Vertex algebras generated by primary fields of low conformal weight

by

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Abstract

We classify a certain class of vertex algebras, finitely generated by a Virasoro field, even primary fields of conformal weight 1 and odd primary fields of conformal weight 3/2. This is the first interesting case to consider when looking at finitely generated vertex algebras containing a Virasoro field (the most interesting from the point of view of physics).

By the axioms of vertex algebras it follows that the space $\mathfrak{g}$ of fields with conformal weight 1 is a Lie algebra, and the space U of fields with conformal weight 3/2 is a $\mathfrak{g}$–module with a symmetric invariant bilinear form.

One of the main observations is that, under the assumption of existence of a quasi–classical limit (which basically translates to the existence of a one parameter family of vertex algebras, the free parameter being the Kac–Moody level k), the complex connected algebraic group G corresponding to the Lie algebra $\mathfrak{g}$ acts transitively on the quadric $S^2 = \{u \in U \text{ s.t. } (u,u) = 1\} \subset U$. This generalizes a similar result of Kac in the case of conformal algebras. Using this observation, we will classify vertex algebras satisfying the above assumptions, by using the classification of connected compact subgroups of SO_N acting transitively on the unit sphere. The solution is given by the following list:

- $\mathfrak{g} = \mathrm{so}_n,\ U = \mathbb{C}^n$, for $n \geq 3$,
- $\mathfrak{g} = \mathrm{gl}_n,\ U = \mathbb{C}^n \oplus \mathbb{C}^{n,*}$, for $n \geq 1, n \neq 2$,
- $\mathfrak{g} = \mathrm{sl}_2,\ U = \mathbb{C}^2 \oplus \mathbb{C}^{2,*}$,
- $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2,\ U = \mathbb{C}^n \otimes \mathbb{C}^2 \quad n \geq 2$,
- $\mathfrak{g} = B_3,\ U = V_{\pi_3} = \mathrm{Spin}_7$,
- $\mathfrak{g} = G_2,\ U = V_{\pi_1}$.

However, if one removes the assumption of existence of quasi–classical limit, the above argument fails and the problem of classification has to be studied using different techniques. In the case in which $\mathfrak{g}$ is a simple Lie algebra and U an irreducible $\mathfrak{g}$–module, we will prove, under some weak technical assumption, that no examples with "discrete" values of the Kac–Moody level appear.

Thesis Supervisor: Victor G. Kac
Title: Professor of Mathematics

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Chapter 1

Introduction

Vertex algebras were introduced about 15 years ago by Richard Borcherds [3]. They provide a rigorous definition of the chiral part of 2–dimensional conformal field theory, intensively studied by physicists. Since then they had important applications to physics, in string theory and conformal field theory, and to mathematics, by providing tools to study the most interesting representations of infinite dimensional Lie algebras.

Conformal algebras were introduced more recently by Victor Kac. They give an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. To some extent, conformal algebras are related to vertex algebras in the same way Lie algebras are related to associative algebras. This is more than just an analogy. In fact, any conformal algebra R has a Lie bracket, and the corresponding universal enveloping algebra $U(R)$ is naturally endowed with a structure of vertex algebra, which is then called the "enveloping vertex algebra" over R.

From a mathematical point of view, the theory of vertex algebras is much richer than the related theory of finite conformal algebras. Indeed in a relatively short period of time, the theory of finite conformal algebras has been extensively developed by Kac and his collaborators, and a complete classification of finite simple Lie conformal superalgebras was found [5, 6]. However, a similar structure theory for vertex algebras is far from being known. This thesis approaches this problem by studying vertex algebras which are generated by a Virasoro field and primary fields of conformal weight 1 and 3/2.

In the next section we will give the definitions of Lie conformal algebra and vertex algebra, needed throughout the thesis. In Section 1.2 we will then describe the problem studied in the thesis and state the main results.

1.1 Definitions of Lie conformal algebras and vertex algebras

Definition 1.1.1. A *Lie conformal superalgebra* is (see [12]) a $\mathbb{Z}_2$–graded $\mathbb{C}[T]$-module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a $\mathbb{C}$-linear map $R \otimes R \to \mathbb{C}[\lambda] \otimes R$ denoted by $a \otimes b \mapsto [a\ {}_\lambda\ b]$

and called λ-bracket, satisfying the following axioms

$$[Ta\ {}_\lambda\ b] = -\lambda[a\ {}_\lambda\ b]\ , \quad [a\ {}_\lambda\ Tb] = (\lambda+T)[a\ {}_\lambda\ b] \qquad \text{(sesquilinearity)}$$

$$[b\ {}_\lambda\ a] = -p(a,b)[a\ {}_{-\lambda-T}\ b] \qquad \text{(skewsymmetry)}$$

$$[a\ {}_\lambda\ [b\ {}_\mu\ c]] - p(a,b)[b\ {}_\mu\ [a\ {}_\lambda\ c]] = [[a\ {}_\lambda\ b]\ {}_{\lambda+\mu}\ c] \qquad \text{(Jacobi identity)}$$

for $a,b,c \in R$. We denoted $p(a,b) = (-1)^{p(a)p(b)}$. Here and further $\otimes$ stands for the tensor product of vector spaces over $\mathbb{C}$. In the skewsymmetry relation, $[a\ {}_{-\lambda-T}\ b]$ means that we have to replace in $[a\ {}_\lambda\ b]$ the indeterminate λ with the operator $(-\lambda-T)$, acting from the left. We call *rank* of a Lie conformal algebra its rank as $\mathbb{C}[T]$ module. For brevity, we will sometimes drop the prefix super in superalgebra.

For a Lie conformal algebra we can define a $\mathbb{C}$-bilinear product $R\otimes R \to R$ for any $n \in \mathbb{Z}_+ = \{0,1,2,\cdots\}$, denoted by $a\otimes b \mapsto a_{(n)}b$ and given by

$$[a\ {}_\lambda\ b] = \sum_{n\in\mathbb{Z}_+} \lambda^{(n)} a_{(n)} b\ , \tag{1.1}$$

where we are using the notation: $\lambda^{(n)} := \frac{\lambda^n}{n!}$.

Particularly important in physics is the *Virasoro conformal algebra*. It is defined as the free $\mathbb{C}[T]$-module of rank 1 generated by an even element L, with λ-bracket defined by

$$[L\ {}_\lambda\ L] = (T+2\lambda)L \tag{1.2}$$

and extended to $\mathbb{C}[T]\otimes L$ using sesquilinearity.

Vertex algebras can be thought of as a special class of infinite–rank Lie conformal algebras.

Definition 1.1.2. A *vertex superalgebra* is a pair $(V,\ |0\rangle)$, where V is a $\mathbb{Z}_2$–graded $\mathbb{C}[T]$–module, called the *space of states*, and $|0\rangle$ is an element of V, called the *vacuum state*. It is endowed with two parity preserving operations: a λ–bracket $V\otimes V \to \mathbb{C}[\lambda]\otimes V$ which makes it a Lie conformal superalgebra, and a *normal order product* $V\otimes V \to V$, denoted by $a\otimes b \mapsto :ab:$, which makes it a unital differential algebra with unity $|0\rangle$ and derivative T. They satisfy the following axioms

(a) quasi–associativity

$$:(:ab:)c: - :a(:bc:): = :\left(\int_0^T d\lambda a\right)[b\ {}_\lambda\ c]: + p(a,b):\left(\int_0^T d\lambda b\right)[a\ {}_\lambda\ c]:\ ,$$

(b) skewsymmetry of the normal ordered product

$$:ab: - p(a,b):ba: = \int_{-T}^0 d\lambda [a\ {}_\lambda\ b]\ ,$$

(c) non commutative Wick formula

$$[a\ {}_\lambda\ :bc:]\ =\ :[a\ {}_\lambda\ b]c: + p(a,b) :b[a\ {}_\lambda\ c]: + \int_0^\lambda d\mu [[a\ {}_\lambda\ b]\ {}_\mu\ c]\ .$$

The normal order product of more than two elements is obtained by taking products starting on the right. In other words, for elements $a, b, c, \cdots \in V$, we will denote $:abc\cdots: = :a(:b(:c\cdots:):):$.

A proof of the equivalence between this definition of vertex algebra and the one given in [12] can be found in reference [1].

Remark 1.1.3. Since $|0\rangle$ is the unit element of the differential algebra (V, T), we have $T|0\rangle = 0$. Moreover, by sesquilinearity of the λ–bracket, it is easy to prove that the torsion of the $\mathbb{C}[T]$–module V is central with respect to the λ–bracket; namely, if $p(T)a = 0$ for some $p(T) \in \mathbb{C}[T]\backslash\{0\}$, then $[a\ {}_\lambda\ b] = 0$ for all $b \in V$. In particular the vacuum element is central: $[a\ {}_\lambda\ |0\rangle] = [|0\rangle\ {}_\lambda\ a] = 0$, $\forall a \in V$.

Remark 1.1.4. Using skewsymmetry (of both the λ–bracket and the normal ordered product) and non commutative Wick formula, one can prove the right Wick formula

$$\begin{aligned}[:ab:\ {}_\lambda\ c]\ &=\ :\left(e^{T\partial_\lambda}a\right)[b\ {}_\lambda\ c]: + p(a,b) :\left(e^{T\partial_\lambda}b\right)[a\ {}_\lambda\ c]: \\ &+\ p(a,b)\int_0^\lambda d\mu [b\ {}_\mu\ [a\ {}_{\lambda-\mu}\ c]]\ .\end{aligned}$$

The proof follows from a straightforward computation. This formula first appeared in [1].

For a vertex algebra we can define n–th products for every $n \in \mathbb{Z}$ in the following way. Since V is a Lie conformal algebra, all n-th products with $n \geq 0$ are already defined by (1.1). For $n \leq -1$ we define n–th product as $(k = -n-1 \geq 0)$

$$a_{(-k-1)}b = :(T^{(k)}a)b:\ . \tag{1.3}$$

By means of the n–th products, for every $n \in \mathbb{Z}$ we have a linear map $V \to \mathrm{End}V$ given by $a \mapsto a_{(n)}$. The following super commutation relation follows by definition of n–th product and by the axioms of vertex algebra (see for example [12]):

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}\ . \tag{1.4}$$

By definition every Vertex algebra is a Lie conformal algebra. On the other hand, given a Lie conformal algebra R there is a canonical way to construct a vertex algebra which contains R and is strongly generated by it (in the sense specified below). This result is stated in the following

Theorem 1.1.5. *Let R be a Lie conformal superalgebra with λ–bracket $[a\ {}_\lambda\ b]$. Let*

R_{Lie} be R considered as a Lie superalgebra with respect to the Lie bracket:

$$[a,b] = \int_{-T}^{0} d\lambda [a\ {}_\lambda\ b]\ , \qquad a,b \in R\ ,$$

and let $V = U(R_{Lie})$ be its universal enveloping algebra. Then there exists a unique structure of a vertex superalgebra on V such that the restriction of the λ–product to $R_{Lie} \times R_{Lie}$ coincides with the λ–product on R and the restriction of the normal ordered product to $R_{Lie} \times V$ coincides with the associative product of $U(R_{Lie})$.

The vertex algebra thus obtained is denoted by $V(R)$ and is called the *enveloping vertex algebra* associated to R. For a proof of this theorem, see [12] and [1, Th 7.12].

Definition 1.1.6. A *Virasoro* (or *conformal*) vector of a vertex algebra V is an even element L such that

(a) $[L\ {}_\lambda\ L] = (T + 2\lambda)L + \frac{c}{2}\lambda^{(3)}|0\rangle$,

(b) $L_{(0)} = T$,

(c) $L_{(1)}$ is diagonalizable on V.

The number c is called the *central charge* of L. A vertex algebra V endowed with a Virasoro vector L of central charge c is called a *conformal vertex algebra* of rank c.

It follows from (1.4) that if L is a Virasoro element, then the endomorphisms $L_n = L_{(n+1)} \in \mathrm{End}V$ satisfy the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}\ , \quad \forall m,n \in \mathbb{Z}\ .$$

In other words any conformal vertex algebra V of rank c is a representation of the Virasoro Lie algebra with central charge c, via the action $L_n(a) = L_{(n+1)}a$.

By definition of conformal vector, we ask that the operator $L_{(1)}$ is diagonalizable on V. Eigenvalues of $L_{(1)}$ are called *conformal weights*. Therefore if $a \in V$ is eigenvector of V with conformal weight Δ, we have the λ–bracket relation

$$[L\ {}_\lambda\ a] = T(a) + \Delta\lambda a + (\text{ terms of higher order in } \lambda\ )\ .$$

Definition 1.1.7. An element $a \in V$ is called *primary* of conformal weight Δ if there are no other terms in the above λ–bracket:

$$[L\ {}_\lambda\ a] = (T + \Delta\lambda)a\ .$$

The following are some obvious but very useful formulas related to conformal weights.

Lemma 1.1.8. *Suppose a and b are eigenvectors of $L_{(1)}$ with conformal weights $\Delta(a)$ and $\Delta(b)$ respectively. Then* *(a) Ta is eigenvector of $L_{(1)}$ with conformal weight*

$$\Delta(Ta) = \Delta(a) + 1 \ ,$$

(b) for every $n \in \mathbb{Z}$, $a_{(n)}b$ is eigenvector of $L_{(1)}$ with conformal weight

$$\Delta(a_{(n)}b) = \Delta(a) + \Delta(b) - n - 1 \ .$$

In particular $\Delta(:ab:) = \Delta(a) + \Delta(b)$.

The proof follows from a straightforward computation.

Definition 1.1.9. A vertex algebra is said to be *strongly generated* by a collection of elements $A = \{a^\alpha,\ \alpha \in \mathcal{A}\}$, if the vectors

$$a^{\alpha_1}_{(-j_1-1)} \cdots a^{\alpha_n}_{(-j_n-1)}|0\rangle \ , \qquad \alpha_i \in \mathcal{A} \ , \ j_i \geq 0 \ , \ i = 1, \dots, n \ ,$$

span the whole space of states V. In other words V is obtained by applying T to, and by taking ordered products of, elements of A.

1.2 Vertex algebras strongly generated by a Virasoro field and primary fields of conformal weight 1 and 3/2

The main purpose of the thesis is to study the following

Problem 1.2.1. Classify vertex algebras V which are strongly generated by the following elements:

1. the vacuum element $|0\rangle$,
2. a finite dimensional space $\mathfrak{g}$ of even primary elements of conformal weight $\Delta = 1$, called *currents*,
3. a finite dimensional space U of odd primary elements of conformal weight $\Delta = \frac{3}{2}$,
4. the Virasoro element L.

The assumptions of Problem 1.2.1 are motivated by the following considerations. From the point of view of physics, it is natural to require that the vertex algebra V is a representation of the Virasoro Lie algebra, namely it is a conformal vertex algebra with a Virasoro element $L \in V$. To say that V is strongly generated by primary elements is equivalent to say that the representation of the Virasoro Lie algebra on V decomposes as direct sum of highest weight representations. In this case, each primary element corresponds to the highest weight vector of a subrepresentation. In particular, this assumption is automatically satisfied if we require that the representation of the Virasoro Lie algebra on V is unitary. Moreover, if we require that the vertex algebra V is unitary (namely the adjoint representation of V on itself is a unitary representation), it follows that the conformal weights have non negative semi–integer values, [15]. Let us now examine the smallest possible values of the conformal weight $\Delta \in \frac{1}{2}\mathbb{Z}_+$. $\Delta = 0$ corresponds to the vacuum vector $|0\rangle \in V$. Primary elements of conformal weight $\Delta = 1/2$ generate a Fermion subalgebra $V_{1/2} \subset V$, [12]. In particular the vertex algebra V becomes a representation of the Clifford algebra. It follows by representation theory of the Clifford algebra, that the space V decomposes as a tensor product $V = V_{1/2} \otimes V'$, where $V' \subset V$ is a vertex subalgebra generated by primary elements of conformal weight $\Delta \neq 1/2$, see [12, Theorem 3.6], [9]. We can thus restrict ourselves to consider vertex algebras strongly generated by a Virasoro element and primary elements of conformal weight $\Delta = 1, 3/2, \dots$. The easiest situation, in which V is generated by L and primary elements of conformal weight $\Delta = 1$, is "trivial". Indeed in this case it follows by simple conformal weight considerations that the generating set $R = \mathbb{C}|0\rangle \oplus \mathbb{C}[T]\mathfrak{g} \oplus \mathbb{C}[T]L$ is the semidirect product of the current Lie conformal algebra and the Virasoro Lie conformal algebra, [12], so that V is a quotient of the enveloping vertex algebra $U(R)$. In conclusion, the first "non trivial" situation is when the vertex algebra V is strongly generated by a Virasoro element and primary elements of conformal weight 1 and 3/2.

A problem similar to Problem 1.2.1, though easier, has been studied in [13]. There is given a complete list of *physical* Lie conformal algebras which, by definition, contain a Virasoro element L and for which the set of even (resp. odd) elements is generated, as $\mathbb{C}[T]$–module, by primary elements of conformal weight $\Delta = 1$ (resp. $\Delta = 1/2, 3/2$).

We want to stress the difficulties arising from formulating Problem 1.2.1 in the context of vertex algebras rather than finite Lie conformal algebras. If we denote by R the $\mathbb{C}[T]$–module

$$R = \mathbb{C}[T](\mathbb{C}|0\rangle \oplus \mathfrak{g} \oplus U \oplus \mathbb{C}L) \ , \tag{1.5}$$

in both situations we want to define a bilinear λ–bracket $[a\ {}_\lambda\ b]$ for element $a, b \in R$, satisfying the axioms of Lie conformal algebra. When classifying physical Lie conformal algebras, one asks that R is itself a Lie conformal algebra, therefore closed under the λ–bracket; namely $[a\ {}_\lambda\ b] \in \mathbb{C}[\lambda]R$. Conversely, in the vertex algebra setting, we just ask that R is a strongly generating set for some vertex algebra V, which is thus obtained from R by taking normal ordered products. Therefore one allows the λ–bracket $[a\ {}_\lambda\ b]$ to take values in a larger space:

$$[a\ {}_\lambda\ b] \in \mathbb{C}[\lambda]V = \mathbb{C}[\lambda](R + :RR: + \cdots) \ .$$

Notice that by asking that R is as in (1.5) we make sure, by conformal weight considerations, that the λ–bracket $[a\ {}_\lambda\ b]$ with $a, b \in R$ is a linear combination of elements of R and normal ordered products of at most two elements of R, namely

$$[a\ {}_\lambda\ b] \in \mathbb{C}[\lambda](R + :RR:) \subset \mathbb{C}[\lambda]V \ . \tag{1.6}$$

In other words Problem 1.2.1 can be viewed as first deformation of the problem of classifying all physical Lie conformal algebras.

A vertex algebra V generated by a space R with a λ–bracket of kind (1.6) is known in physics literature as a conformal algebra with quadratic non linearities. The problem of their classification is discussed in [7] and [8] and a list is provided. There the proof is based on "physical" arguments and in particular on the assumption of existence of classical limit, which, in this paper, will be removed.

As we said, if we want the space R in (1.5) to be a strongly generating set of a vertex algebra V, we need it to be endowed with a λ–bracket of kind (1.6) satisfying sesquilinearity, skewsymmetry and Jacobi identity. It turns out this is all we need to do: the existence of such λ–bracket structure on $R\otimes R$ guarantees the existence of an "enveloping vertex algebra" over R. In other words, we will prove a generalization of Theorem 1.1.5, to the case in which R is not necessarily a Lie conformal algebra, since it might not be closed under the λ–bracket, but it still has a λ–bracket structure $[\ {}_\lambda\ ]:\ R\otimes R\to\mathbb{C}[\lambda](R+\ :RR:)$ satisfying all the axioms of Lie conformal algebras. This is stated in the following

Theorem 1.2.2. *Let $R=R_{\bar 0}\oplus R_{\bar 1}$ be a vector superspace, endowed with an even vector $|0\rangle\in R$ and an even endomorphism $T\in EndR$ such that $T|0\rangle=0$. Assume that R is of finite rank as $\mathbb{C}[T]$–module. Denote by $\mathcal{T}(R)$ the tensor algebra over R. Let $L_\lambda:\ R\otimes R\to\mathbb{C}[\lambda]\otimes\mathcal{T}(R)$ be a Lie λ–bracket of degree 2, namely a parity preserving linear map satisfying sesquilinearity, skewsymmetry and Jacobi identity (in the sense specified in Section 2.1), such that $L_\lambda(a,b)\in\mathbb{C}[\lambda]\otimes R$ if either a or $b\in R_{\bar 0}$ and $L_\lambda(a,b)\in\mathbb{C}[\lambda]\otimes(R\oplus(R_{\bar 0}\otimes R_{\bar 0}))$ if $a,b\in R_{\bar 1}$. Then there exists a vertex algebra, called the enveloping vertex algebra over R and denoted by $U(R)$, together with a surjective map $\pi:\ \mathcal{T}(R)\twoheadrightarrow U(R)$, such that $R\xrightarrow{\sim}\pi(R)$ is a generating set for $U(R)$, the vacuum vector is $\pi(|0\rangle)$, the infinitesimal translation operator of $U(R)$, restricted to R, is given by T, the λ–bracket, restricted to $R\otimes R$, is compatible with L_λ:*

$$[\pi(a)\ {}_\lambda\ \pi(b)]\ =\ \pi(L_\lambda(a,b))\ ,\quad \forall a,b\in R\ ,$$

and the normally ordered product, restricted to $R\otimes R$, is compatible with the associative product in $\mathcal{T}(R)$:

$$:\pi(a)\pi(b):\ =\ \pi(a\otimes b)\ ,\quad \forall a,b\in R\ .$$

Chapter 2 will be entirely devoted to stating and proving Theorem 1.2.2.

In the next chapters we will use Theorem 1.2.2 to partially solve Problem 1.2.1. The main technique used is based on the analysis of the generating space $R\subset V$, defined by (1.5). Apart from the $\mathbb{C}[T]$–module structure, R carries a bilinear product $a\otimes b\mapsto a_{(n)}b$, for every $n\in\mathbb{Z}$. The axioms of vertex algebras are translated into some complicated equations for all n–th products, known as Borcherds identities. By direct inspection of these identities one gets in particular that the 0-th product restricted to $\mathfrak{g}\otimes\mathfrak{g}$ defines a Lie algebra structure on the space $\mathfrak{g}$, and the 0-th product restricted to $\mathfrak{g}\otimes U$ defines a representation of $\mathfrak{g}$ on the space U. Moreover, the 0-th product of two element $u, v \in U$ is of the form

$$u_{(0)}v = Q(u,v)L + TK(u,v) + P_1(u,v)_{(-1)}P_2(u,v) \ , \tag{1.7}$$

where $Q(u,v)$ is a symmetric invariant bilinear form on U, $K \in \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^2 U, \mathfrak{g})$ and $P \in \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g})$. We are using the notation $P = P_1 \otimes P_2$ for an element of $\mathfrak{g} \otimes \mathfrak{g}$.

One of the main observations is that, under the assumption of existence of a quasi–classical limit (which basically translate to the existence of a one parameter family of vertex algebras, the free parameter being the Kac–Moody level k), the complex connected algebraic group G corresponding to the Lie algebra $\mathfrak{g}$ acts transitively on the quadric $S^2 = \{u \in U \ \text{ s.t. } \ (u,u) = 1\} \subset U$. This generalizes a similar result in the case of conformal algebras [13]. Therefore, under the above assumption, we will solve Problem 1.2.1 by using the classification of connected compact subgroups of SO_N acting transitively on the unit sphere. More precisely we will prove the following

Theorem 1.2.3. *Let V be a vertex algebra strongly generated by the space R in (1.5), which admit quasi–classical limit (according to Definition 5.1.5). Assume $\mathfrak{g}$ is a reductive Lie algebra, U is any $\mathfrak{g}$–module, and the bilinear forms $\bar{\varkappa}: \ S^2\mathfrak{g} \to \mathbb{C}, \ \bar{Q}: S^2U \to \mathbb{C}$ (defined in Table 5.1) are non degenerate. Then the pair $(\mathfrak{g}, U)$ is one of the following*

- $\mathfrak{g} = \mathrm{so}_n, \ U = \mathbb{C}^n$, *for* $n \geq 3$,
- $\mathfrak{g} = \mathrm{gl}_n, \ U = \mathbb{C}^n \oplus \mathbb{C}^{n,*}$, *for* $n \geq 1, n \neq 2$,
- $\mathfrak{g} = \mathrm{sl}_2, \ U = \mathbb{C}^2 \oplus \mathbb{C}^{2,*}$,
- $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2, \ U = \mathbb{C}^n \otimes \mathbb{C}^2 \quad n \geq 2$,
- $\mathfrak{g} = B_3, \ U = V_{\pi_3} = \mathrm{Spin}_7$,
- $\mathfrak{g} = G_2, \ U = V_{\pi_1}$.

However, if one removes the assumption of existence of quasi–classical limit, the above argument fails and Problem 1.2.1 has to be studied using different techniques. We will be able to do so, in the special case in which $\mathfrak{g}$ is a simple Lie algebra and U is an irreducible $\mathfrak{g}$–module, and under a technical assumption that the vertex algebra V is "non degenerate". Roughly speaking, such assumption guarantees that the space R generates freely its enveloping vertex algebra $V = U(R)$, namely an analogue of the Poincare-Birkhoff-Witt Theorem holds. We will prove that, under these assumption, every vertex algebra V admits a quasi–classical limit, and thus no examples appear with "discrete" values of the Kac–Moody level k. This is stated in the following

Theorem 1.2.4. *Let V be a non degenerate (according to Definition 3.2.4) vertex algebra strongly generated by the space R in (1.5), and assume that $\mathfrak{g}$ is a simple Lie algebra, U is an irreducible $\mathfrak{g}$–module and the bilinear forms $\varkappa: \ S^2\mathfrak{g} \to \mathbb{C}, \ Q: S^2Q \to \mathbb{C}$ (defined in Table 3.1) are not identically zero. Then the pair $(\mathfrak{g}, U)$ is one of the following:* $(\mathrm{so}_n, \mathbb{C}^n)$ *with* $n \geq 3, \ n \neq 4, \ (B_3, V_{\pi_3}), \ (G_2, V_{\pi_1})$.

Chapter 2

Existence of the enveloping vertex algebra $U(R)$

In this chapter we will define the enveloping vertex algebra $U(R)$ over a space R endowed with a λ–bracket with "quadratic non–linearities".

In the particular case in which R is a Lie conformal algebra, namely it is closed under the λ–bracket, the space $U(R)$ is easily defined as the universal enveloping algebra over the Lie algebra R_{Lie} (defined in Section 1.1). However, if R is not a Lie conformal algebra, namely the λ–bracket admits "quadratic non–linearities", the construction of $U(R)$ is more involved. In Section 2.1 we will describe such construction. The main results are Theorem 2.1.7 and Theorem 2.1.8. Theorem 2.1.7 gives an explicit description of the space $U(R)$ by providing a basis over $\mathbb{C}$. It is the analogue, in this more general setting, of the Poincare-Birkhoff-Witt Theorem for the universal enveloping algebra over a Lie algebra. Finally, Theorem 2.1.8 states that $U(R)$ is naturally endowed with a vertex algebra structure.

The remaining sections of this chapter will be devoted to proving Theorem 2.1.7 and Theorem 2.1.8. In Section 2.2 we will state and prove some technical lemmata, needed in the following sections. In Section 2.3 we will prove Theorem 2.1.7. In order to prove Theorem 2.1.8 we will need to present an equivalent definition of vertex algebra, based on the notion of local fields, and to state the so called "Existence Theorem", [12]. This will be done in Section 2.4. Finally in Section 2.5 we will prove Theorem 2.1.8.

2.1 Enveloping vertex algebra over R

Throughout this chapter we will denote by R a vector superspace, $R = R_{\bar{0}} \oplus R_{\bar{1}}$, where $R_{\bar{0}}$ (respectively $R_{\bar{1}}$) denotes the even (resp. odd) subspace, endowed with an even element $|0\rangle \in R$ and an even endomorphism $T \in \mathrm{End}R$ such that $T|0\rangle = 0$. We will always assume R is of finite rank as $\mathbb{C}[T]$–module. For example R can be as in (1.5). Let $\mathcal{T}(R)$ be the tensor algebra over R

$$\mathcal{T}(R) = \mathbb{C} \oplus R \oplus R^{\otimes 2} \oplus \cdots$$

and we extend the action of T to $\mathcal{T}(R)$ by derivation of the tensor product, namely

$$T(1)=0\ ,\quad T(A\otimes B)=T(A)\otimes B+A\otimes T(B)\ ,\qquad \forall A,B\in\mathcal{T}(R)\ .$$

We want to define a gradation on $\mathcal{T}(R)$ which will allow us to carry induction arguments.

Definition 2.1.1. (a) A monomial $m=a_1\otimes\cdots\otimes a_n\in\mathcal{T}(R)$ is said to be *homogeneous* if each factor a_i is either even or odd.

(b) The *degree* of a homogeneous monomial m is by definition the pair (n,k), where n is the total number of factors of m and k is the number of odd factors of m. More in general an element $A\in\mathcal{T}(R)$ is said to be a *homogeneous polynomial* of degree (n,k) if it is linear combination of homogeneous monomials of degree (n,k). (The element 0 is considered to be of any degree.)

(c) We assign alphabetical ordering to the set of all possible degrees $\mathcal{D}=\{(n,k)\mid 0\leq k\leq n\in\mathbb{Z}_+\}$, namely we say that $(n_1,k_1)\leq(n_2,k_2)$ if either $n_1<n_2$ or $n_1=n_2$ and $k_1\leq k_2$. An element $A\in\mathcal{T}(R)$ is then said to be a *polynomial* of degree (n,k) if it is linear combination of independent homogeneous monomials of degree less then or equal to (n,k) and the highest degree is (n,k).

