Thesis defence

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4:00–5:00 pm

MIT, room 1-150

Vertex algebras generated by primary fields of low conformal weight

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Slides available from

`http://www-math.mit.edu/~desole/`

Outline

- Notation and Definitions.
  - (i) conformal algebras & vertex algebras
  - (ii) conformal weight, primary fields
  - (iii) “generated”, “low”
- State the Problem of Classification
- Main Results
  - (i) Existence of the Enveloping vertex algebra
  - (ii) Classification results

Part 1: Definitions

- I will start definining every word in the title:
- there are left “by” & “of”. I assume the audience knows the meaning of these two words
- Then I will state the problem of classification, and the main results.

Historical remarks

Vertex algebras were introduced in 1986 by R. Borcherds. They provide a rigorous definition of the chiral part of 2–dimensional conformal field theory, intensively studied by physicists. They have important applications to physics, in string theory and conformal field theory, and to mathematics, by providing tools to study the most interesting representations of infinite dimensional Lie algebras.

Conformal algebras were introduced more recently (1995) by V. Kac. They give an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory.

There are several equivalent definitions of vertex algebras.

1. The first definition is due to R. Borcherds, in 1986. It is given in terms of (quite complicated) identities known as "Borcherd's identities".

2. By far the easiest and the best (in my opinion) definition of vertex algebras is in terms of "local fields" (it also has a much more natural physical interpretation). This is due to V. Kac (1996).

3. In the same time, Kac introduced "conformal algebras". They are a generalization of vertex algebras, in the sense that every vertex algebra is automatically a conformal algebra.

   - The structure of conformal algebra is simpler (one asks fewer axioms)

   Note:
   (a) If one considers simple, finite (= finite dimensional) Lie conformal algebras, they have been completely classified.
   (b) Vertex algebras are (almost) always infinite Lie conformal algebras. In this respect they are

more complicated objects to study (the problem of their classification, in any reasonable sense, is far from being solved).

4. There is yet another definition of vertex algebras. (the equivalence was proved last year from B. Bakalov and V. Kac) It shows better the relation between v.a. and L.c.a. It is the best way to think about v.a.'s, in order to understand my thesis.

goal:
We want to define Lie conformal algebras and vertex algebras.

good way to think at them:

- Lie conformal algebras are similar to Lie algebras, (intuitively, they are the same thing as Lie algebras, with a derivation T, and with Lie bracket depending from a parameter l)
- vertex algebras are similar to associative algebras (with a derivation T).

next:

1. we recall the definition of Lie (super)algebra
2. recall the definition of associative algebra
3. define Lie conformal super algebras
4. define vertex algebra

Def's of Lie conformal alg's and vertex algebras

Definition. A Lie superalgebra is

- a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$
- endowed with a Lie–bracket, namely a parity preserving linear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, denoted by,

$$a \otimes b \mapsto [a, b] \ \in \ \mathfrak{g}$$

satisfying the following axioms ($a, b, c \in R$).

(i) skewsymmetry

$$[b, a] \ = \ -p(a, b)[a, b]$$

(ii) Jacobi identity

$$[a, [b, c]] - p(a, b)[b, [a, c]] \ = \ [[a, b], c]$$

* Notation. $p(a)$: parity of a, $p(a, b) = (-1)^{p(a)p(b)}$.

Definition. A superalgebra is

- a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$
- with a parity preserving product $a \cdot b$

A unity is an even element $1 \in V$ such that

$$1 \cdot a \;=\; a \cdot 1 \;=\; a$$

A derivation, is an even endomorphism $T : V \to V$ satisfying Leibniz rule

$$T(a \cdot b) \;=\; (Ta) \cdot b + a \cdot (Tb)$$

In this case, V is called a differential algebra.

V is said to be associative if

$$a \cdot (b \cdot c) \;=\; (a \cdot b) \cdot c \;.$$

Remark. If V is associative, then it is a Lie algebra, with Lie–bracket

$$[a, b] \;=\; a \cdot b - p(a, b) b \cdot a$$

Definition. A Lie conformal superalgebra is

- a vector superspace $R = R_{\bar{0}} \oplus R_{\bar{1}}$
- with an even endomorphism T
- endowed with a parity preserving λ–bracket

$$a \otimes b \mapsto [a\ {}_\lambda\ b] \ \in \ R[\lambda]$$

satisfying the following axioms ($a, b, c \in R$).

