Generalized periods of Kähler manifolds

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A *semicosimplicial (or presimplicial) differential graded Lie algebra* is a covariant functor $\mathbf{\Delta}_{\mathrm{mon}} \to \mathbf{DGLA}$

$\mathbf{\Delta}_{\mathrm{mon}}$: finite ordinal sets with order-preserving injective maps.

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; \mathfrak{g}_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

$$\partial_{k,i} \colon \mathfrak{g}_{i-1} \to \mathfrak{g}_i, \qquad k = 0, \dots, i,$$

such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$.

The maps

$$\partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}$$

endow the vector space $\bigoplus_i \mathfrak{g}_i$ with the structure of a differential complex.

Eeach $\mathfrak{g}_i$ is in particular a differential complex

$$\mathfrak{g}_i = \bigoplus_j \mathfrak{g}_i^j; \qquad d_i \colon \mathfrak{g}_i^j \to \mathfrak{g}_i^{j+1}$$

$$\mathfrak{g}_\bullet^\bullet = \bigoplus_{i,j} \mathfrak{g}_i^j$$

has a natural bicomplex structure.

The associated total complex

$$\mathrm{Tot}(\mathfrak{g}^\Delta) = \bigoplus_i \mathfrak{g}_i[-i], \quad d_{\mathrm{Tot}} = \sum_{i,j} \partial_i + (-1)^j d_j$$

has no canonical DGLA structure.

Yet, it can be endowed with a canonical L_∞-algebra structure via *homotopy transfer* by the Thom-Whitney DGLA.

$$C^{i,j}_{TW}(\mathfrak{g}^\Delta) = \{(x_n)_{n\in\mathbb{N}} \in \bigoplus_n \Omega^i_n \otimes \mathfrak{g}^j_n \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1}\}.$$

where Ω_n are the polynomial differential forms on the n-simplex.

$\chi\colon L \to M$ DGLA morphism.

$$L \underset{0}{\overset{\chi}{\rightrightarrows}} M \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; 0 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

The associated total complex is the *mapping cone* of χ:

$$C_\chi = L \oplus M[-1]; \qquad d(l, m) = (dl, \chi(l) - dm)$$

One finds an explicit expression for the multilinear brackets

$$\mu_n\colon \bigwedge^n C_\chi \to C_\chi[2-n], \qquad n \geq 2,$$

defining the L_∞-algebra structure on C_χ.

$$\mu_1(l,m) = (dl, \chi(l) - dm)$$

$$\mu_2((l_1,m_1)\wedge(l_2,m_2)) = \left([l_1,l_2], \frac{1}{2}[m_1,\chi(l_2)] + \frac{(-1)^{\deg(l_1)}}{2}[\chi(l_1),m_2]\right)$$

and for $n \geq 3$

$$\mu_n((l_1,m_1)\wedge\cdots\wedge(l_n,m_n)) =$$
$$\left(0, \frac{B_{n-1}}{(n-1)!}\sum_{\sigma\in S_n}\pm[m_{\sigma(1)},[\cdots,[m_{\sigma(n-1)},\chi(l)_{\sigma(n)}]\cdots]]\right)$$

where the B_n are the Bernoulli numbers.

Note that $\pi_1\colon C_\chi \to L$ is a linear L_∞-morphism.

L_∞-algebras $\rightsquigarrow$ Deformation functors

$$\mathrm{Def}_{\mathfrak{g}}(A) = \frac{\mathrm{MC}(\mathfrak{g}\otimes\mathfrak{m}_A)}{\text{homotopy equivalence}}$$

$(A, \mathfrak{m}_A)$ local Artin algebra.

If $\mathfrak{g}$ and $\mathfrak{h}$ are quasiisomophic, then $\mathrm{Def}_{\mathfrak{g}} \simeq \mathrm{Def}_{\mathfrak{h}}$

By composing with *DGLA*-morphisms $\rightsquigarrow$ L_∞-algebras we get a functor *DGLA*-morphisms $\rightsquigarrow$ Deformation functors.

