G & \longleftrightarrow \text{ system of weights } \{w_i\} \\
\left\{ \begin{array}{l} \text{metric graphs } G \subset \dot{S} \\ \text{homotopy equiv.} \\ \text{up to isotopy} \end{array} \right\} & \longleftrightarrow \begin{array}{l} \text{topological realization} \\ |\mathfrak{A}^\circ(S, x)|_{\mathbb{R}} := |\mathfrak{A}^\circ(S, x)| \times \mathbb{R}_+ \end{array}
\end{array}$$

See also [10] for a more detailed treatment of arc systems and ribbon graphs.

### 3. – First construction (Penner, Bowditch-Epstein)

Given  $[f : S \rightarrow \Sigma] \in \mathcal{T}(S, x)$ , let us endow  $\dot{\Sigma}$  with the Poincaré metric (namely, the hyperbolic metric obtained from a uniformization  $\mathbb{H} \rightarrow \dot{\Sigma}$ ) and consider the horoball  $H_i$  around the cusp  $x_i$  of circumference  $p_i$ .

Call  $(\Sigma, H_1, \dots, H_n)$  a **decorated surface** (see [13]) and  $\Sigma_p = \Sigma \setminus \cup H_i^\circ$  the corresponding truncated surface.



Just for simplicity (as it is not necessary), assume that  $0 < p_i < 1$ , in such a way that the  $H_i$ 's are pairwise disjoint.

For every arc  $\alpha_i \subset S$ , identify  $f(\alpha_i) \subset \Sigma$  with its unique geodesic representative and call its *reduced length*  $\tilde{a}_i := \ell(\alpha_i \cap \Sigma_p)$ . The following is due to Penner.

**PROPOSITION 1 (Penner [13])** *If  $\underline{\alpha} = \{\alpha_i\} \in \mathfrak{A}^\circ(S, x)$  is an ideal triangulation (i.e. a maximal system of arcs) of  $(S, x)$ , then*

$$\begin{array}{ccc}
\tilde{F}_{\underline{\alpha}} : \mathcal{T}(S, x) \times \mathbb{R}_+^n & \longrightarrow & \mathbb{R}_+^{6g-6+3n} \\
([f : S \rightarrow \Sigma], \underline{p}) & \longmapsto & (\tilde{a}_i)
\end{array}$$

is a real-analytic diffeomorphism.

The problem with the coordinates  $\{\tilde{a}_i\}$  is that they are not  $\Gamma(S, x)$ -invariant, unless one is able to canonically associate to every point in  $\mathcal{T}(S, x) \times \mathbb{R}_+^n$  a system of arcs.

In order to do that, define the **valence** of a point  $q \in \Sigma$  as

$$\mathbf{val}(q) := \# \left\{ \begin{array}{l} \text{paths of minimal length} \\ \text{from } q \text{ to } \partial \Sigma_p \end{array} \right\}$$

and the **spine** of  $\Sigma_p$  (see [2]) to be

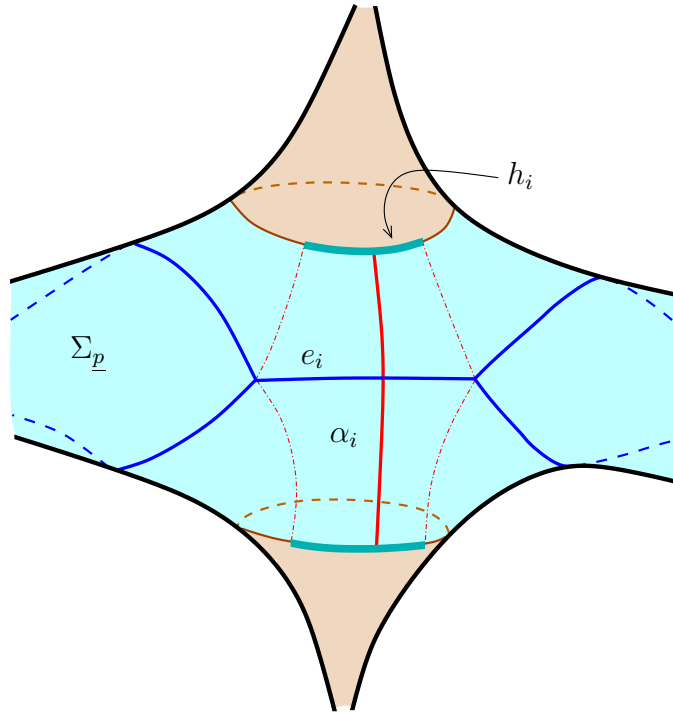
$$G = \mathbf{Sp}(\Sigma_p) := \{q \in \Sigma_p \mid \mathbf{val}(q) \geq 2\}$$

The components of  $\text{val}^{-1}(2)$  are geodesic **edges** of  $G$  and  $\text{val}^{-1}([3, \infty))$  are isolated **vertices**, so that  $G$  is a 1-dimensional CW complex embedded in  $\dot{\Sigma}$  as a deformation retract. In fact, for every  $q \in \dot{\Sigma} \setminus G$ , we can consider the unique shortest geodesic from  $\partial\Sigma_{\underline{p}}$  to  $q$ , prolong it until it meets  $G$  and flow  $q$  along this prolongation.

Now, for every edge  $e_i$  of  $\text{Sp}(\Sigma_{\underline{p}})$ , we can define a **dual arc**  $\alpha_i$  by joining the two shortest geodesics that contribute  $\text{val}(q)$ , where  $q$  is any point internal to  $e_i$ .