(i) sesquilinearity

$$[Ta\ {}_\lambda\ b] \quad = \quad -\lambda[a\ {}_\lambda\ b]$$

$$[a\ {}_\lambda\ Tb] \quad = \quad (\lambda + T)[a\ {}_\lambda\ b]$$

(ii) skewsymmetry

$$[b\ {}_\lambda\ a] \ = \ -p(a,b)[a\ {}_{-\lambda-T}\ b]$$

(iii) Jacobi identity

$$[a\ {}_\lambda\ [b\ {}_\mu\ c]] - p(a,b)[b\ {}_\mu\ [a\ {}_\lambda\ c]]$$

$$= \ [[a\ {}_\lambda\ b]\ {}_{\lambda+\mu}\ c]$$

- from sesquilinearity, T is a derivation of the l-bracket
- In the skewsymmetry relation, [a -l-T b] means that we have
- to replace in [a m b] the indeterminant m with the
- operator (-l-T), acting from the left.
- We call rank of a Lie conformal algebra its rank as C[T] module. R is said finite if it is of finite rank.

Definition. A vertex superalgebra is a superspace V, endowed with two operations:

A) a λ–bracket $[a\ {}_\lambda\ b]$, which makes it a Lie conformal superalgebra,

B) a normal order product $:ab:$, which makes it an algebra, with unity $|0\rangle \in V$ (vacuum) and derivation T (infinitesimal translation operator)

They satisfy the following axioms:

(i) quasi–associativity

$$:(:ab:)c: - :a(:bc:): = :\left(\int_0^T d\lambda a\right)[b\ {}_\lambda\ c]: \\ + p(a,b) :\left(\int_0^T d\lambda b\right)[a\ {}_\lambda\ c]:$$

(ii) skewsymmetry of the normal ordered product

$$:ab: - p(a,b) :ba: = \int_{-T}^{0} d\lambda [a\ {}_\lambda\ b]$$

(iii) non commutative Wick formula

$$[a\ {}_\lambda\ :bc:] = :[a\ {}_\lambda\ b]c: + p(a,b) :b[a\ {}_\lambda\ c]: \\ + \int_0^\lambda d\mu [[a\ {}_\lambda\ b]\ {}_\mu\ c]\ .$$

- A proof of the equivalence between this definition and the more familiar one (in terms of local fields) is due to Bojko Bakalov and Victor Kac (2002)

- Since $T|0\rangle = 0$, we immediately get that $|0\rangle$ is central with respect to l-bracket

- Note: from now on I will drop "super"

Examples of Lie conformal algebras

Example. Let $\mathfrak{g}$ be a Lie algebra, with a simmetric invariant bilinear form (a, b). The Current Lie conf alg is

$$\mathrm{Cur}\mathfrak{g} \;=\; (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}|0\rangle$$

with λ–bracket given by ($|0\rangle$ central)

$$[a\ {}_\lambda\ b] = [a, b] + \lambda k(a, b)|0\rangle\ , \quad a, b \in \mathfrak{g}$$

$k \in \mathbb{C}$ is called Kac–Moody level.

Example. The Virasoro Lie conformal algebra is

$$Vir \;=\; (\mathbb{C}[T]L) \oplus \mathbb{C}|0\rangle$$

with λ–bracket

$$[L\ {}_\lambda\ L] = (T + 2\lambda)L + \frac{c}{12}\lambda^3|0\rangle$$

$c \in \mathbb{C}$ is called central charge.

Remark. We have the corresponding enveloping vertex algebras $U(\mathrm{Cur}\mathfrak{g})$ and $U(Vir)$.

- These are examples of Lie conformal algebras.
- What about vertex algebras? Continuing the analogy with Lie algebras, we know that to every Lie algebra we can associate a universal enveloping algebra, which is an associative unital (infinite dim) algebra. In the same way, given a Lie conformal algebra R, one can construct an "enveloping" vertex algebra U(R). So each example of conf algebra produces an example of vertex algebra.