(d) We then have a *gradation* on $\mathcal{T}(R)$, namely

$$\mathcal{T}(R)=\bigoplus_{(n,k)\in\mathcal{D}}\mathcal{T}(R)[n,k]\ ,$$

where $\mathcal{T}(R)[n,k]$ is the space of homogeneous polynomials of degree (n,k). The corresponding *filtration* is

$$\mathcal{T}_{0,0}(R)\subset\mathcal{T}_{1,0}(R)\subset\mathcal{T}_{1,1}(R)\subset\cdots\subset\mathcal{T}(R),$$

where $\mathcal{T}_{n,k}(R)$ is the space of polynomials of degree less then or equal to (n,k).

Definition 2.1.2. A λ–*bracket* (of degree 2) on R is a linear map

$$L_\lambda\ :\ R\otimes R\to\mathbb{C}[\lambda]\otimes\mathcal{T}(R)\ ,$$

satisfying the following conditions:

1. L_λ is parity preserving and every element $L_\lambda(a,b)$ is a polynomial in λ with coefficients in $\mathcal{T}(R)$ of degree at most $(2,0)$. More precisely

$$L_\lambda\Big|_{R_i\otimes R_j}\ :\ R_i\otimes R_j\longrightarrow R_{i+j}[\lambda]\ ,\qquad\qquad \text{if either } i \text{ or } j=\bar{0}\ ,$$

$$L_\lambda\Big|_{R_{\bar{1}}\otimes R_{\bar{1}}}\ :\ R_{\bar{1}}\otimes R_{\bar{1}}\longrightarrow(R_{\bar{0}}\oplus R_{\bar{0}}^{\otimes 2})[\lambda]\ .$$

2. sesquilinearity conditions hold, namely for $a,b\in R$

$$L_\lambda(Ta,b)=-\lambda L_\lambda(a,b)\ ,\quad L_\lambda(a,Tb)=(T+\lambda)L_\lambda(a,b)\ .$$

We say that the λ–bracket L_λ is *skewsymmetric* (or a skew λ–bracket) if for $a, b \in R$

$$L_\lambda(a,b) = -p(a,b)L_{-\lambda-T}(b,a)\ . \tag{2.1}$$

Lemma 2.1.3. *A) Let R be as above and let $L_\lambda :\ R\otimes R \to \mathbb{C}[\lambda]\otimes\mathcal{T}(R)$ be a λ–bracket on R. One can define uniquely linear maps*

$$L_\lambda :\ \mathcal{T}_{2,2}(R)\otimes\mathcal{T}(R) \longrightarrow \mathbb{C}[\lambda]\otimes\mathcal{T}(R)\ ,$$
$$N :\ \mathcal{T}_{2,2}(R)\otimes\mathcal{T}(R) \longrightarrow \mathcal{T}(R)\ ,$$

such that L_λ, restricted to $R\otimes R$ coincides with the given λ–bracket, and the following conditions hold for $a, b \in R,\ A \in \mathcal{T}_{2,2}(R),\ C \in \mathcal{T}(R)$:

$$L_\lambda(1,C) = 0\ ,\ L_\lambda(A,1) = 0\ , \tag{2.2}$$

$$\begin{aligned} L_\lambda(a, b\otimes C) &= N(L_\lambda(a,b),C) + p(a,b)b\otimes L_\lambda(a,C) \\ &\quad + \int_0^\lambda d\mu\ L_\mu(L_\lambda(a,b),C)\ , \end{aligned} \tag{2.3}$$

$$\begin{aligned} L_\lambda(a\otimes b, C) &= \Big(e^{T\partial_\lambda}a\Big)\otimes L_\lambda(b,C) + p(a,b)\Big(e^{T\partial_\lambda}b\Big)\otimes L_\lambda(a,C) \\ &\quad + p(a,b)\int_0^\lambda d\mu\ L_\mu(b, L_{\lambda-\mu}(a,C))\ , \end{aligned} \tag{2.4}$$

$$N(1,C) = C\ ,\quad N(A,1) = A\ ,\quad N(a,C) = a\otimes C\ , \tag{2.5}$$

$$\begin{aligned} N(a\otimes b, C) &= a\otimes b\otimes C + \Big(\int_0^T d\lambda\ a\Big)\otimes L_\lambda(b,C) \\ &\quad + p(a,b)\Big(\int_0^T d\lambda\ b\Big)\otimes L_\lambda(a,C)\ . \end{aligned} \tag{2.6}$$

B) Moreover L_λ and N satisfy the following grading conditions ($(n_1,k_1)\leq(2,2)$):

$$\begin{aligned} L_\lambda(\mathcal{T}_{n_1,k_1}(R)\otimes\mathcal{T}_{n_2,k_2}(R)) &\subset \mathbb{C}[\lambda]\otimes\mathcal{T}_{n_1+n_2,k_1+k_2-2}(R)\ , \\ N(\mathcal{T}_{n_1,k_1}(R)\otimes\mathcal{T}_{n_2,k_2}(R)) &\subset \mathcal{T}_{n_1+n_2,k_1+k_2}(R)\ . \end{aligned} \tag{2.7}$$

(We use the convention $\mathcal{T}_{n,-1}(R) = \mathcal{T}_{n-1,n-1}(R),\ T_{n,-2}(R) = \mathcal{T}_{n-1,n-2}(R)$).

Proof. For $A\in\mathbb{C}\oplus R$ or $C\in\mathbb{C}$, $N(A,C)$ is given by (2.5) and for $A\in\mathbb{C},\ C\in\mathbb{C}$ or $A, C \in R, L_\lambda(A,C)$ is given by (2.2). Assume then by induction that $N(a\otimes b, C')$, $L_\lambda(a\otimes b, C')$ and $L_\lambda(a, b\otimes C')$ are uniquely defined and they satisfy the grading conditions (2.7), for every $a, b \in R$ and $C' \in \mathcal{T}_{p,q}(R)$ with $(p,q) < (n,k)$. For $C \in \mathcal{T}_{n,k}(R)$ we then have that $N(a\otimes b, C)$ is defined by (2.6), $L_\lambda(a\otimes b, C)$ is defined by (2.4) and $L_\lambda(a, b\otimes C)$ is defined by (2.3). It's easy to check that all terms in the right hand sides of (2.6), (2.4) and (2.3) are well defined by the inductive assumption and they satisfy the grading conditions (2.7). □

Definition 2.1.4. Let R be as above and let $L_\lambda :\ R\otimes R \to \mathbb{C}[\lambda]\otimes\mathcal{T}(R)$ be any

λ–bracket on R. We define the subspace $\mathcal{M}(R) \subset \mathcal{T}(R)$ by

$$\mathcal{M}(R) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{array}{c} |0\rangle - 1 \ ; \quad A \otimes b \otimes c \otimes D - p(b,c) A \otimes c \otimes b \otimes D \\ - A \otimes N\Big(\Big(\int_{-T}^{0} d\lambda \ L_\lambda(b,c) \Big), D\Big) \ , \quad A, D \in \mathcal{T}(R) \ , \ b, c \in R \end{array} \right\}$$

The *universal enveloping algebra* over R is, by definition, the quotient space

$$U(R) = \mathcal{T}(R) / \mathcal{M}(R) \ .$$

If $\pi : \ \mathcal{T}(R) \to U(R)$ is the quotient map onto $U(R)$, we denote the image of monomials of $\mathcal{T}(R)$ by

$$: ab \cdots z : = \ \pi(a \otimes b \otimes \cdots \otimes z) \ .$$

The filtration on $\mathcal{T}(R)$ induces naturally a filtration on $U(R)$

$$U_{0,0}(R) \subset U_{1,0}(R) \subset U_{1,1}(R) \subset \cdots U(R) \ ,$$

defined by $U_{n,k}(R) = \pi(\mathcal{T}_{n,k}(R))$.

Definition 2.1.5. Lemma 2.1.3 allows us to give sense to triple λ–brackets. We say that the λ–bracket $L_\lambda : \ R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ satisfies *Jacobi identity* if for every $a, b, c \in R$ it satisfies

$$L_\lambda(a, L_\mu(b,c)) - p(a,b) L_\mu(b, L_\lambda(a,c)) \equiv L_{\lambda+\mu}(L_\lambda(a,b), c) \mod \mathbb{C}[\lambda,\mu] \otimes \mathcal{M}(R) \ . \tag{2.8}$$

A λ–bracket $L_\lambda : \ R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ is said to be a *Lie λ–bracket* if it satisfies skewsymmetry (2.1) and Jacobi identity (2.8).

Lemma 2.1.6. *Given a λ–bracket $L_\lambda : \ R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ we can define the map $L : \ R \otimes \mathcal{T}_{2,2}(R) \to \mathcal{T}(R)$ given by ($a \in R, \ B \in \mathcal{T}_{2,2}(R)$)*

$$L(a, B) = \int_{-T}^{0} d\lambda \ L_\lambda(a, B) \ .$$

If L_λ is a Lie λ–bracket, then L satisfies skewsymmetry

$$L(a,b) = -p(a,b) L(b,a) \ , \quad \forall a, b \in R$$

and Jacobi identity

$$L(a, L(b,c)) - p(a,b) L(b, L(a,c)) \equiv L(L(a,b), c) \mod \mathcal{M}(R) \ , \quad \forall a, b, c \in R \ .$$

Proof. The proof follows from a straightforward computation. □

The main results of this chapter are the following two theorems.

Theorem 2.1.7. *Let $\mathcal{A} = \{a_0 = |0\rangle, a_1, a_2, \dots\}$ be a countable basis for the space R such that each basis element a_i has given parity. Denote by*

$$\mathcal{B} = \{: a_{i_1} \dots a_{i_n} : \ , \quad 1 \leq i_1 \leq i_2 \leq \dots \leq i_n\} \subset U(R)$$

the collection of ordered monomials of $U(R)$. If L_λ is a Lie λ–bracket (of degree 2) on R, then $\mathcal{B}$ is a basis for $U(R)$. In particular there is a natural embedding

$$R \ \stackrel{\sim}{\to} \ \pi(R) \ \subset \ U(R) \ .$$

Theorem 2.1.8. *Let $L_\lambda : \ R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ be a Lie λ–bracket (of degree 2) on R. There is a unique structure of vertex algebra on $U(R)$ (which is then called the enveloping vertex algebra over R) such that the vacuum vector is $|0\rangle = \pi(1)$, the infinitesimal translation operator is induced by the action of T on $\mathcal{T}(R)$:*

$$T(\pi(A)) = \pi(T(A)) \ , \quad \forall A \in \mathcal{T}(R) \ ,$$

the λ–bracket restricted to $R \otimes U(R)$ is compatible with L_λ:

$$[\pi(a) \ {}_\lambda \ \pi(B)] = \pi(L_\lambda(a, B)) \ , \quad \forall a \in R, \ B \in \mathcal{T}(R) \ ,$$

and the normally ordered product restricted to $R \otimes U(R)$ is compatible with N (namely with the tensor product):

$$: \pi(a)\pi(B) : = \ \pi(N(a, B)) = \pi(a \otimes B) \ , \quad \forall a \in R, \ B \in \mathcal{T}(R) \ .$$

Remark 2.1.9. Consider the particular case in which R is closed under the λ–bracket L_λ, namely $L_\lambda(a, b) \in R[\lambda], \ \forall a, b \in R$.

(a) In this case the map $L : \ R \otimes R \to R$ defined in Lemma 2.1.6 is a Lie bracket on R and $U(R)$ coincides with the universal enveloping algebra over R (viewed as a Lie algebra with respect to L). Therefore Theorem 2.1.7 coincides, in this case, with the PBW Theorem for ordinary Lie superalgebras.

(b) Moreover, to say that R is closed under the Lie λ–bracket L_λ is equivalent to say that R is a Lie conformal algebra. Therefore $U(R)$ coincides with the enveloping vertex algebra over R (as defined in Section 1.1), and Theorem 2.1.8 reduces to Theorem 1.1.5. We can thus view Theorem 2.1.8 as a generalization of Theorem 1.1.5 to the case in which the λ–bracket L_λ is allowed to take values on the larger space $\mathbb{C}[\lambda] \otimes (R \oplus R_{\bar{0}}^{\otimes 2})$.

The remaining sections of this chapter will be devoted to prove Theorem 2.1.7 and Theorem 2.1.8. In the next section we will prove some technical results needed in the proof of both Theorem 2.1.7 and 2.1.8. The proof of Theorem 2.1.7 will be in Section 2.3 and the proof of Theorem 2.1.8 will be in Sections 2.4 and 2.5. The content of the following sections will not be needed for an understanding of the remaining of the thesis, so it can be skipped at first reading.

2.2 Some technical results

In the following we will assume that all hypotheses of Theorems 2.1.7 and 2.1.8 are satisfied. Namely $R = R_{\bar{0}} \oplus R_{\bar{1}}$ is a vector superspace with an even (vacuum) vector $|0\rangle$, an even endomorphism $T: R \to R$ which makes R a $\mathbb{C}[T]$–module of finite rank, and a Lie λ–bracket (of degree 2) $L_\lambda : R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ (see Definitions 2.1.2 and 2.1.5).

For convenience, we want to introduce a notation which will be used throughout this section. For elements $A, B, C \dots \in \mathcal{T}(R)$ for which the right hand side is defined, we will denote

$$
\begin{aligned}
\mathrm{skl}(A,B;\lambda) &= L_\lambda(A,B) + p(A,B)L_{-\lambda-T}(B,A) ,\\
\mathrm{skn}(A,B,C) &= N(A,N(B,C)) - p(A,B)N(B,N(A,C))\\
&- N\Big(\Big(\int_{-T}^{0} d\lambda\ L_\lambda(A,B)\Big),C\Big) ,\\
\mathrm{lW}(A,B,C;\lambda) &= L_\lambda(A,N(B,C)) - \int_0^\lambda d\mu\ L_\mu(L_\lambda(A,B),C)\\
&- N(L_\lambda(A,B),C) - p(A,B)N(B,L_\lambda(A,C)) ,\\
\mathrm{rW}(A,B,C;\lambda) &= L_\lambda(N(A,B),C) - p(A,B)\int_0^\lambda d\mu\ L_\mu(B,L_{\lambda-\mu}(A,C))\\
&- N\Big(\Big(e^{T\partial_\lambda}A\Big),L_\lambda(B,C)\Big) - p(A,B)N\Big(\Big(e^{T\partial_\lambda}B\Big),L_\lambda(A,C)\Big) ,\\
\mathrm{qA}(A,B,C) &= N(N(A,B),C) - N(A,N(B,C))\\
&- N\Big(\Big(\int_0^T d\lambda\ A\Big),L_\lambda(B,C)\Big)\\
&- p(A,B)N\Big(\Big(\int_0^T d\lambda\ B\Big),L_\lambda(A,C)\Big) ,\\
\mathrm{J}(A,B,C;\lambda,\mu) &= L_\lambda(A,L_\mu(B,C)) - p(A,B)L_\mu(B,L_\lambda(A,C))\\
&- L_{\lambda+\mu}(L_\lambda(A,B),C) .
\end{aligned}
$$

Remark 2.2.1. Using the above notation, we can write in a more concise form all the definitions introduced in the previous subsection. The maps $L_\lambda : \mathcal{T}_{2,2}(R) \otimes \mathcal{T}(R) \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ and $N: \mathcal{T}_{2,2}(R) \otimes \mathcal{T}(R) \to \mathcal{T}(R)$ are defined respectively by the equations

$$
\begin{aligned}
\mathrm{lW}(a,b,C;\lambda) &= \mathrm{rW}(a,b,C;\lambda) = 0 ,\\
\mathrm{qA}(a,b,C) &= 0 , \qquad \forall a,b \in R,\ C \in \mathcal{T}(R) .
\end{aligned}
$$

The definition of Lie λ–bracket can be written in terms of the conditions

$$
\begin{aligned}
\mathrm{skl}(a,b\ ;\lambda) &= 0 , & \forall a,b \in R ,\\
\mathrm{J}(a,b,c\ ;\lambda,\mu) &\in \mathbb{C}[\lambda,\mu] \otimes \mathcal{M}(R) , & \forall a,b,c \in R .
\end{aligned}
$$

Finally, the subspace $\mathcal{M}(R) \subset \mathcal{T}(R)$ is defined as

$$\mathcal{M}(R) = \text{span}\,\{|0\rangle - 1\ ;\ A \otimes \text{skn}(b, c, D)\ ,\quad b, c \in R,\ A, D \in \mathcal{T}(R)\}\ .$$

Lemma 2.2.2. *The map $N:\ \mathcal{T}_{2,2}(R) \otimes \mathcal{T}(R) \to \mathcal{T}(R)$ satisfies Leibniz rule with respect to the derivation T, namely, for $A \in \mathcal{T}_{2,2}(R),\ B \in \mathcal{T}(R)$*

$$TN(A, B) = N(TA, B) + N(A, TB)\ , \tag{2.9}$$

The map $L_\lambda:\ \mathcal{T}_{2,2}(R) \otimes \mathcal{T}(R) \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ satisfies sesquilinearity conditions with respect to T, namely, for $A \in \mathcal{T}_{2,2}(R),\ B \in \mathcal{T}(R)$

$$L_\lambda(TA, B) = -\lambda L_\lambda(A, B)\ ,\quad L_\lambda(A, TB) = (\lambda + T)L_\lambda(A, B)\ , \tag{2.10}$$

In particular T is a derivation of L_λ.

Proof. Recall that $1 \in \mathbb{C}$ is a unit element for the normal ordered product N, and it is a central element for the λ–bracket L_λ; this means that equations (2.9) and (2.10) are trivially true if either A or B belongs to $\mathbb{C}$. Since by definition $N(a, B) = a \otimes B$ for $a \in R,\ B \in \mathcal{T}(R)$, we also have that (2.9) is obviously satisfied for $A \in R,\ B \in \mathcal{T}(R)$. Moreover, by definition of λ–bracket, we also know that equations (2.10) are true for $A, B \in R$. The lemma will be proved if we show that, for $a, b \in R,\ C \in \mathcal{T}(R)$

$$L_\lambda(T(a \otimes b), C) = -\lambda L_\lambda(a \otimes b, C)\ , \tag{2.11}$$

$$L_\lambda(Ta, b \otimes C) = -\lambda L_\lambda(a, b \otimes C)\ , \tag{2.12}$$

$$TN(a \otimes b, C) = N(T(a \otimes b), C) + N(a \otimes b, TC)\ , \tag{2.13}$$

$$L_\lambda(a \otimes b, TC) = (\lambda + T)L_\lambda(a \otimes b, C)\ , \tag{2.14}$$

$$L_\lambda(a, T(b \otimes C)) = (\lambda + T)L_\lambda(a, b \otimes C)\ , \tag{2.15}$$

We will prove these equations by induction on $\deg(C)$. By the above arguments, all equations (2.11)–(2.15) are obviously true if $\deg(C) = (0, 0)$. Assume now that equations (2.11)–(2.15) are satisfied if $\deg(C) < (n, k)$, and we want to prove them for $C \in \mathcal{T}_{n,k}(R)$. By definition one has

$$\begin{aligned} L_\lambda(T(a \otimes b), C) &= L_\lambda((Ta) \otimes b, C) + L_\lambda(a \otimes (Tb), C) \\ &= \left(e^{T\partial_\lambda} Ta\right) \otimes L_\lambda(b, C) + p(a, b)\left(e^{T\partial_\lambda} b\right) \otimes L_\lambda(Ta, C) \\ &+ p(a, b) \int_0^\lambda d\mu\ L_\mu(b, L_{\lambda-\mu}(Ta, C)) \\ &+ \left(e^{T\partial_\lambda} a\right) \otimes L_\lambda(Tb, C) + p(a, b)\left(e^{T\partial_\lambda} Tb\right) \otimes L_\lambda(a, C) \\ &+ p(a, b) \int_0^\lambda d\mu\ L_\mu(Tb, L_{\lambda-\mu}(a, C)) \end{aligned} \tag{2.16}$$

By inductive assumption we have

$$L_\lambda(Ta,C) \;=\; -\lambda L_\lambda(a,C)\;, \quad L_\lambda(Tb,C) \;=\; -\lambda L_\lambda(b,C)\;,$$
$$L_\mu(b,L_{\lambda-\mu}(Ta,C)) \;=\; (\mu-\lambda)L_\mu(b,L_{\lambda-\mu}(a,C))\;,$$
$$L_\mu(Tb,L_{\lambda-\mu}(a,C)) \;=\; -\mu L_\mu(b,L_{\lambda-\mu}(a,C))\;.$$

In the last two equations we used the fact that, by Lemma 2.1.3, $\deg L_\lambda(a,C) < \deg(a)+\deg(C)$. Combining these results we can rewrite the right hand side of (2.16) as

$$\begin{aligned}&\Big(e^{T\partial_\lambda}(T-\lambda)a\Big)\otimes L_\lambda(b,C) \;+\; p(a,b)\Big(e^{T\partial_\lambda}(T-\lambda)b\Big)\otimes L_\lambda(a,C)\\ &-\;\lambda p(a,b)\int_0^\lambda d\mu\; L_\mu(b,L_{\lambda-\mu}(a,C)) \;=\; -\lambda L_\lambda(a\otimes b,C)\;.\end{aligned}$$

Here we used the fact that $e^{T\partial_\lambda}\lambda=(\lambda+T)e^{T\partial_\lambda}$. This proves equation (2.11). Similarly one has, by induction

$$\begin{aligned}L_\lambda(Ta,b\otimes C) \;&=\; N(L_\lambda(Ta,b),C) \;+\; p(a,b)b\otimes L_\lambda(Ta,C))\\ &+\;\int_0^\lambda d\mu\; L_\mu(L_\lambda(Ta,b),C) \;=\; -\lambda L_\lambda(a,b\otimes C)\;,\end{aligned}$$

which proves (2.12). By definition of N one has

$$\begin{aligned}TN(a\otimes b,C) = T\Big\{&a\otimes b\otimes C+\Big(\int_0^T d\lambda\; a\Big)\otimes L_\lambda(b,C)\\ &+p(a,b)\Big(\int_0^T d\lambda\; b\Big)\otimes L_\lambda(a,C)\Big\}\;.\end{aligned} \tag{2.17}$$

Recall that T is defined by derivation of the tensor product. Moreover, by inductive assumption, we have $TL_\lambda(a,C)=L_\lambda(Ta,C)+L_\lambda(a,TC)$. Putting together these facts, we can rewrite the right hand side of (2.17) as

$$N(T(a\otimes b),C) \;+\; N(a\otimes b,TC)\;,$$

thus proving (2.13). Similarly we have

$$\begin{aligned}TL_\lambda(a\otimes b,C) \;=\; T\Big\{&\Big(e^{T\partial_\lambda}a\Big)\otimes L_\lambda(b,C) \;+\; p(a,b)\Big(e^{T\partial_\lambda}b\Big)\otimes L_\lambda(a,C)\\ &+\;p(a,b)\int_0^\lambda d\mu\; L_\mu(b,L_{\lambda-\mu}(a,C))\Big\}\;.\end{aligned}$$

Since $\deg\Big(L_\lambda(a,C)\Big) < \deg(a)+\deg(C)$, we can use inductive assumption to write the right hand side as

$$L_\lambda(T(a\otimes b),C) \;+\; L_\lambda(a\otimes b,TC)\;,$$

which, together with (2.11), gives (2.14). Finally we have

$$\begin{aligned} TL_\lambda(a, b\otimes C) &= T\Big\{ N(L_\lambda(a,b),C) + p(a,b) b\otimes L_\lambda(a,C) \\ &\quad + \textstyle\int_0^\lambda d\mu\, L_\mu(L_\lambda(a,b),C)\Big\} \ . \end{aligned} \tag{2.18}$$

We need to use (2.13) and (2.14) to compute respectively the first and the third term of the right hand side. This, combined to inductive assumptions, allows us to rewrite the right hand side of (2.18) as

$$L_\lambda(Ta, b\otimes C) + L_\lambda(a, T(b\otimes C)) \ ,$$

which, together with (2.12), gives (2.15). This completes the proof of the lemma. □

Corollary 2.2.3. *The space $\mathcal{M}(R) \subset \mathcal{T}(R)$ is invariant under the action of T, namely $T\mathcal{M}(R) \subset \mathcal{M}(R)$.*

Proof. It follows immediately by the definition of $\mathcal{M}(R)$ and by Lemma 2.2.2. □

Lemma 2.2.4. *(a) For $a, b, c \in R$, we have*

$$\mathrm{skl}(a, b\otimes c;\lambda) \ = \ \mathrm{skn}(L_\lambda(a,b),c,1) + \int_{-T}^{\lambda} d\nu\ \mathrm{skl}(L_\lambda(a,b),c;\nu) \ . \tag{2.19}$$

In particular, if $b, c \in R_{\bar 0}$ we have

$$\mathrm{skl}(a, b\otimes c;\lambda) \ = \ \mathrm{skn}(L_\lambda(a,b),c,1) \ . \tag{2.20}$$

(b) For $a\in R$, $b, c\in R_{\bar 0}$ and $D \in \mathcal{T}(R)$, we have

$$\begin{aligned} \mathrm{skn}(a, b\otimes c, D) &= b\otimes \mathrm{skn}(a,c,D) + \mathrm{skn}(a,b,c\otimes D) \\ &+ \ \mathrm{skn}\Big(a, \Big(\textstyle\int_0^T d\lambda\ b\Big), L_\lambda(c,D)\Big) \\ &+ \ \mathrm{skn}\Big(a, \Big(\textstyle\int_0^T d\lambda\ c\Big), L_\lambda(b,D)\Big) \\ &+ \ \textstyle\int_0^{T_c} d\lambda\, \mathrm{skn}(L_\lambda(b,a),c,D) \ , \end{aligned} \tag{2.21}$$

where T_c denotes T acting only on the element c.

