Poisson vertex algebras in the theory of Hamiltonian equations

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Joint work with V. Kac & A. Barakat

slides avalable at:

www.mat.uniroma1.it/∼desole

Outline

1. Poisson vertex algebras
   - Definition of a PVA
   - Examples: GFZ, Virasoro, Currents
2. Hamiltonian equations associated to a PVA
   - Hamiltonian operators, Hamiltonian eq's, integrals of motion
   - Lenard scheme for pair of PVA's
   - Examples: KdV, HD, CNW, CNW of HD type
3. Symplectic operators (and Dirac structures)
   - Definition of a symplectic operators, Hamiltonian equations and Lenard scheme
   - Examples: KN equation and NLS system

Outline

1. Poisson vertex algebras
2. Hamiltonian equations associated to a PVA
3. Symplectic operators (and Dirac structures)

Definition A Lie conformal algebra R is a vector space with an endomorphism $\partial \in \mathrm{End}\, R$ and a Lie–bracket

$$a \otimes b \mapsto \{a_\lambda b\} \in R[\lambda]$$

satisying the following axioms:

- sesquilinearity

$$\begin{aligned} \partial\{a_\lambda b\} &= \{\partial a_\lambda b\} + \{a_\lambda \partial b\} \\ \{\partial a_\lambda b\} &= -\lambda\{a_\lambda b\} \end{aligned}$$

- skew-simmetry

$$\{b_\lambda a\} = -\{a_{-\lambda-\partial} b\}$$

- Jacobi identity

$$\{a_\lambda\{b_\mu c\}\} - \{b_\mu\{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}$$

Definition A differential Lie conformal algebra R is a vector space with an endomorphism $\partial \in \mathrm{End}\, R$ and a Lie–bracket

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satisying the following axioms:

- Leibniz rule

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$$\{a_\lambda b\} = \sum_{n=0}^{N} \lambda^n a_n b$$

$$\{a_{-\lambda-\partial} b\} = \sum_{n=0}^{N} (-1)^n (\lambda+\partial)^n (a_n b)$$

Lie conformal algebra

A $\mathbb{C}[\partial]$-module R with a λ-bracket $R \otimes R \to R[\lambda]$ satisfying

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Poisson vertex algebras

Lie conformal algebra ⇝ Poisson vertex algebra

↑ conformal analogue ↑

Lie algebra ⇝ Poisson algebra

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- a commutative associative differential product, $f, g \mapsto f \cdot g \in \mathcal{V}$,
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Remark: this is the *quasi-classical limit* of the notion of a vertex algebra.

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Note: We also have the right Leibniz rule:

$$\{fg_\lambda h\} = \{f_{\lambda+\partial} g\}_\to h + \{h_{\lambda+\partial} g\}_\to f\,.$$

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Def: An algebra of differential funcions is

$$\mathbb{C}[u_i^{(n)} \mid i = 1, \ldots, \ell,\ n \in \mathbb{Z}_+] \subset \mathcal{V}$$

with the partial derivatives

$$\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \to \mathcal{V}\,,$$

and the total derivative

$$\partial = \sum_{i,n} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}\,.$$

We require: $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for $n >> 0$,

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Main formula: $\{f_\lambda g\} =$

$$\sum_{i,j,m,n} \frac{\partial g}{\partial u_j^{(n)}} (\lambda+\partial)^n \{u_{i\,\lambda+\partial}\, u_j\}_{\to} (-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}}$$

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Examples

1. GFZ PVA: $\mathcal{V} = \mathbb{C}[u^{(n)} \,|\, n \in \mathbb{Z}_+]$ with
$$\{u_\lambda u\} = \lambda$$

2. Virasoro-Magri PVA: $\mathcal{V} = \mathbb{C}[u^{(n)} \,|\, n \in \mathbb{Z}_+]$ with
$$\{u_\lambda u\} = u' + 2u\lambda + \frac{c}{12}\lambda^3$$
c: Virasoro central charge
Note: every linear combin. is a PVA!

