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Vertex algebras generated by primary fields of low conformal weight

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Slides available from

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There are several equivalent definitions of vertex algebras.

1. The first definition is due to R. Borcherds, in 1986. It is given in terms of (quite complicated) identities known as “Borcherd’s identities”.

2. We will see a different definition of vertex algebras. (the equivalence was proved last year from B. Bakalov and V. Kac) It shows better the relation between v.a. and L.c.a. It does not enlighten the physics interpretation of vertex algebras, but it is sometimes the most convenient way to think at v.a.’s

Definition. A **Lie conformal superalgebra** is

- $R = R_{\bar{0}} \oplus R_{\bar{1}}$, a vector superspace
- $T: \ R \to R$, an even endomorphism
- a parity preserving λ–bracket

$$a \otimes b \mapsto [a \ {}_\lambda \ b] \ \in \ R[\lambda]$$

satisfying:

(i) **sesquilinearity**

$$\begin{aligned} [Ta \ {}_\lambda \ b] &= -\lambda[a \ {}_\lambda \ b] \\ [a \ {}_\lambda \ Tb] &= (\lambda + T)[a \ {}_\lambda \ b] \end{aligned}$$

(ii) **skewsymmetry**

$$[b \ {}_\lambda \ a] \ = \ -p(a,b)[a \ {}_{-\lambda-T} \ b]$$

(iii) **Jacobi identity**

$$[a \ {}_\lambda \ [b \ {}_\mu \ c]] - p(a,b)[b \ {}_\mu \ [a \ {}_\lambda \ c]]$$

$$= \ [[a \ {}_\lambda \ b] \ {}_{\lambda+\mu} \ c]$$

- from sesquilinearity, T is a derivation of the l-bracket
- In the skewsymmetry relation, [a -l-T b] means that we have
- to replace in [a m b] the indeterminant m with the
- operator (-l-T), acting from the left.
- We call rank of a Lie conformal algebra its rank as C[T] module. R is said finite if it is of finite rank.

<u>Definition</u>. A **vertex superalgebra** is:
- V, a vector superspace (the *space of states*)
- $|0\rangle \in V$, the *vacuum element*,
- $T: \ V \to V$, the *infinitesimal traslation operator*, with two operations

1) a λ–*bracket* $[a \ _\lambda \ b]$, s.t. V is a Lie conformal algebra,

2) a *normal order product* $:ab:$, s.t. V is a unital differential algebra

They satisfy some compatibility conditions:
(i) **skewsymmetry** of $:ab:$

$$:ab: - p(a,b) :ba: = \int_{-T}^{0} d\lambda [a \ _\lambda \ b]$$

(ii) **quasi–associativity**

$$:(:ab:)c: - :a(:bc:): = \quad \text{expr involving } [\ _\lambda \ ]$$

(iii) **non commutative Wick formula**

$$[a \ _\lambda \ :bc:] \ = :[a \ _\lambda \ b]c: + p(a,b) :b[a \ _\lambda \ c]: + ...$$

- A proof of the equivalence between this definition and the more familiar one (in terms of local fields) is due to Bojko Bakalov and Victor Kac (2002)
- good way to think at them:
  - Lie conformal algebras are similar to Lie algebras, (intuitively, they are the same thing as Lie algebras, with a derivation T, and with Lie bracket depending from a parameter l)
  - vertex algebras are similar to associative algebras (with a derivation T).

Examples

Current Lie onformal algebra

$\mathfrak{g}$: Lie algebra, $(,)$: simm invariant bil form on $\mathfrak{g}$. Cur$\mathfrak{g}$ is defined as

$$\mathrm{Cur}\mathfrak{g} \;=\; (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}|0\rangle$$

with λ–bracket ($|0\rangle$ central)

$$[a\ {}_\lambda\ b] = [a,b] + \lambda k(a,b)|0\rangle\ , \quad a,b \in \mathfrak{g}$$

$k \in \mathbb{C}$ is called *Kac–Moody level.*

Virasoro Lie conformal algebra

$$Vir \;=\; (\mathbb{C}[T]L) \oplus \mathbb{C}|0\rangle$$

with λ–bracket

$$[L\ {}_\lambda\ L] = (T + 2\lambda)L + \frac{c}{12}\lambda^3|0\rangle$$

$c \in \mathbb{C}$ is called *central charge.*

Remark. We have the corresponding *enveloping vertex algebras* $U(\mathrm{Cur}\mathfrak{g})$ and $U(Vir)$.

