Seminario di Geometria e Algebra

Università di Roma, "La Sapienza"

Wednesday October 20, 2004

3:00 - 4:00 Aula di Consiglio

# *Conformal algebras and Jacobi polynomials*

Slides available at:

www.math.harvard.edu/~desole

## Outline

- Example of a Lie conformal algebra: *Vir*
- Definition of associative and Lie conformal algebras
- The general linear Lie conformal algebra $\mathrm{gc}_N$
- Problems of Classification
- *Primary* and *quasi-primary* elements
- Relation to *Jacobi polynomials*
- Classification results

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## Lie conformal algebras

**Example:** Virasoro Lie conf alg

$W$: Lie algebra of vector fields on the circle $S^1$,

$$W = \mathbb{C}[x^{\pm 1}]\frac{d}{dx} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_{(n)} \ , \quad L_{(n)} = -x^n \frac{d}{dx}$$

$$[L_{(m)}, L_{(n)}] \ = \ (m-n)L_{(m+n-1)}$$

$L(z)$: formal distribution defined by the $L_{(n)}$'s,

$$L(z) \ = \ \sum_{n\in\mathbb{Z}} L_{(n)} z^{-n-1} \ \in \ W[[z^{\pm 1}]]$$

Basic observation: locality

$$(z-w)^2[L(z), L(w)] \ = \ 0$$

Equivalently, we have the OPE

$$\begin{aligned}[L(z), L(w)] \ &= \ \partial_w L(w)\delta(z-w) \\ &+ \ 2L(w)\partial_w \delta(z-w)\end{aligned}$$

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The $\delta$–distribution is

$$\begin{aligned}\delta(z-w) &:= \sum_{n\in\mathbb{Z}}z^{-n-1}w^n \;\in\; \mathbb{C}[[z^{\pm 1},w^{\pm 1}]]\\ (z-w)\delta(z-w) &= 0\\ \frac{1}{2\pi i}\oint dz f(z)\partial_w^{(k)}\delta(z-w) &= f^{(k)}(w)\end{aligned}$$

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**General situation**

$a(z), b(z), \cdots \in \mathfrak{g}[[z^{\pm 1}]]$, local distributions

$$(z-w)^N[a(z), b(w)] \ = \ 0$$

Equivalently, we have an OPE

$$[a(z), b(w)] \ = \ \sum_{k=0}^{N} c_k(w)\partial_w^{(k)}\delta(z-w)$$

The $\lambda$–bracket is the Fourier transform

$$\begin{aligned}[a\ _\lambda\ b](w) \ &= \ \frac{1}{2\pi i}\oint dz e^{\lambda(z-w)}[a(z), b(w)] \\ &= \ \sum_{k\geq 0} \lambda c_k(w)/k!\ .\end{aligned}$$

The $\delta$–distribution is

$$\delta(z-w) := \sum_{n\in\mathbb{Z}} z^{-n-1}w^n \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$$

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## Properties of the $\lambda$–bracket

1. sesquilinearity

$$[\partial_w a(w)\ {}_\lambda\ b(w)] = -\lambda[a\ {}_\lambda\ b](w)$$

$$[a(w)\ {}_\lambda\ \partial_w b(w)] = (\lambda+\partial_w)[a\ {}_\lambda\ b](w)$$

2. skew–symmetry

$$[b\ {}_\lambda\ a](w) = -[a\ {}_{-\lambda-\partial_w}\ b](w)$$

3. Jacobi identity

$$[a\ {}_\lambda\ [b\ {}_\mu\ c]](w) - [b\ {}_\mu\ [a\ {}_\lambda\ c]](w)$$

$$= [[a\ {}_\lambda\ b]\ {}_{\lambda+\mu}\ c](w)$$

**Definition** An associative conformal algebra is a $\mathbb{C}[\partial]$–module $R$, with a $\lambda$–product $R \times R \to R[\lambda]$ satisfying sesquilinearity

$$\partial a_\lambda b = -\lambda a_\lambda b \ , \quad a_\lambda \partial b = (\lambda + \partial) a_\lambda b$$

and associativity

$$a_\lambda (b_\mu c) \ = \ (a_\lambda b)_{\lambda+\mu} c$$

A module $M$ over an associative conformal algebra $R$ is a $\mathbb{C}[\partial]$–module, with a map $R \times M \to M[\lambda]$ satisfying sesquilinearity