One can notice that the system of arcs  $\underline{\alpha}_{sp}$  dual to the spine  $\text{Sp}(\Sigma_{\underline{p}})$  is proper. More generally, every proper system of arcs is dual to a graph embedded in  $\dot{\Sigma}$  through a homotopy equivalence.

It is not hard to see that, for a general  $\Sigma$ , the spine  $\text{Sp}(\Sigma_{\underline{p}})$  has trivalent vertices and so  $\underline{\alpha}_{sp}$  is an ideal triangulation of  $\Sigma$ .



Define the **weight** of  $e_i$  (and dually of  $\alpha_i$ ) to be equal to the length  $h_i$  of the portion of boundary horocycle obtained by “projecting”  $e_i$  onto  $\partial\Sigma_{\underline{p}}$ .

The result below provides the wished equivariant cellularization of the Teichmüller space (times a contractible  $\Delta^{n-1} \times \mathbb{R}_+$ ).

**THEOREM 1 (Penner [13], Bowditch-Epstein [2])** *The following map*

$$\begin{aligned} \Phi_{BPE} : \mathcal{T}(S, x) \times \Delta^{n-1} \times \mathbb{R}_+ &\longrightarrow |\mathfrak{A}^\circ(S, x)|_{\mathbb{R}} \\ ([f : S \rightarrow \Sigma], \underline{p}) &\longmapsto (\underline{\alpha}_{sp}, \{h_i\}) \end{aligned}$$

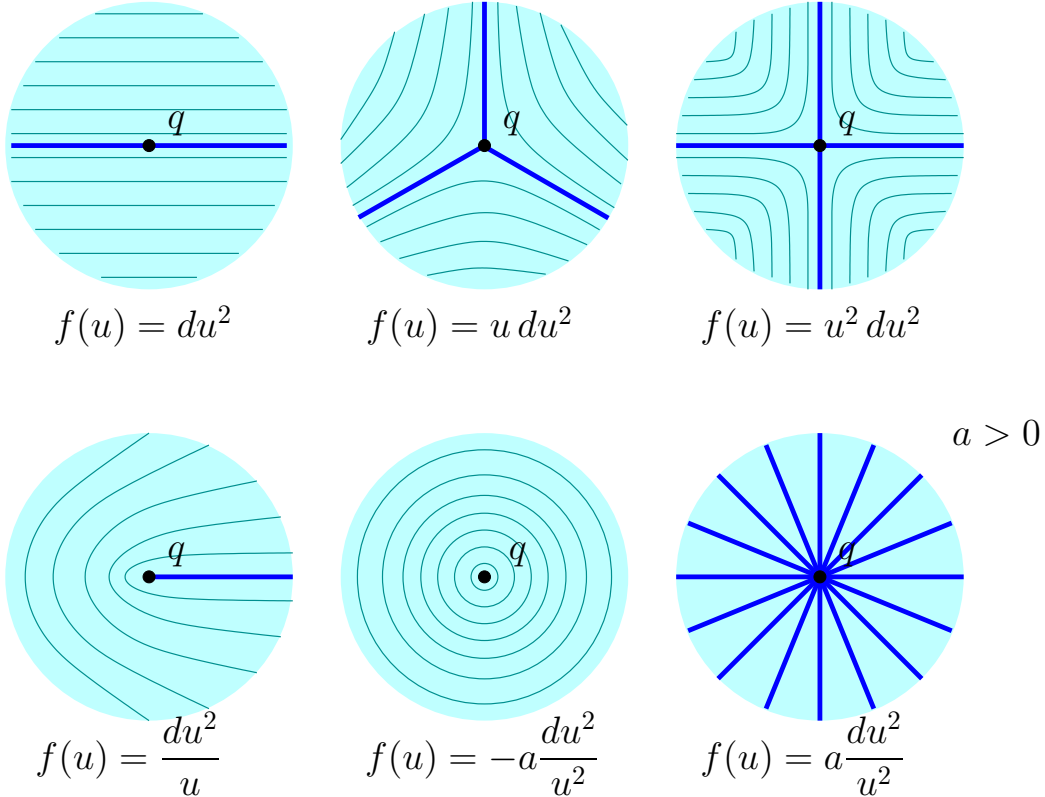
is a  $\Gamma(S, x)$ -equivariant homeomorphism. Moreover, its restriction to every open cell of  $|\mathfrak{A}^\circ(S, x)|_{\mathbb{R}}$  is a real-analytic diffeomorphism.

#### 4. – Second construction (Harer-Mumford-Thurston)

Consider a holomorphic quadratic differential  $\varphi$  on  $\dot{\Sigma}$ , which locally looks like  $\varphi = f(z)dz^2$ . Its absolute value  $|\varphi| = |f(z)|dzd\bar{z}$  induces a flat metric, which has conical singularities at the **critical points** (that is, where the metric degenerates).

In particular, given  $q \in \dot{\Sigma}$  with  $f(q) \neq 0$ , there exists a local holomorphic coordinate  $u$  around  $q$  such that  $u(q) = 0$  and  $\varphi(u) = du^2$ . As  $u$  is uniquely determined up to sign, the **(horizontal) trajectories**  $\text{Im}(u) = \text{const}$  are well-defined. We call **critical** those trajectories that meet a critical point.