- These are examples of Lie conformal algebras.
- What about vertex algebras? Continuing the analogy with Lie algebras, we know that to every Lie algebra we can associate a universal enveloping algebra, which is an associative unital (infinite dim) algebra. In the same way, given a Lie conformal algebra R, one can construct an "enveloping" vertex algebra U(R). So each example of conf algebra produces an example of vertex algebra.

- It is natural from the physics point of view, to ask that a vertex algebra V ("the space of states", in physics) is a representation of the Virasoro Lie algebra, with eigenvalues of L_0 (=energy operator) bounded from below.
- This is guaranteed if we ask that V has a Virasoro element

Next:

1. Definition of Virasoro element, and why it give a repr of Vir Lie algebra,
2. Def. of conformal weight and primary elements.
3. Def of "generated" and "low"

Definition. A **Virasoro element** $L \in V$ is such that

1. $[L\ {}_\lambda\ L] = (T + 2\lambda)L + \frac{c}{2}\lambda^{(3)}|0\rangle$,
2. $L_{-1} = [L\ {}_\lambda\ \cdot]\Big|_{\lambda=0} = T$,
3. $L_0 = \frac{d}{d\lambda}[L\ {}_\lambda\ \cdot]\Big|_{\lambda=0}$ is diagonalizable on V.

c is the *central charge* of L.

Remark. We have an action of the **Virasoro Lie algebra** Vir on V, given by $(m \geq 0)$

$$\begin{aligned} L_{m-1}(a) &= \frac{d^m}{d\lambda^m}[L\ {}_\lambda\ a]\Big|_{\lambda=0} \\ L_{-m-2}(a) &= \frac{1}{m!} :(T^m L)a: \end{aligned}$$

Namely, the operators $L_m,\ m \in \mathbb{Z}$ satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}\mathbb{1}$$

Definition. Eigenvalues of L_0 are called *conformal weights*: a has conformal weight Δ if

$$[L\ {}_\lambda\ a] \ = \ Ta + \Delta\lambda a + O(\lambda^2)$$

$a \in V$ is a *primary element* if

$$[L\ {}_\lambda\ a] \ = \ Ta + \Delta\lambda a$$

Namely $L_n(a) = 0$ for $n > 0$.

Definition. V *(strongly) generated* by a set $A = \{a^\alpha,\ \alpha \in \mathcal{A}\} \subset V$, if

$$V = \mathrm{Span}\Big\{ :(T^{j_1} a^{\alpha_1}) \cdots (T^{j_n} a^{\alpha_n}): \ ,\ j_i \geq 0 \Big\}$$

Remark. By physical requests:

- $\Delta \in \frac{1}{2}\mathbb{Z}_+$
- Δ even (resp. odd) $= a$ even (resp odd)
- $\Delta = 1/2$ can be studied separately

- To say that a is primary, means that $L_n(a) = 0$ for $n > 0$. This means that a is a highest weight vector of Vir, with highest weight (c, Δ).

- V is generated by A, if V is obtained by applying T to, and by taking ordered products of, elements of A. For example, the enveloping vertex algebra $U(R)$ over the conformal algebra R is (strongly) generated by R.

- To say that V is strongly generated by primary elements is equivalent to say that the representation of the Virasoro Lie algebra on V decomposes as direct sum of highest weight representations (it is in cathegory O).

- In particular, this assumption is automatically satisfied if we require that the representation of the Virasoro Lie algebra on V is unitary. (which is a natural assumption from the physics point of view)

- Moreover, if we require that the vertex algebra V is unitary (namely the adjoint representation of V on itself is a unitary representation), it follows that the conformal weights have non negative semi–integer values.

- From spin-statistics theorem, we want

$$\Delta_a \in \mathbb{Z} \to a \text{ even}$$
$$\Delta \in 1/2(2\mathbb{Z}+1) \to a \text{ odd}$$

- Let us now examine the smallest possible values of the conformal weight $\Delta \in 1/2\mathbb{Z}_+$.