A commutative diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{f_L} & L_2 \\ \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} \\ M_1 & \xrightarrow{f_M} & M_2 \end{array}$$

of morphisms of differential graded Lie algebras induces a natural transformation

$$\mathrm{Def}_{\chi_1} \to \mathrm{Def}_{\chi_2}\,.$$

Moreover, if f_L and f_M are quasi-isomorphisms, then $\mathrm{Def}_{\chi_1} \xrightarrow{\sim} \mathrm{Def}_{\chi_2}$ is an isomorphism.

An explicit description of Def_χ.

$$\mathrm{Def}_\chi(A) = \frac{\mathrm{MC}_\chi(A)}{\text{gauge equivalence}},$$

$$\mathrm{MC}_\chi(A) = \Big\{(x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid dx + \frac{1}{2}[x,x] = 0, \; e^a * \chi(x) = 0\Big\},$$

where $*$ is the gauge action of $\exp(M^0 \otimes \mathfrak{m}_A)$ on $M^1 \otimes \mathfrak{m}_A$.

The gauge action

$$(\exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)) \times \mathrm{MC}_\chi(A) \xrightarrow{*} \mathrm{MC}_\chi(A)$$

is

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}).$$

- *Example.* If $M = 0$, then $\mathrm{Def}_\chi = \mathrm{Def}_L$
- *Example.* If $L = 0$ and the differential of M is trivial, then $\mathrm{Def}_\chi = \exp(M^0)$, i.e., $\mathrm{Def}_\chi(A) = \exp(M^0 \otimes \mathfrak{m}_A)$.

Let X be a compact Kähler manifold, and A_X the differential graded commutative algebra of smooth complex differential forms. We have DGLAs

$$L = \{f \in \mathrm{Hom}^*(A_X, A_X) \mid f(\ker\partial) \subseteq \partial A_X\},$$

and

$$M = \{f \in \mathrm{Hom}^*(A_X, A_X) \mid f(\ker\partial) \subseteq \ker\partial \text{ and } f(\partial A_X) \subseteq \partial A_X\}$$

and a commutative diagram

$$\begin{array}{ccccc}
0 & \longleftarrow & L & = & L \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\chi} \\
\mathrm{Hom}^*\left(\frac{\ker\partial}{\partial A_X}, \frac{\ker\partial}{\partial A_X}\right) & \longleftarrow & M & \longrightarrow & \mathrm{Hom}^*(A_X, A_X)
\end{array}$$

By the $\partial\overline{\partial}$-lemma, we have quasi-isomorphisms

$$(A_X, d) \longleftarrow (\ker \partial, d) \longrightarrow \left(\frac{\ker \partial}{\partial A_X}, 0\right).$$

Hence the horizontal arrows in the commutative diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & L & = & L \\ \downarrow \rho & & \downarrow \eta & & \downarrow \chi \\ \mathrm{Hom}^*\left(\frac{\ker\partial}{\partial A_X}, \frac{\ker\partial}{\partial A_X}\right) & \longleftarrow & M & \longrightarrow & \mathrm{Hom}^*(A_X, A_X) \end{array}$$

are quasi-isomorphisms.

We get an isomorphism of deformation functors

$$\mathrm{Def}_\chi \simeq \mathrm{Def}_\eta \simeq \mathrm{Def}_\rho = \mathrm{Aut}^0\left(\frac{\ker \partial}{\partial A_X}\right) \simeq \mathrm{Aut}^0(H^*(X, \mathbb{C})).$$

The isomorphism $\mathrm{Def}_\chi \to \mathrm{Aut}^0(H^*(X,\mathbb{C}))$ has a simple explicit description.

$(\alpha, e^a) \mapsto \psi_a$, where

$$\psi_a([\omega]) = [e^a(\omega_0 + \partial\beta)]$$

for any ∂-closed representative ω_0 of the cohomology class $[\omega]$, and any $\beta \in A_X$ such that $de^a(\omega_0 + \partial\beta) = 0$.