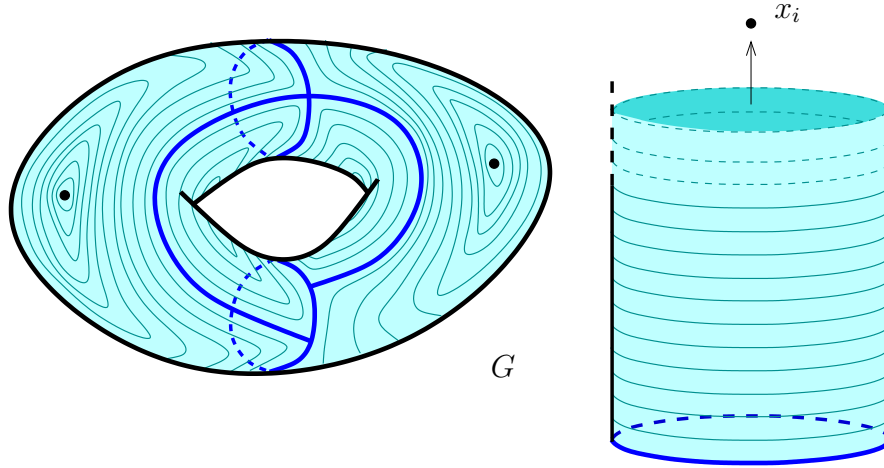
The structure of trajectories of  $\varphi$  is completely classified. In particular, periodic trajectories sweep out **annular domains** (see [16]).



The result below attaches to every surface  $\Sigma$  with weights at its marked points a quadratic differential whose trajectories are as simple as possible.

**THEOREM 2 (Strebel [16])** *Given  $p_1, \dots, p_n \geq 0$  (not all zero),  $\exists!$   $\varphi$  holomorphic quadratic differential on  $\dot{\Sigma}$  such that*

- $\exists$  a local holomorphic coordinate  $z_i$  at  $x_i \in \Sigma$  such that  $\varphi(z_i) = -\left(\frac{p_i}{2\pi}\right)^2 \frac{dz_i^2}{z_i^2}$
- the horizontal trajectories of  $\varphi$  are closed
- the complement of the critical trajectories is a disjoint union of annular domains, each one isometric to a semi-infinite cylinder whose parallels wind around  $x_i$  and have length  $p_i$ .



Let  $G$  be the critical graph of  $\varphi$  and let  $\underline{\alpha}_{JS}$  be an arc system on  $S$  such that  $f(\underline{\alpha}_{JS})$  is dual to  $G$ . An associated system of weights is simply determined using the  $|\varphi|$ -lengths of the edges of  $G$ , namely  $\tilde{w}_i = \ell_{|\varphi|}(e_i)$ .

The following is due to Harer-Mumford-Thurston.

**THEOREM 3 ([3])** *The map*

$$\begin{aligned} \Phi_{JS} : \mathcal{T}(S, x) \times \Delta^{n-1} \times \mathbb{R}_+ &\longrightarrow |\mathfrak{A}^\circ(S, x)|_{\mathbb{R}} \\ ([f : S \rightarrow \Sigma], \underline{p}) &\longmapsto (\underline{\alpha}_{JS}, \{\tilde{w}_i\}) \end{aligned}$$

is a  $\Gamma(S, x)$ -equivariant homeomorphism.

Hubbard-Masur [4] showed that the restriction of  $\Phi_{JS}$  to each open simplex is a real-analytic diffeomorphism.

Notice that one can act by  $\mathbb{R}_+$  rescaling both sides  $\mathcal{T}(S, x) \times \Delta^{n-1} \times \mathbb{R}_+$  and  $|\mathfrak{A}^\circ(S, x)|_{\mathbb{R}}$ . It is immediate to see that both constructions are  $\mathbb{R}_+$ -equivariant.

## 5. – Symplectic structure

The space  $\mathcal{T}(S, x)$  can be made into a complex variety of (complex) dimension  $3g-3+n$ . In fact, its holomorphic cotangent space at  $[f : S \rightarrow \Sigma]$  can be identified to  $H^0(\Sigma, K_\Sigma^{\otimes 2}(x))$  and its holomorphic tangent space to  $H^{0,1}(\Sigma, T_\Sigma(-x))$ . The natural pairing between them is

$$\begin{aligned} H^0(\Sigma, K_\Sigma^{\otimes 2}(x)) \times H^{0,1}(\Sigma, T_\Sigma(-x)) &\longrightarrow \mathbb{C} \\ (\varphi, \mu) &\longmapsto \int_\Sigma \varphi \mu \end{aligned}$$

The **Weil-Petersson Kähler** metric is defined on the cotangent space as

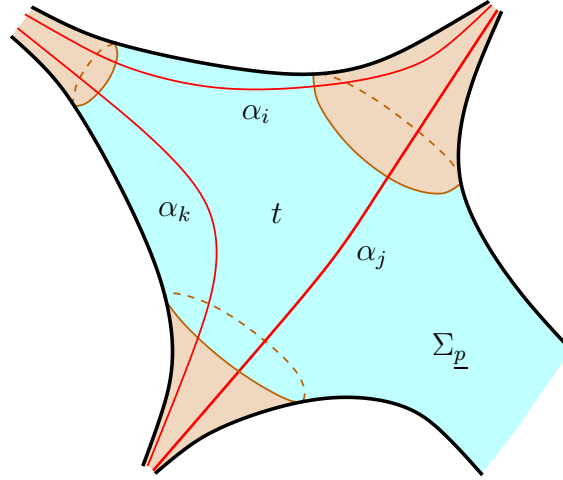
$$h^*(\varphi, \psi) = \int_\Sigma \frac{\varphi \bar{\psi}}{\lambda}$$

where  $\varphi, \psi \in T_{[f]}^* \mathcal{T}(S, x)$  and  $\lambda$  is the hyperbolic metric; the corresponding  $h$  on the tangent space is obtained using the pairing above. If  $h = g + i\omega$ , then  $\omega$  is called **Weil-Petersson symplectic form**.