- It is natural from the physics point of view, to ask that a vertex algebra V ("the space of states", in physics) is a representation of the Virasoro Lie algebra, with eigenvalues of L_0 (=energy operator) bounded from below.
- This is guaranteed if we ask that V has a Virasoro element

Next:

1. Definition of Virasoro element, and why it give a repr of Vir Lie algebra,
2. Def. of conformal weight and primary elements.
3. Def of "generated" and "low"

Other definitions

Definition. A Virasoro element of a vertex algebra V is an even element L such that

1. $[L\ {}_\lambda\ L] = (T + 2\lambda)L + \frac{c}{2}\lambda^{(3)}|0\rangle$,
2. $L_{-1} = [L\ {}_\lambda\ \cdot]\Big|_{\lambda=0} = T$,
3. $L_0 = \frac{d}{d\lambda}[L\ {}_\lambda\ \cdot]\Big|_{\lambda=0}$ is diagonalizable on V.

The number c is called the central charge of L.

Remark. Let L be a Virasoro element of a vertex algebra V. Then we have an action of the Virasoro Lie algebra Vir on V, given by $(m \geq 0)$

$$\begin{aligned} L_{m-1}(a) &= \frac{d^m}{d\lambda^m}[L\ {}_\lambda\ a]\Big|_{\lambda=0} \\ L_{-m-2}(a) &= \frac{1}{m!} :(T^m L)a: \end{aligned}$$

Namely, the operators $L_m,\ m \in \mathbb{Z}$ satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}\mathbb{1}$$

Definition. Eigenvalues of L_0 are called conformal weights. Namely a has conformal weight Δ if

$$[L\ {}_\lambda\ a] \;=\; Ta + \Delta\lambda a + O(\lambda^2)$$

An element $a \in V$ is called a primary element of conformal weight Δ if

$$[L\ {}_\lambda\ a] \;=\; Ta + \Delta\lambda a$$

Or, equivalently, $L_n(a) = 0$ for $n > 0$.

Definition. A vertex algebra V is said to be (strongly) generated by a collection of elements $A = \{a^\alpha,\ \alpha \in \mathcal{A}\} \subset V$, if it is spanned by $|0\rangle$ and elements

$$:(T^{j_1}a^{\alpha_1})\cdots(T^{j_n}a^{\alpha_n}): , \qquad \alpha_i \in \mathcal{A}\ ,\ j_i \geq 0$$

Remark. By physical requests: $\Delta \in \frac{1}{2}\mathbb{Z}$. Moreover, the case $\Delta = 1/2$ can be studied separately.

Definition. "low" means 1 or 3/2.

- To say that a is primary, means that $L_n(a) = 0$ for $n > 0$. This means that a is a highest weight vector of Vir, with highest weight (c, Δ).

- V is generated by A, if V is obtained by applying T to, and by taking ordered products of, elements of A. For example, the enveloping vertex algebra $U(R)$ over the conformal algebra R is (strongly) generated by R.

- To say that V is strongly generated by primary elements is equivalent to say that the representation of the Virasoro Lie algebra on V decomposes as direct sum of highest weight representations (it is in cathegory O).

- In particular, this assumption is automatically satisfied if we require that the representation of the Virasoro Lie algebra on V is unitary. (which is a natural assumption from the physics point of view)

- Moreover, if we require that the vertex algebra V is unitary (namely the adjoint representation of V on itself is a unitary representation), it follows that the conformal weights have non negative semi–integer values.

- From spin-statistics theorem, we want

$$\Delta_a \in \mathbb{Z} \to a \text{ even}$$
$$\Delta \in 1/2(2\mathbb{Z}+1) \to a \text{ odd}$$

- Let us now examine the smallest possible values of the conformal weight $\Delta \in 1/2\mathbb{Z}_+$.

  1. $\Delta = 0$ corresponds to the vacuum vector $|0\rangle$.

  2. Primary elements of conformal weight $\Delta = 1/2$ generate a Fermion subalgebra $V_{1/2}$. In particular the vertex algebra V becomes a representation of the Clifford algebra. It follows by representation theory of the Clifford algebra, that the space V decomposes as a tensor product $V = V_{1/2} x V'$, for some vertex subalgebra V' generated by primary elements of conformal weight $\Delta \neq 1/2$.