L and M DGLAs, $\mathbf{i}\colon L \to M[-1]$ linear map. Let $\mathbf{l} = [d, \mathbf{i}]\colon L \to M$, i.e.,

$$\mathbf{l}_a = d\mathbf{i}_a + \mathbf{i}_{da}.$$

The map $\mathbf{i}$ is called a *Cartan homotopy* if for every $a, b \in L$ we have:

$$\mathbf{i}_{[a,b]} = [\mathbf{i}_a, \mathbf{l}_b], \qquad [\mathbf{i}_a, \mathbf{i}_b] = 0.$$

If $\mathbf{i}\colon L \to M[-1]$ is a Cartan homotopy, then $\mathbf{l}\colon L \to M$ is a DGLA morphism.

Example. Let M be a differential manifold, $\mathcal{X}(M)$ be the Lie algebra of vector fields on M, and $\mathcal{E}nd^*(\Omega^*(M))$ be the Lie algebra of endomorphisms of the de Rham algebra of M. The Lie algebra $\mathcal{X}(M)$ can be seen as a DGLA concentrated in degree zero, and the graded Lie algebra $\mathcal{E}nd^*(\Omega^*(M))$ has a degree one differential given by $[d_{\mathrm{dR}}, -]$, where d_{dR} is the de Rham differential. Then the contraction

$$\mathbf{i}\colon \mathcal{X}(M) \to \mathcal{E}nd^*(\Omega^*(M))[-1]$$

is a Cartan homotopy and its differential is the Lie derivative

$$[d, \mathbf{i}] = \mathcal{L}\colon \mathcal{X}(M) \to \mathcal{E}nd^*(\Omega^*(M)).$$

When the second equation $[\mathbf{i}_a, \mathbf{i}_b] = 0$ is replaced by the weaker condition $\sum_{\sigma \in S_3} \pm[\mathbf{i}_{x_{\sigma(1)}}, [\mathbf{i}_{x_{\sigma(2)}}, \mathbf{l}_{x_{\sigma(3)}}]] = 0$ we say that $\mathbf{i}$ is a *weak Cartan homotopy*.

Now recall $C_{\mathbf{l}} = L \oplus M[-1]$ and that $\pi_1 \colon C_{\mathbf{l}} \to L$ is a linear L_∞-morphism. If $\mathbf{i} \colon L \to M[-1]$, then $(id, \mathbf{i}) \colon L \to C_{\mathbf{l}}$ is a linear map lifting $id \colon L \to L$

$(id, \mathbf{i}) \colon L \to C_{\mathbf{l}}$ is a linear L_∞ morphism if and only if $\mathbf{i} \colon L \to M[-1]$ is a weak Cartan homotopy.

Let $\mathbf{i}\colon N \to M[-1]$ be a (weak) Cartan homotopy for $\mathbf{l}\colon N \to M$, let L be a subDGLA of M such that $\mathbf{l}(N) \subseteq L$, and let $\chi\colon L \hookrightarrow M$ be the inclusion. Then the linear map

$$\Phi\colon N \to C_\chi, \qquad \Phi(a) = (\mathbf{l}_a, \mathbf{i}_a)$$

is a linear L_∞-morphism. In particular, the map $a \mapsto (\mathbf{l}_a, e^{\mathbf{i}_a})$ induces a natural transformation of Maurer-Cartan functors $\mathrm{MC}_N \to \mathrm{MC}_\chi$, and consequently a natural transformation of deformation functors $\mathrm{Def}_N \to \mathrm{Def}_\chi$.

Proof. We have a commutative diagram of differential graded Lie algebras

$$\begin{array}{ccc} N & \xrightarrow{\mathbf{l}} & L \\ {\scriptstyle \mathbf{l}}\downarrow & & \downarrow{\scriptstyle \chi} \\ M & = & M \end{array}$$

Let X be a complex manifold. For any integer (a, b) with $a \leq 0$ and $b \geq 0$, let $\mathcal{G}erst_X^{a,b}$ be the sheaf

$$\mathcal{G}erst_X^{a,b} = \mathcal{A}_X^{0,b}(\wedge^{-a} T_X).$$

The direct sum $\mathcal{G}erst_X^{*} = \oplus_k \oplus_{a+b=k} \mathcal{G}erst_X^{a,b}$ is a sheaf of differential Gerstenhaber algebras, with the wedge product as graded commutative product, the Dolbeault differential $\overline{\partial}$ as differential and the Schouten-Nijenhuis bracket as odd graded Lie bracket.