The following result expresses the Weil-Petersson form in the  $\alpha$ -coordinates.

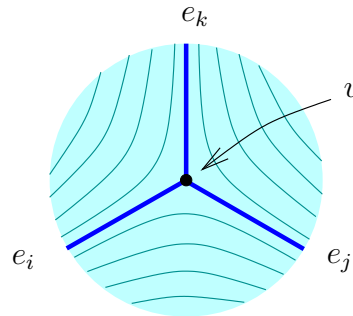
**THEOREM 4 (Penner [14])** *Let  $\underline{\alpha} \in \mathfrak{A}^\circ(S, x)$  be an ideal triangulation. Then*

$$\tilde{F}_{\underline{\alpha}}^* \omega = -\frac{1}{2} \sum_t (da_i \wedge da_j + da_j \wedge da_k + da_k \wedge da_i)$$



On the other hand, Kontsevich [5] defined a symplectic piecewise-linear form  $\Omega$  on  $|\mathfrak{A}^\circ(S, x)|_{\mathbb{R}}$ , which is an incarnation (up to some coefficient) of the cohomology class  $\sum_k p_k^2 \psi_k$  on  $\mathcal{M}(S, x)$ . The dual  $\beta$  of  $\Omega$  can be written as

$$\beta = \sum_v \left( \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j} + \frac{\partial}{\partial e_j} \wedge \frac{\partial}{\partial e_k} + \frac{\partial}{\partial e_k} \wedge \frac{\partial}{\partial e_i} \right)$$



The aim of the following sections is to relate  $\omega$  and  $\Omega$ .

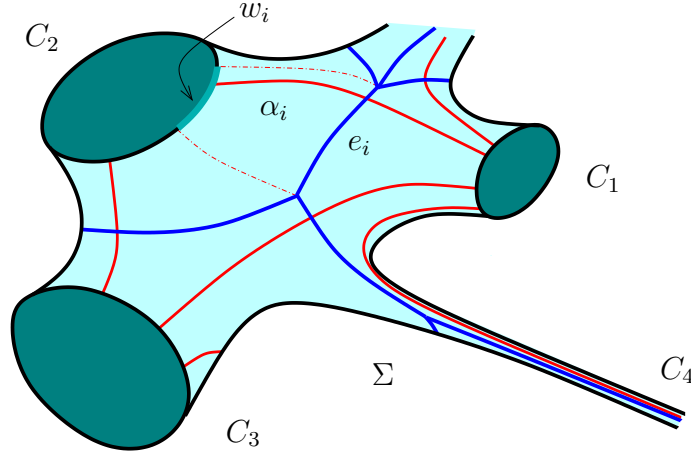
## 6. – Surfaces with boundary

The key idea (see [11]) is to interpolate between the two constructions using hyperbolic surfaces with geodesic boundary.

This way, we will recover BPE's construction as  $\underline{p} \rightarrow 0$  and HTM's as  $\underline{p} \rightarrow \infty$ .

Let  $S$  be a hyperbolic surface with boundary  $\partial S = C_1 \cup \dots \cup C_n$  consisting of  $n$  geodesic circles and let  $\mathcal{T}(S)$  be the Teichmüller space of  $S$ . Its cotangent space  $T_{[f:S \rightarrow \Sigma]}^* \mathcal{T}(S)$  can be naturally identified to the *real* vector space of holomorphic quadratic differentials on  $\Sigma$ , whose restriction to  $\partial \Sigma$  is real.

Thus,  $\mathcal{T}(S)$  is a *real-analytic* manifold of dimension  $6g - 6 + 3n$ .



The geometry of hyperbolic hexagons shows that, if  $\underline{\alpha} = \{\alpha_i\}$  is a maximal system of arcs on  $S$  and  $a_i = \ell_{\alpha_i}$ , then

$$\begin{array}{ccc} F_{\underline{\alpha}} : \mathcal{T}(S) & \longrightarrow & \mathbb{R}_+^{6g-6+3n} \\ [f : S \rightarrow \Sigma] & \longmapsto & (a_i) \end{array}$$

is a real-analytic diffeomorphism.

Similarly to the BPE construction, the spine  $\text{Sp}(\Sigma)$  induces a system of arcs  $\underline{\alpha}_{sp} = \{\alpha_i\} \in \mathfrak{A}(S)$ . The weight  $w_i$  of  $\alpha_i$  is the length of the projection of  $e_i$  (i.e. the edge of  $\text{Sp}(\Sigma)$  transverse to  $\alpha_i$ ) to the boundary of  $\Sigma$ . Extend the Teichmüller space  $\mathcal{T}(S) \subset \tilde{\mathcal{T}}(S)$  in order to contain those  $[f : S \rightarrow \Sigma]$ , which collapse some  $C_i$  to a cusp of  $\Sigma$ .