3. Affine PVA: **Data**: Lie algebra $\mathfrak{g}$, $p \in \mathfrak{g}$, symm. invariant bil. form $(\,|\,)$, $k \in \mathbb{C}$
$V^{cl}(\mathfrak{g}, p, (\,|\,)) = \mathbb{C}\big[\mathbb{C}[\partial]\mathfrak{g}\big]$ with $(a, b \in \mathfrak{g})$
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Examples

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$\mathcal{V} = \mathbb{C}[u^{(n)} \mid n \in \mathbb{Z}_+]$ with
$\{u_\lambda u\} = \lambda$

2. Virasoro-Magri PVA:
$\mathcal{V} = \mathbb{C}[u^{(n)} \mid n \in \mathbb{Z}_+]$ with

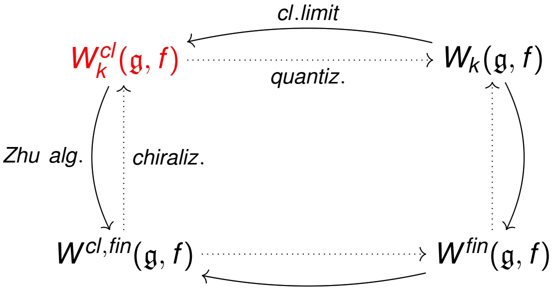
$$\{u_\lambda u\} = u' + 2u\lambda + \frac{c}{12}\lambda^3$$

c: Virasoro central charge

3. Affine PVA: **Data**: Lie algebra $\mathfrak{g}$, $p \in \mathfrak{g}$, symm. invariant bil. form $(\,|\,)$, $k \in \mathbb{C}$
$V^{cl}(\mathfrak{g}, p, (\,|\,)) = \mathbb{C}\big[\mathbb{C}[\partial]\mathfrak{g}\big]$,

$$\{a_\lambda b\} = [a,b] + (p|[a,b]) + k(a|b)\lambda\,.$$

4. W-algberas, obtained by quantum Hamiltonian reduction:



Outline

1. Poisson vertex algebras
2. Hamiltonian equations associated to a PVA
3. Symplectic operators (and Dirac structures)

How are P.V.A.'s related to the theory of Hamiltonian equations?

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Basic Lemma: Let $\mathcal{V}$ be a P.V.A. Then:

- $\mathcal{V}/\partial\mathcal{V}$ is a Lie algebra with Lie bracket

$$\{\textstyle\int f, \int g\} = \int \{f_\lambda g\}\big|_{\lambda=0} \in \mathcal{V}/\partial\mathcal{V}$$

  (Notation: $\int f = f + \partial\mathcal{V} \in \mathcal{V}/\partial\mathcal{V}$.)

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Integrability: $\exists\, \infty$ sequence, lin. indep. $\int h_0 = \int h, \int h_1 \int h_2 \dots$ s.t.

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Note: having two *compatible* Hamiltonian forms is a key point for proving integrability via the *Lenard scheme*.

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Assumptions:

1. $\mathcal{V}$ is a normal algebra of differential functions.
2. Two compatible PVA stcuctures on $\mathcal{V}$: $\{\cdot\,_\lambda\,\cdot\}_H$ and $\{\cdot\,_\lambda\,\cdot\}_K$.
3. $K(\partial)=\left(\{u_{j\,\partial}u_i\}_K\right)_{i,j=1,\dots,\ell}$ is non-degenerate.
4. Let $\int h_0, \int h_1 \in \mathcal{V}/\partial\mathcal{V}$ be s.t. $\{\int h_0, \cdot\}_H = \{\int h_1, \cdot\}_K$.
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It implies that the complex of variational calculus is exact [DSK08]. In particular: $D_F(\partial) = D_F^*(\partial) \Leftrightarrow F = \frac{\delta f}{\delta u}$, where: the Frechet derivative and its adjoint are:

$$\big(D_F(\partial)P\big)_i = \sum_{j,n} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n P_j \quad , \qquad \big(D_F^*(\partial)P\big)_i = \sum_{j,n} (-\partial)^n \Big(\frac{\partial F_j}{\partial u_i^{(n)}} P_j \Big) \,.$$