Let

$$\mathbf{i}\colon \mathcal{G}erst_X^{a,b} \to \mathcal{H}om^{a,b}(\mathcal{A}_X, \mathcal{A}_X), \qquad \xi \mapsto \mathbf{i}_\xi, \quad \mathbf{i}_\xi(\omega) = \xi \lrcorner \omega,$$

be the *contraction map*.

$\mathbf{i}$ is a morphism of sheaves of bigraded associative algebras:

$$\mathbf{i}_{\xi\wedge\eta} = \mathbf{i}_\xi \mathbf{i}_\eta$$

Since $(\mathcal{G}erst_X^*, \wedge)$ is a graded commutative algebra, $[\mathbf{i}_\xi, \mathbf{i}_\eta] = 0$, so iterated contractions give a symmetric map

$$\mathbf{i}^{(n)}\colon \bigodot^{n} \mathcal{G}erst_X^* \to \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)$$
$$\xi_1 \odot \xi_2 \odot \cdots \odot \xi_n \mapsto \mathbf{i}_{\xi_1}\mathbf{i}_{\xi_2}\cdots\mathbf{i}_{\xi_n}$$

$$\mathcal{P}oly_X^* = \mathcal{G}erst[-1]_X^*$$

is a sheaf of differential graded Lie algebras. Note that, due to the shift, the differential D in $\mathcal{P}oly_X^*$ is $-\overline{\partial}$.

The sheaf $\mathcal{KS}_X^* = \mathcal{A}^{0,*}(T_X)$ is a sheaf of subDGLAs of $\mathcal{P}oly_X^*$, called the *Kodaira-Spencer* sheaf of X

The contraction map $\mathbf{i}\colon \mathcal{G}erst^*_X \to \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)$ can be seen as a linear map

$$\mathbf{i}\colon \mathcal{P}oly^*_X \to \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)[-1].$$

- It is a Cartan homotopy
- The induced morphism $\mathbf{l}$ of sheaves of differential graded Lie algebras is the *holomorphic Lie derivative* $\mathbf{l}_\xi = [\partial, \mathbf{i}_\xi]$.

$$\mathbf{l}_\xi = d_{\mathrm{Hom}}\mathbf{i}_\xi + \mathbf{i}_{D\xi} = [d, \mathbf{i}_\xi] - \mathbf{i}_{\overline{\partial}\xi} = [d, \mathbf{i}_\xi] - [\overline{\partial}, \mathbf{i}_\xi].$$

$$L = \{f \in \mathrm{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \partial A_X\},$$
$$\chi \colon L \hookrightarrow \mathrm{Hom}^*(A_X, A_X)$$

$$\mathbf{i} \colon \mathrm{Poly}^*_X \to \mathrm{Hom}^*(A_X, A_X)[-1]; \qquad \mathbf{l}(\mathrm{Poly}^*_X) \subseteq L$$

So we have a natural transformation of deformation functors $\mathrm{Def}_{\mathrm{Poly}^*_X} \to \mathrm{Def}_\chi$ induced, at the Maurer-Cartan level, by the map $\xi \mapsto (\mathbf{l}_\xi, e^{\mathbf{i}_\xi})$.

$\mathrm{Def}_{\mathrm{Poly}^*_X} \simeq \widetilde{\mathrm{Def}}_X$ (generalized deformation of X)

$\mathrm{Def}_\chi \simeq \mathrm{Aut}^0(H^*(X; \mathbb{C}))$

Theorem. The linear map

$$\mathrm{Poly}_X^* \to \widetilde{C}(\chi), \qquad \xi \mapsto (\mathbf{l}_\xi, \mathbf{i}_\xi)$$

is a linear L_∞-morphism and induces a natural transformation of functors

$$\Phi\colon \widetilde{\mathrm{Def}}_X \to \mathrm{Aut}^0(H^*(X;\mathbb{C})),$$

given at the level of Maurer-Cartan functors by the map $\xi \mapsto \psi_{\mathbf{i}_\xi}$.