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**Definition** A Lie conformal algebra is a $\mathbb{C}[\partial]$–module $R$, with a $\lambda$–bracket $R \times R \to R[\lambda]$ satisfying sesquilinearity

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## Facts about associative and Lie algebras

- $\mathrm{Mat}_n\mathbb{C}$ = space of all linear endom's of $\mathbb{C}^n$.
- Associative algebra, $\mathrm{End}_n = \mathrm{Mat}_n\mathbb{C}$.

  **Burnside Theorem**

  if $A \subset \mathrm{End}_n$ acts irreducibly on $\mathbb{C}^n$, then $A = \mathrm{End}_n$.

- Lie algebra, $\mathrm{gl}_n = \mathrm{Mat}_n\mathbb{C}$.

  **Theorem 1**

  if $\mathfrak{g} \subset \mathrm{gl}_n$ acts irreducibly on $\mathbb{C}^n$, then $\mathfrak{g}$ is either semisimple, or semisimple plus scalars.

  **Theorem 2**

  If $\mathfrak{g}$ is semisimple, then it is direct sum of simple ideals.

  **Theorem 3**

  A complete list of simple finite dimensional Lie algebras is:

  1. classical simple Lie algebras: $\mathrm{sl}_n, \mathrm{so}_n, \mathrm{sp}_n$,
  2. exceptional Lie alg's: $E_6, E_7, E_8, F_4, G_2$.

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## Analogues facts about conformal algebras

A conformal endomorphism of $M = \mathbb{C}[\partial]^N$ is a map $a_\lambda : M \to M[\lambda]$ such that

$$a_\lambda(\partial m) = (\partial + \lambda)a_\lambda m$$

**Proposition**

The space of all conformal endom's of $\mathbb{C}[\partial]^N$ is

$$\mathrm{Mat}_N\mathbb{C}[\partial, x]$$

The $\lambda$–action of $\mathrm{Mat}_N\mathbb{C}[\partial, x]$ on $\mathbb{C}[\partial]$ is given by

$$A(\partial, x)_\lambda v(\partial) = A(-\lambda, \lambda + \partial) \cdot v(\lambda + \partial)$$

(*Note*: it is of infinite rank over $\mathbb{C}[\partial]$).

It is an associative conformal algebra, $\mathrm{CEnd}_N$, with $\lambda$–product

$$A(\partial, x)_\lambda B(\partial, x) = A(-\lambda, \lambda + \partial + x) \cdot B(\lambda + \partial, x)$$

It is a Lie conformal algebra, $\mathrm{gc}_N$, with $\lambda$–bracket

$$\begin{aligned} [A(\partial, x)\ {}_\lambda\ B(\partial, x)] &= A(\partial, x)_\lambda B(\partial, x) \\ &- B(\partial, x)_{-\lambda-\partial} A(\partial, x) \end{aligned}$$

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## Associative case

**Problem**

Classify all associative conformal subalgebras $R \subset \mathrm{CEnd}_N$ acting irreducibly on $\mathbb{C}[\partial]^N$.

**Observations**

1. Burnside Thm holds for finite conf. alg's: there is a unique (up to conjugation) finite associative conformal subalgebra of $\mathrm{CEnd}_N$ acting irreducibly on $\mathbb{C}[\partial]^N$,

$$\mathrm{Mat}_N\mathbb{C}[\partial] \subset \mathrm{CEnd}_N$$

2. Burnside Thm fails in the conformal case: for every $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ with $\det P(x) \neq 0$, there is a proper infinite associative conformal subalgebra acting irreducibly on $\mathbb{C}[\partial]^N$,

$$\mathrm{CEnd}_{N,P} = \mathrm{CEnd}_N P(x) \subset \mathrm{CEnd}_N$$

**Results** (B.K.L., 2002)

For $N = 1$ there is nothing else. (For $N \geq 2$ the problem is open).