  1. $\Delta = 0$ corresponds to the vacuum vector $|0\rangle$.

  2. Primary elements of conformal weight $\Delta = 1/2$ generate a Fermion subalgebra $V_{1/2}$. In particular the vertex algebra V becomes a representation of the Clifford algebra. It follows by representation theory of the Clifford algebra, that the space V decomposes as a tensor product $V = V_{1/2} x V'$, for some vertex subalgebra V' generated by primary elements of conformal weight $\Delta \neq 1/2$.

  3. We can thus restrict ourselves to consider vertex algebras strongly generated by a Virasoro element and primary elements of conformal weight $\Delta = 1, 3/2, \ldots$.

4. The easiest situation, in which V is generated by L and primary elements of conformal weight $\Delta = 1$, is “trivial”. Indeed R is semidirect product of the current Lie conformal algebra and Vir, and V is a quotient of the enveloping vertex algebra $U(R)$.
5. In conclusion, the first “non trivial” situation is when the vertex algebra V is strongly generated by a Virasoro element and primary elements of conformal weight 1 and 3/2.

- This is the situation we want to consider. For us “low” means $\Delta = 1, 3/2$.

Main Problem

Classify vertex algebras V finitely generated by the space

$$R = \mathbb{C}[T](\mathbb{C}|0\rangle \oplus \mathfrak{g} \oplus U \oplus \mathbb{C}L)$$

where

1. $|0\rangle$ is the vacuum element
2. $\mathfrak{g}$ is a space of even primary els of $\Delta = 1$,
3. U is a space of odd primary els of $\Delta = \frac{3}{2}$,
4. L is a Virasoro element.

Basic Remark. Suppose V is a vertex algebra generated by R.

- In general, the λ–bracket, restricted to R, is

$$[R\ {}_\lambda\ R] \subset V[\lambda] = (R+ : RR : + \dots)[\lambda]$$

- If we allow only $\Delta = 1$, we just have

$$[R\ {}_\lambda\ R] \subset V[\lambda] = R[\lambda]$$

  Namely, R is a Lie conformal algebra.

- If we allow $\Delta = 1, 3/2$ we have

$$[R\ {}_\lambda\ R] \subset (R+ : RR :)[\lambda]$$

  Namely, the λ–bracket on R has *quadratic "non linearities"*.

Comments:

We introduce the notion of Lie conformal algebras with quadratic non linearities.

Then we need to prove two things:

1. classifying vertex algebras with above properties, is equivalent to classifying Lie conformal algebras with quadr non lin's. (This is almost true).
2. Classify Lie conformal algebras with quadr non lin's. (This is duable, in some special case, namely under assumptions on $\mathfrak{g}$ and U)

Definition. Let R be as above. A *Lie λ–bracket of degree 2* (or with "quadratic non linearities") on R is a map

$$[\ _\lambda\ ]\ :\quad R \times R \to (R \oplus R^{\otimes 2})[\lambda]$$

satisfying sesquilinearity, skewsymmetry and Jacobi identity^{Note}.

R is then called **Lie conf alg with quadr non lin's**.

Theorem1. *If R is a Lie conf alg with quadr non lin's, then there is an enveloping vertex algebra $U(R)$, generated by R, with compatible λ–bracket.*

Theorem2. *There exists a non degenerate vertex alg V generated by R if and only if R is a Lie conf alg with quadr non lin's.*

- Note: the triple λ–bracket in the Jacobi identity is not defined. We define it by using, when needed, the non commutative Wick formula.

- Theorem 1 is a generalization of a result for Lie conformal algebras: given any Lie conformal algebra there is an enveloping vertex algebra associated to it.

  Note: PBW theorem holds (one has a basis for $U(R)$).

- Theorem 1 solves Prob B in one direction: if we have a Lie λ–bracket of degree 2 on R, then we have a vertex algebra V generated by R.

  What about the other direction? If V is a vertex algebra generated by R, do we have a Lie λ–bracket of degree 2 on R?

  Obviously we have a map

  $$R \times R \to (R + :RR:)[\lambda]$$

  satisfying skewsymmetry and Jacobi identity

  But this is not quite a Lie λ–bracket of degree 2, since in general $:RR:$ is a quotient space of $R \otimes R$.