Proof. Equations (2.19) and (2.21) follow by rather lengthy computations. The details are provided in the Appendix. Equation (2.20) follows immediately from (2.19), after noticing that $L_\lambda(a,b) \in R[\lambda]$ and using (2.1). □

Corollary 2.2.5. *The following conditions hold*

$$\mathrm{skl}(\mathcal{T}_{1,1}(R), \mathcal{T}_{2,0}(R);\lambda) \ \subset \ \mathbb{C}[\lambda]\otimes\mathcal{M}(R) \ ,$$

$$\mathrm{skn}(\mathcal{T}_{1,1}(R), \mathcal{T}_{2,0}(R), \mathcal{T}(R)) \ \subset \ \mathcal{M}(R) \ .$$

Proof. We need to prove that $\mathrm{skl}(A, B; \lambda) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R)$ and $\mathrm{skn}(A, B, D) \in \mathcal{M}(R)$ for every $A, B, D \in \mathcal{T}(R)$ such that $\deg(A) \leq (1,1)$ and $\deg(B) \leq (2,0)$. If $\deg(A) = 0$ or $\deg(B) = 0$, we obviously have $\mathrm{skl}(A, B; \lambda) = 0$ and $\mathrm{skn}(A, B, D) = 0$, so we can assume $A = a \in R$ and $B \in R \oplus R_{\bar{0}}^{\otimes 2}$. If $B = b \in R$, then $\mathrm{skl}(a, b; \lambda) = 0$ by (2.1) and $\mathrm{skn}(a, b, D) \in \mathcal{M}(R)$ by definition of $\mathcal{M}(R)$. Finally, if $B = b \otimes c$ with $b, c \in R_{\bar{0}}$, we have $\mathrm{skl}(a, b \otimes c; \lambda) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R)$ by (2.20) and $\mathrm{skn}(a, b \otimes c, D) \in \mathcal{M}(R)$ by (2.21). □

Lemma 2.2.6. *For $a, b, c \in R,\ D \in \mathcal{T}(R)$, the following equations hold*

$$\begin{aligned}
\mathrm{J}(a, b, c \otimes D; \lambda, \mu) \;=\;& \mathrm{lW}(a, L_\mu(b, c), D; \lambda) \;-\; p(a, b)\mathrm{lW}(b, L_\lambda(a, c), D; \mu) \\
-\;& \mathrm{lW}(L_\lambda(a, b), c, D; \lambda + \mu) \;+\; N(\mathrm{J}(a, b, c; \lambda, \mu), D) \\
+\;& p(a, c)p(b, c)c \otimes \mathrm{J}(a, b, D; \lambda, \mu) \\
+\;& \textstyle\int_0^{\lambda+\mu} d\nu L_\nu(\mathrm{J}(a, b, c; \lambda, \mu), D) \\
+\;& \textstyle\int_0^{\mu} d\nu\ \mathrm{J}(a, L_\mu(b, c), D; \lambda, \nu) \\
-\;& p(a, b) \textstyle\int_0^{\lambda} d\nu\ \mathrm{J}(b, L_\lambda(a, c), D; \mu, \nu)
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\mathrm{J}(a, b \otimes c, D; \lambda, \mu) \;=\;& \left(e^{T\partial_\mu} b\right) \otimes \mathrm{J}(a, c, D; \lambda, \mu) + \left(e^{T\partial_\mu} c\right) \otimes \mathrm{J}(a, b, D; \lambda, \mu) \\
+\;& \textstyle\int_0^{\lambda} d\nu\ \mathrm{J}(L_\lambda(a, b), c, D; \nu, \lambda + \mu - \nu) \\
+\;& \textstyle\int_0^{\mu} d\nu\ \mathrm{J}(a, c, L_{\mu-\nu}(b, D); \lambda, \nu) \\
+\;& \textstyle\int_0^{\mu} d\nu\ L_\nu(c, \mathrm{J}(a, b, D; \lambda, \mu - \nu)) \quad , \qquad \text{for } b, c \in R_{\bar{0}}
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
\mathrm{lW}(a, b \otimes c, D; \lambda) \;=\;& \textstyle\int_0^{\lambda} d\mu\ \int_0^{\lambda-\mu} d\nu\ \mathrm{J}(L_\lambda(a, b), c, D; \mu, \nu) \\
+\;& \left(\textstyle\int_0^{T} d\mu\ b\right) \otimes \mathrm{J}(a, c, D; \lambda, \mu) \\
+\;& \left(\textstyle\int_0^{T} d\mu\ c\right) \otimes \mathrm{J}(a, b, D; \lambda, \mu) \quad , \qquad \text{for } b, c \in R_{\bar{0}}
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
\mathrm{lW}(a \otimes b, c, D; \lambda) \;=\;& \textstyle\int_0^{\lambda} d\mu\ \int_0^{\lambda-\mu} d\nu\ \mathrm{J}(b, L_{\lambda-\mu}(a, c), D; \mu, \nu) \\
+\;& \mathrm{skn}(\left(e^{T\partial_\lambda} a\right), c, L_\lambda(b, D)) \\
+\;& \mathrm{skn}(\left(e^{T\partial_\lambda} b\right), c, L_\lambda(a, D)) \quad , \qquad \text{for } a, b \in R_{\bar{0}}\ .
\end{aligned} \tag{2.25}$$

Proof. All above equations follow directly from the definitions, by straightforward though rather lengthy computations. The details are provided in the Appendix. □

Lemma 2.2.7. *For $a, b, c \in R,\ D \in \mathcal{T}(R)$, the following equations hold*

$$
\begin{aligned}
N(\mathrm{skn}(a,b,1),D) &= \mathrm{skn}(a,b,D)\ , & (2.26)\\
L_\lambda(\mathrm{skn}(a,b,1),D) &= -\int_0^\lambda d\mu\ \mathrm{J}(a,b,D;\mu,\lambda-\mu)\ , & (2.27)\\
L_\lambda(a,\mathrm{skn}(b,c,D)) &= -\mathrm{lW}(a,\Big(\int_{-T}^0 d\mu\ L_\mu(b,c)\Big),D;\lambda) \\
&+\int_0^\lambda d\mu\ \Big(\mathrm{lW}(L_\lambda(a,b),c,D;\mu)-p(b,c)\mathrm{lW}(L_\lambda(a,c),b,D;\mu)\Big)\\
&-p(a,c)p(b,c)\mathrm{skn}(c,L_\lambda(a,b),D)+p(a,b)\mathrm{skn}(b,L_\lambda(a,c),D)\\
&+p(a,b)p(a,c)\mathrm{skn}(b,c,L_\lambda(a,D)) & (2.28)\\
&+\int_0^\lambda d\mu\ N(\mathrm{skl}(L_\lambda(a,b),c;\mu)-p(b,c)\mathrm{skl}(L_\lambda(a,c),b;\mu),D)\\
&-N(\Big(\int_{-\lambda-T}^0 d\mu\ \mathrm{skl}(a,L_\mu(b,c);\lambda)\Big),D)\\
&-p(a,b)\int_0^\lambda d\nu\ \int_\nu^\lambda d\mu\ L_\nu(\mathrm{skl}(L_{\mu-\lambda}(b,a),c;\mu),D)\\
&+p(a,c)p(b,c)\int_0^\lambda d\nu\ \int_\nu^\lambda d\mu\ L_\nu(\mathrm{skl}(L_{\mu-\lambda}(c,a),b;\mu),D)\\
&-\int_0^\lambda d\nu\ \int_\nu^\lambda d\mu\ L_\nu(\mathrm{skl}(a,L_{\mu-\lambda}(b,c);\lambda),D)\\
&-p(a,b)p(a,c)N(\Big(\int_{-\lambda-T}^0 d\mu\ \mathrm{J}(b,c,a;\mu,-\lambda-\mu-T)\Big),D)\\
&-p(a,b)p(a,c)\int_0^\lambda d\nu\int_\nu^\lambda d\mu\ L_\nu(\mathrm{J}(b,c,a;\nu-\mu,\mu-\lambda),D)\ .
\end{aligned}
$$

Proof. Equation (2.26) follows immediately by quasi–associativity (2.6). For (2.27), we use the right Wick formula (2.4) to get

$$
\begin{aligned}
&L_\lambda(\mathrm{skn}(a,b,1),D)\\
&\quad= L_\lambda(a\otimes b,D)\ -\ p(a,b)L_\lambda(b\otimes a,D)+\textstyle\int_0^\lambda d\mu\ L_\lambda(L_\mu(a,b),D)\\
&\quad= -\ \textstyle\int_0^\lambda d\mu\ L_\mu(a,L_{\lambda-\mu}(b,D))\ +\ p(a,b)\int_0^\lambda d\mu\ L_\mu(b,L_{\lambda-\mu}(a,D))\\
&\qquad+\ \textstyle\int_0^\lambda d\mu\ L_\lambda(L_\mu(a,b),D)\ =\ -\ \int_0^\lambda d\mu\ \mathrm{J}(a,b,D;\mu,\lambda-\mu)\ .
\end{aligned}
$$

Equation (2.28) follows by a straightforward but quite lengthy computation. The proof is provided in the Appendix. □

Corollary 2.2.8. *The following condition holds*

$$N(\mathcal{T}_{2,2}(R)\cap\mathcal{M}(R),\mathcal{T}(R))\ \subset\ \mathcal{M}(R)\ . \qquad (2.29)$$

Proof. Notice that $\mathcal{T}_{1,1}(R) \cap \mathcal{M}(R) = 0$, so that $\mathcal{T}_{2,2}(R) \cap \mathcal{M}(R)$ is spanned by elements $\mathrm{skn}(a, b, 1)$ with $a, b \in R$. Therefore condition (2.29) follows immediately by equation (2.26). ☐

Corollary 2.2.9. *The following conditions hold*

$$L_\lambda(\mathcal{T}_{2,2}(R) \cap \mathcal{M}(R), \mathcal{T}(R)) \subset \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \tag{2.30}$$

$$N(\mathcal{T}_{2,2}(R), \mathcal{M}(R)) \subset \mathcal{M}(R) , \tag{2.31}$$

$$L_\lambda(\mathcal{T}_{2,2}(R), \mathcal{M}(R)) \subset \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \tag{2.32}$$

$$\mathrm{lW}(\mathcal{T}_{1,1}(R), \mathcal{T}_{2,0}(R), \mathcal{T}(R); \lambda) \subset \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \tag{2.33}$$

$$\mathrm{lW}(\mathcal{T}_{2,0}(R), \mathcal{T}_{1,1}(R), \mathcal{T}(R); \lambda) \subset \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \tag{2.34}$$

$$\mathrm{J}(\mathcal{T}_{1,1}(R), \mathcal{T}_{2,0}(R), \mathcal{T}(R); \lambda, \mu) \subset \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}(R) . \tag{2.35}$$

In particular the space $\mathcal{M}(R)$ is a two sided ideal with respect to the maps N and L_λ.

Proof. All conditions (2.30)–(2.35) will be proved simultaneously with an induction argument based on Lemmata 2.2.6 and 2.2.7. In order to prove condition (2.31) we need to prove that $N(A, E) \in \mathcal{M}(R)$ for every $A \in \mathcal{T}_{2,2}(R)$, $E \in \mathcal{M}(R)$. On the other hand, for $A \in \mathcal{T}_{1,1}(R)$ this is automatically satisfied. It is therefore enough to consider $A = a \otimes b \in R^{\otimes 2}$. To prove (2.32) we need to show that $L_\lambda(A, E) \in \mathcal{M}(R)$ for every $A \in \mathcal{T}_{2,2}(R)$ and $E \in \mathcal{M}(R)$. We will consider separately the two cases $A = a \in R$ and $A = a \otimes b \in R^{\otimes 2}$. To prove (2.33) and (2.34) we need to show that $\mathrm{lW}(A, B, D; \lambda) \in \mathcal{M}(R)$ in the two cases $A \in \mathcal{T}_{1,1}(R)$, $B \in \mathcal{T}_{2,0}(R)$, $D \in \mathcal{T}(R)$, and $A \in \mathcal{T}_{2,0}(R)$, $B \in \mathcal{T}_{1,1}(R)$, $D \in \mathcal{T}(R)$. If either $\deg(A) = 0$, or $\deg(B) = 0$ or $A \in R$ and $B \in R$, we have $\mathrm{lW}(A, B, D; \lambda) = 0$ thanks to the left Wick formula (2.3). We thus only need to consider the two cases $A = a \in R$, $B = b \otimes c \in R_{\bar{0}}^{\otimes 2}$ and $A = a \otimes b \in R_{\bar{0}}^{\otimes 2}$, $B = c \in R$. Finally, to prove (2.35) we need to show that $\mathrm{J}(A, B, D; \lambda, \mu) \in \mathcal{M}(R)$ for $A = a \in R$, $B \in \mathcal{T}_{2,0}(R)$ and $D \in \mathcal{T}(R)$. We will consider separately the two cases $B = b \otimes c \in R_{\bar{0}}^{\otimes 2}$ and $B = b \in R$. In conclusion, the corollary will follow once we will have proved the following conditions

$$L_\lambda(\mathrm{skn}(a, b, 1), D) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \qquad \forall a, b \in R,\ D \in \mathcal{T}(R) , \tag{2.36}$$

$$N(a \otimes b, E) \in \mathcal{M}(R) , \qquad \forall a, b \in R,\ E \in \mathcal{M}(R) , \tag{2.37}$$

$$L_\lambda(a \otimes b, E) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \qquad \forall a, b \in R,\ E \in \mathcal{M}(R) , \tag{2.38}$$

$$L_\lambda(a, F) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \qquad \forall a \in R,\ F \in \mathcal{M}(R) , \tag{2.39}$$

$$\mathrm{lW}(a, b \otimes c, D; \lambda) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \qquad \forall a \in R,\ b, c \in R_0,\ D \in \mathcal{T}(R) , \tag{2.40}$$

$$\mathrm{lW}(a \otimes b, c, D; \lambda) \in \mathbb{C}[\lambda] \otimes \mathcal{M}(R) , \qquad \forall a, b \in R_{\bar{0}},\ c \in R,\ D \in \mathcal{T}(R) , \tag{2.41}$$

$$J(a, b \otimes c, D; \lambda, \mu) \;\in\; \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}(R) \;, \quad \forall a \in R,\; b, c \in R_{\bar{0}},\; D \in \mathcal{T}(R) \;, \tag{2.42}$$

$$J(a, b, c \otimes D; \lambda, \mu) \;\in\; \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}(R) \;, \quad \forall a, b, c \in R,\; D \in \mathcal{T}(R) \;. \tag{2.43}$$

Notice that if $\deg(D) = \deg(E) = 0$ and $\deg(F) \leq (1, 1)$, all conditions (2.36)–(2.43) are satisfied. For (2.43) this is true due to assumption (2.8) on the λ–bracket. Let's assume then, by induction, that (2.36)–(2.43) are satisfied for $D \in \mathcal{T}_{p,q}(R)$ with $(p, q) < (n, k)$, $E \in \mathcal{T}_{p,q}(R) \cap \mathcal{M}(R)$ with $(p, q) < (n, k)$ and $F \in \mathcal{T}_{p,q}(R) \cap \mathcal{M}(R)$ with $(p, q) < (n + 1, k + 1)$. We need to prove that (2.36)–(2.43) are satisfied for $D \in \mathcal{T}_{n,k}(R)$, $E \in \mathcal{T}_{n,k}(R) \cap \mathcal{M}(R)$ and $F \in \mathcal{T}_{n+1,k+1}(R) \cap \mathcal{M}(R)$. Condition (2.36) follows by equation (2.27) and inductive assumption on (2.43). By quasi–associativity relation (2.6) we have

$$N(a \otimes b, E) = a \otimes b \otimes E + \Big(\int_0^T d\lambda\; a\Big) \otimes L_\lambda(b, E) + \Big(\int_0^T d\lambda\; b\Big) \otimes L_\lambda(a, E) \;,$$

and every term in the right hand side is in $\mathcal{M}(R)$ by inductive assumption on (2.39). This proves (2.37). Similarly, by right Wick formula (2.4) we have

$$\begin{aligned} L_\lambda(a \otimes b, E) \;&=\; \Big(e^{T\partial_\lambda} a\Big) \otimes L_\lambda(b, E) \;+\; p(a, b)\Big(e^{T\partial_\lambda} b\Big) \otimes L_\lambda(a, E) \\ &+\; p(a, b) \textstyle\int_0^\lambda d\mu\; L_\mu(b, L_{\lambda-\mu}(a, E)) \;. \end{aligned}$$

Since $\deg(L_\lambda(a, E)) < (n+1, k+1)$, we can use again inductive assumption on (2.39) to deduce that the right hand side belongs to $\mathcal{M}(R)$, thus proving (2.38). In order to prove (2.39) we will consider separately the following two situations:

(a) $\quad F = b \otimes E$ for $b \in R,\; E \in \mathcal{M}(R) \cap \mathcal{T}_{n,k}(R)$,

(b) $\quad F = \mathrm{skn}(b, c, D) \in \mathcal{M}(R) \cap \mathcal{T}_{n+1,k+1}(R)$, $\quad$ (and $D \in \mathcal{T}_{n-1,k-1}(R)$) .

In the first case we have, by left Wick formula (2.3)

$$L_\lambda(a, b \otimes E) = N(L_\lambda(a, b), E) + p(a, b) b \otimes L_\lambda(a, E) + \int_0^\lambda d\mu\; L_\mu(L_\lambda(a, b), E) \;,$$

and all terms in the right hand side are in $\mathcal{M}(R)$ by inductive assumption and by (2.37) and (2.38). In the second case, $L_\lambda(a, F)$ is given by equation (2.28), and (2.39) follows from Corollaries 2.2.5 and 2.2.8, from equation (2.36) and from inductive assumption on (2.40) and (2.41). Conditions (2.40) and (2.41) follow respectively by equations (2.24) and (2.25) in Lemma 2.2.6 and inductive assumption on (2.43). Similarly, (2.42) follows by equation (2.23), by inductive assumption on (2.43) and by (2.39). Finally, (2.43) follows by Corollary 2.2.8, by equations (2.36), (2.40), (2.41), (2.42) and by inductive assumption. This concludes the proof of the corollary. □

2.3 Proof of Theorem 2.1.7

In this section we want to use the results of Section 2.2 to prove Theorem 2.1.7. Let us denote by $\tilde{\mathcal{B}}$ the collection of homogeneous ordered monomials in $\mathcal{T}(R)$:

$$\tilde{\mathcal{B}} = \Big\{ a_{i_1} \otimes \cdots \otimes a_{i_n} \ , \quad 1 \leq i_1 \leq \cdots \leq i_n, \ n \in \mathbb{Z}_+ \Big\}$$

and let $\tilde{\mathcal{B}}[n,k]$ be the collection of homogeneous ordered monomials of degree (n,k); we then have

$$\tilde{\mathcal{B}} = \bigsqcup_{(n,k)\in\mathcal{D}} \tilde{\mathcal{B}}[n,k] \ .$$

We denote the corresponding filtration by

$$\tilde{\mathcal{B}}_{n,k} = \bigcup_{(p,q)\leq(n,k)} \tilde{\mathcal{B}}[p,q] \ .$$

By definition (see Theorem 2.1.7) we have $\mathcal{B} = \pi(\tilde{\mathcal{B}})$ and we denote

$$\mathcal{B}_{n,k} = \pi(\tilde{\mathcal{B}}_{n,k}) = \left\{ \begin{array}{c} :a_{i_1}\dots a_{i_p}: \ , \quad 1 \leq i_1 \leq \cdots \leq i_p \\ \text{s.t. either } p<n, \text{ or } p=n \text{ and } \sum_j p(a_{i_j}) \leq k \end{array} \right\}$$

Lemma 2.3.1. *1. Any element $E \in \mathcal{T}_{n,k}(R)$ can be decomposed as*

$$E = P + M \ , \tag{2.44}$$

where $P \in span_{\mathbb{C}}\tilde{\mathcal{B}}_{n,k}$ and $M \in \mathcal{M}(R)$.

2. The space $U_{n,k}(R)$ is spanned by the set $\mathcal{B}_{n,k}$. In particular

$$U(R) = span_{\mathbb{C}}\mathcal{B} \ .$$

Proof. We want to prove part 1 by induction on (n,k). For $(n,k) = (0,0)$ we have $\mathcal{T}_{(0,0)}(R) = \mathbb{C}$ and the statement is trivial. Let then $(n,k) \geq (1,0)$. It suffices to prove the statement for monomials $E = a_{i_1} \otimes \cdots \otimes a_{i_n} \in \mathcal{T}(R)$ of degree (n,k) (They obviously form a basis of $\mathcal{T}(R)$, if $a_{i_j} \in \mathcal{A}$). Let us define the *disorder* of E by

$$d(E) = \#\Big\{ (p,q) \quad \text{s.t. } 1 \leq p < q \leq n \ , \ i_p > i_q \Big\} \ .$$

We will prove that E decomposes as in (2.44) by induction on $d(E)$. To say $d(E) = 0$ is equivalent to say that E is an ordered monomial and $E \in \tilde{\mathcal{B}}_{n,k}$, so there is nothing to prove. Suppose then $d(E) \geq 1$ and let $p \in \{1,\dots,n-1\}$ be such that $i_p > i_{p+1}$.

By definition of $\mathcal{M}(R)$ we have

$$\begin{aligned} E &\equiv p(a_{i_p}, a_{i_{p+1}}) a_{i_1} \otimes \cdots \otimes a_{i_{p+1}} \otimes a_{i_p} \otimes \cdots \otimes a_{i_n} \\ &+ a_{i_1} \otimes \cdots \otimes a_{i_{p-1}} \otimes N\Big(L(a_{i_p}, a_{i_{p+1}}), a_{i_{p+2}} \otimes \cdots \otimes a_{i_n}\Big) \qquad \text{mod } \mathcal{M}(R) \ . \end{aligned}$$

Recall $L(a,b)$ was defined in Lemma 2.1.6. The first term in the right hand side has disorder $d(E)-1$, so it can be decomposed as in (2.44) by induction. By Lemma 2.1.3, the second term of the right hand side belongs to $\mathcal{T}_{n,k-1}(R)$, therefore by inductive assumption it also decomposes as in (2.44). This concludes the proof of the first part of the lemma. Part 2 follows trivially from part (a) and from the definition of $U(R)$. □

Lemma 2.3.2. *Let us denote by $\tilde{\mathcal{U}}$ a vector space with basis $\tilde{\mathcal{B}}$, namely*

$$\tilde{\mathcal{U}} = \bigoplus_{A \in \tilde{\mathcal{B}}} \mathbb{C}A \ .$$

There is a unique linear map

$$\sigma \ : \ \mathcal{T}(R) \ \longrightarrow \ \tilde{\mathcal{U}}$$

such that

1. $\sigma(1) = 1$,
2. $\sigma(A) = A, \ \forall A \in \tilde{\mathcal{B}}$,
3. $\mathcal{M}(R) \subset \ker \sigma$.

Proof. We want to prove that there is a unique sequence of linear maps $\sigma_{n,k} \ : \mathcal{T}_{n,k}(R) \ \longrightarrow \ \tilde{\mathcal{U}}$, such that

1. $\sigma_{0,0}(1) = 1 \ , \quad \sigma_{n,k}\big|_{\mathcal{T}_{p,q}(R)} = \sigma_{p,q}, \ \forall (p,q) \leq (n,k)$,
2. $\sigma_{n,k}(A) = A, \ \forall A \in \tilde{\mathcal{B}}_{n,k}$,
3. $\mathcal{M}(R) \cap \mathcal{T}_{n,k}(R) \subset \ker \sigma_{n,k}$.

This obviously proves the lemma, since we can then define σ with the conditions $\sigma\big|_{\mathcal{T}_{n,k}(R)} = \sigma_{n,k}, \ \forall (n,k) \in \mathcal{D}$. The condition $\sigma_{0,0}(1) = 1$ defines completely $\sigma_{0,0}$. Let then $(n,k) \geq (1,0)$ and suppose by induction that $\sigma_{p,q}$ is uniquely defined and it satisfies all conditions (1)–(3) for every $(p,q) < (n,k)$.

A) *Uniqueness of $\sigma_{n,k}$*

Given $E = a_{i_1} \otimes \cdots \otimes a_{i_n} \in \mathcal{T}_{n,k}(R)$ with $a_{i_j} \in \mathcal{A}$, we will prove that $\sigma_{n,k}(E)$ is uniquely defined by induction on the disorder $d(E)$ (see the definition above). For $d(E) = 0$ we have $E \in \tilde{\mathcal{B}}$, so it must be $\sigma_{n,k}(E) = E$ by condition (2). Let then

$d(E) \geq 1$ and let (i_p, i_{p+1}) be the "left most disorder", namely $p \in \{1, \dots, n\}$ is the smallest integer such that $i_p > i_{p+1}$. By condition (3) we have

$$\begin{aligned} \sigma_{n,k}(E) &= p(a_{i_p}, a_{i_{p+1}})\sigma_{n,k}\Big(a_{i_1} \otimes \cdots \otimes a_{i_{p-1}} \otimes a_{i_{p+1}} \otimes a_{i_p} \otimes a_{i_{p+2}} \otimes \cdots \otimes a_{i_n}\Big) \\ &+ \sigma_{n,k}\Big(a_{i_1} \otimes \cdots \otimes a_{i_{p-1}} \otimes N(L(a_{i_p}, a_{i_{p+1}}), a_{i_{p+2}} \otimes \cdots \otimes a_{i_n})\Big) . \end{aligned}$$

Since $(a_{i_1} \otimes \cdots \otimes a_{i_{p+1}} \otimes a_{i_p} \otimes \cdots \otimes a_{i_n})$ has disorder $d(E) - 1$, the first term in the right hand side is uniquely defined by inductive assumption. Moreover, by Lemma 2.1.3 we have $(a_{i_1} \otimes \cdots \otimes a_{i_{p-1}} \otimes N(L(a_{i_p}, a_{i_{p+1}}), a_{i_{p+2}} \otimes \cdots \otimes a_{i_n})) \in \mathcal{T}_{n,k-2}(R)$, so that the second term of the right hand side is uniquely defined by condition (1) and inductive assumption.

B) *Existence of* $\sigma_{n,k}$

The above prescription defines uniquely a linear map $\sigma_{n,k} : \mathcal{T}_{n,k}(R) \to \tilde{\mathcal{U}}$, which by construction satisfies conditions (1) and (2). We are left to show that condition (3) holds. By definition of $\mathcal{M}(R)$ it suffices to prove that, for any homogeneous monomial $E = a_{i_1} \otimes \cdots \otimes a_{i_n} \in \mathcal{T}_{n,k}(R)$ $(a_{i_j} \in \mathcal{A})$ of degree (n, k), and for any $q = 1, \dots, n$, one has

$$\sigma_{n,k}\Big(a_{i_1} \otimes \dots a_{i_{q-1}} \otimes \mathrm{skn}(a_{i_q}, a_{i_{q+1}}, a_{i_{q+2}} \otimes \cdots \otimes a_{i_n})\Big) = 0 . \tag{2.45}$$

Without loss of generality $i_q > i_{q+1}$, so that $d(E) \geq 1$. We will prove equation (2.45) by induction on $d(E)$. Let (i_p, i_{p+1}) be the "left most disorder". For $p = q$ equation (2.3.2) holds by construction, so there is nothing to prove. We will consider separately the cases $p < q - 1$ and $p = q - 1$.

1) Assume $p < q - 1$. For simplicity we rewrite

$$E = A \otimes c \otimes b \otimes D \otimes f \otimes e \otimes H ,$$

where $A = a_{i_1} \otimes \cdots \otimes a_{i_{p-1}}$ $(i_1 \leq \cdots \leq i_{p-1} \leq i_p)$, $c = a_{i_p}$, $b = a_{i_{p+1}}$ $(i_p > i_{p+1})$, $D = a_{i_{p+2}} \otimes \cdots \otimes a_{i_{q-1}}$, $f = a_{i_q}$, $e = a_{i_{q+1}}$ $(i_q > i_{q+1})$, $H = a_{i_{q+2}} \otimes \cdots \otimes a_{i_n}$. The left hand side of equation (2.45) then takes the form

$$\begin{aligned} &\sigma_{n,k}\Big(A \otimes c \otimes b \otimes D \otimes f \otimes e \otimes H\Big) - p(e, f)\sigma_{n,k}\Big(A \otimes c \otimes b \otimes D \otimes e \otimes f \otimes H\Big) \\ &\qquad - \sigma_{n,k}\Big(A \otimes c \otimes b \otimes D \otimes N(L(f, e), H)\Big) . \end{aligned} \tag{2.46}$$

By definition of $\sigma_{n,k}$, the first term of (2.46) can be written as

$$\sigma_{n,k}\Big(p(b, c)A \otimes b \otimes c \otimes D \otimes f \otimes e \otimes H + A \otimes N(L(c, b), D \otimes f \otimes e \otimes H)\Big) . \tag{2.47}$$

Since $d(A \otimes c \otimes b \otimes D \otimes e \otimes f \otimes H) = d(E) - 1$, we can use inductive assumption to rewrite the second term of (2.46) as

$$-p(e, f)\sigma_{n,k}\Big(p(b, c)A \otimes b \otimes c \otimes D \otimes e \otimes f \otimes H + A \otimes N(L(c, b), D \otimes e \otimes f \otimes H)\Big) . \tag{2.48}$$

By Lemma 2.1.3 we know that $A \otimes c \otimes b \otimes D \otimes N(L(f,e),H) \in \mathcal{T}_{n,k-2}(R)$, so that we can use the fact that $\sigma_{n,k}\big|_{\mathcal{T}_{n,k-2}(R)} = \sigma_{n,k-2}$ and inductive assumption to rewrite the third term of (2.46) as

$$-\sigma_{n,k}\Big(p(b,c)A\otimes b\otimes c\otimes D\otimes N(L(f,e),H)+A\otimes N(L(c,b),D\otimes N(L(f,e),H))\Big)\ . \quad (2.49)$$

Combining (2.47), (2.48) and (2.49) we can rewrite (2.46) as

$$\sigma_{n,k}\Big(p(b,c)A\otimes b\otimes c\otimes D\otimes \mathrm{skn}(f,e,H)+A\otimes N(L(c,b),D\otimes \mathrm{skn}(f,e,H))\Big)\ . \quad (2.50)$$

The first term of (2.50) is zero by inductive assumption, since $d(A\otimes b\otimes c\otimes D\otimes f\otimes e\otimes H) = d(E)-1$. Consider the second term in the argument of $\sigma_{n,k}$ in (2.50). By Lemma 2.1.3 we know that it is an element of $\mathcal{T}_{n,k-2}(R)$; moreover by Corollary 2.2.9 we know that it is in $\mathcal{M}(R)$. It follows by induction that also the second term of (2.50) is zero. We thus proved, as we wanted, that (2.46) is zero.

2) We are left to consider the case $p = q-1$. For simplicity we rewrite

$$E = A\otimes c\otimes b\otimes a\otimes D\ ,$$

where $A = a_{i_1}\otimes\cdots\otimes a_{i_{p-1}}$ $(i_1 \le \cdots \le i_{p-1} \le i_p)$, $c = a_{i_p}$, $b = a_{i_{p+1}}$, $a = a_{i_{p+2}}$ $(i_p > i_{p+1} > i_{p+2})$, $D = a_{i_{p+3}}\otimes\cdots\otimes a_{i_n}$. The left hand side of equation (2.45) then takes the form

$$\sigma_{n,k}\Big(A\otimes c\otimes b\otimes a\otimes D - p(a,b)A\otimes c\otimes a\otimes b\otimes D - A\otimes c\otimes N(L(b,a),D)\Big)\ . \quad (2.51)$$

After some manipulations based on inductive assumption similar to the ones used above, we can rewrite (2.51) as

$$\begin{aligned}\sigma_{n,k}\Big(A\otimes\Big\{&N(L(c,b),a\otimes D) - p(a,b)p(a,c)a\otimes N(L(c,b),D)\\ &+p(b,c)b\otimes N(L(c,a),D) - p(a,b)N(L(c,a),b\otimes D)\\ &+p(b,c)p(a,c)N(L(b,a),c\otimes D) - c\otimes N(L(b,a),D)\Big\}\Big)\ .\end{aligned} \quad (2.52)$$

It follows by Corollary 2.2.5 that

$$\begin{aligned}A\otimes a\otimes N(L(c,b),D) &- p(a,b)p(a,c)A\otimes N(L(c,b),a\otimes D)\\ &\equiv A\otimes N(L(a,L(c,b)),D) \qquad \mathrm{mod}\ \mathcal{M}(R)\ ,\\ A\otimes b\otimes N(L(c,a),D) &- p(a,b)p(b,c)A\otimes N(L(c,a),b\otimes D)\\ &\equiv A\otimes N(L(b,L(c,a)),D) \qquad \mathrm{mod}\ \mathcal{M}(R)\ ,\\ A\otimes c\otimes N(L(b,a),D) &- p(b,c)p(a,c)A\otimes N(L(b,a),c\otimes D)\\ &\equiv A\otimes N(L(c,L(b,a)),D) \qquad \mathrm{mod}\ \mathcal{M}(R)\ .\end{aligned}$$

Since every term in the above expressions is in $\mathcal{T}_{n,k-2}(R)$, we can use inductive assumption to rewrite (2.52) as

$$\sigma_{n,k}\Big(A\otimes N\Big(\Big(p(b,c)L(b,L(c,a))-p(a,b)p(a,c)L(a,L(c,b))-L(c,L(b,a))\Big),D\Big)\Big). \quad (2.53)$$

By Lemma 2.1.6, L satisfies skewsymmetry. A simple computation based on the definition of $L(A,B)$, of $\mathrm{J}(A,B,C;\lambda,\mu)$ and of $\mathrm{skl}(A,B;\lambda)$ allows us to rewrite (2.53) as

$$\begin{aligned} p(a,b)p(a,c)p(b,c)\sigma_{n,k}\Big(A\otimes N\Big(\Big(\int_{-T}^{0}d\lambda\int_{-\lambda-T}^{0}d\mu\ \mathrm{J}(a,b,c;\lambda,\mu)\Big) \\ +\Big(\int_{-T}^{0}d\lambda\ \mathrm{skl}(L(a,b),c;\lambda)\Big),D\Big)\Big)\ . \end{aligned} \quad (2.54)$$

By definition of Lie λ–bracket, we know that $\mathrm{J}(a,b,c;\lambda,\mu)\in\mathbb{C}[\lambda,\mu]\otimes(\mathcal{T}_{2,2}(R)\cap\mathcal{M}(R))$, and by Corollary 2.2.5 $\mathrm{skl}(L(a,b),c;\lambda)\in\mathbb{C}[\lambda]\otimes(\mathcal{T}_{2,2}(R)\cap\mathcal{M}(R))$. We finally notice that, by Corollary 2.2.8, the argument of $\sigma_{n,k}$ in (2.54) is an element of $\mathcal{M}(R)$, and by part B) of Lemma 2.1.3 it belongs to $\mathcal{T}_{n-1,n-1}(R)$. Therefore, by inductive assumption, it is in the kernel of $\sigma_{n,k}$, namely (2.54) is zero. We thus proved, as we wanted, that (2.51) is zero. □

Lemma 2.3.3. *If σ is as in Lemma 2.3.2, the induced map*

$$\hat{\sigma}\ :\ U(R)=\mathcal{T}(R)/\mathcal{M}(R)\ \xrightarrow{\sim}\ \tilde{\mathcal{U}}$$

is an isomorphism of vector spaces (namely $\mathcal{M}(R)=\ker\sigma$).