  3. We can thus restrict ourselves to consider vertex algebras strongly generated by a Virasoro element and primary elements of conformal weight $\Delta = 1, 3/2, \ldots$.

4. The easiest situation, in which V is generated by L and primary elements of conformal weight $\Delta = 1$, is "trivial". Indeed R is semidirect product of the current Lie conformal algebra and Vir, and V is a quotient of the enveloping vertex algebra $U(R)$.
5. In conclusion, the first "non trivial" situation is when the vertex algebra V is strongly generated by a Virasoro element and primary elements of conformal weight 1 and 3/2.

- This is the situation we want to consider. For us "low" means $\Delta = 1, 3/2$.

Main Problem

Classify vertex algebras V finitely generated by the space

$$R = \mathbb{C}[T](\mathbb{C}|0\rangle \oplus \mathfrak{g} \oplus U \oplus \mathbb{C}L)$$

where

1. $|0\rangle$ is the vacuum element
2. $\mathfrak{g}$ is a space of even primary els of $\Delta = 1$,
3. U is a space of odd primary els of $\Delta = \frac{3}{2}$,
4. L is a Virasoro element.

Part 2: Explain the Problem

Basic Remark. Suppose V is a vertex algebra generated by R.

- In general, the λ–bracket, restricted to R, has values

$$[R\ {}_\lambda\ R] \subset V[\lambda] = (R + :RR: + \dots)[\lambda]$$

- By our assumptions on conformal weights, it follows that

$$[R\ {}_\lambda\ R] \subset (R + :RR:)[\lambda]$$

Namely, there are at most quadratic "non linearities".

Note: In the easiest situation $[R\ {}_\lambda\ R] \subset R[\lambda]$, R is a "physical" Lie conformal algebra. (Classified by V. Kac in 1997). We consider the first generalization.

We want to stress the differences between

1. The “Main Problem”,
2. and the problem of classifying “physical” Lie conformal algebra

(It’s not a straightforward generalization)

To better understand the difference between these two problems, we consider again the analogy with Lie algebras (which is problably more familiar to the audience).

1. The problem of classifying “Physical” Lie conformal algebras, is analogous to the problem of classifying fin dim Lie algebras $\mathfrak{g}$, (analogue to Lie conf alg) satisfying certain assumptions (For example, of given dimension).
    - This is usually duable. To solve: write down Jacobi identity, and see what constraints it gives on the elements of $\mathfrak{g}$.
    - It can be hard in practice, but at least it’s easy to understand what to do.
2. Consider no the analogue of the “Main Problem”.

- is analogous to the problem of classifying spaces $\mathfrak{g}$ (analogue to the generating set R) which are embedded in an infinite dim associative unital algebra A (analogue to the vertex algebra V), and such that the Lie bracket on A, restricted to $\mathfrak{g}$, has at most "quadratica non linearities": $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} + \mathfrak{g}\mathfrak{g}$.
- In other words, $\mathfrak{g}$ is not quite a Lie algebra, but almost, since it has a Lie bracket, but the values of $[a,b]$ for $a, b \in \mathfrak{g}$ is an expression of degree 2 in $\mathfrak{g}$.
- Main idea: forget about A and think only about $\mathfrak{g}$. What can we say about $\mathfrak{g}$? It is endowed with a Lie bracket (satisfying skewsymmetry and Jacobi identity), but it's not closed under the Lie bracket:

  $$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} + \mathfrak{g}\mathfrak{g}$$

  If we don't have A, what do we mean by $\mathfrak{g}\mathfrak{g}$. We can think at it as $\mathfrak{g}^{\otimes 2}$.

In conclusion, to classify such objects, we need to do two things:

1. Classify all spaces $\mathfrak{g}$ endowed with a Lie bracket

of degree 2, namely such that

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$$

and satisfying skewsymmetry and Jacobi idenitity.

Remarks:

- This is a practical problem: write down Jacobi idenity and see what restrictions we get.
- It's much harder than in the Lie algebra setting, since in the Jacobi idenity there are many more terms.

2. Prove that the following facts are equivalent:

    (a) there exists of an associative unital algebra A generated by $\mathfrak{g}$ and such that the Lie bracket of A, restricted to $\mathfrak{g}$ has at most quadratic non linearities,

    (b) $\mathfrak{g}$ admits a Lie bracket of degree 2.