Then, the boundary length function extends to  $\mathcal{L} : \tilde{\mathcal{T}}(S) \longrightarrow \mathbb{R}_{\geq 0}^n$  and  $\tilde{\mathcal{T}}(S)(0)$  identifies to the Teichmüller space of a pointed surface.

**THEOREM 5 (Luo [6])** *The map*

$$\begin{array}{ccc} \Phi : \tilde{\mathcal{T}}(S) \setminus \tilde{\mathcal{T}}(S)(0) & \longrightarrow & |\mathfrak{A}^\circ(S)|_{\mathbb{R}} \\ [f : S \rightarrow \Sigma] & \longmapsto & (\underline{\alpha}_{sp}, \{w_i\}) \end{array}$$

is a  $\Gamma(S)$ -equivariant homeomorphism. Moreover, the restriction of  $\Phi$  to each open simplex is a real-analytic diffeomorphism.

A Weil-Petersson pairing can still be defined on  $T^*\mathcal{T}(S)$  as

$$h^*(\varphi, \psi) = \int_S \frac{\varphi \bar{\psi}}{\lambda}$$

and we can write  $h = g + i\omega$ . However,  $g$  is not Kähler and  $\omega$  is degenerate.

More precisely,  $\eta = \text{Im}(h^*)$  defines a Poisson structure and  $\mathcal{T}(S)(\underline{p}) := \mathcal{L}^{-1}(\underline{p})$  is a symplectic leaf.

In this context, the analogue of Penner's result is the following.

**THEOREM 6 (Mondello [11])** *If  $\underline{\alpha} \in \mathfrak{A}^\circ(S)$  is a maximal system of arc on  $S$ , then*

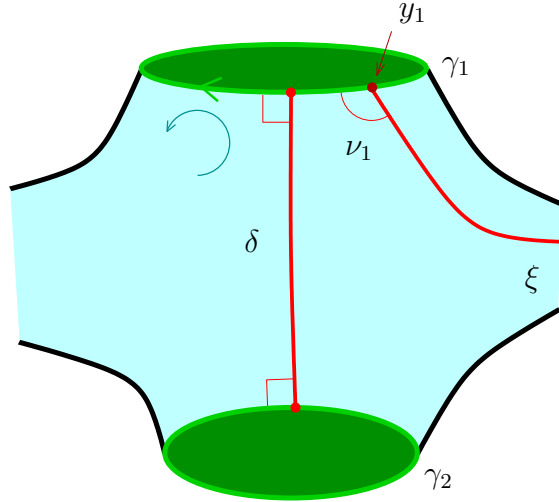
$$\eta = \frac{1}{4} \sum_C \sum_{\substack{y_i \in \alpha_i \cap C \\ y_j \in \alpha_j \cap C}} \frac{\sinh(p_C/2 - d_C(y_i, y_j))}{\sinh(p_C/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

where  $d_C$  is the distance (with sign) along  $C$  and  $p_C$  is the length of  $C$ .

The proof uses Wolpert's formula  $\omega = \sum_i dl_i \wedge d\tau_i$  for a compact hyperbolic surface (see [17]) and the following.

**THEOREM 7 (Mondello [11])** *Let  $R$  be a closed hyperbolic surface and let  $\gamma_1, \gamma_2, \xi \subset R$  be simple closed geodesics. Let  $\delta$  be a geodesic between  $\gamma_1$  and  $\gamma_2$ , that meets the  $\gamma_i$ 's perpendicularly. For simplicity, suppose that no portion of  $\xi$  is homotopic to  $\delta$  and that they do not intersect each other. Then*

$$\frac{\partial \ell(\delta)}{\partial \tau_\xi} = \frac{1}{2} \sum_{i=1}^2 \sum_{y_i \in \xi \cap \gamma_i} \frac{\sinh(p_i/2 - d_{\gamma_i}(\delta, y_i))}{\sinh(p_i/2)} \sin(\nu_{y_i})$$



## 7. – Limits for long/short boundary components

Let  $\mathbb{R}_+^n \hookrightarrow \Delta^{n-1} \times [0, \infty]$  as  $\underline{p} \mapsto (\underline{p}/\mathbf{p}, \mathbf{p})$ , where  $\mathbf{p} = p_1 + \dots + p_n$ .

We want to complete the family of Teichmüller spaces  $\mathcal{L} : \tilde{\mathcal{T}}(S) \rightarrow \Delta^{n-1} \times [0, \infty]$  over  $\Delta^{n-1} \times \{0\}$  and  $\Delta^{n-1} \times \{\infty\}$ .

In order to do that, we interpret boundaries of zero length as cusps, and so we identify the space of surfaces with projective decoration  $\tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1}$  with  $\mathcal{L}^{-1}(\Delta^{n-1} \times \{0\})$ .

Thus,  $\Phi : \tilde{\mathcal{T}}(S) \setminus \tilde{\mathcal{T}}(S)(0) \rightarrow |\mathfrak{A}^\circ(S)| \times (0, \infty)$  extends over 0 by setting  $\Phi_0 := \Phi_{BPE}$  and  $\eta$  limits to

$$\eta_0 = \frac{1}{4} \sum_H \sum_{\substack{y_i \in \alpha_i \cap H \\ y_j \in \alpha_j \cap H}} \left( 1 - \frac{2d_H(y_i, y_j)}{p_H} \right) \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

To understand the limit  $\mathbf{p} \rightarrow \infty$ , we introduce the space of measured laminations.