Proof. By definition the map $\hat{\sigma}:\ U(R)\to\tilde{\mathcal{U}}$ is surjective. On the other hand, we have a natural map $\hat{\pi}:\ \tilde{\mathcal{U}}\to U(R)$ which maps every basis element $A=a_{i_1}\otimes\cdots\otimes a_{i_n}\in\tilde{\mathcal{B}}$ to the corresponding $\hat{\pi}(A)=:a_{i_1}\dots a_{i_n}:\in\mathcal{B}\subset U(R)$. By Lemma 2.3.1 this map is also surjective. The composition map

$$\tilde{\mathcal{U}}\ \stackrel{\hat{\pi}}{\longrightarrow}\ U(R)\ \stackrel{\hat{\sigma}}{\longrightarrow}\ \tilde{\mathcal{U}}$$

is the identity map (by definition of $\hat{\pi}$ and $\hat{\sigma}$.) This of course implies that both $\hat{\pi}$ and $\hat{\sigma}$ are isomorphisms of vector spaces. □

The last lemma is basically saying that $\mathcal{B}$ is a basis for the space $U(R)$, thus concluding the proof of Theorem 2.1.7.

2.4 Existence Theorem for vertex algebras

In order to prove Theorem 2.1.8 we will need to use, together with the results in Section 2.2, the so called Existence Theorem for vertex algebras [12], which we want to state here.

In order to state the Existence Theorem, we need an equivalent definition of vertex algebra (see [12]), based on the notion of fields.

Definition 2.4.1. A *field* on a vector superspace V is an expression

$$a(z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}\in \mathrm{End}V[[z^{\pm 1}]]\ ,$$

such that for every $v\in V$ one has $a_{(n)}v=0$ for $n>>0$ (namely $a(z)v$ is a Laurent series in z, with coefficients in V). We say that $a(z)$ has parity $p(a)$ if each $a_{(n)}\in \mathrm{End}V$ has parity $p(a)$. We denote by $\mathrm{glf}(V)$ the space of fields on V.

For each $n\in\mathbb{Z}$, one defines the n–th product of fields as

$$a(z)_{(n)}b(z)=\mathrm{Res}_x\Big(a(x)b(z)i_{x,z}(x-z)^n-p(a,b)b(z)a(x)i_{z,x}(x-z)^n\Big)\ . \qquad (2.55)$$

Here Res_x denotes the coefficient of x^{-1} and $i_{x,z}$ (respectively $i_{z,x}$) stands for the series expansion in the domain $|x|>|z|$ (resp. $|z|>|x|$); namely

$$\begin{aligned} i_{x,z}(x-z)^n &= \sum_{j\geq 0}\binom{n}{j}x^{n-j}(-z)^j\ ,\\ i_{z,x}(x-z)^n &= \sum_{j\geq 0}\binom{n}{j}x^{j}(-z)^{n-j}\ . \end{aligned}$$

It is easy to see that the space $\mathrm{glf}(V)$ is closed under all n–th products, and also under derivation ∂_z by the indeterminate z. Formula (2.55) is equivalent to the following two formulas for $n\in\mathbb{Z}_+$

$$\begin{aligned} a(z)_{(n)}b(z) &= \mathrm{Res}_x[a(x),b(z)](x-z)^n\ ,\\ a(z)_{(-n-1)}b(z) &= :(\partial_z^{(n)}a(z))b(z):\ . \end{aligned}$$

We denoted by $[a(z),b(w)]$ the super commutator of fields

$$[a(z),b(w)]=\sum_{m,n\in\mathbb{Z}}[a_{(m)},b_{(n)}]z^{-m-1}w^{-n-1}\ ,$$

and by $:a(z)b(z):$ the normal ordered product of fields, defined as

$$:a(z)b(z):=a(z)_+b(z)+p(ab)b(z)a(z)_-\ ,$$

where

$$a(z)_+=\sum_{n\leq -1}a_{(n)}z^{-n-1}\ ,\quad a(z)_-=\sum_{n\geq 0}a_{(n)}z^{-n-1}\ .$$

Definition 2.4.2. A pair of fields $a(z),b(z)\in \mathrm{glf}(V)$ is said to be *local* if

$$(z-w)^N[a(z),b(w)]=0\ ,\quad \text{for } N>>0\ .$$

It is not hard to prove the following (see [12])

Lemma 2.4.3. *A pair of fields* $(a(z), b(z))$ *is local if and only if*

$$[a(z), b(w)] = \sum_{\substack{j \geq 0 \\ \text{(finite)}}} (a(w)_{(j)}b(w))\partial_w^{(j)}\delta(z-w) \ .$$

Recall the definition of the formal δ–distribution

$$\delta(z-w) = (i_{z,w} - i_{w,z})(z-w)^{-1} = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \ .$$

It is characterized by the property

$$\text{Res}_z v(z)\delta(z-w) = v(w) \ , \quad \forall v(z) \in V[[z^{\pm 1}]] \ .$$

Definition 2.4.4. Let $(V, |0\rangle)$ be a pointed vector superspace, with an even endomorphism T, and let $R \subset V$ be a $\mathbb{Z}_2$–graded $\mathbb{C}[T]$–submodule containing $|0\rangle$. A *state–field correspondence* from R to V is a parity preserving linear map $Y: \ R \to \text{glf}(V)$, denoted by $a \mapsto Y(a,z)$, such that the following axioms hold:

1. vacuum axioms: $\quad Y(|0\rangle, z) = \mathbb{1}_V \ , \quad Y(a,z)|0\rangle = a + Ta + \cdots \in V[[z]]$,
2. translation invariance $\quad [T, Y(a,z)] = Y(Ta, z) = \partial_z Y(a,z)$.

The state–field correspondence $Y: \ R \to \text{glf}(V)$ is said to be *local* if every pair of fields $(Y(a,z), Y(b,z))$, with $a, b \in R$, is local.

The above definition was introduced in [1], together with the following

Lemma 2.4.5. *Giving a state–field correspondence* Y *from* R *to* V *is equivalent to providing* $R \otimes V$ *with two parity preserving operations:*

1. *a* λ*–bracket* $R \otimes V \to V[\lambda]$*, denoted* $a \otimes B \mapsto [a \ _\lambda \ B]$*, satisfying sesquilinearity*

   $$[Ta \ _\lambda \ B] = -\lambda[a \ _\lambda \ B] \ , \ [a \ _\lambda \ TB] = (\lambda + T)[a \ _\lambda \ B] \ , \quad \forall a \in R, \ B \in V \ ,$$

2. *a normal ordered product* $R \otimes V \to V$*, denoted* $a \otimes B \mapsto \, :aB:$*, such that* $|0\rangle$ *is a unity and* T *is a derivation.*

More precisely, this correspondence is obtained in the following way. If Y *is a state–field correspondence from* R *to* V*, then we define* $(a \in R, \ B \in V)$

$$[a \ _\lambda \ B] = Res_z e^{\lambda z} Y(a,z)B \ , \quad :aB: = a_{(-1)}B \ .$$

Conversely, given a λ*–bracket and a normal ordered product on* $R \otimes V$*, we get a state field correspondence by defining* $(a \in R, \ B \in V)$

$$Y(a,z)_- B = [a \ _{-\partial_z} \ B]z^{-1} \ , \quad Y(a,z)_+ B = \, :(e^{zT}a)B: \ . \tag{2.56}$$

Proof. The proof of this statement is straightforward. □

Thanks to Lemma 2.4.5 we can reformulate the definition of vertex algebra in terms of the state–field correspondence Y. We have the following

Definition 2.4.6. A *vertex algebra* is a vector superspace V, with an even vacuum vector $|0\rangle \in V$, an even infinitesimal translation operator $T \in \mathrm{End}V$ and a local state–field correspondence $Y:\ V \to \mathrm{glf}(V)$.

This definition of vertex algebra was first given in [12]. The equivalence between this definition and the one given in Section 1.1 is far from obvious; a proof can be found in [1].

The following fact is known as Existence Theorem, and it states that, given a suitable collection of pairwise local fields, one can construct a vertex algebra.

Theorem 2.4.7. *Let V be a vector superspace, let $|0\rangle$ be an even vector of V and T an even endomorphism on V. Let $\{a^\alpha(z),\ \alpha \in A\}$ (A an index set) be a collection of fields such that*

(i) $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$, *for all* $\alpha \in A$,

(ii) $T|0\rangle = 0,\ a^\alpha(z)|0\rangle \in V[[z]]$, *and denote* $a^\alpha = a^\alpha(z)|0\rangle|_{z=0} \in V$,

(iii) every pair $(a^\alpha(z), a^\beta(z))$, *for* $\alpha, \beta \in A$, *is local,*

(iv) the vectors $a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_s}_{(j_s)}|0\rangle$ *with* $s \in \mathbb{Z}_+,\ j_n \in \mathbb{Z},\ \alpha_n \in A$, *span* V.

Then the formula

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_s}_{(j_s)}|0\rangle, z) = a^{\alpha_1}(z)_{(j_1)}(a^{\alpha_2}(z)_{(j_2)}(\cdots a^{\alpha_s}(z)_{(j_s)}\mathbb{I}_V))$$

defines a unique vertex algebra structure on V such that $|0\rangle$ is the vacuum vector, T is the infinitesimal translation operator and $Y(a^\alpha, z) = a^\alpha(z)$ for all $\alpha \in A$.

The proof of the Existence Theorem can be found in [12]. In the next subsection we will use the Existence Theorem (or rather Corollary 2.4.9 below) to prove Theorem 2.1.8.

Remark 2.4.8. In reference [12] the Existence Theorem is stated with one additional "non degeneracy" assumption:

(v) the linear map $\sum_\alpha \mathbb{C}a^\alpha(z) \to \sum_\alpha \mathbb{C}a^\alpha$, *defined by* $a^\alpha(z) \mapsto a^\alpha$, *is injective.*

This hypothesis is actually not needed.

Proof. of the Remark. Suppose $(V, |0\rangle, T, \mathcal{A} = \{a^\alpha(z), \alpha \in A\})$ satisfy assumptions (i), (ii), (iii) and (iv) of Existence Theorem. We want to prove that the non degeneracy condition (v) automatically holds. For this, let $a(z) \in \mathrm{span}_{\mathbb{C}}\mathcal{A}$ be a field such that $a = a(z)|0\rangle|_{z=0} = 0$. (Namely $a(z)$ is in the kernel of the linear map defined in (v)). We want to prove that $a(z) = 0$. Clearly $a(z)|0\rangle = e^{zT}a = 0$. Assume, by induction on s, that $a(z)b = 0$ for every $b \in W_s = \text{span}_{\mathbb{C}}\left\{ a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_s}_{(j_s)}|0\rangle,\ \alpha_i \in A,\ j_i \in \mathbb{Z},\ i = 1 \dots s \right\}$. We then have by locality ($N >> 0$)

$$(z-w)^N a(z) a^\alpha(w) b = p(a, a^\alpha)(z-w)^N a^\alpha(w) a(z) b = 0 \ , \quad \forall \alpha \in A \ .$$

Since $a^\alpha(z) a^\beta(w) b$ involves only finitely many negative powers of w we conclude that

$$a(z) a^\alpha_{(j)} b = 0,\ \forall j \in \mathbb{Z} \ .$$

In other words $a(z)v = 0,\ \forall v \in V$, as we wanted. □

Corollary 2.4.9. *Let V be a vector superspace with an even vector $|0\rangle \in V$ and an even endomorphism $T \in End_{\mathbb{C}} V$, let $R \subset V$ be a $\mathbb{Z}_2$–graded T–invariant subspace containing $|0\rangle$, and let $Y: \ R \to \text{glf}(V)$ be a local state field correspondence such that vectors $a_{(j_1)} b_{(j_2)} \cdots |0\rangle$ with $s \in \mathbb{Z}_+,\ j_n \in \mathbb{Z},\ a, b, \dots \in R$, span V (we are using the notation $Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$). Then V has a unique vertex algebra structure compatible with Y.*

Proof. This corollary follows immediately from the Existence Theorem by choosing $\{Y(a,z),\ a \in R\}$ as collection of local fields. □

2.5 Proof of Theorem 2.1.8

The following lemma is immediate consequence of the results in Section 2.2.

Lemma 2.5.1. *(a) The space $U(R)$ is endowed with an endomorphism (infinitesimal translation operator) $T: \ U(R) \to U(R)$, defined by*

$$T(\pi(A)) = \pi(TA) \ , \quad \forall A \in \mathcal{T}(R) \ ,$$

with a λ–bracket $[\ {}_\lambda\ ]: \ U_{2,2}(R) \otimes U(R) \to \mathbb{C}[\lambda] \otimes U(R)$, given by

$$[\pi(A)\ {}_\lambda\ \pi(B)] = \pi(L_\lambda(A,B)) \ , \quad \forall A \in \mathcal{T}_{2,2}(R),\ B \in \mathcal{T}(R) \ ,$$

and with a normally ordered product $: \ , \ : \ U_{2,2}(R) \otimes U(R) \to U(R)$, given by

$$: \pi(A)\pi(B) : = \pi(N(A,B)) \ , \quad \forall A \in \mathcal{T}_{2,2}(R),\ B \in \mathcal{T}(R) \ .$$

(b) All above maps are parity preserving (we assign to $U(R)$ the parity induced by $\mathcal{T}(R)$). The λ–bracket satisfies sesquilinearity with respect to T

$$[TA\ {}_\lambda\ B] = -\lambda[A\ {}_\lambda\ B] \ , \ [A\ {}_\lambda\ TB] = (\lambda + T)[A\ {}_\lambda\ B] \ , \quad \forall A \in U_{2,2}(R),\ B \in U(R) \ .$$

The normally ordered product satisfies Leibniz rule with respect to T

$$T(:AB:) \ = :(TA)B: + :A(TB): \ , \quad \forall A \in U_{2,2}(R),\ B \in U(R) \ .$$

(c) For $a \in R,\ B \in U_{2,0}(R)$ the following skewsymmetry conditions hold

$$\begin{aligned} [a\ {}_\lambda\ B] &= -p(a,B)[B\ {}_{-\lambda-T}\ a] \\ :aB: &= p(a,B):Ba: + \textstyle\int_{-T}^0 d\lambda\ [a\ {}_\lambda\ B]\ . \end{aligned}$$

(d) For $a,b \in R,\ C \in U(R)$

$$[a\ {}_\lambda\ [b\ {}_\mu\ C]] = p(a,b)[b\ {}_\mu\ [a\ {}_\lambda\ C]] + [[a\ {}_\lambda\ b]\ {}_{\lambda+\mu}\ C]\ , \tag{2.57}$$

$$\begin{aligned} [a\ {}_\lambda\ :bC:] &= :[a\ {}_\lambda\ b]C: + p(a,b):b[a\ {}_\lambda\ C]: \\ &+ \int_0^\lambda d\mu\ [[a\ {}_\lambda\ b]\ {}_\mu\ C]\ , \end{aligned} \tag{2.58}$$

$$\begin{aligned} [:ab:\ {}_\lambda\ C] &= :\left(e^{T\partial_\lambda}a\right)[b\ {}_\lambda\ C]: + p(a,b):\left(e^{T\partial_\lambda}b\right)[a\ {}_\lambda\ C]: \\ &+ p(a,b)\int_0^\lambda d\mu\ [b\ {}_\mu\ [a\ {}_{\lambda-\mu}\ C]]\ , \end{aligned} \tag{2.59}$$

$$\begin{aligned} :(:ab:)C: &= :a(:bC:): + :\left(\int_0^T d\lambda\ a\right)[b\ {}_\lambda\ C]: \\ &+ p(a,b):\left(\int_0^T d\lambda\ b\right)[a\ {}_\lambda\ C]:\ . \end{aligned} \tag{2.60}$$

Remark 2.5.2. Quasi–associativity (2.60) implies that the expression $:(:ab:)C: - :a(:bC:):$ is super symmetric under the exchange of a and b. It follows that

$$:a(:bC:) - p(a,b):b(:aC:) :=: (:ab:)C: -p(a,b):(:ba:)C:\ . \tag{2.61}$$

Thanks to Lemma 2.4.5, with the λ–bracket and the normally ordered product we can define a state–field correspondence

$$Y:\ U_{2,2}(R) \to \mathrm{glf}(U(R))\ ,$$

given by (2.56). According to Corollary 2.4.9, in order to prove that $U(R)$ has a vertex algebra structure (compatible with the λ–bracket $[\ {}_\lambda\ ]$ and the normally ordered product $:\ :$), it suffices to show that the restriction of the state–field correspondence Y to R, namely $Y|_R:\ R \to \mathrm{glf}(U(R))$, is local.

Locality of $Y|_R$ will be proved with a sequence of two lemmata.

Lemma 2.5.3. *For $A \in U_{2,2}(R)$ and $n \in \mathbb{Z}$, let $A_{(n)} \in End\ U(R)$ be defined by $Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)}z^{-n-1}$, where $Y(A,z)$ is given by (2.56). The following super commutation relations hold:*

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}\ , \quad \forall\ a,b \in R,\ m,n \in \mathbb{Z}\ . \tag{2.62}$$

Proof. First, notice that the endomorphisms $A_{(n)}$, for $A \in U_{2,2}(R)$, are equivalently defined by ($B \in U(R)$)

$$\begin{aligned} [A\ {}_\lambda\ B] &= \textstyle\sum_{n\geq 0} \lambda^{(n)} A_{(n)} B\ , \\ A_{(-n-1)} B &= \ :(T^{(n)} A) B:\ , \qquad \forall n \in \mathbb{Z}_+\ . \end{aligned} \tag{2.63}$$

By translation invariance of the state–field correspondence Y, we have ($A \in U_{2,2}(R)$)

$$[T, A_{(n)}] = -n A_{(n-1)} = (TA)_{(n)}\ , \qquad \forall n \in \mathbb{Z}\ . \tag{2.64}$$

Equation (2.57) is equivalent to (2.62) for $m, n \in \mathbb{Z}_+$. From (2.58) we get, after substituting b with $T^{(j)} b$

$$\begin{aligned} &[a\ {}_\lambda\ :(T^{(j)} b) C:] \ - \ p(a,b) :(T^{(j)} b)[a\ {}_\lambda\ C]: \\ &= :\Big((\lambda + T)^{(j)} [a\ {}_\lambda\ b]\Big) C: \ + \ \textstyle\int_0^\lambda d\mu\ (\lambda-\mu)^{(j)} [[a\ {}_\lambda\ b]\ {}_\mu\ C]\ . \end{aligned} \tag{2.65}$$

The coefficient of $\lambda^{(m)}$ of the left hand side is

$$[a_{(m)}, b_{(-j-1)}] C\ .$$

The coefficient of $\lambda^{(m)}$ of the first term in the right hand side of (2.65) is

$$\sum_{\sup\{0,m-j\}\leq i\leq m} \binom{m}{i} (a_{(i)} b)_{(m-j-1-i)} C\ .$$

The coefficient of $\lambda^{(m)}$ of the second term in the right hand side of (2.65) is

$$\sum_{0\leq i\leq m-j-1} \binom{m}{i} (a_{(i)} b)_{(m-j-i-1)} C\ .$$

In order to get this, we used the following identity

$$\int_0^\lambda d\mu\ (\lambda-\mu)^{(p)} \mu^{(q)} \ = \ \lambda^{(p+q+1)}\ , \quad \forall p, q \in \mathbb{Z}_+\ ,$$

which follows by the relation $B(t,s) = \Gamma(t)\Gamma(s)/\Gamma(t+s)$ between the Beta–function and the Gamma–function. Combining the previous results, we get that (2.62) holds for $m \in \mathbb{Z}_+$, $n = -j-1 \in -\mathbb{Z}_+ - 1$. From skewsymmetry of the λ–bracket on R, we have

$$b_{(j)} a = p(a,b) \sum_{k\geq 0} (-1)^{k+j+1} T^{(k)} (a_{(k+j)} b)\ , \quad \forall a, b \in R,\ j \in \mathbb{Z}_+\ . \tag{2.66}$$

Using (2.64), (2.66) and the previous result, we get, for $a, b \in R$, $n \in \mathbb{Z}_+$, $m =$

$-i-1 \in -\mathbb{Z}_+ - 1$

$$[a_{(-i-1)}, b_{(n)}] = \sum_{p,q\geq 0} (-1)^p \binom{n}{p} \binom{n-p-i-1}{q} (a_{(p+q)}b)_{(n-i-p-q-1)} \ .$$

This equation is equivalent to (2.62) for $m = -i-1 \in -\mathbb{Z}_+ - 1,\ n \in \mathbb{Z}_+$, provided the following combinatorial identity

$$\sum_{0\leq p\leq j} (-1)^p \binom{n}{p} \binom{n-i-p-1}{j-p} = \binom{-i-1}{j} \ ,$$

which is not hard to prove. We are left to prove (2.62) for $m, n \in -\mathbb{Z} - 1$. From (2.61) we get

$$\begin{aligned} [a_{(-1)}, b_{(-1)}]C &= \ :(:ab: - p(a,b) :ba:)C: \\ &= \ :\left(\int_{-T}^{0} d\lambda\ [a\ {}_\lambda\ b]\right) C: \\ &= \ \sum_{j\geq 0} (-1)^j (a_{(j)}b)_{(-j-2)} C \ , \end{aligned}$$

which is (2.62) for $m = n = -1$. More in general, if we change a with $T^{(i)}a$ and b with $T^{(j)}b$ in (2.61) we get

$$\begin{aligned} [a_{(-i-1)}, b_{(-j-1)}] &= \left(\int_{-T}^{0} d\lambda\ (-\lambda)^{(i)} (\lambda+T)^{(j)} [a\ {}_\lambda\ b]\right)_{(-1)} \\ &= \sum_{k\geq 0} \binom{-i-1}{k} (a_{(k)}b)_{(-i-j-k-2)} \ , \end{aligned}$$

which is (2.62) for generic $m, n \in -\mathbb{Z}_+ - 1$. □

Lemma 2.5.4. *For $a, b \in R$ one has*

$$[Y(a,z), Y(b,w)] = \sum_{j\geq 0} Y(a_{(j)}b, w) \partial_w^{(j)} \delta(z-w) \ . \tag{2.67}$$

In particular all pairs $(Y(a,z), Y(b,z))$ for $a, b \in R$ are local.

Proof. (2.67) follows, with a straightforward computation, from (2.62). The second part of the statement is obvious. □

This concludes the proof of Theorem 2.1.8.

Chapter 3

Main ideas and techniques used in the classification

As we explained in Chapter 1, we want to study vertex algebras V which are strongly generated by a subspace

$$R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L) \ , \tag{3.1}$$

where $|0\rangle$ is the vacuum vector, L is a Virasoro element, $\mathfrak{g}$ is a finite dimensional space of even primary elements (with respect to L) of conformal weight $\Delta = 1$ and U is a finite dimensional space of odd primary elements of conformal weight $\Delta = 3/2$. We denote as usual $R_{\bar{0}} = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T]L)$ and $R_{\bar{1}} = \mathbb{C}[T] \otimes U$.

According to Theorem 2.1.8, if a space R as in (3.1) is endowed with a Lie λ–bracket of degree 2, namely an even linear map

$$L_\lambda \ : \ R_i \otimes R_j \ \longrightarrow \ (R_{i+j} \oplus \delta_{i=j=\bar{1}} R_{\bar{0}}^{\otimes 2})[\lambda] \ , \quad i,j \in \mathbb{Z}/2\mathbb{Z} \ ,$$

satisfying sesquilinearity, skewsymmetry and Jacobi identity (in the sense specified in Section 2.1), then there is a vertex algebra $V = U(R)$ which is strongly generated by R, with λ–bracket structure compatible with L_λ.

We would like to prove a converse statement to Theorem 2.1.8. Namely, if V is a vertex algebra strongly generated by a subspace $R \subset V$ as in (3.1), we want to find under which assumption the space R is endowed with a Lie λ–bracket L_λ of degree 2. Since V is a vertex algebra, by definition it is endowed with a Lie λ–bracket $[\ _\lambda\ ] \ : \ V \otimes V \ \to \ V[\lambda]$. It is easy to check that this λ–bracket, restricted to $\mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g} \otimes U$, defines a Lie algebra structure on $\mathfrak{g}$ and a structure of $\mathfrak{g}$–module on U. This will be discussed in Section 3.1. Moreover, by simple conformal weight considerations, the restriction of $[\ _\lambda\ ]$ to R is such that

$$[\ _\lambda\ ]\Big|_{R_i \otimes R_j} \ : \ R_i \otimes R_j \ \longrightarrow \ R_{i+j}[\lambda] \quad \text{if either } i = \bar{0} \text{ or } j = \bar{0} \ ,$$

$$[\ _\lambda\ ]\Big|_{R_{\bar{1}} \otimes R_{\bar{1}}} \ : \ R_{\bar{1}} \otimes R_{\bar{1}} \ \longrightarrow \ (R_{\bar{0}} + :R_{\bar{0}} R_{\bar{0}}:)[\lambda] \ .$$

We are denoting $:R_{\bar{0}}R_{\bar{0}}:$ the linear span of elements $:ab:$ with $a,b \in R_{\bar{0}}$. We can then define L_λ to coincide with $[\ {}_\lambda\ ]$ when restricted to $R_i \otimes R_j$, if either $i = \bar{0}$ or $j = \bar{0}$. The only problem is to define $L_\lambda(a,b)$ when both a and b are odd elements. Let π denote the quotient map

$$\pi\ :\ R_{\bar{0}} \oplus R_{\bar{0}}^{\otimes 2} \ \longrightarrow\ R_{\bar{0}} + :R_{\bar{0}}R_{\bar{0}}:\ ,$$

given by $a \otimes b \mapsto :ab:$ for $a,b \in R_{\bar{0}}$. We need to lift the map $[\ {}_\lambda\ ]\Big|_{R_{\bar{1}} \otimes R_{\bar{1}}}\ :\ R_{\bar{1}} \otimes R_{\bar{1}} \ \rightarrow\ (R_{\bar{0}} + :R_{\bar{0}}R_{\bar{0}}:)[\lambda]$ to a map $L_\lambda\ :\ R_{\bar{1}} \otimes R_{\bar{1}}\ \rightarrow\ (R_{\bar{0}} \oplus R_{\bar{0}}^{\otimes 2})[\lambda]$, so that the following diagram commutes

$$\begin{array}{ccc} & \overset{L_\lambda}{\nearrow} & (R_{\bar{0}} \oplus R_{\bar{0}}^{\otimes 2})[\lambda] \\ & & \big\downarrow \pi \\ R_{\bar{1}} \otimes R_{\bar{1}} & \underset{[\ {}_\lambda\ ]}{\longrightarrow} & (R_{\bar{0}} + :R_{\bar{0}}R_{\bar{0}}:)[\lambda] \end{array} \tag{3.2}$$

and in such a way that the resulting linear map L_λ on R is a Lie λ–bracket of degree 2 (according to Definition 2.1.5). We will see in Section 3.2 that such Lie λ–bracket L_λ on R exists under a non degeneracy condition on the vertex algebra V. Loosely speaking, this condition guarantees that the quotient map $\pi:\ (\mathfrak{g} \otimes U)\ \rightarrow\ :\mathfrak{g}U:$ is an isomorphism.

In Section 3.3 we will look more closely at this non degeneracy assumption. In particular we will show that, in the special case in which $\mathfrak{g}$ is a simple Lie algebra and U is an irreducible $\mathfrak{g}$–module, it holds for all but finitely many values of the Kac–Moody level k.

In conclusion, the problem of classifying vertex algebras which are strongly generated by the space R as in (3.1), and which are "non degenerate" (in the sense specified above) reduces to the problem of classifying triples $(\mathfrak{g}, U, L_\lambda)$, where $\mathfrak{g}$ is a Lie algebra, U is a $\mathfrak{g}$–module and L_λ is a Lie λ–bracket of degree 2 on the space R given by (3.1). In Section 3.4 we will find necessary and sufficient conditions on $\mathfrak{g}$ and U in order for R to admit such a Lie λ–bracket L_λ. This will be used in Chapter 4 to completely solve the problem of classification in the special case when $\mathfrak{g}$ is a simple Lie algebra and U is an irreducible $\mathfrak{g}$–module.

3.1 Preliminary results

Let V be any vertex algebra strongly generated by a subspace $R \subset V$ as in (3.1). We will assume that R is a minimal generating set for V, namely V is not generated by any T–invariant proper subspace $R' \subset R$. This is equivalent to ask that the Virasoro element L does not lie in the space $T\mathfrak{g} + :\mathfrak{g}\mathfrak{g}:$. In other words, we are ruling out the situation in which L is obtained by Sugawara construction taking normal ordered products of elements in $\mathfrak{g}$.