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## Lie case

**Problem**

Classify all infinite rank Lie conformal subalg's $R \subset \mathrm{gc}_N$ acting irreducibly on $\mathbb{C}[\partial]^N$.

**Partial results:**

- In the finite case, the problem has been solved (D.K., 98).
- Examples: "classical" infinite conf. alg's,
  1. for $P(x) \in \mathrm{Mat}_N[x]$, $\det P(x) \neq 0$,
  $$\mathrm{gc}_{N,P} = \mathrm{Mat}_N\mathbb{C}[\partial, x]P(x)$$
  2. if $P^T(-x) = P(x)$,
  $$\mathrm{oc}_{N,P} = \{a \in \mathrm{gc}_{N,P} \mid a^* = -a\}$$
  $$(A(\partial, x)P(x))^* = A^T(\partial, -\partial - x)P(x)$$
  3. if $P^T(-x) = -P(x)$,
  $$\mathrm{spc}_{N,P} = \{a \in \mathrm{gc}_{N,P} \mid a^* = -a\}$$
  $$(A(\partial, x)P(x))^* = -A^T(\partial, -\partial - x)P(x)$$

**Question**: are there exceptional Lie conf. alg.'s?

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## Main result

**Conjecture** (K.,97)

A complete list of subalgebras $R \subset \mathrm{gc}_N$ which act irreducibly on $\mathbb{C}[\partial]^N$ is given (up to conjugation) by: $\mathrm{gc}_{N,P}$, $\mathrm{oc}_{N,P}$, $\mathrm{spc}_{N,P}$.

**Theorem** (DS.K.,02)

There are no other subalgebras $R \subset \mathrm{gc}_N$ which are normalized by a Virasoro element $L = (x + \alpha\partial)\mathbb{1} \in \mathrm{gc}_N$.

## The reduced space $\widehat{\mathrm{gc}}_N$

Fix a Virasoro element

$$L = (x + \alpha\partial)\mathbb{I} \in \mathrm{gc}_N$$

namely such that $[L\ {}_\lambda\ L] \;=\; (\partial + 2\lambda)L$.

**Observations**

1. There is a representation of the Lie algebra of polynomial vector fields,

$$W^0 = \bigoplus_{n\geq 0} \mathbb{C}L_{(n)},\ L_{(n)} = -x^n \frac{d}{dx}$$

   on $\mathrm{gc}_N$ given by

$$L_{(n)}A \;=\; \frac{1}{n!}\frac{d^n}{d\lambda^n}[L\ {}_\lambda\ A]\Big|_{\lambda=0}$$

2. There is an $\mathrm{sl}_2$ subalgebra,
   $\mathrm{sl}_2 = \{E, H, F\} \subset W^0$, given by

$$E = L_{(2)}\ ,\quad H = -2L_{(1)}\ ,\quad F = -L_{(0)}$$

3. We have the following identities

$$L_{(0)} \;=\; \partial\ ,\quad L_{(1)} \;=\; \text{degree operator} + 1$$

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## Definition

A normalized subalgebra $R \subset gc_N$ is such that

$$L_{(n)}R \subset R \;, \quad n = 0, 1, 2$$

Namely $R$ is invariant under the $sl_2$–action.

## Other Definitions

A primary element $a \in gc_N$ of conformal weight $\Delta$ is such that

$$[L \,{}_\lambda\, a] \;=\; (\partial + \Delta\lambda)a$$

Equivalently, $a$ is eigenvector of $L_{(1)}$ with eigenvalue $\Delta$ and $L_{(n)}a = 0$ for $n \geq 2$.