Let  $R$  be a closed hyperbolic surface and let  $\mathcal{S}(R)$  be the set of (nontrivial isotopy classes of) simple closed curves on  $R$ .

The hyperbolic length function  $\mathcal{T}(R) \times \mathcal{S}(R) \rightarrow \mathbb{R}_+$  induces an embedding

$$\begin{array}{ccc} \mathcal{T}(R) & \xrightarrow{\quad} & \mathbb{R}_+^{\mathcal{S}(R)} \\ & \searrow j & \downarrow \\ & & \mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)}) \end{array}$$

of  $\mathcal{T}(R)$  inside  $\mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)})$ .

Thurston's compactification of  $\mathcal{T}(R)$  is the closure  $\overline{\mathcal{T}}^{Th}(R) = \mathcal{T}(R) \cup \partial^{Th}\mathcal{T}(R)$  of  $j(\mathcal{T}(R))$ . Incidentally, we recall that  $\overline{\mathcal{T}}^{Th}(R)$  is homeomorphic to a closed ball, whose boundary sphere identifies to  $\partial^{Th}\mathcal{T}(R)$ .

A **measured lamination**  $m$  on  $R$  is a closed  $F \subset R$  which is foliated by complete geodesics together with a transverse measure. Though not evident, the space of measured lamination does not depend on the specific hyperbolic metric of  $R$  but only on its diffeomorphism type.

Notice that  $\mathcal{ML}(R) \hookrightarrow \mathbb{R}_+^{\mathcal{S}(R)}$  as  $m \mapsto \iota(m, \cdot)$ , where  $\iota(\cdot, \cdot)$  is the geometric intersection pairing, and so  $\mathbb{P}\mathcal{ML}(R) \hookrightarrow \mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)})$ . Moreover, the boundary  $\partial^{Th}\mathcal{T}(R)$  can be identified to the space  $\mathbb{P}\mathcal{ML}(R)$  of projective measured laminations.

An example of measured lamination is a **multi-curve**, namely a (nonzero) weighted sum of disjoint simple closed curves  $m = w_1\gamma_1 + \dots + w_n\gamma_n$ .

When  $S$  is a hyperbolic surface with geodesic boundary and  $dS$  is its double (with real involution  $\sigma$ ), we define  $\mathcal{ML}(S) := \mathcal{ML}(dS)^\sigma$ , so that  $|\mathfrak{A}(S)| \hookrightarrow \mathbb{P}\mathcal{ML}(S)$ .

We also define  $\overline{\mathcal{T}}^{Th}(S) := \overline{\mathcal{T}}^{Th}(dS)^\sigma$ , which is thus equal to  $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ . Then, we can compare  $|\mathfrak{A}(S)| \hookrightarrow \mathbb{P}\mathcal{ML}(S) \hookrightarrow \overline{\mathcal{T}}^{Th}(S)$  with  $\mathcal{L}^{-1}(\Delta^{n-1} \times \{\infty\})$ .

**THEOREM 8 (Mondello [12])** *The map*

$$\Phi^{-1} : |\mathfrak{A}^\circ(S)| \times (0, \infty) \rightarrow \overline{\mathcal{T}}^{Th}(S)$$

*extends over  $\infty$  by setting  $\Phi_\infty^{-1}$  to be equal to the natural embedding*

$$|\mathfrak{A}^\circ(S)| \times \{\infty\} \hookrightarrow \mathbb{P}\mathcal{ML}(S)$$

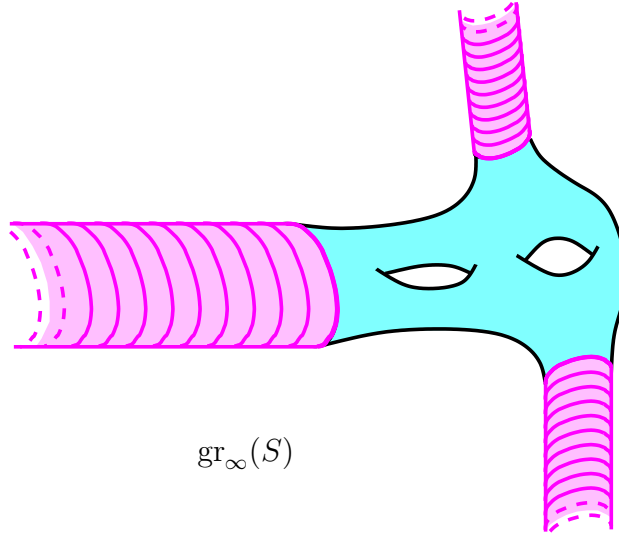
*Moreover,  $\tilde{\eta} = (\mathbf{p}/2)^2\eta \rightarrow \tilde{\beta}/2 = (\mathbf{p}/2)^2\beta/2$  as  $\mathbf{p} \rightarrow \infty$ .*

Consider the  $S^1$ -bundle  $\mathcal{C}_i \rightarrow \overline{\mathcal{M}}(S)$  associated to the  $i$ -th boundary component of  $S$ , and let  $\psi_i = c_1(\mathcal{C}_i)$ .