Since by assumption L is a Virasoro element in V, we have

$$[L\ {}_\lambda\ L] = (T+2\lambda)L + \frac{c}{12}\lambda^3|0\rangle\ ,$$

for some $c \in \mathbb{C}$. Moreover, by our assumptions on $\mathfrak{g}$ and U we also have

$$\begin{aligned}
[L\ {}_\lambda\ a] &= (T+\lambda)a\ , & [a\ {}_\lambda\ L] &= \lambda a\ , & \forall a \in \mathfrak{g}\ ,\\
[L\ {}_\lambda\ v] &= (T+\tfrac{3}{2}\lambda)v\ , & [v\ {}_\lambda\ L] &= (\tfrac{1}{2}T+\tfrac{3}{2}\lambda)v\ , & \forall v \in U\ .
\end{aligned}$$

We want to write the most general λ–bracket between any two elements of R. For this we will use conformal weight considerations, based on Lemma 1.1.8. Since V is strongly generated by R, the spectrum of $L_{(1)}$ is contained in $\frac{1}{2}\mathbb{Z}_+$. Moreover, if $V[\Delta]$ denotes the eigenspace of $L_{(1)}$ with conformal weight Δ, then we necessarily have

$$V[0] = \mathbb{C}|0\rangle\ ,\ V[1/2] = 0\ ,\ V[1] = \mathfrak{g}\ ,\ V[3/2] = U\ ,\ V[2] = (\mathbb{C}L \oplus T\mathfrak{g}) + (:\mathfrak{g}\mathfrak{g}:)\ ,$$

where $:\mathfrak{g}\mathfrak{g}$ denotes the linear span of elements $:ab:$ with $a,b \in \mathfrak{g}$. As a consequence of the above considerations, we have the following restrictions on the λ–bracket of two elements of R $(a,b \in \mathfrak{g},\ u,v \in U)$:

$$[a\ {}_\lambda\ b] = a_{(0)}b + \lambda a_{(1)}b\ , \quad [a\ {}_\lambda\ u] = a_{(0)}u\ =\ -[u\ {}_\lambda\ a]\ ,$$
$$[u\ {}_\lambda\ v] = u_{(0)}v + \lambda u_{(1)}v + \lambda^{(2)}u_{(2)}v\ ,$$

where

$$\begin{gathered}
a_{(0)}b \in \mathfrak{g}\ , \quad a_{(1)}b \in \mathbb{C}|0\rangle\ , \quad a_{(0)}u \in U\ ,\\
u_{(0)}v \in V[2]\ , \quad u_{(1)}v \in \mathfrak{g}\ , \quad u_{(2)}v \in \mathbb{C}|0\rangle\ .
\end{gathered} \tag{3.3}$$

By imposing skewsymmetry and Jacobi identity to the λ–bracket of elements of R, we get additional restrictions. In particular, by skewsymmetry of $[a\ {}_\lambda\ b]$ with $a,b \in \mathfrak{g}$ we get

$$a_{(0)}b = -b_{(0)}a\ , \qquad a_{(1)}b = b_{(1)}a\ ,$$

and by imposing Jacobi identity we get

$$\begin{aligned}
a_{(0)}(b_{(0)}c) &= b_{(0)}(a_{(0)}c)\ +\ (a_{(0)}b)_{(0)}c\ ,\\
a_{(1)}(b_{(0)}c) &= (a_{(0)}b)_{(1)}c\ .
\end{aligned}$$

In other words, $\mathfrak{g}$ is a Lie algebra with respect to the Lie bracket $[,]:\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ given by

$$[a,b] = a_{(0)}b\ , \quad \forall a,b \in \mathfrak{g}\ ,$$

with a symmetric, invariant, bilinear form $\varkappa(,):\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ given by

$$a_{(1)}b = \varkappa(a,b)|0\rangle\ , \quad \forall a,b \in \mathfrak{g}\ .$$

Moreover, by Jacobi identity of the triple λ–bracket $[a\ {}_\lambda\ [b\ {}_\mu\ v]]$, with $a, b \in \mathfrak{g},\ v \in U$, we get

$$a_{(0)}(b_{(0)}v) - b_{(0)}(a_{(0)}v) = [a,b]_{(0)}v\ .$$

In other words, there is a representation $\rho:\ \mathfrak{g} \to \mathrm{gl}(U)$ of the Lie algebra $\mathfrak{g}$ on the space U, given by

$$(\rho(a))(v) = a_{(0)}v\ , \quad \forall a \in \mathfrak{g},\ v \in U\ .$$

We will denote in the following $(\rho(a))(v) = av$.

We now want to find conditions on the λ–bracket $[u\ {}_\lambda\ v]$ of elements in U. By (3.3), for every $u, v \in U$ we can find $Q(u,v) \in \mathbb{C},\ K(u,v) \in \mathfrak{g}$ and $P(u,v) \in \mathfrak{g}^{\otimes 2}$ such that

$$u_{(0)}v \ = \ Q(u,v)L + TK(u,v) + :P(u,v):\ . \tag{3.4}$$

Recall that by assumption R is a minimal generating set for V, and therefore $\mathbb{C}L \cap (T\mathfrak{g} + :\mathfrak{g}\mathfrak{g}:) = 0$. It follows that (3.4) uniquely defines the bilinear map $Q:\ U \times U \to \mathbb{C}$. On the contrary $K(u,v)$ and $P(u,v)$ are not uniquely defined. In fact, by skewsymmetry of the normal order product, it is very easy to check that, for $a, b \in \mathfrak{g}$, we can redefine

$$K'(u,v) \ = \ K(u,v) - [a,b]\ , \quad P'(u,v) \ = \ P(u,v) + a \otimes b - b \otimes a\ , \tag{3.5}$$

and the expression (3.4) of $u_{(0)}v$ still holds after replacing K with K' and P with P'. In particular, after an appropriate rescaling of kind (3.5), we can make sure that $P(u,v) \in S^2\mathfrak{g} \subset \mathfrak{g}^{\otimes 2}$.

Lemma 3.1.1. *(i) Let* $:S^2\mathfrak{g}: \subset :\mathfrak{g}\mathfrak{g}:$ *denote the linear span of elements* $:ab: + :ba:$*, with* $a, b \in \mathfrak{g}$*. Then*

$$T\mathfrak{g} \cap\ :S^2\mathfrak{g}: = \ 0\ .$$

(ii) In particular

$$\mathbb{C}L + T\mathfrak{g} + :\mathfrak{g}\mathfrak{g}: = \ \mathbb{C}L \oplus T\mathfrak{g} \oplus :S^2\mathfrak{g}:\ .$$

Proof. Let $A = \sum_i (:a_ib_i: + :b_ia_i:) = Tc\ \in\ :S^2\mathfrak{g}: \cap T\mathfrak{g}$. By taking λ–bracket with L we get, on one hand

$$[L\ {}_\lambda\ A] = (T + 2\lambda)\sum_i (:a_ib_i: + :b_ia_i:) + \frac{1}{3}\lambda^3 \sum_i \varkappa(a_i, b_i)\ ,$$

and on the other hand

$$[L\ {}_\lambda\ A] = (T + 2\lambda)Tc + \lambda^2 c\ .$$

By comparing the coefficient of λ^2 in both these expressions, we get $c = 0$. This proves part (i). Part (ii) follows immediately from part (i) and the assumption that $L \notin :\mathfrak{g}\mathfrak{g}:$. □

Since we are assuming $P(u,v) \in S^2\mathfrak{g}$, it follows by Lemma 3.1.1 that $K(u,v)$ is uniquely defined by the decomposition (3.4). In other words, equation (3.4) uniquely defines the bilinear map $K: \ U\times U\to\mathfrak{g}$. Let us denote by π_0 the quotient map $\pi_0: \ \mathfrak{g}^{\otimes 2}\to :\mathfrak{g}\mathfrak{g}:$, defined by $\pi_0(a\otimes b)=:ab:$, and by $\bar{\pi}_0$ its restriction to $S^2\mathfrak{g}\subset\mathfrak{g}^{\otimes 2}$.

Lemma 3.1.2. *(i) Consider $\mathfrak{g}^{\otimes 2}$ as a $\mathfrak{g}$–module via adjoint action. The space $\mathcal{K}=Ker(\pi_0)\subset\mathfrak{g}^{\otimes 2}$ is a $\mathfrak{g}$–submodule.*

(ii) In particular $\bar{\mathcal{K}}=Ker(\bar{\pi}_0)=\mathcal{K}\cap S^2\mathfrak{g}$ is a $\mathfrak{g}$–submodule of $S^2\mathfrak{g}$.

(iii) The space $:S^2\mathfrak{g}:$ is naturally endowed with the structure of $\mathfrak{g}$–module via the isomorphism

$$:S^2\mathfrak{g}:\ \simeq\ S^2\mathfrak{g}/\bar{\mathcal{K}}\ .$$

Proof. Let $A=\sum_i a_i\otimes b_i\in\mathcal{K}$. By Wick formula we then have, for every $c\in\mathfrak{g}$

$$0=c_{(0)}\sum_i :a_ib_i:=\sum_i(:[c,a_i]b_i:+:a_i[c,b_i]:)\ .$$

We then have

$$\mathrm{ad}(c)A=\sum_i[c,a_i]\otimes b_i+a_i\otimes[c,b_i]\in\mathcal{K}\ ,$$

thus proving part (i) of the lemma. Parts (ii) and (iii) follow trivially from part (i). □

It follows from Lemma 3.1.1 and the definition of $\bar{\mathcal{K}}$ that the decomposition (3.4) defines uniquely a bilinear map $P: \ U\times U\to S^2\mathfrak{g}/\bar{\mathcal{K}}\simeq :S^2\mathfrak{g}:$.

We can use Jacobi identity for the triple λ–bracket $[L\ _\lambda\ [u\ _\mu\ v]]$ with $u,v\in U$, to express $u_{(1)}v$ and $u_{(2)}v$ in terms of $Q(u,v)$, $K(u,v)$ and $P(u,v)$. By Jacobi identity

$$[L\ _\lambda\ [u\ _\mu\ v]]=[u\ _\mu\ [L\ _\lambda\ v]]+[[L\ _\lambda\ u]\ _{\lambda+\mu}\ v]\ . \tag{3.6}$$

By non commutative Wick formula and simple algebraic manipulations, we can rewrite the left hand side of (3.6) as

$$\begin{aligned}&(T+2\lambda)u_{(0)}v+\lambda^2K(u,v)+\mu(T+\lambda)u_{(1)}v\\&+\tfrac{1}{12}\lambda^3\Big(cQ(u,v)+2\varkappa(P_1(u,v),P_2(u,v))\Big)|0\rangle\ .\end{aligned} \tag{3.7}$$

Here and further we use the convenient notation

$$P(u,v)=P_1(u,v)\otimes P_2(u,v)+\bar{\mathcal{K}}\ \in\ S^2\mathfrak{g}/\bar{\mathcal{K}}\ ,$$

not forgetting that the right hand side actually denotes, in general, a linear combination of monomials in $S^2\mathfrak{g}\subset\mathfrak{g}^{\otimes 2}$. We also used the fact that, since $P_1\otimes P_2\in S^2\mathfrak{g}$, then $[P_1(u,v),P_2(u,v)]=0$. The right hand side of (3.6) is

$$(T+2\lambda)u_{(0)}v+\Big(\frac{1}{2}\lambda^2+\mu(T+\lambda)\Big)u_{(1)}v+\frac{1}{4}\lambda^3u_{(2)}v\ . \tag{3.8}$$

Comparing (3.7) and (3.8) we then have

$$\begin{aligned} u_{(1)}v &= 2K(u,v) \ , \\ u_{(2)}v &= \tfrac{1}{3}\Big(cQ(u,v) + 2\varkappa(P_1(u,v), P_2(u,v))\Big)|0\rangle \ . \end{aligned} \tag{3.9}$$

By skewsymmetry of the λ–bracket, we have

$$[u \ _\lambda \ v] \ = \ [v \ _{-\lambda-T} \ u] \ , \quad \forall u, v \in U \ .$$

By comparing separately the coefficients of each power of λ in the above equation, we get

$$u_{(0)}v = v_{(0)}u - Tv_{(1)}u \ , \quad u_{(1)}v = -v_{(1)}u \ , \quad u_{(2)}v = v_{(2)}u \ .$$

We can then use (3.4), (3.9) and Lemma 3.1.1 to conclude that, for every $u, v \in U$,

$$Q(u,v) = Q(v,u) \ , \quad K(u,v) = -K(v,u) \ , \quad :P(u,v): = :P(v,u): \ .$$

In other words, Q, K and P can be thought of as linear maps

$$\begin{aligned} Q \ &: \ S^2U \ \longrightarrow \ \mathbb{C} \ , \\ K \ &: \ \Lambda^2U \ \longrightarrow \ \mathfrak{g} \ , \\ P \ &: \ S^2U \ \longrightarrow \ S^2\mathfrak{g}/\bar{\mathcal{K}} \simeq :S^2\mathfrak{g}: \ . \end{aligned} \tag{3.10}$$

We finally want to use Jacobi identity for the triple λ–bracket $[a \ _\lambda \ [u \ _\mu \ v]]$ with $a \in \mathfrak{g}$ and $u, v \in U$, to prove that the linear maps in (3.10) are actually $\mathfrak{g}$–module homomorphisms. By Jacobi identity

$$[a \ _\lambda \ [u \ _\mu \ v]] = [u \ _\mu \ [a \ _\lambda \ v]] + [[a \ _\lambda \ u] \ _{\lambda+\mu} \ v] \ . \tag{3.11}$$

The left hand side of (3.11) is, by Wick formula

$$\begin{aligned} &\lambda Q(u,v)a + (T + \lambda + 2\mu)[a, K(u,v)] + \lambda(\lambda + 2\mu)\varkappa(a, K(u,v))|0\rangle \\ &+ :[a, P_1(u,v)]P_2(u,v): + :P_1(u,v)[a, P_2(u,v)]: \\ &+2\lambda\varkappa(a, P_1(u,v))P_2(u,v) + \lambda[[a, P_1(u,v)], P_2(u,v)] \ . \end{aligned} \tag{3.12}$$

Similarly, the right hand side of (3.11) is

$$\begin{aligned} &Q(u,av)L + Q(au,v)L + (T + 2\mu)K(u,av) + (T + 2\lambda + 2\mu)K(au,v) \\ &+ :P(u,av): + :P(au,v): + \mu^{(2)}u_{(2)}(av) + (\lambda + \mu)^{(2)}(au)_{(2)}v \ . \end{aligned} \tag{3.13}$$

If we put $\lambda = \mu = 0$ in both (3.12) and (3.13) and we compare the resulting expressions, we get the following equation

$$(Q(au,v)+Q(u,av))L+T(K(au,v)+K(u,av)-[a,K(u,v)])+:P(au,v):$$
$$+:P(u,av):=:[a,P_1(u,v)]P_2(u,v):+:P_1(u,v)[a,P_2(u,v)]:\ .$$

It then follows by part (ii) of Lemma 3.1.1 that, for every $a \in \mathfrak{g},\ u,v,\in U$,

$$\begin{aligned} Q(au,v)+Q(u,av) &= 0 \\ K(au,v)+K(u,av) &= [a,K(u,v)] \\ :P(au,v):+:P(u,av): &= :\mathrm{ad}(a)P(u,v):\ . \end{aligned}$$

In other words, the linear maps $Q,\ K$ and P defined in (3.10) are $\mathfrak{g}$–module homomorphisms.

Remark 3.1.3. By comparing the coefficients of every other power of λ and μ in (3.12) and (3.13), we get that the following additional conditions hold

$$Q(u,v)a+2\varkappa(a,P_1(u,v))P_2(u,v)+[[a,P_1(u,v)],P_2(u,v)]=K(au,v)-K(u,av)\ , \tag{3.14}$$

and

$$6\varkappa(a,K(u,v))=cQ(au,v)+2\varkappa(P_1(au,v),P_2(au,v))\ . \tag{3.15}$$

We can summarize all the results obtained so far in the following

Proposition 3.1.4. *Let V be any vertex algebra strongly generated by the space R in (3.1), and assume $L \notin :\mathfrak{g}\mathfrak{g}:$. Then $\mathfrak{g}$ is a Lie algebra, U is a $\mathfrak{g}$–module, and they are endowed with $\mathfrak{g}$–module homomorphisms $\varkappa:\ S^2\mathfrak{g}\to\mathbb{C}$, $Q:\ S^2U\to\mathbb{C}$, $K:\ \Lambda^2U\to\mathfrak{g}$ and $P:\ S^2U\to S^2\mathfrak{g}/\bar{\mathcal{K}}$, where $\bar{\mathcal{K}}\subset S^2\mathfrak{g}$ is the $\mathfrak{g}$–submodule defined by $\bar{\mathcal{K}}=Ker(\bar{\pi}_0:\ S^2\mathfrak{g}\to :S^2\mathfrak{g}:)$. The above structures are uniquely defined by the λ–bracket on R, which is given in the following table*

Table 3.1:

	L	b	v
L	$(T+2\lambda)L$ $+\frac{c}{2}\lambda^{(3)}\|0\rangle$	$(T+\lambda)b$	$(T+\frac{3}{2}\lambda)v$
a	λa	$[a,b]+\lambda\varkappa(a,b)\|0\rangle$	av
u	$(\frac{1}{2}T+\frac{3}{2}\lambda)u$	$-bu$	$(T+2\lambda)K(u,v)+:P(u,v):$ $+Q(u,v)\left(L+\frac{c}{3}\lambda^{(2)}\|0\rangle\right)$ $+\frac{2}{3}\lambda^{(2)}\varkappa(P_1(u,v),P_2(u,v))\|0\rangle$

So far we studied the conditions on $\mathfrak{g}$ and U obtained by imposing skewsymmetry and Jacobi identity of the λ–bracket of elements of R. The only non trivial condition left to impose is Jacobi identity for the triple λ–bracket $[u\ {}_\lambda\ [v\ {}_\mu\ w]]$ of elements in U. In Section 3.3 we will study this last condition. In the next section we will use Proposition 3.1.4 to prove a converse statement to Theorem 2.1.8.

3.2 Converse statement to Theorem 2.1.8

In this section we want to state and prove a converse statement to Theorem 2.1.8. In other words, we will prove that if V is any vertex algebra strongly generated by a subspace $R \subset V$ as in (3.1), then R is naturally endowed with a Lie λ–bracket of degree 2. This holds under some "non degeneracy" assumption on the vertex algebra V.

In order to formulate precisely what this assumption consists of, we need to introduce some notation. By Proposition 3.1.4, $\mathfrak{g}$ is a Lie algebra and U is a $\mathfrak{g}$–module. We consider the space $S^2\mathfrak{g}$ naturally embedded in $\mathfrak{g}^{\otimes 2}$

$$S^2\mathfrak{g} \ = \ \mathrm{span}_{\mathbb{C}}\{a\otimes b+b\otimes a\ ,\quad a,b\in\mathfrak{g}\}\subset\mathfrak{g}^{\otimes 2}\ .$$

By definition we have

$$\mathfrak{g}\otimes U \ = \ \mathrm{span}_{\mathbb{C}}\{a\otimes u\ ,\quad a\in\mathfrak{g},\ u\in U\}\ ,$$

and we define the space

$$S^2(\mathfrak{g},U)_2 \;=\; \mathrm{span}_{\mathbb{C}}\{2a\otimes u+u\otimes a\ ,\quad a\in\mathfrak{g},\ u\in U\}\subset(\mathfrak{g}\otimes U)\oplus(U\otimes\mathfrak{g})\ .$$

Lemma 3.2.1. *The spaces $S^2\mathfrak{g}$, $\mathfrak{g}\otimes u$ and $S^2(\mathfrak{g},U)_2$ are $\mathfrak{g}$–modules. Moreover, there is a $\mathfrak{g}$–module isomorphism*

$$\sigma\ :\ \mathfrak{g}\otimes U\ \stackrel{\sim}{\longrightarrow}\ S^2(\mathfrak{g},U)_2\ ,$$

given by $\sigma(a\otimes u)=2a\otimes u+u\otimes a$.

Proof. The proof of this lemma is straightforward. □

If $\mathcal{T}(R)$ denotes the tensor algebra over R, consider the surjective map $\mathcal{T}(R)\twoheadrightarrow V$, defined by $a_1\otimes a_2\otimes\cdots\otimes a_n\mapsto\ :a_1a_2\cdots a_n:$, for every $a_1,\dots,a_n\in R$. Recall that we denote the normal order product of several elements by taking normal ordering starting to the right, $:a_1a_2\cdots a_n:\ =\ :a_1(:a_2(:a_3\cdots a_n)\ :)\ :$. Notice that this map is surjective since, by assumption, V is strongly generated by R. We denote by $\bar\pi_0,\ \pi_1,\ \pi_2$ the restrictions of this map to the spaces $S^2\mathfrak{g}$, $\mathfrak{g}\otimes U$ and $S^2(\mathfrak{g},U)_2$ and by $\bar{\mathcal{K}},\ \mathcal{K}_1,\ \mathcal{K}_2$ their kernels. Namely, if we let

$$\begin{aligned}
:S^2\mathfrak{g}:\ &=\ \mathrm{span}_{\mathbb{C}}\{:ab:+:ba:\ ,\quad a,b\in\mathfrak{g}\}\ \subset V\ ,\\
:\mathfrak{g}U:\ &=\ \mathrm{span}_{\mathbb{C}}\{:au:\ ,\quad a\in\mathfrak{g},\ u\in U\}\ \subset V\ ,\\
:S^2(\mathfrak{g},U)_2:\ &=\ \mathrm{span}_{\mathbb{C}}\{2:au:+:ua:\ ,\quad a\in\mathfrak{g},\ u\in U\}\ \subset V\ ,
\end{aligned}$$

then we are denoting

$$\begin{aligned}
\bar{\mathcal{K}}\ &=\ \mathrm{Ker}\Big(\bar\pi_0\ :\ S^2\mathfrak{g}\twoheadrightarrow :S^2\mathfrak{g}\Big)\ ,\\
\mathcal{K}_1\ &=\ \mathrm{Ker}\Big(\pi_1\ :\ \mathfrak{g}\otimes U\twoheadrightarrow :\mathfrak{g}U:\Big)\ ,\\
\mathcal{K}_2\ &=\ \mathrm{Ker}\Big(\pi_2\ :\ S^2(\mathfrak{g},U)\twoheadrightarrow :S^2(\mathfrak{g},U)_2:\Big)\ .
\end{aligned}$$

Lemma 3.2.2. *The spaces $\bar{\mathcal{K}}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ are $\mathfrak{g}$–modules.*

Proof. We already proved in Lemma 3.1.2 that $\bar{\mathcal{K}}$ is a $\mathfrak{g}$–module. The same argument proves that $\mathcal{K}_1$ and $\mathcal{K}_2$ are $\mathfrak{g}$–modules. □

As immediate consequence of Lemma 3.2.2, the spaces $:S^2\mathfrak{g}:$, $:\mathfrak{g}U:$ and $:S^2(\mathfrak{g},U)_2:$ are naturally endowed with the structure of $\mathfrak{g}$–module, by asking that the maps $\bar\pi_0$, π_1 and π_2 are $\mathfrak{g}$–module homomorphisms. Later we will need the following

Lemma 3.2.3. *Let us denote*

$$\Lambda^2(\mathfrak{g},U)=span_{\mathbb{C}}\{a\otimes u-u\otimes a\ ,\quad a\in\mathfrak{g},\ u\in U\}\ ,$$

and let $\hat{\pi}$ be the restriction of the quotient map $\mathcal{T}(R) \twoheadrightarrow V$ to $\Lambda^2(\mathfrak{g}, U)$.

(i) We have an isomorphism of vector spaces

$$(\mathfrak{g} \otimes U) \oplus (U \otimes \mathfrak{g}) \;=\; \Lambda^2(\mathfrak{g}, U) \oplus S^2(\mathfrak{g}, U)_2 \ .$$

(ii) The image under $\hat{\pi}$ of $\Lambda^2(\mathfrak{g}, U)$ is

$$\hat{\pi}(\Lambda^2(\mathfrak{g}, U)) \subset TU \ .$$

(iii) The space $: S^2(\mathfrak{g}, U)_2 : \;=\; \pi_2(S^2(\mathfrak{g}, U)_2)$ is such that

$$: S^2(\mathfrak{g}, U)_2 : \;\cap\; (TU) \;=\; 0 \ .$$

(iv) As immediate consequence

$$TU + :\mathfrak{g}U: + :U\mathfrak{g}: \;=\; TU \oplus :S^2(\mathfrak{g}, U)_2: \ .$$

Proof. Part (i) is immediate consequence of the following obvious identities

$$\begin{aligned} a \otimes u &= \tfrac{1}{3}(a \otimes u - u \otimes a) + \tfrac{1}{3}(2a \otimes u + u \otimes a) \ , \\ u \otimes a &= -\tfrac{2}{3}(a \otimes u - u \otimes a) + \tfrac{1}{3}(2a \otimes u + u \otimes a) \ . \end{aligned}$$

For (ii), just notice that, by skewsymmetry of the normal order product,

$$\hat{\pi}(a \otimes u - u \otimes a) = :au: - :ua: = \int_{-T}^{0} d\lambda \ [a \ {}_\lambda \ u] = T(au) \in TU \ .$$

We now want to prove (iii). Suppose $A = \sum_i (2a_i \otimes u_i + u_i \otimes a_i) = Tv \in :S^2(\mathfrak{g}, U)_2: \cap (TU)$. By taking the λ–bracket with L we have, on one hand

$$[L \ {}_\lambda \ A] \;=\; (T + \frac{5}{2}\lambda) \sum_i (2a_i \otimes u_i + u_i \otimes a_i) \ ,$$

and on the other hand

$$[L \ {}_\lambda \ A] \;=\; (T + \lambda)(T + \frac{3}{2}\lambda) v \ .$$

Comparing the coefficient of λ^2 in the above expressions, we get $v = 0$, as we wanted. We are left to prove (iv). By (i) we have

$$TU \oplus (\mathfrak{g} \otimes U) \oplus (U \otimes \mathfrak{g}) \simeq TU \oplus \Lambda^2(\mathfrak{g}, U) \oplus S^2(\mathfrak{g}, U)_2 \ .$$

Taking the image in V of both sides and using (ii) we get

$$TU + :\mathfrak{g}U: + :U\mathfrak{g}: \;=\; TU + :S^2(\mathfrak{g}, U)_2: \ ,$$

which, together with (iii), gives (iv). □

We can now define the non degeneracy condition needed to prove the converse statement of Theorem 2.1.8.

Definition 3.2.4. A vertex algebra V strongly generated by a space R as in (3.1) is said to be *non degenerate* if the following three conditions hold:

(i) R is a minimal generating set for V, namely $L \notin :\mathfrak{g}\mathfrak{g}: + T\mathfrak{g}$.

(ii) There is an embedding of $\mathfrak{g}$–modules : $S^2\mathfrak{g}: \simeq S^2\mathfrak{g}/\bar{\mathcal{K}} \hookrightarrow S^2\mathfrak{g}$. Namely $\bar{\mathcal{K}} \subset S^2\mathfrak{g}$ admits a complementary submodule, so that $S^2\mathfrak{g}$ can be decomposed as direct sum of submodules

$$S^2\mathfrak{g} \simeq \bar{\mathcal{K}} \oplus :S^2\mathfrak{g}: .$$

(iii) The spaces $S^2(\mathfrak{g}, U)_2$ and $:S^2(\mathfrak{g}, U)_2$ are isomorphic, namely $\mathcal{K}_2 = 0$.

Remark 3.2.5. If $\mathfrak{g}$ is a reductive Lie algebra, condition (ii) is automatically satisfied by complete reducibility. In Section 3.3 we will study in more detail condition (iii). In particular we will prove that, in the special case in which $\mathfrak{g}$ is a simple Lie algebra and U an irreducible $\mathfrak{g}$–module, condition (iii) is automatically satisfied for all but finitely many values of the Kac–Moody level k.