Note: it's not clear how to do this. And it's not true! (one needs some extra assumptions)

Problems:

- In one direction: once you have $\mathfrak{g}$, how do you reconstruct A? (Idea: a construction similar to that of the universal enveloping algebra $U(\mathfrak{g})$)
- In the other direction, $\mathfrak{g}\mathfrak{g}$ is NOT the same thing as $\mathfrak{g}^{\otimes 2}$. (One needs some non degenracy assumption)

Analogy with Lie algebras

Problem 1. Classify fin dim Lie algebras $\mathfrak{g}$ (analogue of R, Lie conf alg) of some kind.

Problem 2. Classify spaces $\mathfrak{g}$ (analogue of R) such that $\mathfrak{g} \subset A$ where:

- A is an infinite dimensional unital associative algebra, (analogue to the v.a. V)
- A is generated by $\mathfrak{g}$, namely $A = \mathcal{T}(\mathfrak{g})/(\text{ideal})$,
- the Lie bracket on A ($[a,b] = ab - ba$), restricted to $\mathfrak{g}$, is such that

  $$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} + \mathfrak{g}\mathfrak{g}$$

  Namely there are "quadratic non–linearities" ($\mathfrak{g}$ is NOT a Lie algebra).

Ideas:

- Forget A, consider the algebra structure of $\mathfrak{g}$. It has a "Lie bracket" of kind $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} + \mathfrak{g}\mathfrak{g}$.
- Replace $\mathfrak{g}\mathfrak{g}$ by $\mathfrak{g}^{\otimes 2}$. We define a Lie bracket of degree 2 on $\mathfrak{g}$ to be a map

$$[\ ,\ ]: \ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$$

  satisfying skewsymmetry and Jacobi identity.

Things to do:

Problem 2A Classify all spaces $\mathfrak{g}$ which admit a Lie bracket of degree 2. (Write down Jacobi identity and see what restrictions you get)

Problem 2B Prove that:

- there exists A satisfying all assumptions of Problem 2, if and only if
- $\mathfrak{g}$ admits a Lie bracket of degree 2.

Part 3: Results

First, let's see the solution to Problem 2B.

Existence of the Enveloping Vertex Algebra

Definition. Let R be as above. A Lie λ–bracket of degree 2 (or with "quadratic non linearities") on R is a map

$$[\ {}_\lambda\ ]\ :\quad R \times R \to (R \oplus R^{\otimes 2})[\lambda]$$

satisfying sesquilinearity, skewsymmetry and Jacobi identity ^{Note}.

R is then called Lie conf alg with quadr non lin's.

Theorem1. *If R is a Lie conf alg with quadr non lin's, then there is an enveloping vertex algebra $U(R)$, generated by R, with compatible λ–bracket.*

Theorem2. *There exists a non degenerate vertex alg V generated by R if and only if R is a Lie conf alg with quadr non lin's.*

- Note: the triple λ–bracket in the Jacobi identity is not defined. We define it by using, when needed, the non commutative Wick formula.

- Theorem 1 is a generalization of a result for Lie conformal algebras: given any Lie conformal algebra there is an enveloping vertex algebra associated to it.

  Note: PBW theorem holds (one has a basis for $U(R)$).

- Theorem 1 solves Problem 2B in one direction: if we have a Lie λ–bracket of degree 2 on R, then we have a vertex algebra V generated by R.

  What about the other direction? If V is a vertex algebra generated by R, do we have a Lie λ–bracket of degree 2 on R?

  Obviously we have a map

$$R \times R \to (R + :RR:)[\lambda]$$

  satisfying skewsymmetry and Jacobi identity

  But this is not quite a Lie λ–bracket of degree 2, since in general $:RR:$ is a quotient space of $R \otimes R$.

We need to ask a "non degeneracy" condition on V, which says the quotient map $R \otimes R \to : RR :$ has kernel not too large.

Classification results

Proposition. *R admits a Lie λ–bracket of deg 2 if and only if:*

- *$\mathfrak{g}$ is a Lie algebra,*
- *U is a $\mathfrak{g}$–module,*
- *there exist $\mathfrak{g}$-module homomorphisms*

$$\varkappa \;:\; S^2\mathfrak{g} \to \mathbb{C}\ , \quad Q \;:\; S^2U \to \mathbb{C}\ ,$$
$$K \;:\; \Lambda^2 U \to \mathfrak{g}\ , \quad P \;:\; S^2U \to S^2\mathfrak{g}\ ,$$

satisfying a bunch of equations.