Mirzakhani [8] observed that Wolpert's  $[\omega_0] = \pi^2 \kappa_1$  (see [18]) and standard facts of symplectic geometry imply  $[\omega_{\underline{p}}] = \pi^2 \kappa_1 + \sum_i (p_i^2/2) \psi_i$ . Hence,  $(2/\underline{p})^2 [\omega] \rightarrow 2 \sum_i (p_i/\underline{p})^2 \psi_i$ , which proves that *intersection numbers of  $\psi$  classes are Weil-Petersson symplectic volumes* in the large boundary limit. Thus, for this purpose, a detailed analysis of  $\partial\mathcal{M}(S)$  is not essential. (About the problems that can emerge when dealing with  $\partial\mathcal{M}(S)$ , see [1] and [9], for instance.)

## 8. – Grafting

Let  $S$  be a hyperbolic surface with geodesic boundary. The **grafted surface**  $\text{gr}_\infty(S)$  is obtained from  $S$  by glueing semi-infinite cylinders at  $\partial S$ . Notice that  $\text{gr}_\infty(S)$  is biholomorphic to a pointed surface.



Thus, grafting defines the following real-analytic map

$$\text{gr}_\infty : \mathcal{T}(S) \longrightarrow \mathcal{T}(S)(0)$$

**THEOREM 9 (Mondello [12])** *For every  $\underline{p}$ , the map  $\text{gr}_\infty : \mathcal{T}(S)(\underline{p}) \rightarrow \mathcal{T}(S)(0)$  is a real-analytic diffeomorphism.*

We remark that continuity and properness of  $\text{gr}_\infty$  are not difficult. To prove that the differential  $d\text{gr}_\infty$  is injective, we mimick the proof of Scannell-Wolf [15] with suitable variations.

Intuitively, as  $\underline{p} \rightarrow \infty$ , the collar of  $\partial S$  becomes thinner and thinner. If we rescale the metric so that  $\underline{p} = 1$  (and so the area goes to zero), the hyperbolic portion shrinks to a graph  $G$ .

Clearly, one can obtain the surface  $\text{gr}_\infty(S)$  simply glueing semi-infinite cylinders along  $G$ , which immediately defines a Strebel differential (up to multiples) on  $\text{gr}_\infty(S)$ . We recall that the exceptional locus of the real-oriented blow-up  $\text{Bl}_0 \tilde{\mathcal{T}}(S)$  is made of projectively decorated surfaces and can be identified to  $\tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1}$ .

Hence, we can finally summarize our previous conclusions in the following commutative diagram of homeomorphisms

$$\begin{array}{ccc}
\tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \times [0, \infty] & \xleftarrow{\text{gr}_\infty} & \widehat{\mathcal{T}}(S) \\
& \searrow \Psi & \downarrow \Phi \\
& & |\mathfrak{A}^\circ(S)| \times [0, \infty]
\end{array}$$

in which  $\widehat{\mathcal{T}}(S) := \text{Bl}_0 \tilde{\mathcal{T}}(S) \cup |\mathfrak{A}^\circ(S)|$  and the two limits for short/long boundaries are given by

$$\Psi_0 = \Phi_{BPE} : \tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \rightarrow |\mathfrak{A}^\circ(S)|$$

and

$$\Psi_\infty = \Phi_{JS} : \tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \rightarrow |\mathfrak{A}^\circ(S)|.$$

## 9. – A remark on the Riemannian volume

Regarding  $\mathcal{T}(S)(\underline{p})$  as a family of moduli spaces with a Riemannian metric, obtained restricting the Weil-Petersson metric on  $\mathcal{T}(S)$  to each leaf, the situation is more complicated, because  $\mathcal{T}(S)(\underline{p})$  is not totally geodesic (except for  $\underline{p} = 0$ ).

In any case, when the arc  $\alpha_i$  is short, Masur [7] provided the expansion  $\|\nabla a_i\|^2 \approx 2\pi a_i$ . Moreover, Wolpert showed that, if  $\alpha_i, \alpha_j$  are two disjoint short arcs, then  $\langle a_i, a_j \rangle = O(a_i^2 a_j^2)$ .

As  $w_i \approx 2 \log(2/a_i)$  for  $a_i$  short, we get  $\|\nabla w_i\| \approx 2\sqrt{\pi} \exp(w_i/4)$  in the former case, and  $\langle \nabla w_i, \nabla w_j \rangle = O(\exp(-w_i/2 - w_j/2))$  in the latter.

Thus, given a maximal system of arcs  $\underline{\alpha} = \{\alpha_i\}$  for  $S$ , the coordinates  $\{\nabla w_i\}$  become orthogonal as  $\underline{p} \rightarrow \infty$ .

In fact, the leading term of the associated Riemannian volume form looks like

$$\text{vol} \approx (4\pi)^{-N/2} \exp(-\underline{p}/2) dw_1 \wedge \cdots \wedge dw_N$$

where  $N = \dim_{\mathbb{R}} \mathcal{T}(S)$ .

## 10. – Bordification of arcs

The Weil-Petersson completion  $\overline{\mathcal{T}}(S)(\underline{p})$  of  $\mathcal{T}(S)(\underline{p})$  is obtained by adding hyperbolic surfaces with nodes. Namely, a point of  $\overline{\mathcal{T}}(S)$  is a class of maps  $f : S \rightarrow \Sigma$  from  $S$  to a hyperbolic surface, such that  $f$  is a homeomorphism everywhere, except at some simple closed curves of  $S$ , which are shrunk to form nodes.

The family  $(\overline{\mathcal{T}}(S)(\underline{p}), (2/\underline{p})^2 \omega_{WP})$  of symplectic spaces can be completed by a “piecewise-linear symplectic space”  $(|\mathfrak{A}(S)|, \tilde{\Omega}/2)$  over  $\underline{p} = \infty$ .