The main result of this section is the following

Theorem 3.2.6. *Let V be any non degenerate vertex algebra strongly generated by a space R as in (3.1). Then R is endowed with a Lie λ–bracket of degree 2, $L_\lambda : R \otimes R \to \mathcal{T}(R)[\lambda]$, compatible with the λ–bracket structure of the vertex algebra V, namely such that the following diagram commutes*

$$\begin{array}{ccc} & \overset{L_\lambda}{\nearrow} & \mathcal{T}(R)[\lambda] \\ & & \downarrow \\ R\otimes R & \xrightarrow[{[\ {}_\lambda\ ]}]{} & V[\lambda] \end{array} \tag{3.16}$$

Proof. We want to define a Lie λ–bracket of degree 2 on R. Since L_λ has to be compatible with $[\ {}_\lambda\ ]$, and since $[R_i\ {}_\lambda\ R_j] \subset R_{i+j}[\lambda]$ if either $i = \bar{0}$ or $j = \bar{0}$, we necessarily have $L_\lambda|_{\mathbb{R}_i \otimes R_j} = [\ {}_\lambda\ ]|_{\mathbb{R}_i \otimes R_j}$ if $ij = \bar{0}$. Thanks to sesquilinearity, we are left to define the λ–bracket $L_\lambda(u, v)$ for elements $u, v \in U$. By Proposition 3.1.4, there are $\mathfrak{g}$–module homomorphisms $\varkappa :\ S^2\mathfrak{g} \to \mathbb{C}$, $Q :\ S^2U \to \mathbb{C}$, $K :\ \Lambda^2 U \to \mathfrak{g}$, $P : S^2U \to S^2\mathfrak{g}/\bar{\mathcal{K}} \simeq: S^2\mathfrak{g}:$ such that

$$\begin{aligned} [u\ {}_\lambda\ v] \ &= \ Q(u,v)L \ + \ (T + 2\lambda)K(u,v) + \ :P(u,v): \\ &+ \ \tfrac{1}{6}\lambda^2 \Big(cQ(u,v) + 2\varkappa(P_1(u,v), P_2(u,v)) \Big) |0\rangle \ . \end{aligned}$$

We can rewrite the above equation as

$$[u\ {}_\lambda\ v] \ = \ [u\ {}_\lambda\ v]_1 + \ :P(u,v): ,$$

where $[u\ {}_\lambda\ v]_1 \in R[\lambda]$. By non degeneracy assumption on V, there is an embedding

$S^2\mathfrak{g}/\bar{\mathcal{K}} \hookrightarrow S^2\mathfrak{g} \subset \mathfrak{g}^{\otimes 2}$. We then denote by P the composition map

$$P \ : \quad S^2U \ \xrightarrow{P} \ S^2\mathfrak{g}/\bar{\mathcal{K}} \ \hookrightarrow \ \mathfrak{g}^{\otimes 2} \ ,$$

and we define

$$L_\lambda(u,v) \ = \ [u \ _\lambda \ v]_1 \ + \ P(u,v) \ .$$

We want to prove that the map $L_\lambda : \ R \times R \to (R \oplus \mathbb{C}[T]\mathfrak{g}^{\otimes 2})[\lambda]$ defined above is a Lie λ–bracket of degree 2, namely it satisfies skewsymmetry and Jacobi identity (in the sense specified in Definition 2.1.5). Skewsymmetry condition is obvious, and the same is true for Jacobi identity of the triple λ–bracket $[x \ _\lambda \ [y \ _\mu \ z]]$, if x, y, z are homogeneous and at most one of them is in $R_{\bar{1}}$. We thus are left to prove the following three conditions ($a \in \mathfrak{g}, \ u, v, u_i \in U, \ i = 1, 2, 3$)

$$L_\lambda(L, L_\mu(u,v)) \ - \ L_\mu(u, L_\lambda(L,v)) \ - \ L_{\lambda+\mu}(L_\lambda(L,u),v) \ \in \mathcal{M}(R)[\lambda,\mu] \ , \qquad (3.17)$$

$$L_\lambda(a, L_\mu(u,v)) \ - \ L_\mu(u, L_\lambda(a,v)) \ - \ L_{\lambda+\mu}(L_\lambda(a,u),v) \ \in \mathcal{M}(R)[\lambda,\mu] \ , \qquad (3.18)$$

$$L_\lambda(u_1, L_\mu(u_2,u_3)) + L_\mu(u_2, L_\lambda(u_1,u_3)) - L_{\lambda+\mu}(L_\lambda(u_1,u_2),u_3) \in \mathcal{M}(R)[\lambda,\mu] \ , \ (3.19)$$

where $\mathcal{M}(R)$ was defined in Section 2.1. Notice the plus sign in the last equation, due to the fact that both u_1 and u_2 are odd elements. With a computation similar to the one used to study equation (3.6), we can prove that the expression in (3.17) is identically zero. Moreover, with a computation similar to the one used for equation (3.11) we can rewrite the first two terms of (3.18) as

$$\lambda Q(u,v)a + (T+\lambda+2\mu)[a, K(u,v)] + \lambda(\lambda+2\mu)\varkappa(a, K(u,v))|0\rangle$$
$$+ \, ad(a)P(u,v) + \lambda[[a, P_1(u,v)], P_2(u,v)] + 2\lambda\varkappa(a, P_1(u,v))P_2(u,v) \ ,$$

and the last term of (3.18) as

$$-(T+2\mu)(K(u,av) + K(au,v)) - 2\lambda K(au,v) - P(u,av) - P(au,v)$$
$$- \tfrac{1}{6}\lambda(\lambda+2\mu)\Big(cQ(au,v) + 2\varkappa(P_1(au,v), P_2(au,v))\Big)|0\rangle \ .$$

Using the fact that $\varkappa$, Q, K and P are $\mathfrak{g}$–module homomorphisms and comparing the above two expressions, we get that the expression in (3.18) is identically zero, thanks to equations (3.14) and (3.15). We are left to prove condition (3.19). First notice that, by conformal weight considerations based on Lemma 1.1.8, every term in (3.19) leaves in the space $V[5/2] \oplus (\mathbb{C}[\lambda,\mu]_1 \otimes V[3/2])$, where $\mathbb{C}[\lambda,\mu]_1$ denotes the space of homogeneous polynomials in λ and μ of degree 1. Recall that $V[3/2] = U$. Moreover L_λ is compatible with $[\ _\lambda \ ]$, namely $[\ _\lambda \ ]$ is obtained via the composition map $R \otimes R \xrightarrow{L_\lambda} \mathcal{T}(R)[\lambda] \twoheadrightarrow V[\lambda]$. Since the quotient map $\mathcal{T}(R) \twoheadrightarrow V$, restricted to U, is an isomorphism, we have that the part of expression (3.19) which lies in $\mathbb{C}[\lambda,\mu]_1 \otimes V[3/2]$ is the same if we replace L_λ with $[\ _\lambda \ ]$ everywhere. On the other hand, $[\ _\lambda \ ]$ satisfies Jacobi identity. This means that the part of expression (3.19) which lies in $\mathbb{C}[\lambda,\mu]_1 \otimes V[3/2]$ is identically zero. We then only have to check condition (3.19) after putting $\lambda = \mu = 0$. Using Wick formula we get that the expression in (3.19) with $\lambda = \mu = 0$ is

$$T\Big(\tfrac{1}{2}Q(u_2,u_3)u_1 + \tfrac{1}{2}Q(u_1,u_3)u_2 - Q(u_1,u_2)u_3 - K(u_2,u_3)u_1 - K(u_1,u_3)u_2\Big)$$
$$- (P_1(u_2,u_3)u_1) \otimes P_2(u_2,u_3) - P_1(u_2,u_3) \otimes (P_2(u_2,u_3)u_1)$$
$$-(P_1(u_1,u_3)u_2) \otimes P_2(u_1,u_3) - P_1(u_1,u_3) \otimes (P_2(u_1,u_3)u_2)$$
$$- 2P_1(u_1,u_2) \otimes (P_2(u_1,u_2)u_3)\ .$$

Notice that, by definition of $\mathcal{M}(R)$

$$\begin{aligned} P_1(u_1,u_2) \otimes (P_2(u_1,u_2)u_3) \ &\equiv\ (P_1(u_1,u_2)u_3) \otimes P_2(u_1,u_2) \\ &+\ T\Big(P_1(u_1,u_2)(P_2(u_1,u_2)u_3)\Big) \qquad \text{mod } \mathcal{M}(R)\ . \end{aligned}$$

Using this and the above expression, we conclude that condition (3.19) is equivalent to

$$\begin{aligned} &T\Big(\tfrac{1}{2}Q(u_2,u_3)u_1 + \tfrac{1}{2}Q(u_1,u_3)u_2 - Q(u_1,u_2)u_3 - K(u_2,u_3)u_1 - K(u_1,u_3)u_2 \\ &+ P_1(u_1,u_2)(P_2(u_1,u_2)u_3)\Big) - \textstyle\sum_{\sigma\in C_3} \Big((P_1(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}) \otimes P_2(u_{\sigma_1},u_{\sigma_2}) \\ &+P_1(u_{\sigma_1},u_{\sigma_2}) \otimes (P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})\Big)\ \in\ \mathcal{M}(R)\ , \end{aligned} \tag{3.20}$$

where C_3 denotes the group of cyclic permutations of $(1,2,3)$. Since $[\ {}_\lambda\ ]$ satisfies Jacobi identity, we know that if we replace the tensor products in (3.20) with normal order products, we get zero:

$$\begin{aligned} &T\Big(\tfrac{1}{2}Q(u_2,u_3)u_1 + \tfrac{1}{2}Q(u_1,u_3)u_2 - Q(u_1,u_2)u_3 - K(u_2,u_3)u_1 - K(u_1,u_3)u_2 \\ &+ P_1(u_1,u_2)(P_2(u_1,u_2)u_3)\Big) - \textstyle\sum_{\sigma\in C_3} \Big(:(P_1(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})P_2(u_{\sigma_1},u_{\sigma_2}): \\ &+:P_1(u_{\sigma_1},u_{\sigma_2})(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}):\Big)\ =\ 0\ . \end{aligned} \tag{3.21}$$

On the other hand, we can decompose the left hand side of (3.21) according to the decomposition (see Lemma 3.2.3)

$$TU + :\mathfrak{g}U: + :U\mathfrak{g}: =\ TU \oplus :S^2(\mathfrak{g},U)_2:\ ,$$

and conclude that both components in TU and in $:S^2(\mathfrak{g},U)_2:$ have to be zero. In particular

$$\sum_{\sigma\in C_3} \Big(2:P_1(u_{\sigma_1},u_{\sigma_2})(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}): + :(P_1(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})P_2(u_{\sigma_1},u_{\sigma_2}):\Big)\ =\ 0\ .$$

By non degeneracy assumption on V, we know that $\mathcal{K}_2 = 0$, so that $S^2(\mathfrak{g}, U)_2 \xrightarrow{\sim} : S^2(\mathfrak{g}, U)_2 :$. It then follows

$$\sum_{\sigma \in C_3} \Big(2P_1(u_{\sigma_1}, u_{\sigma_2}) \otimes (P_2(u_{\sigma_1}, u_{\sigma_2}) u_{\sigma_3}) + (P_1(u_{\sigma_1}, u_{\sigma_2}) u_{\sigma_3}) \otimes P_2(u_{\sigma_1}, u_{\sigma_2}) \Big) \;=\; 0 \;,$$

which clearly implies, since $(\mathfrak{g} \otimes U) \cap (U \otimes \mathfrak{g}) = 0$, that

$$\sum_{\sigma \in C_3} P_1(u_{\sigma_1}, u_{\sigma_2}) \otimes (P_2(u_{\sigma_1}, u_{\sigma_2}) u_{\sigma_3}) \;= 0 \;.$$

We then have that the last term in (3.20) is zero. This, together with equation (3.21), implies that the whole expression in (3.20) is zero. This concludes the proof of the theorem. □

3.3 Non degeneracy condition in the special case: $\mathfrak{g}$ simple, U irreducible

In the previous section we were able to prove the converse statement of Theorem 2.1.8 under the assumption that V is non degenerate (see Definition 3.2.4). Condition (i) in Definition 3.2.4 means that L is not obtained, via Sugawara construction, taking normal order products of elements in $\mathfrak{g}$. Condition (ii) is automatically satisfied as soon as the Lie algebra $\mathfrak{g}$ is reductive. In this section we want to study condition (iii).

Throughout this section we will denote by σ the $\mathfrak{g}$–module isomorphism $\sigma : \; \mathfrak{g} \otimes U \xrightarrow{\sim} S^2(\mathfrak{g}, U)_2$, given by $\sigma(a \otimes u) = (2a \otimes u + u \otimes a)$. Moreover, let $\rho : \; \mathfrak{g} \otimes U \to U$ be the $\mathfrak{g}$–module homomorphism defined by $\rho(a \otimes u) = au$. The image through σ of $\mathrm{Ker}(\rho)$ is a $\mathfrak{g}$–submodule of $S^2(\mathfrak{g}, U)_2$, given by

$$\mathrm{Ker}(\rho \circ \sigma^{-1}) = \left\{ \sum_i (2a_1 \otimes u_i + u_i \otimes a_i) \in S^2(\mathfrak{g}, U)_2 \;, \quad \text{s.t.} \; \sum_i a_i u_i = 0 \right\}$$

Lemma 3.3.1. *The following condition holds*

$$\sigma(\mathcal{K}_1) \;=\; \mathcal{K}_2 \cap Ker(\rho \circ \sigma^{-1}) \;.$$

Proof. Let $A = \sum_i a_i \otimes u_i \in \mathcal{K}_1$. By definition of $\mathcal{K}_1$ we have

$$\sum_i : a_i u_i : \;=\; 0 \;. \tag{3.22}$$

Taking the λ–bracket of both sides of (3.22) with L we get

$$0 = [L \;_\lambda \; \sum_i : a_i u_i :] \;=\; \frac{1}{2} \lambda^2 \sum_i a_i u_i \;,$$

so that $A \in \mathrm{Ker}(\rho)$. This implies, by skewsymmetry of the normal order product

$$\sum_i (: a_i u_i : - : u_i a_i :) \ = \ \int_{-T}^0 d\lambda \ \sum_i [a_i \ {}_\lambda \ u_i] \ = \ T \sum_i a_i u_i \ = \ 0 \ . \qquad (3.23)$$

It follows from (3.22) and (3.23) that $\sum_i (2 : a_i u_i : + : u_i a_i :) = 0$, namely $\sigma(A) \in \mathcal{K}_2$. We thus proved $\sigma(\mathcal{K}_1) \subset \mathcal{K}_2 \cap \sigma(\mathrm{Ker}(\rho))$. For the opposite inclusion, let $A = \sum_i (2a_i \otimes u_i + u_i \otimes a_i) \in \mathcal{K}_2 \cap \mathrm{Ker}(\rho \circ \sigma^{-1})$. Since $\sigma^{-1}(A) \in \mathrm{Ker}(\rho)$ we have

$$\sum_i (: a_i u_i : - : u_i a_i :) \ = \ T \sum_i a_i u_i \ = \ 0 \ ,$$

and since $A \in \mathcal{K}_2$ we have

$$\sum_i (2 : a_i u_i : + : u_i a_i :) \ = \ 0 \ .$$

It then follows $\sum_i : a_i u_i : = 0$, namely $\sigma^{-1}(A) \in \mathcal{K}_1$, as we wanted. This concludes the proof of the lemma. □

Consider now the special case in which $\mathfrak{g}$ is a simple Lie algebra and U is an irreducible $\mathfrak{g}$–module. We denote by $(,)$ the normalized Killing form on $\mathfrak{g}$ (defined so that the square length of long root is equal to 2), so that the bilinear form $\varkappa : S^2\mathfrak{g} \to \mathbb{C}$ defined in Proposition 3.1.4 is $\varkappa(a, b) = k(a, b)$, $\forall a, b \in \mathfrak{g}$, for some constant $k \in \mathbb{C}$, known as Kac–Moody level. Let $\{J_\alpha, \ J^\alpha, \ \alpha = 1, \dots, \dim\mathfrak{g}\}$ be a dual basis of $\mathfrak{g}$ with respect to the Killing form. Namely $(J_\alpha, J^\beta) = \delta_{\alpha,\beta}$. We use the convention of summing over repeated indices. Then $\Omega_2 = J_\alpha \otimes J^\alpha \in \mathfrak{g}^{\otimes 2}$ is a $\mathfrak{g}$–invariant element, namely

$$ad(a)(\Omega_2) \ = \ [a, J_\alpha] \otimes J^\alpha + J_\alpha \otimes [a, J^\alpha] \ = \ 0 \ , \quad \forall a \in \mathfrak{g} \ .$$

Moreover the Casimir element $\Omega = J_\alpha J^\alpha \in U(\mathfrak{g})$ acts as a scalar on any irreducible $\mathfrak{g}$–module Π, namely

$$\Omega|_\Pi \ = \ \Omega_\Pi 1\!\!1_\Pi \ ,$$

where $1\!\!1_\Pi$ denotes the identity operator on Π and $\Omega_\Pi \in \mathbb{C}$.

Let us first study the space $\mathcal{K}_1$. We want to prove the following

Lemma 3.3.2. *Let $\mathfrak{g}$ be a simple Lie algebra and U an irreducible $\mathfrak{g}$–module. Then $\mathcal{K}_1 = 0$, unless the Kac–Moody level k is*

$$k \ = \ \frac{1}{2}(\Omega_\Pi - \Omega_\mathfrak{g} - \Omega_U) \ ,$$

for some irreducible component $\Pi \subset \mathfrak{g} \otimes U$.

Proof. If $A = \sum_i a_i \otimes u_i \in \mathcal{K}_1$, we have

$$0 = [b \ {}_\lambda \ \sum_i : a_i u_i :] \ = \ \sum_i (: [b, a_i] u_i : + : a_i(b u_i) :) + \lambda \sum_i (k(b, a_i) u_i + [b, a_i] u_i) \ .$$

By looking at the coefficient of λ in the above equation, we get, after replacing b with J^α, taking the tensor product with J_α and summing over α,

$$0 = J_\alpha \otimes \sum_i (k(J^\alpha, a_i)u_i + [J^\alpha, a_i]u_i) = k \sum_i a_i \otimes u_i + \sum_i J_\alpha \otimes [J^\alpha, a_i]u_i \ . \quad (3.24)$$

In the second equality we used the obvious identity $J_\alpha(J^\alpha, a) = a$. Moreover, using the fact that Ω_2 is $\mathfrak{g}$–invariant, we can rewrite the last term in the right hand side of (3.24) as

$$-\sum_i [J_\alpha, a_i] \otimes J^\alpha u_i \ . \quad (3.25)$$

By definition of the Casimir operator Ω, we can rewrite (3.25) as

$$-\frac{1}{2}\Omega(\sum_i a_i \otimes u_i) + \frac{1}{2}(\Omega_\mathfrak{g} + \Omega_U)(\sum_i a_i \otimes u_i) \ .$$

Substituting back in (3.24) we thus get

$$\Omega(A) \ = \ (2k + \Omega_\mathfrak{g} + \Omega_U)A \ .$$

In other words, $A \in \mathfrak{g} \otimes U$ is eigenvector of Ω with eigenvalue $(2k + \Omega_\mathfrak{g} + \Omega_U)$. On the other hand, the eigenvalues of Ω in $\mathfrak{g} \otimes U$ are Ω_Π, for irreducible components $\Pi \subset \mathfrak{g} \otimes U$. This completes the proof of the lemma. □

We can now study the space $\mathcal{K}_2$. We want to prove the following

Lemma 3.3.3. *Let $\mathfrak{g}$ be a simple Lie algebra, U an irreducible $\mathfrak{g}$–module, and assume $\mathcal{K}_1 = 0$. Then $\mathcal{K}_2 = 0$, unless the Kac–Moody level k is*

$$k = \frac{1}{6}(2\Omega_U - 3\Omega_\mathfrak{g}) \ .$$

Proof. Since U is irreducible, $\rho(\mathfrak{g} \otimes U) = U$. Moreover, since $\mathfrak{g}$ is simple, by complete reducibility we have

$$\mathfrak{g} \otimes U \simeq \mathrm{Ker}(\rho) \oplus U \ ,$$

where the natural embedding $U \hookrightarrow \mathfrak{g} \otimes U$ is given by $v \mapsto \frac{1}{\Omega_\mathfrak{g}} J_\alpha \otimes J^\alpha v$. Taking the image of this decomposition through the isomorphism $\sigma : \ \mathfrak{g} \otimes U \xrightarrow{\sim} S^2(\mathfrak{g}, U)_2$, we get

$$S^2(\mathfrak{g}, U)_2 \ \simeq \ \mathrm{Ker}(\rho \circ \sigma^{-1}) \oplus U \ ,$$

where now U is embedded in $S^2(\mathfrak{g}, U)_2$ via the map $v \mapsto \frac{1}{\Omega_\mathfrak{g}} \sigma(2J_\alpha \otimes J^\alpha v + J_\alpha v \otimes J^\alpha)$. By Lemma 3.3.1 and our assumptions, we have

$$\mathcal{K}_2 \cap \mathrm{Ker}(\rho \circ \sigma^{-1}) = 0 \ .$$

It follows that $\mathcal{K}_2 \subset \tilde{U}$, where $\tilde{U} \subset S^2(\mathfrak{g}, U)_2$ is the isotipic component with the same highest weight as U. (Namely it is direct sum of a finite number of copies of U).

Notice that $\Omega|_{\tilde{U}} = \Omega_U \mathbb{1}_{\tilde{U}}$. Let then $A = \sum_i (2a_i \otimes u_i + u_i \otimes a_i) \in \mathcal{K}_2 \cap \tilde{U}$. Since $A \in \mathcal{K}_2$, we have

$$\begin{aligned} 0 &= [b\ {}_\lambda\ \textstyle\sum_i (2 :a_i u_i: + :u_i a_i:)] = \sum_i \Big(2 :[b,a_i]u_i: + 2 :a_i(bu_i): \\ &+ \ :(bu_i)a_i: + :u_i[b,a_i]: \Big) + \lambda \textstyle\sum_i \Big(3k(b,a_i)u_i + 3[b,a_i]u_i - ba_iu_i \Big) \ . \end{aligned}$$

As we did in the proof of Lemma 3.3.2, we now look at the coefficient of λ in the above equation, we replace b by J^α, we take the tensor product with J_α and we sum over α. The result is

$$\sum_i \Big(3ka_i \otimes u_i + 3J_\alpha \otimes [J^\alpha, a_i]u_i - J_\alpha \otimes J^\alpha a_i u_i \Big) \ = \ 0 \ . \tag{3.26}$$

Since $A \in \tilde{U}$, we know that $\Omega(A) = \Omega_U A$, and therefore $\Omega(\sigma^{-1}(A)) = \Omega_U \sigma^{-1}(A)$. This implies

$$\sum_i J_\alpha \otimes [J^\alpha, a_i]u_i \ = \ -\sum_i [J_\alpha, a_i] \otimes J^\alpha u_i \ = \ \frac{1}{2}\Omega_{\mathfrak{g}} \sum_i (a_i \otimes u_i) \ .$$

Substituting back in (3.26) we then get

$$3(k + \frac{1}{2}\Omega_{\mathfrak{g}}) \sum_i a_i \otimes u_i \ = \ J_\alpha \otimes J^\alpha \sum_i a_i u_i \ .$$

Taking the image of both sides of this equation under $\rho :\ \mathfrak{g} \otimes U \twoheadrightarrow U$, we get

$$3(k + \frac{1}{2}\Omega_{\mathfrak{g}}) \sum_i a_i u_i \ = \ \Omega_U \sum_i a_i u_i \ . \tag{3.27}$$

Suppose that $\sum_i a_i u_i = 0$. This is equivalent to say $\rho(\sigma^{-1}(A)) = 0$. On the other hand we assumed $A \in \mathcal{K}_2$. We would then have $A \in \mathcal{K}_2 \cap \mathrm{Ker}(\rho \circ \sigma^{-1})$, which implies $A = 0$ thanks to the assumption $\mathcal{K}_1 = 0$ and Lemma 3.3.2. If $A \neq 0$, it must then be $\sum_i a_i u_i \neq 0$, so that equation (3.27) implies

$$3k + \frac{3}{2}\Omega_{\mathfrak{g}} - \Omega_U \ = \ 0 \ .$$

This concludes the proof of the lemma. □

We can summarize the results of Lemma 3.3.2 and Lemma 3.3.3 in the following

Theorem 3.3.4. *Let V be any vertex algebra such that the space R in (3.1) is a minimal strongly generating set. Suppose $\mathfrak{g}$ is a simple Lie algebra and U is an irreducible $\mathfrak{g}$–module. Then V is non degenerate, unless the Kac–Moody level k takes values*

$$k = \frac{1}{6}(2\Omega_U - 3\Omega_{\mathfrak{g}}) \ , \quad \textit{or} \quad k = \frac{1}{2}(\Omega_\Pi - \Omega_{\mathfrak{g}} - \Omega_U) \ ,$$

for some irreducible component $\Pi \hookrightarrow \mathfrak{g} \otimes U$.

3.4 Conditions on R coming from Jacobi identity

In this section we want to find necessary and sufficient conditions on the space R in (3.1), in order for it to admit a Lie λ–bracket of degree 2 (see Definition 2.1.5)

Due to sesquilinearity, it suffices to define $L_\lambda(x, y)$ for $x, y \in \mathbb{C}L \oplus \mathfrak{g} \oplus U$. Moreover, the same computations done in Section 3.1 lead to the following

Lemma 3.4.1. *(i) Let* $L_\lambda : R \otimes R \to (R \oplus R_{\bar{0}}^{\otimes 2})[\lambda]$ *be a Lie* λ*–bracket of degree 2 on the space* $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)$*, such that* L *is a Virasoro element of central charge* c*, namely*

$$L_\lambda(L, L) \;=\; (T + 2\lambda)L + \frac{c}{12}\lambda^3|0\rangle \ ,$$

$\mathfrak{g}$ *is a finite dimensional space of even primary elements of conformal weight* $\Delta = 1$*, namely*

$$L_\lambda(L, a) \;=\; (T + \lambda)a \ , \quad \forall a \in \mathfrak{g} \ ,$$

and U *is a finite dimensional space of odd primary elements of conformal weight* $\Delta = 3/2$*, namely*

$$L_\lambda(L, u) \;=\; (T + \frac{3}{2}\lambda)u \ , \quad \forall u \in U \ .$$

Then $\mathfrak{g}$ *is a Lie algebra and* U *is a* $\mathfrak{g}$*–module. Moreover there exist* $\mathfrak{g}$*–module homomorphisms* $\varkappa : S^2\mathfrak{g} \to \mathbb{C}$, $Q : S^2U \to \mathbb{C}$, $K : \Lambda^2U \to \mathfrak{g}$, $P : S^2U \to S^2\mathfrak{g}$*, such that the* λ*–bracket structure on* R *is given by Table 3.1 for all products except* $L_\lambda(u, v)$ *with* $u, v \in U$*. In this case we have*

$$\begin{aligned} L_\lambda(u, v) \;&=\; Q(u, v)L + (T + 2\lambda)K(u, v) + P(u, v) \\ &+\; \tfrac{1}{6}\lambda^2\Big(cQ(u, v) + 2\varkappa(P_1(u, v), P_2(u, v))\Big)|0\rangle \ . \end{aligned}$$

Recall we are using the notation $A = A_1 \otimes A_2$ *for a generic element* $A \in \mathfrak{g} \otimes \mathfrak{g}$*, namely* $A_1 \otimes A_2$ *stands for a linear combination of monomials in* $\mathfrak{g} \otimes \mathfrak{g}$.

(ii) Conversely, let $\mathfrak{g}$ *be a Lie algebra,* U *be a* $\mathfrak{g}$*–module and* $\varkappa, Q, K, P$ *be* $\mathfrak{g}$*–module homomorphisms as in (i). Define on the space* $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)$ *a map* $L_\lambda : R \otimes R \to (R \oplus R_{\bar{0}}^{\otimes 2})[\lambda]$ *as described in (i). Then* L_λ *is a Lie* λ*–bracket of degree 2 on* R *if and only if equations (3.14) and (3.15) hold and Jacobi identity is satisfied for the triple* λ*–bracket* $L_\lambda(u_1, L_\mu(u_2, u_3))$, $u_i \in U$, $i = 1, 2, 3$*, namely*

$$L_\lambda(u_1, L_\mu(u_2, u_3)) + L_\lambda(u_2, L_\mu(u_1, u_3)) \equiv L_{\lambda+\mu}(L_\lambda(u_1, u_2), u_3) \quad \bmod \mathcal{M}(R) \ . \quad (3.28)$$

Proof. The computations are the same as the ones needed to prove Proposition 3.1.4. The details of the proof are left to the reader. □

According to Lemma 3.4.1, we are left to impose condition (3.28). By Definition 2.1.5 and Lemma 2.1.3, in order to compute the triple λ–brackets we need to use, when needed, the left and right Wick formulas (2.3) and (2.4). Therefore the two terms in the right hand side of (3.28) are respectively

$$\begin{aligned} &\tfrac{1}{2}Q(u_2,u_3)(T+3\lambda)u_1 - (T+\lambda+2\mu)K(u_2,u_3)u_1 + \lambda P_1(u_2,u_3)(P_2(u_2,u_3)u_1) \\ &\quad - P_1(u_2,u_3)\otimes(P_2(u_2,u_3)u_1) - (P_1(u_2,u_3)u_1)\otimes P_2(u_2,u_3)\ , \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &\tfrac{1}{2}Q(u_1,u_3)(T+3\mu)u_2 - (T+2\lambda+\mu)K(u_1,u_3)u_2 + \mu P_1(u_1,u_3)(P_2(u_1,u_3)u_2) \\ &\quad - P_1(u_1,u_3)\otimes(P_2(u_1,u_3)u_2) - (P_1(u_1,u_3)u_2)\otimes P_2(u_1,u_3)\ , \end{aligned} \tag{3.30}$$

and the right hand side of (3.28) is

$$\begin{aligned} &\tfrac{1}{2}Q(u_1,u_2)(2T+3\lambda+3\mu)u_3 + (\lambda-\mu)K(u_1,u_2)u_3 \\ &+ (\lambda+\mu)P_1(u_1,u_2)(P_2(u_1,u_2)u_3) + 2P_1(u_1,u_2)\otimes(P_2(u_1,u_2)u_3)\ . \end{aligned} \tag{3.31}$$

The coefficients of λ in (3.29), (3.30) and (3.31) are elements of $U \subset R$. On the other hand $R \cap \mathcal{M}(R) = 0$. We thus get, from condition (3.28), that

$$\begin{aligned} &\tfrac{3}{2}Q(u_1,u_2)u_3 - \tfrac{3}{2}Q(u_3,u_2)u_1 + K(u_1,u_2)u_3 - K(u_3,u_2)u_1 + 2K(u_1,u_3)u_2 \\ &\quad + P_1(u_1,u_2)(P_2(u_1,u_2)u_3) - P_1(u_3,u_2)(P_2(u_3,u_2)u_1) \ = \ 0\ . \end{aligned} \tag{3.32}$$

The coefficient of μ in (3.28) is obtained from the coefficient of λ by replacing u_1 with u_2, therefore we don't get a new condition from it. We are left to impose condition (3.28) with $\lambda = \mu = 0$. By (3.29), (3.30) and (3.31) this is equivalent to

$$\begin{aligned} &T\Big(\tfrac{1}{2}Q(u_2,u_3)u_1 + \tfrac{1}{2}Q(u_1,u_3)u_2 - Q(u_1,u_2)u_3 - K(u_2,u_3)u_1 - K(u_1,u_3)u_2\Big) \\ &\quad - \Big(P_1(u_2,u_3)\otimes(P_2(u_2,u_3)u_1) + (P_1(u_2,u_3)u_1)\otimes P_2(u_2,u_3) \\ &\quad + P_1(u_1,u_3)\otimes(P_2(u_1,u_3)u_2) + (P_1(u_1,u_3)u_2)\otimes P_2(u_1,u_3) \\ &\quad + 2P_1(u_1,u_2)\otimes(P_2(u_1,u_2)u_3)\Big) \ \in \mathcal{M}(R)\ . \end{aligned} \tag{3.33}$$

By taking the sum of the expression in (3.33) over all permutations of u_1, u_2, u_3 the first part disappear, so that condition (3.33) implies

$$\sum_{\sigma\in C_3}\Big(2P_1(u_{\sigma_1},u_{\sigma_2})\otimes(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}) + (P_1(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})\otimes P_2(u_{\sigma_1},u_{\sigma_2})\Big) \ \in\ \mathcal{M}(R)\ , \tag{3.34}$$

where C_3 denotes the set of cyclic permutations of $(1,2,3)$. By definition of $\mathcal{M}(R)$, (3.34) is equivalent to

$$\sum_{\sigma\in C_3}\Big(3P_1(u_{\sigma_1},u_{\sigma_2})\otimes(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})-T(P_1(u_{\sigma_1},u_{\sigma_2})(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}))\Big)\ \in\ \mathcal{M}(R)\ . \tag{3.35}$$

Since $((\mathfrak{g}\otimes U)\oplus R)\cap\mathcal{M}(R)=0$, it follows that condition (3.35) is equivalent to the following equation

$$\sum_{\sigma\in C_3} P_1(u_{\sigma_1},u_{\sigma_2})\otimes(P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3})\ =\ 0\ . \tag{3.36}$$

Using this equation, it follows that condition (3.33) is equivalent to

$$\begin{aligned} &T\Big(\tfrac{1}{2}Q(u_2,u_3)u_1+\tfrac{1}{2}Q(u_1,u_3)u_2-Q(u_1,u_2)u_3-K(u_2,u_3)u_1-K(u_1,u_3)u_2\Big)\\ &\quad+(P_1(u_1,u_2)u_3)\otimes P_2(u_1,u_2)-P_1(u_1,u_2)\otimes(P_2(u_1,u_2)u_3)\ \in\ \mathcal{M}(R)\ . \end{aligned} \tag{3.37}$$

By definition of $\mathcal{M}(R)$, we can replace the last two terms of (3.37) by

$$-T(P_1(u_1,u_2)(P_2(u_1,u_2)u_3))\ .$$

Since $R\cap\mathcal{M}(R)=0$, we conclude that (3.37) is equivalent to

$$\begin{aligned} &\tfrac{1}{2}Q(u_2,u_3)u_1+\tfrac{1}{2}Q(u_1,u_3)u_2-Q(u_1,u_2)u_3\\ &=K(u_2,u_3)u_1+K(u_1,u_3)u_2+P_1(u_1,u_2)(P_2(u_1,u_2)u_3)\ . \end{aligned} \tag{3.38}$$

We then conclude that Jacobi identity (3.28) is equivalent to the three equations (3.32), (3.36) and (3.38). Notice that equations (3.32) and (3.38) are equivalent. Indeed (3.38) is obtained from (3.32) by taking symmetrization in $u_1,\ u_2$ and using (3.36). Conversely (3.32) is obtained from (3.38) by taking anti-symmetrization in $u_1,\ u_3$.