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Theorem: (Lenard scheme of integrability)

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Def: $K(\partial)$ non-degenerate: $K(\partial)\big(D_F(\partial) - D_F^*(\partial)\big)K(\partial) = 0 \Rightarrow D_F(\partial) = D_F^*(\partial)$.

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Note: it gives an evolution equation which can be written in two Hamiltonian forms:

$$\frac{du_i}{dt} = \{\textstyle\int h_0, u_i\}_H = \{\int h_1, u_i\}_K$$

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Notation: the orthogonality relation is: $F \perp P \Leftrightarrow \int F \cdot P = 0$.

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Concl.: the hierarchy of Hamiltonian eq.'s $\frac{du_i}{dt_n} = \{\int h_n, u_i\}_H$ is integrable.

Lenard scheme

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Examples.

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Space of functions: $\mathcal{V} = \mathbb{C}[u, u', u'', \dots]$.
Equation:

$$\frac{du}{dt} = 3uu' + cu''' = \underbrace{\partial}_{K(\partial)} \frac{\delta}{\delta u} \underbrace{\frac{1}{2}(u^3 + cuu'')}_{\int h_2}$$

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Example 2: Harry-Dym equation

Space of funct's: $\mathcal{V} = \mathbb{C}[u^{\pm\frac{1}{2}}, u', u'', \dots]$.
Equation:

$$\frac{du}{dt} = (u^{-1/2})''' = \underbrace{\partial^3}_{H(\partial)} \frac{\delta}{\delta u} \underbrace{2u^{1/2}}_{\int h_0}$$

$$= \underbrace{u' + 2u\partial}_{K(\partial)} \frac{\delta}{\delta u} \underbrace{-2u^{-1/4}\partial^2 u^{-1/4}}_{\int h_1}\,.$$

Orthogonality condition:

$$\left(\mathbb{C}\frac{\delta h_0}{\delta u} + \mathbb{C}\frac{\delta h_1}{\delta u}\right)^\perp \subset (u^{-1/2})^\perp = u^{1/2}\partial\mathcal{V} = \mathrm{Im}\,(K(\partial))\,.$$

Conclusion: the HD eq is integrable.

Lenard scheme

Assumptions:

1. $\mathcal{V}$ is normal.
2. $\{\cdot\,{}_\lambda\,\cdot\}_H$ and $\{\cdot\,{}_\lambda\,\cdot\}_K$ are compatible PVA structures.
3. $K(\partial) = \big(\{u_j\,{}_\partial\,u_i\}_K\big)_{i,j=1,\dots,\ell}$ is non-degenerate.
4. $\int h_0, \int h_1 \in \mathcal{V}/\partial\mathcal{V}$ are s.t. $\{\int h_0, \cdot\}_H = \{\int h_1, \cdot\}_K$.
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$\exists \int h_0, \int h_1, \int h_2, \dots \in \mathcal{V}/\partial\mathcal{V}$, such that

$$\{\textstyle\int h_n, \cdot\}_H = \{\textstyle\int h_{n+1}, \cdot\}_K\,, \quad \forall n \geq 0\,.$$

Hence: integrable hierarchy

$$\frac{du_i}{dt_n} = \{\textstyle\int h_n, u_i\}_H\,.$$

Example 2: Harry-Dym equation

Space of funct's: $\mathcal{V} = \mathbb{C}[u^{\pm\frac{1}{2}}, u', u'', \dots]$.
Equation:

$$\frac{du}{dt} = (u^{-1/2})''' = \underbrace{\partial^3}_{H(\partial)} \frac{\delta}{\delta u} \underbrace{2u^{1/2}}_{\int h_0}$$