A quasi–primary element $a \in gc_N$ of c.w. $\Delta$ is such that

$$[L \,{}_\lambda\, a] \;=\; (\partial + \Delta\lambda)a + O(\lambda^3)$$

Equivalently, $a$ is eigenvector of $L_{(1)}$ with eigenvalue $\Delta$ and $L_{(2)}a = 0$. Namely it is highest weight vector of the $sl_2$–action with $h = -2\Delta$.

The reduced space $\widehat{gc}_N \subset gc_N$ is the collection of all quasi–primari elements $a \in gc_N$.

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## Remarks

- $\mathrm{gc}_N$ is direct sum of irreducible Verma modules of $\mathrm{sl}_2$.
- Every element $a \in \mathrm{gc}_N$ decomposes uniquely as

$$a = \sum_{i \geq 0} \partial^i a^i \ , \quad a^i \in \widehat{\mathrm{gc}}_N$$

There is a natural projection,

$$\hat{\ } \ : \ \mathrm{gc}_N \to \widehat{\mathrm{gc}}_N \ , \quad \hat{a} = a^0$$

$\widehat{\mathrm{gc}}_N$ is endowed with a $\lambda$–product $\widehat{\mathrm{gc}}_N \times \widehat{\mathrm{gc}}_N \to \widehat{\mathrm{gc}}_N[\lambda]$, induced by the $\lambda$–bracket in $R$,

$$[a \ _{<\lambda>} \ b] \ = \ \widehat{[a \ _\lambda \ b]}$$

## Proposition

There is a bijection between all subalgebra $R \subset \mathrm{gc}_N$ normalized by $L$ and all subalgebras $\hat{R} \subset \widehat{\mathrm{gc}}_N$. The correspondence is given by

$$R \mapsto \pi(R) \ , \quad \hat{R} \mapsto \mathbb{C}[\partial]\hat{R}$$

## Review: the Jacobi polynomials

Given $\alpha, \beta \in \mathbb{C}$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(y), \; n \geq 0$ are defined by the following properties

- $P_0^{(\alpha,\beta)}(y) \;=\; 1,$
- $P_n^{(\alpha,\beta)}(y)$ is a polyn of degree $n$,
- the leading coefficient is 1,
- they form an orthogonal system in $[-1, 1]$ with respect to the weight function $w(y) = (1-y)^\alpha (1+y)^\beta$, namely

$$\int_1^1 dy \; w(y) P_m^{(\alpha,\beta)}(y) P_n^{(\alpha,\beta)}(y) \;=\; \delta_{m,n}$$

They solve the following differential equation

$$(1-y^2)u'' + (\beta - \alpha - (\alpha+\beta+2)y)u' + n(n+\alpha+\beta+1)u \;=\; 0$$

They satisfy the following symmetry relation

$$P_n^{(\alpha,\beta)}(y) \;=\; (-1)^n P_n^{(\beta,\alpha)}(-y)$$

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## Main results

**Theorem**

(a) The reduced space $\widehat{gc}_N$ has basis over $\mathbb{C}$

$$Q_n^{(\sigma)}(\partial, x) E_{ij} \ , \quad n \geq 0 \ , \ 1 \leq i, j \leq N \ ,$$

where $\sigma = 1 - 2\alpha$, $E_{ij}$ is the standard basis $\mathrm{Mat}_N\mathbb{C}$ and $Q_n^{(\sigma)}(\partial, x)$ is defined by

$$Q_n^{(\sigma)}(\partial, x) = \partial^n P_n^{(-\sigma,\sigma)}\left(\frac{2x+\partial}{\partial}\right)$$

Namely, with $y = 2x + \partial$, $Q_n^{(\sigma)}(\partial, x)$ are the Jacobi polyn's $P^{(-\sigma,\sigma)}(y)$ in homogeneus form.