We can summarize all the above results in the following

Theorem 3.4.2. *Let R be a space $R=\mathbb{C}|0\rangle\oplus(\mathbb{C}[T]\otimes\mathfrak{g})\oplus(\mathbb{C}[T]\otimes U)\oplus(\mathbb{C}[T]L)$, where $\mathfrak{g}$ and U are finite dimensional vector spaces. Then R admits a Lie λ–bracket of degree 2 (see Definition 2.1.5) such that $|0\rangle$ is central, L is a Virasoro element and $\mathfrak{g}$ (respectively U) is an even (resp. odd) space of primary elements of conformal weight 1 (resp. 3/2), if and only if $\mathfrak{g}$ is a Lie algebra, U is a $\mathfrak{g}$–module, and there exist $\mathfrak{g}$-module homomorphisms $\varkappa:\ S^2\mathfrak{g}\to\mathbb{C},\ Q:\ S^2U\to\mathbb{C},\ K:\ \Lambda^2U\to\mathfrak{g},\ P:\ S^2U\to S^2\mathfrak{g}$, such that the following conditions hold ($a\in\mathfrak{g},\ u,v,u_i\in U,\ i=1,2,3$)*

$$\begin{aligned} &Q(u,v)a+2\varkappa(a,P_1(u,v))P_2(u,v)+[[a,P_1(u,v)],P_2(u,v)]\\ &\qquad=K(au,v)-K(u,av)\ , \end{aligned} \tag{3.39}$$

$$6\varkappa(a, K(u,v)) = cQ(au,v) + 2\varkappa(P_1(au,v), P_2(au,v)) \ , \tag{3.40}$$

$$\tfrac{1}{2}Q(u_2,u_3)u_1 + \tfrac{1}{2}Q(u_1,u_3)u_2 = Q(u_1,u_2)u_3 + K(u_2,u_3)u_1$$
$$+K(u_1,u_3)u_2 + P_1(u_1,u_2)(P_2(u_1,u_2)u_3) \ , \tag{3.41}$$

$$\textstyle\sum_{\sigma\in C_3} P_1(u_{\sigma_1},u_{\sigma_2}) \otimes (P_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}) \ = \ 0 \ , \tag{3.42}$$

where C_3 denotes the group of cyclic permutations of $(1,2,3)$.

3.5 Equivalent formulations of Problem 1.2.1

As we said in Chapter 1, the main goal of the thesis is to study Problem 1.2.1. In Section 3.2 we have seen that, for technical reasons, it is convenient to consider vertex algebras which are non degenerate (according to Definition 3.2.4). We therefore formulate the following weaker version of Problem 1.2.1

Problem 3.5.1. Classify all vertex algebras V which are non degenerate and strongly generated by a Virasoro element L and finitely many even (respectively odd) primary elements of conformal weight 1 (resp. 3/2).

Thanks to Theorems 2.1.8 and 3.2.6, Problem 3.5.1 is equivalent to the following

Problem 3.5.2. Classify all spaces $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T]\otimes\mathfrak{g}) \oplus (\mathbb{C}[T]\otimes U) \oplus (\mathbb{C}[T]L)$, which admit a Lie λ–bracket of degree 2, for which L is a Virasoro element and $\mathfrak{g}$ (resp. U) is an even (resp. odd) finite dimensional space of primary elements of conformal weight 1 (resp. 3/2).

Finally we can use Theorem 3.4.2 to rewrite Problem 3.5.2 in the following equivalent form

Problem 3.5.3. Classify all 7-tuples $(\mathfrak{g},\ U,\ c,\ \varkappa,\ Q,\ K,\ P)$ where $\mathfrak{g}$ is a Lie algebra, U is a $\mathfrak{g}$–module, $c\in\mathbb{C}$, and $\varkappa:\ S^2\mathfrak{g}\to\mathbb{C},\ Q:\ S^2U\to\mathbb{C},\ K:\ \Lambda^2U\to\mathfrak{g},\ P:\ S^2U\to S^2\mathfrak{g}$ are $\mathfrak{g}$–module homomorphisms such that equations (3.39), (3.40), (3.41) and (3.42) are satisfied.

In the next chapter we will completely solve Problem 3.5.3 in the special case in which $\mathfrak{g}$ is a simple Lie algebra, U is an irreducible $\mathfrak{g}$–module and $\varkappa$, Q are not identically zero.

Chapter 4

Classification in the special case: $\mathfrak{g}$ simple, U irreducible

In this chapter we will completely solve Problem 3.5.3 in the special case in which $\mathfrak{g}$ is simple, U is irreducible, $\varkappa$ and Q are not identically zero. In Section 4.1 we will see that, as a consequence of equations (3.39), (3.40), (3.41) and (3.42), the Casimir operator Ω has to have the same eigenvalue on every irreducible component $\Pi \subset S^2U$ such that $\Pi \not\subset S^2\mathfrak{g}$. This turns out to be a very strong restriction on $\mathfrak{g}$ and U. In Section 4.2 we will classify all pairs $(\mathfrak{g}, U)$ satisfying this condition. The answer is contained in Table 4.5. Therefore we will have restricted ourselves to consider a finite (relatively small) list of pairs (g, U), for which we will have to check whether the assumptions of Problem 3.5.3 are satisfied. This will be done in the remaining Sections 4.3, 4.4 and 4.5, basically with a case by case check. The complete solution of Problem 3.5.3 in the special case in which $\mathfrak{g}$ is simple, U is irreducible, $\varkappa$ and Q are not zero, is provided in Theorem 4.6.1 and Table 4.6.

To conclude, a remark about the notation used in this chapter. With an abuse of notation, we will denote by $u \otimes v$ any element of the space $U^{\otimes 2}$, not necessarily a monomial. In other words $u \otimes v$ has to be understood as a generic linear combination of monomials in $U^{\otimes 2}$.

4.1 Preliminary restrictions

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra (over $\mathbb{C}$) and let U be an irreducible finite dimensional $\mathfrak{g}$–module. We will denote by $(,)$ the "normalized" Killing form on $\mathfrak{g}$, defined so that the square length of long roots is equal to 2. Since $\mathfrak{g}$ is simple, any symmetric invariant bilinear form on $\mathfrak{g}$ has to be proportional to the killing form; in particular $\varkappa(a, b) = k(a, b)$, $\forall a, b \in \mathfrak{g}$, for some constant $k \in \mathbb{C} - \{0\}$, known as Kac–Moody level. Since U is irreducible, by Schur's Lemma there is a unique (up to scalar multiplication) invariant bilinear form on U, which we denote by $(,)$. Since by assumption $Q: S^2U \to \mathbb{C}$ is non zero, after an appropriate choice of the normalization factor, we can set $Q(u, v) = (u, v)$, $\forall u, v \in U$. Throughout this chapter we will denote by $\{J^\alpha,\ \alpha = 1, \dots, \dim \mathfrak{g}\}$ a basis of $\mathfrak{g}$ orthonormal with respect to the Killing form: $(J^\alpha, J^\beta) = \delta^{\alpha\beta}$, and by $\{e^i,\ i = 1, \ldots, \dim U\}$ an orthonormal basis of U: $(e^i, e^j) = \delta^{ij}$. Recall that the Casimir operator is the following central element of the universal enveloping algebra of $\mathfrak{g}$:

$$\Omega = J^\alpha J^\alpha \in U(\mathfrak{g}) \ ,$$

where we are using the convention of summing over repeated indices. In particular it acts as a scalar in any irreducible $\mathfrak{g}$–module Π, namely $\Omega|_\Pi = \Omega_\Pi 1\!\!\mathrm{I}_\Pi$, where $\Omega_\Pi \in \mathbb{C}$ and $1\!\!\mathrm{I}_\Pi$ is the identity operator on Π.

Our goal is to classify all such pairs $(\mathfrak{g}, U)$ which admit $\mathfrak{g}$–module homomorphisms $K:\ \Lambda^2 U \to \mathfrak{g}$, $P:\ S^2 U \to S^2\mathfrak{g}$, satisfying all conditions (3.39), (3.40), (3.41) and (3.42).

The composition of maps $S^2U \to S^2\mathfrak{g} \to \mathbb{C}$ given by $u \otimes v \mapsto P(u,v) \mapsto (P_1(u,v), P_2(u,v))$ defines a symmetric invariant bilinear form on U, which thus has to be proportional to $(,)$; namely

$$(P_1(u,v), P_2(u,v)) = \rho(u,v) \ , \qquad \forall u, v \in U \tag{4.1}$$

from some constant $\rho \in \mathbb{C}$, which will be fixed by condition (3.39). Using (4.1), equation (3.40) turns out to be equivalent to

$$K(u,v) = \sigma(J^\alpha u, v) J^\alpha \ , \tag{4.2}$$

where σ is given by

$$6k\sigma = 2k\rho + c \ , \tag{4.3}$$

and ρ is defined by (4.1). In order to get (4.2), just replace a by J^α in (3.40), multiply both sides by J^α, and sum over the repeated index $\alpha = 1, \ldots, \dim\mathfrak{g}$.

We can now use the expression (4.2) of $K(u,v)$ and condition (3.39) to get

$$[[a, P_1(u,v)], P_2(u,v)] + 2k(a, P_1(u,v))P_2(u,v) = \sigma(\{J^\alpha, a\}u, v)J^\alpha - (u,v)a \ . \tag{4.4}$$

We denote by $\{A, B\}$ the anti-commutator of A and B, namely $\{A, B\} = AB + BA$. After replacing a by J^β in (4.4), taking the tensor product with J^β and summing over $\beta = 1, \ldots, \dim\mathfrak{g}$, the right hand side of (4.4) becomes

$$\sigma(\{J^\alpha, J^\beta\}u, v) J^\alpha \otimes J^\beta - (u,v) J^\alpha \otimes J^\alpha \ ,$$

the second term of the left hand side of (4.4) gives simply $2kP(u,v)$, while from the fist term of the left hand side of (4.4) we get

$$[[J^\alpha, P_1(u,v)], P_2(u,v)] \otimes J^\alpha = -[J^\alpha, P_1(u,v)] \otimes [J^\alpha, P_2(u,v)] \ .$$

In the last identity we used the fact that $\Omega_2 = \sum_\alpha J^\alpha \otimes J^\alpha \in \mathfrak{g} \otimes \mathfrak{g}$ defines a one dimensional trivial submodule of $\mathfrak{g} \otimes \mathfrak{g}$, namely

$$[a, J^\alpha] \otimes J^\alpha = -J^\alpha \otimes [a, J^\alpha] \ . \tag{4.5}$$

Suppose now that $u \otimes v$ (or rather a linear combination of such monomials) is an element of an irreducible component $\Pi \subset S^2U$. Since $P: S^2U \to S^2\mathfrak{g}$ is a $\mathfrak{g}$–module homomorphism, we get that either $P(u,v) = 0$ or there is an embedding $\Pi \subset S^2\mathfrak{g}$ such that $P(u,v) \in \Pi \subset S^2\mathfrak{g}$. Using the fact the the Casimir operator $\Omega \in U(\mathfrak{g})$ acts by scalar multiplication by $\Omega_\mathfrak{g}$ on $\mathfrak{g}$ and by scalar multiplication by Ω_Π on Π, we get

$$[J^\alpha, P_1(u,v)] \otimes [J^\alpha, P_2(u,v)] = \left(\frac{1}{2}\Omega_\Pi - \Omega_\mathfrak{g}\right) P(u,v) \ .$$

In conclusion (4.4) gives, for $u \otimes v \in \Pi \subset S^2U$,

$$(2k - \frac{1}{2}\Omega_\Pi + \Omega_\mathfrak{g})P(u,v) = \sigma(\{J^\alpha, J^\beta\}u, v)J^\alpha \otimes J^\beta - (u,v)J^\alpha \otimes J^\alpha \ . \qquad (4.6)$$

Notice that if $2k \neq \frac{1}{2}\Omega_\Pi - \Omega_\mathfrak{g}$ for all irreducible components $\Pi \subset S^2U$, equation (4.6) defines completely $P(u,v)$ for every $u, v \in U$. Equation (4.6) is equivalent to the following two equations for $P(u,v)$. For $u \otimes v \in \mathbb{C} \subset S^2U$

$$P(u,v) = \alpha_\mathbb{C}(u,v)J^\alpha \otimes J^\alpha \ , \qquad (4.7)$$

where $\alpha_\mathbb{C} \in \mathbb{C}$ is given by the equation

$$(2k + \Omega_\mathfrak{g}) \dim \mathfrak{g} \ \alpha_\mathbb{C} \ = \ 2\sigma\Omega_U - \dim \mathfrak{g} \ .$$

For $u \otimes v \in \Pi$, where Π is any non trivial irreducible submodule of S^2U,

$$P(u,v) = \alpha_\Pi(\{J^\alpha, J^\beta\}u, v)J^\alpha \otimes J^\beta \ , \qquad (4.8)$$

where α_Π is given by

$$(2k - \frac{1}{2}\Omega_\Pi + \Omega_\mathfrak{g})\alpha_\Pi \ = \ \sigma \ .$$

To get (4.7) just notice that, if $u \otimes v \in \mathbb{C} \subset S^2U$, then $u \otimes v = \frac{(u,v)}{\dim U} e^i \otimes e^i$, and that for every $a, b \in \mathfrak{g}$ one has

$$(abe^i, e^i) \ = \ \mathrm{Tr}_U(a,b) \ = \ \frac{\Omega_U \dim U}{\dim \mathfrak{g}}(a,b) \ . \qquad (4.9)$$

Equation (4.8) follows immediately by (4.6). We can use (4.7) to find an equation for the constant ρ defined by (4.1). After taking the image via $(,): S^2\mathfrak{g} \to \mathbb{C}$ of both sides of (4.7) we get (4.1):

$$\rho = \dim \mathfrak{g} \ \alpha_\mathbb{C} \ . \qquad (4.10)$$

So far we used equations (3.39) and (3.40) to find expressions for the homomorphisms $K: \Lambda^2U \to \mathfrak{g}$ and $P: S^2U \to S^2\mathfrak{g}$ depending only on the values of the parameters c and k. We are left to impose conditions (3.41) and (3.42) to get restrictions on the values of c and k and on the choice of $\mathfrak{g}$ and U.

Since $P: S^2U \to S^2\mathfrak{g}$ is a $\mathfrak{g}$–module homomorphism, if $\Pi \subset S^2U$ is an irreducible component which is not embedded in $S^2\mathfrak{g}$, it must be $P|_\Pi = 0$. Therefore if we choose $u \otimes v \in \Pi \subset S^2U$, equation (3.41) gives

$$\sigma\Big((J^\alpha u, w)J^\alpha v + (J^\alpha v, w)J^\alpha u\Big) = \frac{1}{2}\Big((u,w)v + (v,w)u\Big) \ . \tag{4.11}$$

Since by assumption $u \otimes v \in \Pi \subset S^2U$, and since the Casimir operator acts as a scalar on U and Π, we have

$$\Omega_\Pi u \otimes v = 2\Omega_U u \otimes v + 2J^\alpha u \otimes J^\alpha v \ . \tag{4.12}$$

Substituting into (4.11) we thus get

$$\sigma\left(\frac{1}{2}\Omega_\Pi - \Omega_U\right) = \frac{1}{2} \ . \tag{4.13}$$

Notice that (4.13) gives a very strong restriction on $\mathfrak{g}$ and U, namely for every irreducible component $\Pi \subset S^2U$ such that $\Pi \not\subset S^2\mathfrak{g}$ the Casimir operator must take the same value.

In the next section we will find a complete list of pairs $(\mathfrak{g}, U)$ satisfying the above condition.

4.2 Restrictions on $(\mathfrak{g}, U)$ coming from (4.13)

According to the observations at the end of Section 4.1, we need to solve the following:

Problem 4.2.1. Classify all pairs $(\mathfrak{g}, U)$ where $\mathfrak{g}$ is a simple Lie algebra, U is an irreducible, orthogonal, self–contragradient $\mathfrak{g}$–module such that S^2U admits a decomposition $S^2U = \Sigma^a \oplus \Sigma^b$, where Σ_a and Σ_b are such that there is an embedding $\Sigma^a \subset S^2\mathfrak{g}$ and $\Omega|_{\Sigma^b}$ is constant.

For simplicity, we introduce the following notation, which will be used throughout this section. If $\mathfrak{g}$ is a Lie algebra of rank r, $\mathfrak{h}$ its Cartan subalgebra, then $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ denote its simple roots and $\pi_1, \dots, \pi_r \in \mathfrak{h}^*$ its fundamental weights. An irreducible $\mathfrak{g}$–module will be denoted by its highest weight λ. Therefore $\pi_1, \dots, \pi_r$ denote the fundamental representations, and any other irreducible $\mathfrak{g}$–module (or, equivalently, any positive dominant weight), will be $\lambda = \sum_{k=1}^r \lambda_k \pi_k \in P_+$, for arbitrary $\lambda_k \in \mathbb{Z}_+$. λ_k are called the labels of λ, and we define the length of λ as $|\lambda| := \sum_{k=1}^r \lambda_k$. The trivial one dimensional representation will be both denoted as 0 (according to the above notation) or as $\mathbb{C}$. Given an irreducible $\mathfrak{g}$–module $\lambda \in P_+$, the Casimir operator $\Omega \in U(\mathfrak{g})$ acts as scalar on λ, and we denote by $\Omega_\lambda \in \mathbb{R}_+$ its value: $\Omega|_\lambda = \Omega_\lambda \ \mathbb{I}_\lambda$. In the space of dominant weights $P = \{\mu = \sum_{k=1}^r n_k \pi_k, \ n_k \in \mathbb{Z}\}$ we have a partial ordering. We say that $\mu > \nu$ if and only if $(\mu - \nu)$ can be written as non zero linear combination of simple roots α_i with non negative integer coefficients. In particular if λ is an irreducible $\mathfrak{g}$–module and μ is any weight of λ, namely the weight subspace of λ with weight μ is non zero, then $\lambda \geq \mu$.

We will need the following simple

Lemma 4.2.2. *If $\lambda > \mu$, then $\Omega_\lambda > \Omega_\mu$.*

Proof. Recall that $\Omega_\lambda = (\lambda, \lambda + 2\rho)$, where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{k=1}^{r}\pi_k$ is the semi-sum of all positive roots, or equivalently the sum of all fundamental weights. By assumption $\lambda = \mu + \gamma$, where $\gamma = \sum_{k=1}^{r} n_k\alpha_k \neq 0$ for some $\mathfrak{n}_i \in \mathbb{Z}_+,\ i = 1\ldots, r$. Then

$$\Omega_\lambda - \Omega_\mu = 2(\mu + \rho, \gamma) + (\gamma, \gamma) > 0\ .$$

□

Immediate consequence of Lemma 4.2.2 is the following:

Corollary 4.2.3. *Let $\lambda \in P_+$ be an irreducible $\mathfrak{g}$–module. For any irreducible component $\Pi \subset S^2\lambda,\ \Pi \neq 2\lambda$ we have $\Omega_{2\lambda} > \Omega_\Pi$.*

It follows that if the pair $(\mathfrak{g}, \lambda)$ satisfies the assumptions of Problem 4.2.1, either there is an embedding $(2\lambda) \subset S^2\mathfrak{g}$, or there is an embedding $S^2\lambda/(2\lambda) \subset S^2\mathfrak{g}$. We will consider these two cases separately.

4.2.1 Case 1: $(2\lambda) \subset S^2\mathfrak{g}$

The following table shows the decomposition of $S^2\mathfrak{g}$ into irreducible components for every simple Lie algebra $\mathfrak{g}$ (see [18]):

Table 4.1:

$\mathfrak{g}$	$S^2\mathfrak{g}$
$A_r,\ r \geq 1$	$(2\pi_1 + 2\pi_r) \oplus (\pi_2 + \pi_{r-1})\delta_{r\geq 3} \oplus (\pi_1 + \pi_r)\delta_{r\geq 2} \oplus \mathbb{C}$
$B_r,\ r \geq 3$	$(2\pi_2) \oplus \begin{cases} (2\pi_3) \text{ for } r = 3 \\ (2\pi_4) \text{ for } r = 4 \\ (\pi_4) \text{ for } r \geq 5 \end{cases} \oplus (2\pi_1) \oplus \mathbb{C}$
$C_r,\ r \geq 2$	$(4\pi_1) \oplus (2\pi_2) \oplus \pi_2 \oplus \mathbb{C}$
$D_r,\ r \geq 4$	$(2\pi_2) \oplus (2\pi_1) \oplus \begin{cases} (2\pi_3) \oplus (2\pi_4) \text{ for } r = 4 \\ (\pi_4 + \pi_5) \text{ for } r = 5 \\ (\pi_4) \text{ for } r \geq 6 \end{cases} \oplus \mathbb{C}$
E_6	$(2\pi_6) \oplus (\pi_1 + \pi_5) \oplus \mathbb{C}$
E_7	$(2\pi_6) \oplus \pi_2 \oplus \mathbb{C}$
E_8	$(2\pi_1) \oplus \pi_7 \oplus \mathbb{C}$
F_4	$(2\pi_4) \oplus (2\pi_1) \oplus \mathbb{C}$
G_2	$(2\pi_2) \oplus (2\pi_1) \oplus \mathbb{C}$

Obviously $\lambda \simeq \mathfrak{g}$ satisfies the condition $(2\lambda) \subset S^2\mathfrak{g}$ and all the assumptions of Problem 4.2.1. If $\lambda \not\simeq \mathfrak{g}$, since we are asking that $\lambda \in P_+$ and $(2\lambda) \subset S^2\mathfrak{g}$, only the following possible pairs $(\lambda, \mathfrak{g})$ are allowed:

Table 4.2:

$\mathfrak{g}$	λ	$S^2\lambda$
A_3	π_2	$(2\pi_2) \oplus \mathbb{C}$
$B_r,\ r \geq 3$	π_1	$(2\pi_1) \oplus \mathbb{C}$
B_3	π_3	$(2\pi_3) \oplus \mathbb{C}$
B_4	π_4	$(2\pi_4) \oplus \pi_1 \oplus \mathbb{C}$
$C_r,\ r \geq 2$	π_2	$(2\pi_2) \oplus \pi_2\delta_{r\geq 3} \oplus \pi_4\delta_{r\geq 4} \oplus \mathbb{C}$
$D_r,\ r \geq 4$	π_1	$(2\pi_1) \oplus \mathbb{C}$
D_4	π_i	$(2\pi_i) \oplus \mathbb{C},\ \forall i = 1, \ldots, 4$
F_4	π_1	$(2\pi_1) \oplus \pi_1 \oplus \mathbb{C}$
G_2	π_1	$(2\pi_1) \oplus \mathbb{C}$

By looking at the decomposition of $S^2\lambda$ into irreducible components (in the third column) for each case in the above list, we see that all the other assumptions of Problem 4.2.1 are satisfied.

4.2.2 Case 2: $S^2\lambda/(2\lambda) \subset S^2\mathfrak{g}$

In order to impose condition $S^2\lambda/(2\lambda) \subset S^2\mathfrak{g}$ we will need the following:

Lemma 4.2.4. *Let $\mathfrak{g}$ be a simple Lie algebra an $\lambda \in P_+$ be any non trivial irreducible $\mathfrak{g}$–module, with $|\lambda| \geq 2$. If $\lambda_i \geq 2$, then*

$$(2\lambda - 2\alpha_i) \subset S^2\lambda \ ,$$

and if $\lambda_i,\ \lambda_j \geq 1$ for $i \neq j$, then

$$(2\lambda - \alpha_i - \alpha_j) \subset S^2\lambda \ .$$

Remark 4.2.5. One can also prove (see [16, 19]) that if $\lambda_i \geq 1$, then

$$(2\lambda - \alpha_i) \subset \Lambda^2\lambda \ .$$

But we will not need this fact and therefore we will not prove it.