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Algorithm: for the integrals of motion $\int h_n,\ n \geq 0$:

$$\int h_0 = 2\int u^{1/2} \xrightarrow{\frac{\delta}{\delta u}} \frac{\delta h_0}{\delta u} = u^{-1/2} \xrightarrow{K(\partial)=u'+2u\partial} 0$$

$$\frac{\delta h_0}{\delta u} = u^{-1/2} \xrightarrow{H(\partial)=\partial^3} \partial^3 u^{-1/2}$$

$$\int h_1 = -\int u^{-1/4}\partial^2 u^{-1/4} \overset{\frac{\delta}{\delta u}}{\rightsquigarrow} F^1 = u^{-5/2}\partial^2 u^{-1/4} \overset{K(\partial)}{\rightsquigarrow} \partial^3 u^{-1/2}$$

$$F^1 \xrightarrow{H(\partial)} \partial^3 u^{-1/2}$$

$$\int h_2 = -\frac{1}{9}\int u^{-3/4}\partial^4 u^{-3/4} \overset{\frac{\delta}{\delta u}}{\rightsquigarrow} F^2 = \frac{1}{6}u^{-7/4}\partial^4 u^{-3/4} \overset{K(\partial)}{\rightsquigarrow} \partial^3 u^{-1/2}$$

$$F^2 \xrightarrow{H(\partial)} \dots$$

Example 3: CNW system

Space of funct's: $\mathcal{V} = \mathbb{C}[u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_+]$. Compatible PVA structures:

$$\{u_\lambda u\}_H = (\partial + 2\lambda)u + c\lambda^3\,,\ \{v_\lambda v\}_H = 0\,,\ \{u_\lambda v\}_H = (\partial + \lambda)v\,,\ \{v_\lambda u\}_H = \lambda v\,,$$
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Corresp. Hamiltonian operators: $H(\partial) = \begin{pmatrix} c\partial^3 + 2u\partial + u' & v\partial \\ v\partial + v' & 0 \end{pmatrix}$, $K(\partial) = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}$.

Let $h_0 = v$, $h_1 = u$. We have: $H(\partial)\frac{\delta h_0}{\delta u} = K(\partial)\frac{\delta h_1}{\delta u} = 0$, and $\left(\mathbb{C}\frac{\delta h_0}{\delta u} \oplus \mathbb{C}\frac{\delta h_1}{\delta u}\right)^\perp = \mathrm{Im}\,(\partial)$.
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$$\frac{du}{dt} = H(\partial)\frac{\delta h_{n-1}}{\delta u} = K(\partial)\frac{\delta h_n}{\delta u}\,,\ n \geq 0\,.$$

For example, we have:

$$\textstyle\int h_2 = \frac{1}{2}\int(u^2 + v^2)\,,\qquad \int h_3 = \frac{1}{2}\int(cuu'' + u^3 + uv^2)\qquad ,\cdots$$
$$\begin{cases} \frac{du}{dt_2} = u' \\ \frac{dv}{dt_2} = v' \end{cases},\qquad \begin{cases} \frac{du}{dt_2} = cu''' + 3uu' + vv' \\ \frac{dv}{dt_2} = \partial(uv) \end{cases}\qquad ,\cdots$$

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Example 4: CNW system of HD type

Space of funct's: $\mathcal{V} = \mathbb{C}[u, v^{\pm 1}, u', v', u'', v'']$. Three compatible PVA structures:

$$\{u_\lambda u\}_1 = (\partial + 2\lambda)u\,,\ \{v_\lambda v\}_1 = 0\,,\ \{u_\lambda v\}_1 = (\partial + \lambda)v\,,\ \{v_\lambda u\}_1 = \lambda v\,,$$
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$$\frac{du}{dt_1} = (\partial + c\partial^3)\left(\frac{1}{v}\right) \quad , \qquad \frac{dv}{dt_1} = -\partial\left(\frac{u}{v^2}\right)$$