(b) By denoting $X^n = Q_n^{(\sigma)}(\partial, x)$, we can identify

$$\widehat{gc}_N \simeq \mathrm{Mat}_N\mathbb{C}[X]$$

(c) The projection $\widehat{\ }: gc_N \to \widehat{gc}_N$ is given by:

$$\widehat{A(\partial, x)}) = A(0, X)$$

(d) The $\lambda$–product in $\widehat{gc}_N$ is

$$[X^m A \ {}_{<\lambda>} \ X^n B] = Q_m^{(-\sigma)}(\lambda, X) Q_n^{(\sigma)}(\lambda, X) AB - Q_m^{(\sigma)}(-\lambda, X) Q_n^{(-\sigma)}(-\lambda, X) BA$$

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$$-Q_m^{(\sigma)}(-\lambda, X)Q_n^{(-\sigma)}(-\lambda, X)BA$$



**Theorem**

(a) For $\sigma \notin \mathbb{Z}$, $\widehat{gc}_1$ has no proper $\infty$–dim subalg's. For $\sigma \in \mathbb{Z}$, let $\sigma = \pm S$ with $S \in \mathbb{Z}_+$. A complete list of $\infty$–dim subalgebras of $\widehat{gc}_1$ is:

$$\hat{R}_S^{(\pm)} \ = \ X^S\mathbb{C}[X]$$
$$\hat{R}_{*,S}^{(\pm)} \ = \ \mathrm{span}_{\mathbb{C}}\{X^n \mid n \geq S, n \in 2\mathbb{Z}+1\}$$

(b) A complete list of infinite rank, normalized proper subalgebras of $gc_1$ is ($S \in \mathbb{Z}_+$),

$$R_S^{(+)} = x^S\mathbb{C}[\partial, x] \ , \quad R_S^{(-)} = (x+\partial)^S\mathbb{C}[\partial, x]$$
$$R_{*,S}^{(+)} = x^S\mathbb{C}_*[\partial, x] \ , \quad R_{*,S}^{(-)} = (x+\partial)^S\mathbb{C}_*[\partial, x]$$

where

$$\mathbb{C}_*[\partial, x] = \left\{ \begin{array}{c} x^S(p(\partial, x) + (-1)^{S+1}p(\partial, -\partial - x)) \\ \text{with } p(\partial, x) \in \mathbb{C}[\partial, x] \end{array} \right\}$$

The corresponding Virasoro element is

$$L_S^{(\pm)} = x + \frac{1 \mp S}{2}\partial$$

The irreducible subalg's are $R_S^{(+)}$ and $R_{*,S}^{(+)}$.

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## Corollary 1

For $S \in \mathbb{Z}_+$, we have

$$\tilde{P}_n^{(S)}(y) := \frac{P_n^{(-S,S)}(y)}{(y-1)^S} = \frac{P_n^{(S,-S)}(y)}{(y+1)^S},$$
$$\tilde{P}_n^{(S)}(y) = (-1)^{n-S}\tilde{P}_n^{(S)}(-y)$$

**Notation**$(y = 2x+1)$

$$P_m^{(\sigma,-\sigma)}(y)P_n^{(-\sigma,\sigma)}(y) = \textstyle\sum_{l=0}^{m+n} D(\sigma;\ m,n,l)x^l$$
$$\mathcal{A}_{l,n} = \{l+1,\ldots,n\},\ \text{if } l < n,\ \ \mathcal{A}_{l,n} = \emptyset,\ \text{if } l \geq n$$

## Corollary 2

(i) If $n - m \leq l \leq m+n$, then

$$D(\sigma; m, n, l) = \prod_{i \in \mathcal{A}_{l,n}} (i^2 - \sigma^2)R(\sigma),\ \ R(\sigma) = R(-\sigma)$$

(ii) If $l < n - m$, then:

$$D(\sigma, m, n, l) = \textstyle\prod_{i \in \mathcal{A}_{l,m}}(i^2-\sigma^2)\prod_{i \in \mathcal{A}_{m,n}}(i-\sigma)R(\sigma)$$
$$R(\sigma) = \frac{\prod_{i \in \mathcal{A}_{m,n}}(i+\sigma)}{\prod_{i \in \mathcal{A}_{m,n}}(i-\sigma)}R(-\sigma)\ ,\ i \geq 0$$

The End