Quotienting by the action of  $\Gamma(S)$ , we obtain a family of moduli spaces  $(\overline{\mathcal{M}}(S)(\underline{p}), (2/\underline{p})^2 \omega_{WP})$  homeomorphic to the Deligne-Mumford compactification. This family can be compactified at  $\infty$  by  $(|\mathfrak{A}(S)|/\Gamma(S), \tilde{\Omega}/2)$ .

In order to describe this compactification using quantities associated to arcs, we introduce the **transverse length**  $t_\alpha$  of an arc  $\alpha \subset S$ , defined as  $t_\alpha := T(\ell_\alpha)$ , where

$$T(x) = 2\operatorname{arcsinh}\left(\frac{1}{\sinh(x/2)}\right)$$

The function  $T : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and strictly decreasing, and  $T \circ T = id$ . Moreover,  $T(x) \approx 4e^{-x/2}$  as  $x \rightarrow \infty$ , and  $T(x) \approx 2 \log(4/x)$  as  $x \rightarrow 0$ .

The following map is an embedding

$$\begin{aligned} t_\bullet : \mathcal{T}(S) &\longrightarrow \mathbb{P}(\mathcal{A}(S)) \times [0, \infty) \\ [f] &\longmapsto (t_\bullet(f), \|t_\bullet(f)\|_\infty) \end{aligned}$$

where  $\mathcal{A}(S) = \mathfrak{A}_0(S)$  is the set of arcs of  $S$  and  $\mathbb{P}(\mathcal{A}(S))$  is the projectivization of  $L^\infty(\mathcal{A}(S))$ .

In analogy to Thurston's compactification, we call **bordification of arcs**  $\overline{\mathcal{T}}^a(S)$  the closure of  $t_\bullet(\mathcal{T}(S))$  inside  $\mathbb{P}(\mathcal{A}(S)) \times [0, \infty)$  (see [12]). We have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Bl}_0 \overline{\mathcal{T}}(S) \cup |\mathfrak{A}(S)| & \xrightarrow{\pi} & \overline{\mathcal{T}}^a(S) \\ \downarrow & \nearrow \operatorname{tr} & \\ |\mathfrak{A}(S)| \times [0, \infty) & & \end{array}$$

in which  $\pi$  “forgets” the hyperbolic metric on the components without boundary (and without positively weighted cusps).

**THEOREM 10 (Mondello [12])** *The map  $\operatorname{tr} : |\mathfrak{A}(S)| \times [0, \infty) \rightarrow \overline{\mathcal{T}}^a(S)$  is a  $\Gamma(S)$ -equivariant homeomorphism.*

We have thus recovered a different interpretation of Kontsevich's compactification  $\mathcal{M}(S) \subset |\mathfrak{A}(S)|/\Gamma(S) \times [0, \infty)$  (see [5]).

## REFERENCES

- [1] ARBARELLO E. and CORNALBA M., *Combinatorial and algebro-geometric cohomology classes on the moduli space of curves*, J. Algebraic Geom. **5** (1996), no. 4, 705–749.
- [2] BOWDITCH B.H. and EPSTEIN D. B. A., *Natural triangulations associated to a surface*, Topology **27** (1988), no. 1, 91–117.
- [3] HARER J. L., *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176.

- [4] HUBBARD J. and MASUR H., *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274.
- [5] KONTSEVICH M., *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.
- [6] LUO F., *On the Teichmüller Space of Surfaces with Boundary*, Duke Math. J. **139** (2007), no. 3, 463–482.
- [7] MASUR H., *Extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. J. **43** (1976), no. 3, 623–635.
- [8] MIRZAKHANI M., *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20** (2007), no. 1, 1–23.
- [9] MONDELLO G., *Combinatorial classes on  $\mathcal{M}_{g,n}$  are tautological*, Int. Math. Res. Not. (2004), no. 44, 2329–2390.
- [10] MONDELLO G., *Riemann surfaces, ribbon graphs and the combinatorial classes*, to appear in “Handbook of Teichmüller theory, vol.2”, European Mathematical Society, ed. Athanase Papadopoulos.
- [11] MONDELLO G., *Triangulated Riemann surfaces with boundary and the Weil-Petersson Poisson structure*, submitted paper.
- [12] MONDELLO G., *Riemann surfaces with boundary and natural triangulation of the Teichmüller space*, to appear.
- [13] PENNER R. C., *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), no. 2, 299–339.
- [14] PENNER R. C., *Weil-Petersson volumes*, J. Differential Geom. **35** (1992), no. 3, 559–608.
- [15] SCANNELL K. P. and WOLF M., *The grafting map of Teichmüller space*, J. Amer. Math. Soc. **15** (2002), no. 4, 893–927.
- [16] STREBEL K., *Quadratic differentials*, Springer-Verlag, Berlin, 1984.
- [17] WOLPERT S. A., *The Fenchel-Nielsen deformation*, Ann. of Math. (2) **115** (1982), no. 3, 501–528.
- [18] WOLPERT S. A., *On the homology of the moduli space of stable curves*, Ann. of Math. (2) **118** (1983), no. 3, 491–523.

IMPERIAL COLLEGE OF LONDON  
 SOUTH KENSINGTON CAMPUS  
 HUXLEY BUILDING - ROOM 639  
 LONDON SW7 2AZ, UK  
 E-MAIL: g.mondello@imperial.ac.uk