We need to ask a "non degeneracy" condition on V, which says the quotient map $R \otimes R \to : RR :$ has kernel not too large.

Classification results

Proposition. *R admits a Lie λ–bracket of deg 2 if and only if:*

- *$\mathfrak{g}$ is a Lie algebra,*
- *U is a $\mathfrak{g}$–module,*
- *there exist $\mathfrak{g}$-module homomorphisms*

$$\varkappa \;:\; S^2\mathfrak{g} \to \mathbb{C}\,, \quad Q \;:\; S^2U \to \mathbb{C}\,,$$
$$K \;:\; \Lambda^2 U \to \mathfrak{g}\,, \quad P \;:\; S^2U \to S^2\mathfrak{g}\,,$$

satisfying a bunch of equations.

Next: Need to look at the equations and classify all possible $(\mathfrak{g}, U)$. Solution in two special cases:

case 1 $\mathfrak{g}$ simple, U irreducible, $\varkappa$, Q non degenerate,

case 2 $\mathfrak{g}$ reductive, U any repr, $\varkappa$, Q non degenerate, if R admits a "quasi–classical limit".

We can now look at the analogue of Problem 2A: classify Lie conformal algebras with quadratic non linearities

- Just by looking at Jacobi identity, one gets Proposition (it's just a simple computation). I didn't write the equations, since they are a bit complicated and not enlightening.

- We have very explicit conditions to look at. We can then classify all pairs $(\mathfrak{g}, U)$ satisfying all conditions in the Proposition.

- I found a solution in two special cases:

    1. $\mathfrak{g}$ simple and U irreducible (Theorem 3)
    2. $\mathfrak{g}$ reductive, U any, and R admits a quasi–classical limit

Case 1

Theorem3. *A complete classification of conformal algebras with quadratic non linearities R, such that $\mathfrak{g}$ is a simple Lie algebra, U is an irreducible $\mathfrak{g}$–module and the bilinear forms $\varkappa$ and Q are non degenerate, is given by the following list*

$\mathfrak{g}$	U
so_n	$\mathbb{C}^n$
B_3	π_3
G_2	π_1

Case 2

Definition. We say that a Lie conf alg with quadr non lin's R admits a quasi–classical limit if $S(R)$ has a compatible Poisson vertex algebra structure.

Proposition. *If R admits quasi–classical limit (and $\varkappa$ and Q are non degenerate), then the connected complex algebraic group G associated to $\mathfrak{g}$ acts transitively on the quadric $S^2 = \{v \in U \mid Q(v,v) = 1\} \subset U$.*

Theorem4. *A complete classification of conformal algebras with quadratic non linearities R, such that $\mathfrak{g}$ is a reductive Lie algebra, U is any $\mathfrak{g}$–module and the bilinear forms $\varkappa$ and Q are non degenerate, and such that R admits quasiclassical limit, is given by the following list*

$\mathfrak{g}$	U
so_n	$\mathbb{C}^n,\ n \geq 3, n \neq 4$
B_3	π_3
G_2	π_1
sl_2	$\mathbb{C}^2 \oplus C^2$
gl_n	$\mathbb{C}^n \oplus \mathbb{C}^{n,*},\ n \neq 2$
$\mathrm{sp}_n \oplus \mathrm{sp}_2$	$\mathbb{C}^n \otimes \mathbb{C}^2,\ n \geq 2$

- We now consider the special case in which the vertex algebra $U(R)$ admits a quasi–classical limit.

  Remarks:

  1. A poisson vertex algebra is the same as a vertex algebra, for which the normal order product is associative (not quasi–associative), symmetric (not skewsymmetric) and satisfies Leibniz rule (not non commutative Wick formula).
  2. The requirement that R admits quasi–classical limit is (in some sense) equivalent to say that the value of the Kac–Moody level is arbitrary.
  3. If R is a Lie conformal algebra, then it admits quasi–classical limit (it's a simple fact). But if there are quadratic non linearities, this is not always the case.

- The main observation is that, if R admits quasi classical limit, then $\mathfrak{g}$ acts transitively on the quadric $S^2 \subset U$. This fact was observed for physical conformal algebras by V.Kac

- Using this fact, we can classify Lie conformal algebras with quadr non lin's admitting quasi classical limit (Theorem 4)