Via the natural identifications $H^1(\mathrm{Poly}_X^*) = \oplus_{i\geq 0} H^i(\wedge^i T_X)$ and $H^*(X;\mathbb{C}) = \oplus_{p,q} H^q(X;\Omega_X^p)$ given by the Dolbeault's theorem and the $\partial\overline{\partial}$-lemma, the differential of Φ,

$$d\Phi\colon H^1(\mathrm{Poly}_X^*) \to \mathrm{Hom}^0(H^*(X;\mathbb{C}), H^*(X;\mathbb{C}))$$

is identified with the contraction

$$\left(\oplus_{i\geq 0} H^i(\wedge^i T_X)\right) \otimes \left(\oplus_{p,q} H^q(X;\Omega_X^p)\right) \to \oplus_{i,p,q} H^{q+i}(X;\Omega_X^{p-1})$$

The linear map $\xi \mapsto (\mathbf{l}_\xi, \mathbf{i}_\xi)$ induces a morphism of obstruction theories $H^2(\mathrm{Poly}^*) \to \mathrm{Hom}^1(H^*(X;\mathbb{C}), H^*(X;\mathbb{C}))$ which is naturally identified with the contraction

$$\big(\oplus_{i\geq 0} H^{i+1}(\wedge^i T_X)\big) \otimes \big(\oplus_{p,q} H^q(X;\Omega_X^p)\big) \to \oplus_{i,p,q} H^{q+i+1}(X;\Omega_X^{p-1}).$$

Since the deformation functor $\mathrm{Aut}^0(H^*(X;\mathbb{C}))$ is smooth we get:

- The obstructions to extended deformations of a compact Käheler manifold X are contained in the subspace

$$\bigoplus_{i\geq 0}\bigcap_{p,q}\ker\left(H^{i+1}(\wedge^i T_X)\xrightarrow{\mathbf{i}}\mathrm{Hom}\left(H^q(X;\Omega_X^p),H^{q+i+1}(X;\Omega_X^{p-1}\right)\right)$$

of $H^2(\mathrm{Poly}_X^*)$. (Kodaira principle: ambient cohomology annihilates obstruction)

Corollary. Extended deformations of compact Calabi-Yau manifolds are unobstructed (Bogomolov-Tian-Todorov's Lemma)

Proof. If X is an n-dimensional compact Calabi-Yau manifold, then for any $i \geq 0$ the contraction pairing

$$H^{i+1}(\wedge^i T_X) \otimes H^{n-i-1}(X;\Omega^i) \to H^n(X;\mathcal{O}_X) \simeq H^n(X;\Omega^n_X)$$

is nondegenerate.

Barannikov-Kontsevich's generalized periods.
Let X be a Calabi-Yau manifold with volume element Ω. Represent a generalized deformation of X by a Maurer-Cartan element ξ in Poly_X^* chosen in the Tian's gauge: $\partial(\xi \lrcorner \Omega) = 0$.
Then

$$\Phi_\xi([\Omega]) = [e^{\mathbf{i}_\xi}(\Omega)],$$

the generalized period of (X, ω).

Let X be a compact Kähler manifold, $A_X = F^0 \supseteq F^1 \supseteq \cdots$ be the Hodge filtration of differential forms on X.

The inclusion of DGLAs $KS_X \hookrightarrow \mathrm{Poly}^*_X$ induces $\mathrm{Def}_X \hookrightarrow \widetilde{\mathrm{Def}}_X$.

The Grassmannian $\mathrm{Grass}(H^*(F^p), H^*(X;\mathbb{C}))$ is a homogeneous space for $\mathrm{Aut}^0(H^*(X;\mathbb{C})$ so we have a natural projection

$$\mathrm{Aut}^0(H^*(X;\mathbb{C})) \to \mathrm{Grass}(H^*(F^p), H^*(X;\mathbb{C}))$$

$$\begin{array}{ccc} \widetilde{\mathrm{Def}}_X & \longrightarrow & \mathrm{Aut}^0(H^*(X;\mathbb{C})) \\ \uparrow & & \downarrow \\ \mathrm{Def}_X & \xrightarrow{\mathcal{P}^p} & \mathrm{Grass}(H^*(F^p), H^*(X;\mathbb{C})) \end{array}$$

$\mathcal{P}^p$ is the p-th period map.