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Outline

Definition A symplectic structure S on $\mathcal{V}$ is a λ-bracket

$$f \otimes g \mapsto \langle f_\lambda g \rangle^S \in \mathcal{V}[\lambda]$$

satisying the following axioms:

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where $\{\cdot\,_\lambda\,\cdot\}_B$ is the Beltrami λ-bracket, defined by $\{u_{i\lambda} u_j\}_B = \delta_{i,j}$ and extended to $\mathcal{V}$ by sesquilinearity, symmetry and Leibniz rules.

Note: it suffices to define the λ-bracket on generators: $\langle u_{i\lambda} u_j \rangle^S = S_{ji}(\lambda)$ (this is the symplectic operator) and extend it by sesquilinearity and the Leibniz rules.

Definition A symplectic structure S on $\mathcal{V}$ is a λ-bracket

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Basic Lemma:

we have a Lie algebra structure on the space of Hamiltonian functionals

$$\mathcal{F}(S) = \left\{ \int f \in \mathcal{V}/\partial\mathcal{V} \,\middle|\, \frac{\delta f}{\delta u} = S(\partial)P\,, \quad \text{for some } P \in \mathcal{V}^\ell \right\},$$

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Definitions:

- Hamiltonian equation (w.r.t. sympl. str. $\langle \cdot\,_\lambda \cdot \rangle^S$ and Ham.. funct. $\int h$):

$$\frac{du}{dt} = P, \quad \text{where } S(\partial)P = \frac{\delta h}{\delta u}.$$

- Integral of motion (for a Hamilt. eq.): $\int f \in \mathcal{F}(S)$ s.t.

$$\frac{d}{dt}\int f = \{\textstyle\int h, \int f\}_S = 0.$$

- Integrability: $\exists$ infinite sequence $\int h_0 = \int h, \int h_1 \int h_2 \dots \in \mathcal{F}(S)$ s.t.

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Theorem: (Lenard scheme for symplectic operators)

Assumptions:

1. $\mathcal{V}$ is a normal algebra of differential functions.
2. Two compatible symplectic stcuctures on $\mathcal{V}$: $\langle \cdot\,{}_\lambda\,\cdot \rangle^S$ and $\langle \cdot\,{}_\lambda\,\cdot \rangle_T$.
3. non-degeneracy condition holds for S, T.
4. Let $\int h_0, \int h_1 \in \mathcal{F}(S)$ be s.t. $\frac{\delta h_0}{\delta u} = S(\partial)P^0$, $\frac{\delta h_1}{\delta u} = T(\partial)P^0$.
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compatible:

$$(i) \quad {\textstyle\int} F \cdot P = {\textstyle\int} F' \cdot P' \; \forall P, P' \text{ s.t. } S(\partial)P = T(\partial)P'$$

$$\Rightarrow \exists Q \text{ s.t. } F = S(\partial)Q, \; F' = T(\partial)Q$$

$$(ii) \quad {\textstyle\int} \big(L_{P_1} S(\partial) P_2\big) P_3' + {\textstyle\int} \big(L_{P_1'} S(\partial) P_2\big) P_3 + (\text{cyclic permut's}) = 0$$

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Note: it gives an evolution equation

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Notation: the orthogonality relation is: $F \perp P \Leftrightarrow \int F \cdot P = 0$.