Next: Need to look at the equations and classify all possible $(\mathfrak{g}, U)$. Solution in two special cases:

case 1 $\mathfrak{g}$ simple, U irreducible, $\varkappa$, Q non degenerate,

case 2 $\mathfrak{g}$ reductive, U any repr, $\varkappa$, Q non degenerate, if R admits a "quasi–classical limit".

We can now look at the analogue of Problem 2A: classify Lie conformal algebras with quadratic non linearities

- Just by looking at Jacobi identity, one gets Proposition (it's just a simple computation). I didn't write the equations, since they are a bit complicated and not enlightening.
- We have very explicit conditions to look at. We can then classify all pairs $(\mathfrak{g}, U)$ satisfying all conditions in the Proposition.
- I found a solution in two special cases:
    1. $\mathfrak{g}$ simple and U irreducible (Theorem 3)
    2. $\mathfrak{g}$ reductive, U any, and R admits a quasi–classical limit

Case 1

Theorem3. *A complete classification of conformal algebras with quadratic non linearities R, such that $\mathfrak{g}$ is a simple Lie algebra, U is an irreducible $\mathfrak{g}$–module and the bilinear forms $\varkappa$ and Q are non degenerate, is given by the following list*

$\mathfrak{g}$	U
so_n	$\mathbb{C}^n$
B_3	π_3
G_2	π_1

Case 2

Definition. We say that a Lie conf alg with quadr non lin's R admits a quasi–classical limit if $S(R)$ has a compatible Poisson vertex algebra structure.

Proposition. *If R admits quasi–classical limit (and $\varkappa$ and Q are non degenerate), then the connected complex algebraic group G associated to $\mathfrak{g}$ acts transitively on the quadric $S^2 = \{v \in U \mid Q(v, v) = 1\} \subset U$.*

<u>Theorem4</u>. *A complete classification of conformal algebras with quadratic non linearities R, such that $\mathfrak{g}$ is a reductive Lie algebra, U is any $\mathfrak{g}$–module and the bilinear forms $\varkappa$ and Q are non degenerate, and such that R admits quasiclassical limit, is given by the following list*

$\mathfrak{g}$	U
so_n	$\mathbb{C}^n,\ n \geq 3, n \neq 4$
B_3	π_3
G_2	π_1
sl_2	$\mathbb{C}^2 \oplus C^2$
gl_n	$\mathbb{C}^n \oplus \mathbb{C}^{n,*},\ n \neq 2$
$\mathrm{sp}_n \oplus \mathrm{sp}_2$	$\mathbb{C}^n \otimes \mathbb{C}^2,\ n \geq 2$

- We now consider the special case in which the vertex algebra $U(R)$ admits a quasi–classical limit.

  Remarks:

  1. A poisson vertex algebra is the same as a vertex algebra, for which the normal order product is associative (not quasi–associative), symmetric (not skewsymmetric) and satisfies Leibniz rule (not non commutative Wick formula).
  2. The requirement that R admits quasi–classical limit is (in some sense) equivalent to say that the value of the Kac–Moody level is arbitrary.
  3. If R is a Lie conformal algebra, then it admits quasi–classical limit (it's a simple fact). But if there are quadratic non linearities, this is not always the case.

- The main observation is that, if R admits quasi classical limit, then $\mathfrak{g}$ acts transitively on the quadric $S^2 \subset U$. This fact was observed for physical conformal algebras by V.Kac

- Using this fact, we can classify Lie conformal algebras with quadr non lin's admitting quasi classical limit (Theorem 4)

Notice that all the examples listed in Theorem 3 admit quasi classical limit, namely they are also listed in Theorem 4.

It is natural to ask:

question Is there any vertex algebra V generated by R which does not admit quasi classical limit?

answer Yes (if we remove the "non degeneracy" assumption)

(But this is still work in progress...)

Question: Is there any vertex algebra V generated by R which does not admit quasi classical limit?

Answer: Yes (if we remove the "non degeneracy" assumption)