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Example: KN equation

Space of functions: $\mathbb{C}[u, u', u'', \dots] \subset \mathcal{V}$. Equation:

$$\frac{du}{dt} = u''' - \frac{3}{2}\frac{(u'')^2}{u'} = P^1 .$$

It is associated to two symplectic structures:

$$\langle u_\lambda u\rangle^S = \frac{1}{u'}(\partial+\lambda)\frac{1}{u'} \quad , \qquad \langle u_\lambda u\rangle^T = (\partial+\lambda)\frac{1}{u'}(\partial+\lambda)\frac{1}{u'}\lambda$$

Indeed, we have:

$$S(\partial)P = \frac{\delta h_0}{\delta u}, \ \textstyle\int h_0 = \frac{1}{2}\int \frac{(u'')^2}{(u')^2}, \ T(\partial)P = \frac{\delta h_1}{\delta u}, \ \int h_1 = \int\Big(-\frac{1}{2}\frac{(u''')^2}{(u')^2} + \frac{3}{8}\frac{(u'')^4}{(u')^4}\Big).$$

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$$\langle u_\lambda u\rangle^S = \frac{1}{u'}(\partial+\lambda)\frac{1}{u'} \quad , \qquad \langle u_\lambda u\rangle^T = (\partial+\lambda)\frac{1}{u'}(\partial+\lambda)\frac{1}{u'}\lambda$$

Indeed, we have:

$$S(\partial)P = \frac{\delta h_0}{\delta u}\,,\ \int h_0 = \frac{1}{2}\int\frac{(u'')^2}{(u')^2}\,,\ T(\partial)P = \frac{\delta h_1}{\delta u}\,,\ \int h_1 = \int\Big(-\frac{1}{2}\frac{(u''')^2}{(u')^2} + \frac{3}{8}\frac{(u'')^4}{(u')^4}\Big)\,.$$

We also have, letting $P^0 = u'$,

$$S(\partial)P^0 = 0 \quad , \qquad T(\partial)P^0 = S(\partial)P^1 = \frac{\delta}{\delta u}\int h_0\,.$$

All assumptions of the Theorem hold. In particular, the orthogonality condition reads:

$$\left(\mathbb{C}P^0\right)^\perp = (u')^\perp = \frac{1}{u'}\partial\mathcal{V} = \mathrm{Im}\,\left(S(\partial)\right).$$

Conclusion: the KN equation is integrable.

Example: KN equation

Space of functions: $\mathbb{C}[u, u', u'', \dots] \subset \mathcal{V}$. Equation:

$$\frac{du}{dt} = u''' - \frac{3}{2}\frac{(u'')^2}{u'} = P^1 .$$

It is associated to two symplectic structures:

$$\langle u_\lambda u \rangle^S = \frac{1}{u'}(\partial + \lambda)\frac{1}{u'} \quad , \qquad \langle u_\lambda u \rangle^T = (\partial + \lambda)\frac{1}{u'}(\partial + \lambda)\frac{1}{u'}\lambda$$

Indeed, we have:

$$S(\partial)P = \frac{\delta h_0}{\delta u}\,,\ \int h_0 = \frac{1}{2}\int \frac{(u'')^2}{(u')^2}\,,\ T(\partial)P = \frac{\delta h_1}{\delta u}\,,\ \int h_1 = \int\Big(-\frac{1}{2}\frac{(u''')^2}{(u')^2} + \frac{3}{8}\frac{(u'')^4}{(u')^4}\Big)\,.$$

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All assumptions of the Theorem hold. In particular, the orthogonality condition reads:

$$(\mathbb{C}P^0)^\perp = (u')^\perp = \frac{1}{u'}\partial\mathcal{V} = \mathrm{Im}\,\big(S(\partial)\big)\,.$$

Conclusion: the KN equation is integrable.

Dirac structures and the KN equation

The notion of Dirac structure is a generalization which includes both the Hamiltonian and the symplectic structures.

Also for Dirac structures one can develop the whole theory, including the Lenard scheme of integrability.

One proves integrability of the Non-Linear Schroedinger system:

$$\begin{cases} \frac{du}{dt} & = & v'' + 2v(u^2 + v^2)\,, \\ \frac{dv}{dt} & = & -u'' - 2u(u^2 + v^2)\,. \end{cases}$$

The End

Auguri, Corrado!