In order to prove Lemma 4.2.4 we will use the following Freudenthal's formula (see, for example, [10]):

Lemma 4.2.6. *Suppose* $\mu = \lambda - \sum_{k=1}^{r} n_i \alpha_1$ *is a weight of the irreducible* $\mathfrak{g}$*–module* $\lambda \in P_+$*, and denote by* $m_\lambda[\mu]$ *its multiplicity, namely the dimension of the weight subspace of weight* μ*. Then*

$$\Big((\lambda+\rho,\lambda+\rho)-(\mu+\rho,\mu+\rho)\Big)m_\lambda[\mu] = 2\sum_{\alpha>0}\sum_{i\geq 1} m_\lambda[\mu+i\alpha](\mu+i\alpha,\alpha)\ . \qquad (4.14)$$

Proof of Lemma 4.2.4. Consider first the case $\lambda_i \geq 2$. Under this assumption we have $m_\lambda[\lambda] = m_\lambda[\lambda-\alpha_i] = m_\lambda[\lambda - 2\alpha_i] = 1$. Therefore, just by dimension counting, we get $m_{S^2\lambda}[\lambda - 2\alpha_i] = 2$. On the other hand the multiplicity of $(2\lambda - 2\alpha_i)$ in the highest component $(2\lambda) \subset S^2\lambda$ is $m_{2\lambda}[2\lambda - 2\alpha_i] = 1$. This of course implies that there must be in $S^2\lambda$ a singular vector of weight $2\lambda - 2\alpha_i$. In other words $(2\lambda - 2\alpha_i) \subset S^2\lambda$. Consider now the case $\lambda_i,\ \lambda_j \geq 1$, for some $i \neq j = 1,\ldots,r$. Under this assumption we have $m_\lambda[\lambda] = m_\lambda[\lambda-\alpha_i] = m_\lambda[\lambda-\alpha_j] = 1$. We are also interested in $m_\lambda[\lambda-\alpha_i-\alpha_j]$. Since $(\alpha_i,\alpha_j) \leq 0$ for $i \neq j$, we know that $m_\lambda[\lambda-\alpha_i-\alpha_j] \geq 1$. To compute its actual value we can use Freudenthal's formula. By putting $\mu = \lambda - \alpha_i - \alpha_j$ into (4.14), and using the obvious identities $(\lambda,\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)\lambda_i$, $(\rho,\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)$, the left hand side becomes

$$\Big((\alpha_i,\alpha_i)\lambda_i + (\alpha_j,\alpha_j)\lambda_j - 2(\alpha_i,\alpha_j)\Big)m_\lambda[\lambda-\alpha_i-\alpha_j]\ . \qquad (4.15)$$

For the right hand side of (4.14), the sum over the positive roots $\alpha > 0$ will be different depending whether $\alpha_i + \alpha_j$ is a positive root or not. If $\alpha_i+\alpha_j$ is not a positive root (which is equivalent to say $(\alpha_i,\alpha_j) = 0$), we get

$$(\alpha_i,\alpha_i)\lambda_i + (\alpha_j,\alpha_j)\lambda_j\ , \qquad (4.16)$$

and if $\alpha_i + \alpha_j$ is a root (namely $(\alpha_i,\alpha_j) < 0$), we get

$$2\Big((\alpha_i,\alpha_i)\lambda_i + (\alpha_j,\alpha_j)\lambda_j - 2(\alpha_i,\alpha_j)\Big)\ . \qquad (4.17)$$

By comparing (4.15), (4.16) and (4.17) we finally get

$$m_\lambda[\lambda-\alpha_i-\alpha_j] = \begin{cases} 1 \text{ if } (\alpha_i,\alpha_j) = 0 \\ 2 \text{ if } (\alpha_i,\alpha_j) < 0 \end{cases}\ .$$

By dimension counting the multiplicity of $(2\lambda - \alpha_i - \alpha_j)$ in $S^2\lambda$ is

$$m_{S^2\lambda}[2\lambda-\alpha_i-\alpha_j] = 1 + m_\lambda[\lambda-\alpha_i-\alpha_j] = \begin{cases} 2 \text{ if } (\alpha_i,\alpha_j) = 0 \\ 3 \text{ if } (\alpha_i,\alpha_j) < 0 \end{cases}\ .$$

We now want to compute the multiplicity of $(2\lambda-\alpha_i-\alpha_j)$ in the highest component $(2\lambda) \subset S^2\lambda$. Again we can use Freudenthal's formula (4.14), by changing λ with 2λ and taking $\mu = 2\lambda - \alpha_i - \alpha_j$. The left hand side becomes

$$2\Big((\alpha_i,\alpha_i)\lambda_i + (\alpha_j,\alpha_j)\lambda_j - (\alpha_i,\alpha_j)\Big)m_{2\lambda}[2\lambda-\alpha_i-\alpha_j]\ , \qquad (4.18)$$

and the right hand side is, if $\alpha_i + \alpha_j$ is not a root

$$2\Big((\alpha_i, \alpha_i)\lambda_i + (\alpha_j, \alpha_j)\lambda_j\Big) \ , \tag{4.19}$$

while if $\alpha_i + \alpha_j$ is a root

$$4\Big((\alpha_i, \alpha_i)\lambda_i + (\alpha_j, \alpha_j)\lambda_j - (\alpha_i, \alpha_j)\Big) \ . \tag{4.20}$$

By comparing (4.18), (4.19) and (4.20) we get

$$m_{2\lambda}[2\lambda - \alpha_i - \alpha_j] = \begin{cases} 1 \text{ if } (\alpha_i, \alpha_j) = 0 \\ 2 \text{ if } (\alpha_i, \alpha_j) < 0 \end{cases} \ .$$

In all cases we proved that $m_{2\lambda}[2\lambda - \alpha_i - \alpha_j] < m_{S^2\lambda}[2\lambda - \alpha_i - \alpha_j]$, which obviously implies that there is a singular vector in $S^2\lambda$ of weight $(2\lambda - \alpha_i - \alpha_j)$, namely $(2\lambda - \alpha_i - \alpha_j) \subset S^2\lambda$. □

Lemma 4.2.4 gives a powerful tool to solve Problem 4.2.1. We will consider separately the two situations $|\lambda| = 1$ and $|\lambda| \geq 2$.

Case $|\lambda| = 1$

The first case is $|\lambda| = 1$, namely λ is a fundamental representation. We want to check case by case to see when the assumptions of Problem 4.2.1 are satisfied (by looking, for example, at the tables in [18] for the decomposition of $S^2\pi_i$). For $\mathfrak{g} = A_r, \ r \geq 1$, the condition that $\lambda = \pi_i$ is self contragradient (namely $\mathbb{C} \subset S^2\lambda$) is satisfied only for $r = 4k + 3, \ k \in \mathbb{Z}_+$ and $\lambda = \pi_{2k+2}$. On the other hand

$$S^2\pi_{2k+2} = (2\pi_{2k+2}) \oplus (\pi_{2k} + \pi_{2k+4}) \oplus \cdots \oplus (\pi_2 + \pi_{4k+2}) \oplus \mathbb{C} \ .$$

Comparing this with the decomposition of $S^2\mathfrak{g}$ in Table 4.1 we get that $(\mathfrak{g} = A_{4k+3}, \lambda = \pi_{2k+2})$ meets the assumptions of Problem 4.2.1 only for $k = 0, 1$. For $\mathfrak{g} = B_r, \ r \geq 3$ we have that

$$S^2\pi_p = \bigoplus_{\left(\begin{array}{c} 0 \leq y \leq x \\ x + y \leq 2p \\ x \equiv y \mod 4 \end{array}\right)} (\hat{\pi}_x + \hat{\pi}_y) \ , \quad \text{for } 1 \leq p \leq r - 1$$

$$S^2\pi_r = \bigoplus_{\left(\begin{array}{c} 0 \leq 1 \leq r \\ i \equiv 0, 3 \mod 4 \end{array}\right)} \hat{\pi}_{r-i}$$

where we denoted (following [18])

$$\hat{\pi}_p = \begin{cases} 0 & \text{for} \quad p = 0, 2r+1 \\ \pi_p & \text{for} \quad 1 \le p \le r-1 \\ 2\pi_r & \text{for} \quad p = r \\ \pi_{2r+1-p} & \text{for} \quad r+1 \le p \le 2r \end{cases}$$

From the first decomposition, if $p \geq 3$ and $p \neq r$, both $(2\pi_p)$ and $(2\pi_3)$ are irreducible components of $S^2\pi_p$ but not of $S^2\mathfrak{g}$ (see Table 4.1); therefore $\lambda = \pi_p$ does not meet the assumptions of Problem 4.2.1. Similarly, for $p = r \neq 3, 4$ we have $(2\pi_r),\ \pi_{r-3},\ \pi_{r-4} \subset S^2\pi_r$ and $2\pi_r,\ \pi_{r-i} \not\subset S^2\mathfrak{g}$ for either $i = 3$ or 4; so that also $\lambda = \pi_r$ is ruled out, unless $r = 3, 4$. It follows that the assumptions of Problem 4.2.1 are satisfied only for $\lambda = \pi_1, \pi_2,\ \forall r \geq 3$ and $\lambda = \pi_r$ for $r = 3, 4$. For $\mathfrak{g} = C_r,\ r \geq 2$, the condition $\mathbb{C} \subset S^2\pi_p$ is satisfied if and only if $p \in 2\mathbb{Z}_+$. Moreover

$$S^2\pi_p = \bigoplus_{\begin{pmatrix} 0 \le y \le x \\ x+y \le 2p \\ x-y \le 2(r-p) \\ x \equiv y \equiv p \mod 2 \end{pmatrix}} (\pi_x + \pi_y) \ .$$

From this decomposition we have $2\pi_p \subset S^2\pi_p$ for all p, $2\pi_4 \subset S^2\pi_p$ for $p \geq 5$ and $(\pi_2 + \pi_4) \subset S^2\pi_p$ for $p \neq r$. On the other hand if $p \neq 2$ (see Table 4.1) $2\pi_p$, $2\pi_4$ and $\pi_2 + \pi_4$ are not components of $S^2\mathfrak{g}$. This means that $\lambda = \pi_p$ does not satisfy the assumptions of Problem 4.2.1, unless $p = 2$ or $p = r = 4$. For $\mathfrak{g} = D_r$ we have

$$S^2\pi_p = \bigoplus_{\begin{pmatrix} 0 \le y \le x \\ x+y \le 2p \\ x \equiv y \mod 4 \end{pmatrix}} (\hat{\pi}_x + \hat{\pi}_y) \ , \qquad \text{for } 1 \le p \le r-2$$

$$S^2\pi_p = (2\pi_p) \oplus \left(\bigoplus_{i \ge 1} \pi_{r-4i} \right) \ , \qquad \text{for } p = r-1, r \ ,$$

where we denoted $\pi_0 = \mathbb{C}$ and

$$\hat{\pi}_p = \begin{cases} 0 & \text{for} \quad p = 0, 2r \\ \pi_p & \text{for} \quad 1 \le p \le r-2 \\ (\pi_{r-1} + \pi_r) & \text{for} \quad p = r-1, r+1 \\ (2\pi_{r-1}) \oplus (2\pi_r) & \text{for} \quad p = r \\ \pi_{2r-p} & \text{for} \quad r+2 \le p \le 2r-1 \end{cases} \ .$$

From the first decomposition we have that, if $p \geq 3$ and $p \neq r-1, r$, then $(2\pi_p)$ and $(\hat{\pi}_5 + \hat{\pi}_1)$ are embedded in $S^2\pi_p$ but not in $S^2\mathfrak{g}$, therefore this case is ruled out. If $p = r-1$ or r, π_p admits a symmetric invariant bilinear form if and only if $r \equiv 0$ mod 4. If $r \geq 12$, then $(2\pi_p)$ and π_{r-4} are embedded in $S^2\pi_p$ but not in $S^2\mathfrak{g}$, so that $\lambda = \pi_{r-1}$ and π_r are ruled out unless $r = 4$ or 8. In conclusion the only $\mathfrak{g}$–modules $\lambda = \pi_p$ which satisfy all the assumptions of Problem 4.2.1 are $\lambda = \pi_p$ for all p if $r = 4$, $\lambda = \pi_1,\ \pi_2$ for all values of $r \geq 5$, $\lambda = \pi_7,\ \pi_8$ for $r = 8$. We are left to consider the exceptional simple Lie algebras $\mathfrak{g} = E_r, r = 6,7,8,\ F_4,\ G_2$. With a case by case check (based on the tables in [18]) one finds that the only pairs $(\mathfrak{g}, \lambda)$ satisfying the assumptions of Problem 4.2.1 where $\mathfrak{g}$ is an exceptional Lie algebra and λ is a fundamental representation are all the adjoint representations plus the two cases $(\mathfrak{g} = F_4, \lambda = \pi_1),\ (\mathfrak{g} = G_2, \lambda = \pi_1)$.

Case $|\lambda| \geq 2$

Consider now the case $|\lambda| \geq 2$. By Lemma 4.2.4 we have $(2\lambda - \alpha_i - \alpha_j) \subset S^2\lambda$ for i, j such that $\lambda_i, \lambda_j \geq 1$ if $i \neq j$ and $\lambda_i \geq 2$ if $i = j$. Therefore, since we are assuming $S^2\lambda/(2\lambda) \subset S^2\mathfrak{g}$, we must have $(2\lambda - \alpha_i - \alpha_j) \subset S^2\mathfrak{g}$. In other words we can restrict ourselves to look for dominant weights $\lambda \in P_+$ which can be written as $\lambda = \frac{1}{2}(\mu + \alpha_i + \alpha_j)$, for some $i, j = 1, \ldots, r$ and some irreducible component $\mu \subset S^2\mathfrak{g}$. By direct inspection, the only quadruples $(\mathfrak{g}, \mu, \alpha_i, \alpha_j)$ such that $\mu \subset S^2\mathfrak{g}$ and $\lambda = \frac{1}{2}(\mu + \alpha_i + \alpha_j) \in P_+$ are the ones listed in the Table 4.3.

Table 4.3:

$\mathfrak{g}$	μ	$\alpha_i + \alpha_j$	λ
A_1	$\mathbb{C}$	$2\alpha_1 = 4\pi_1$	$2\pi_1 \simeq \mathfrak{g}$
A_1	$4\pi_1$	$2\alpha_1 = 4\pi_1$	$4\pi_1$
A_2	$\pi_1 + \pi_2$	$\alpha_1 + \alpha_2 = \pi_1 + \pi_2$	$(\pi_1 + \pi_2) \simeq \mathfrak{g}$
A_2	$2\pi_1 + 2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$3\pi_1$
A_2	$2\pi_1 + 2\pi_2$	$2\alpha_2 = -2\pi_1 + 4\pi_2$	$3\pi_2$
A_3	$2\pi_1 + 2\pi_3$	$2\alpha_2 = -2\pi_1 + 4\pi_2 - 2\pi_3$	$2\pi_2$
$A_r,\ r \geq 3$	$\pi_2 + \pi_{r-1}$	$\alpha_1 + \alpha_r$ $= 2\pi_1 - \pi_2 - \pi_{r-1} + 2\pi_r$	$(\pi_1 + \pi_r) \simeq \mathfrak{g}$
B_3	$2\pi_2$	$\alpha_1 + \alpha_3 = 2\pi_1 - 2\pi_2 + 2\pi_3$	$\pi_1 + \pi_3$
B_3	$2\pi_2$	$2\alpha_3 = -2\pi_2 + 4\pi_3$	$2\pi_3$
$B_r,\ r \geq 3$	$2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$2\pi_1$
C_2	π_2	$\alpha_1 + \alpha_2 = \pi_2$	π_2
C_2	$2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$2\pi_1 \simeq \mathfrak{g}$
C_2	$4\pi_1$	$2\alpha_2 = -4\pi_1 + 4\pi_2$	$2\pi_2$
$C_r,\ r \geq 3$	$2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$2\pi_1 \simeq \mathfrak{g}$
D_4	$2\pi_2$	$\alpha_i + \alpha_j = 2\pi_i + 2\pi_j - 2\pi_2,$ $i, j = 1, 3, 4$	$\pi_i + \pi_j$
$D_r,\ r \geq 5$	$2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$2\pi_1$
G_2	$2\pi_2$	$2\alpha_1 = 4\pi_1 - 2\pi_2$	$2\pi_1$

Among these pairs $(\mathfrak{g}, \lambda)$ we have to select those for which all assumptions of Problem 4.2.1 are satisfied. By imposing $\mathbb{C} \subset S^2\lambda$ we rule out $\mathfrak{g} = A_2$, $\lambda = 3\pi_1$ and $3\pi_2$. According to Lemma 4.2.2 we can then exclude all pairs $(\mathfrak{g}, \lambda)$ such that $2\lambda \not\subset S^2\mathfrak{g}$ and there is another component $\Pi \subset S^2\lambda$ such that $\Pi \not\subset S^2\mathfrak{g}$. In this way we rule out the following cases:

Table 4.4:

$\mathfrak{g}$	λ	$\Pi \subset S^2\lambda,\ \not\subset S^2\mathfrak{g}$
B_3	$\pi_1 + \pi_3$	$\pi_1 + 2\pi_3$
B_3	$2\pi_3$	$\pi_1 + \pi_2$
D_4	$\pi_i + \pi_j$ $i \neq j = 1, 3, 4$	$\pi_1 + \pi_3 + \pi_4$
G_2	$2\pi_1$	$\pi_1 + \pi_2$

4.2.3 Solution of Problem 4.2.1

We can summarize all the results we have gotten so far in the following Table 4.5, which gives a complete solution to Problem 4.2.1.

Table 4.5:

$\mathfrak{g}$	λ	$S^2\lambda$	d	n
All	$\lambda \simeq \mathfrak{g}$	see Table 4.1		
A_1	$4\pi_1$	$(8\pi_1) \oplus (4\pi_1) \oplus \mathbb{C}$	2	1
A_3	π_2	$(2\pi_2) \oplus \mathbb{C}$	2	0
A_3	$2\pi_2$	$(4\pi_2) \oplus (2\pi_1 + 2\pi_3) \oplus (2\pi_2) \oplus \mathbb{C}$	3	1
A_7	π_4	$(2\pi_4) \oplus (\pi_2 + \pi_6) \oplus \mathbb{C}$	2	1
B_3	π_3	$(2\pi_3) \oplus \mathbb{C}$	2	0
B_4	π_4	$(2\pi_4) \oplus \pi_1 \oplus \mathbb{C}$	2	1
$B_r,\ r \geq 3$	π_1	$(2\pi_1) \oplus \mathbb{C}$	2	0
$B_r,\ r \geq 3$	$2\pi_1$	$(4\pi_1) \oplus (2\pi_2) \oplus (2\pi_1) \oplus \mathbb{C}$	3	1
C_2	$2\pi_2$	$(4\pi_2) \oplus (4\pi_1) \oplus (2\pi_2) \oplus \mathbb{C}$	3	1
C_4	π_4	$(2\pi_4) \oplus (2\pi_2) \oplus \mathbb{C}$	2	1
C_2	π_2	$(2\pi_2) \oplus \mathbb{C}$	2	0
C_3	π_2	$(2\pi_2) \oplus \pi_2 \oplus \mathbb{C}$	3	0
$C_r,\ r \geq 4$	π_2	$(2\pi_2) \oplus \pi_4 \oplus \pi_2 \oplus \mathbb{C}$	3	1
D_4	$\pi_i,\ i = 1, \ldots, 4$	$(2\pi_i) \oplus \mathbb{C}$	2	0
D_4	$2\pi_i,\ i = 1, \ldots, 4$	$(4\pi_i) \oplus (2\pi_2) \oplus (2\pi_i) \oplus \mathbb{C}$	3	1
D_8	$\pi_p,\ p = 7, 8$	$(2\pi_p) \oplus \pi_4 \oplus \mathbb{C}$	2	1
$D_r,\ r \geq 5$	π_1	$(2\pi_1) \oplus \mathbb{C}$	2	0
$D_r,\ r \geq 5$	$2\pi_1$	$(4\pi_1) \oplus (2\pi_2) \oplus (2\pi_1) \oplus \mathbb{C}$	3	1
F_4	π_1	$(2\pi_1) \oplus \pi_1 \oplus \mathbb{C}$	2	1
G_2	π_1	$(2\pi_1) \oplus \mathbb{C}$	2	0

The second last column shows the value of $d = \dim \mathrm{Hom}_{\mathfrak{g}}(S^2\lambda, S^2\mathfrak{g})$. In other words, if we decompose $S^2\lambda = \Sigma^a \oplus \Sigma^b$ where Σ^a is a maximal submodule embedded in $S^2\mathfrak{g}$ and $\Sigma^b \simeq S^2\lambda/\Sigma^a$, then d is the number of irreducible components of Σ^a. It can be obtained immediately by comparing the decomposition of $S^2\lambda$ in the third column of Table 4.5 with the corresponding decomposition of $S^2\mathfrak{g}$, given in Table 4.1. The last column gives the number of irreducible components of Σ^b, denoted by n.

So far we proved that all the pairs $(\mathfrak{g}, \lambda = U)$ satisfying the assumptions of Problem 3.5.3, and such that $\mathfrak{g}$ is simple, U is irreducible, $\varkappa$ and Q are non zero, have to be listed in Table 4.5. In the following Sections we will find out which of the pairs $(\mathfrak{g}, U)$ listed in Table 4.5 do satisfy all conditions of Problem 3.5.3, namely admit homomorphisms $K: \Lambda^2 U \to \mathfrak{g}$, $P: S^2U \to S^2\mathfrak{g}$ satisfying equations (3.39), (3.40), (3.41) and (3.42). We will consider separately the following three situations:

a) adjoint representation: $U \simeq \mathfrak{g}$,

b) $(\mathfrak{g}, U)$ such that $U \neq \mathfrak{g}$ and $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 3$,

c) $(\mathfrak{g}, U)$ such that $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 2$.

4.3 Case $U \simeq \mathfrak{g}$

Assume $U \simeq \mathfrak{g}$. In this case the equation (4.2) for $K(a,b)$ reduces to

$$K(a,b) = \sigma[a,b] \ . \tag{4.21}$$

Moreover, if $a_1 \otimes a_2$ (or rather a linear combination of such monomials) is an element of an irreducible component $\Pi \subset S^2\mathfrak{g}$, then

$$([J^\alpha, [J^\beta, a_1]], a_2) J^\alpha \otimes J^\beta = -[J^\alpha, a_1] \otimes [J^\alpha, a_2] = -\left(\frac{1}{2}\Omega_\Pi - \Omega_{\mathfrak{g}}\right) a_1 \otimes a_2 \ .$$

Therefore the expressions (4.7) and (4.8) for $P(a_1, a_2)$ become

$$P(a_1, a_2) = \beta_\Pi \ a_1 \otimes a_2 \ , \qquad \text{if } a_1 \otimes a_2 \in \Pi \subset S^2\mathfrak{g} \ , \tag{4.22}$$

where

$$\begin{aligned} \beta_{\mathbb{C}} &= \dim \mathfrak{g} \ \alpha_{\mathbb{C}} \ , \\ \beta_\Pi &= -2\left(\tfrac{1}{2}\Omega_\Pi - \Omega_{\mathfrak{g}}\right) \alpha_\Pi \ , \qquad \text{for } \Pi \subset S^2\mathfrak{g}, \ \Pi \neq \mathbb{C} \ . \end{aligned}$$

For $b \in \mathfrak{g}$ and $a_1 \otimes a_2 \in \Pi \subset S^2\mathfrak{g}$, condition (3.41) gives

$$[P_1(a_{,1}, a_2), [P_2(a_1, a_2), b]] + 2[K(a_1, b), a_2] = (a_1, b)a_2 - (a_1, a_2)b \ .$$

By using (4.21) and (4.22) this equation gives

$$(\beta_\Pi - 2\sigma)[a_1, [a_2, b]] = (a_1, b)a_2 - (a_1, a_2)b \ . \tag{4.23}$$

In the left hand side we can substitute

$$[a_1,[a_2,b]] = -\left(\frac{1}{2}\Omega_\Pi - \Omega_{\mathfrak{g}}\right)(a_1,b)a_2 \ ,$$

which follows by the obvious identity $[a,b] = (J^\alpha,[a,b])J^\alpha$, $\mathfrak{g}$–invariance of the Killing form, and equations (4.5) and (4.12). In the right hand side of (4.23) we notice that $(a_1,a_2) = 0$ if $\Pi \neq \mathbb{C}$ and $(a_1,a_2)b = \dim \mathfrak{g}\ (a_1,b)a_2$ if $\Pi = \mathbb{C}$. Equation (4.23) thus reduces to the identity

$$(\beta_\Pi - 2\sigma)\left(\frac{1}{2}\Omega_\Pi - \Omega_{\mathfrak{g}}\right) + (1 - \delta_{\Pi,\mathbb{C}}\ \dim \mathfrak{g}) = 0 \ . \tag{4.24}$$

Assuming $\Pi \neq \mathbb{C}$ is any irreducible component of $S^2\mathfrak{g}$, we can use the definition (4.22) of β_Π and the definition (4.8) of α_Π to get from (4.24)

$$(4k\sigma + 1)\left(\frac{1}{2}\Omega_\Pi - \Omega_{\mathfrak{g}}\right) = 2k \ . \tag{4.25}$$

Clearly (4.25) can not be satisfied for all non trivial irreducible components $\Pi \subset S^2\mathfrak{g}$, since the left hand side is different for different choices of Π, while the right hand side does not depend on Π.

In conclusion, the case $U \simeq \mathfrak{g}$ is always ruled out, unless $\mathfrak{g}$ is such that $S^2\mathfrak{g} =$ (irreducible)$\oplus$(scalar). This happens only for $\mathfrak{g} = A_1$. We will consider this example in Section 4.5, together with all other cases for which $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 2$.

4.4 Case $\dim \mathbf{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 3$

Let us consider now the pairs $(\mathfrak{g}, U)$ in Table 4.5 for which $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 3$. According to the isomorphisms $C_2 \simeq \mathrm{so}_5$, $A_3 \simeq \mathrm{so}_6$, $B_r \simeq \mathrm{so}_{2r+1}, r \geq 3$, $D_r \simeq \mathrm{so}_{2r}, r \geq 4$, $C_r \simeq \mathrm{sp}_{2r}, r \geq 3$, all such pairs are included in one of the following two cases:

1. $\mathfrak{g} = \mathrm{so}_n \ , \qquad U \subset S^2\mathbb{C}^n/\mathbb{C} \ , \ n \geq 5$
2. $\mathfrak{g} = \mathrm{sp}_n \ , \qquad U \subset \Lambda^2\mathbb{C}^n/\mathbb{C} \ , \ n \geq 6$

For each of these two cases we will prove by direct computation that equations (3.39), (3.40), (3.41) and (3.42) can not be simultaneously satisfied, for any choice of homomorphisms $\varkappa :\ S^2\mathfrak{g} \to \mathbb{C}$, $Q :\ S^2U \to \mathbb{C}$, $K :\ \Lambda^2U \to \mathfrak{g}$, $P :\ S^2U \to \mathfrak{g}$. In other words, all pairs $(\mathfrak{g}, U)$ with $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) \neq 2$ are ruled out.

We have an explicit realization of the Lie algebra $\mathfrak{g}$ and its representation U in the following way. Consider first $\mathfrak{g} = \mathrm{so}_n$, $U = S^2\mathbb{C}^n/\mathbb{C}$. By definition

$$\mathfrak{g} = \mathrm{so}_n = \left\{X \in \mathrm{Mat}_n\mathbb{C} \mid X^T = -X\right\} \ .$$

Its defining representation is the space $\mathbb{C}^n$ of n–vectors. If we naturally identify $\mathbb{C}^n \otimes \mathbb{C}^n \simeq \mathrm{Mat}_n\mathbb{C}$, the action of so_n on $\mathrm{Mat}_n\mathbb{C}$ is given by commutator: $\pi(X)(M) =$

$[X, M] = XM - MX$, $\forall X \in \mathrm{so}_n$, $M \in \mathrm{Mat}_n\mathbb{C}$. The decomposition $\mathbb{C}^n \otimes \mathbb{C}^n = \Lambda^2\mathbb{C}^n \oplus S^2\mathbb{C}^n$ translates into $\mathrm{Mat}_n\mathbb{C} = \mathrm{so}_n \oplus \mathcal{K}$, $\mathcal{K} = U \oplus \mathbb{C}\mathbb{I}$, where

$$U = \left\{ A \in \mathrm{Mat}_n\mathbb{C} \mid A^T = A,\ \mathrm{Tr}A = 0 \right\} .$$

Consider now $\mathfrak{g} = \mathrm{sp}_n$, $U = \Lambda^2\mathbb{C}^n/\mathbb{C}$. Let J be the matrix

$$J = \begin{pmatrix} \mathrm{O} & & & \cdot^{\cdot^{\cdot^{1}}} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot^{-1}}} & & \mathrm{O} \\ -1 & & & \end{pmatrix} \tag{4.26}$$

Then

$$\mathfrak{g} = \mathrm{sp}_n = \left\{ X \in \mathrm{Mat}_n\mathbb{C} \mid XJ + JX^T = 0 \right\} ,$$

and

$$U = \left\{ A \in \mathrm{Mat}_n\mathbb{C} \mid AJ - JA^T = 0,\ \mathrm{Tr}A = 0 \right\} .$$

Throughout this section we denote $n = 2r$, where $r = \mathrm{rank}\mathfrak{g} \geq 3$.

The unique (up to scalar multiplication) symmetric invariant bilinear form on $\mathfrak{g}$ is the trace form $(X, Y) = \mathrm{Tr}(XY)$, $\forall X, Y \in \mathfrak{g}$. Similarly, the unique symmetric invariant bilinear form on U is $(A, B) = \mathrm{Tr}(AB)$, $\forall A, B \in U$. Therefore the $\mathfrak{g}$–module homomorphisms $\varkappa: S^2\mathfrak{g} \to \mathbb{C}$ and $Q: S^2U \to \mathbb{C}$ are necessarily of the form $(X, Y \in \mathfrak{g},\ A, B \in U)$

$$\varkappa(X, Y) = k(X, Y) \quad \text{and} \quad Q(A, B) = \varepsilon(A, B) ,$$

for some $k, \varepsilon \in \mathbb{C}$. Notice that the choice of the normalization factor ε can be modified arbitrarily with a $\mathfrak{g}$–module isomorphism $U \xrightarrow{\sim} U$, $A \mapsto \delta A$ for any $\delta \in \mathbb{C} - \{0\}$. We can thus fix $\varepsilon = 1$.

Remark 4.4.1. (a) We can fix a dual basis $\{J_\alpha,\ J^\alpha,\ \alpha \in \mathcal{A}\}$ on so_n as follows. We take $\mathcal{A} = \{<i, j> \mid 1 \leq i < j \leq n\}$ and

$$J_{<i,j>} = -J^{<i,j>} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}) .$$

(b) Similarly, we have a dual basis $\{J_\alpha,\ J^\alpha,\ \alpha \in \mathcal{A}\}$ on sp_n as follows.

$$\begin{aligned} A_{ij} &= \tfrac{1}{\sqrt{2}}(E_{ij} - E_{n+1-j,n+1-i}) , & A^{ij} &= A_{ji} , \quad 1 \leq i, j \leq r , \\ B_{ij} &= \tfrac{1}{\sqrt{2}}(E_{i,n+1-j} + E_{j,n+1-i}) , & B^{ij} &= C_{ij} , \quad 1 \leq i < j \leq r , \\ C_{ij} &= \tfrac{1}{\sqrt{2}}(E_{n+1-i,j} + E_{n+1-j,i}) , & C^{ij} &= B_{ij} , \quad 1 \leq i < j \leq r , \end{aligned}$$

$$
\begin{array}{rclrcl}
B_{ii} & = & E_{i,n+1-i} & B^{ii} & = & C_{ii}\ , \quad 1 \le i \le r\ , \\
C_{ii} & = & E_{n+1-i,i} & C^{ii} & = & B_{ii}\ , \quad 1 \le i \le r\ .
\end{array}
$$

It is immediate to check that, indeed, $\mathrm{Tr}(J_\alpha J^\beta) = \delta_{\alpha\beta}\ , \quad \forall \alpha, \beta \in \mathcal{A}$.

Lemma 4.4.2. *Let the pair $(\mathfrak{g}, U)$ be either $(\mathrm{so}_n,\ S^2\mathbb{C}^n/\mathbb{C})$, $n \ge 5$, or $(\mathrm{sp}_n,\ \Lambda^2\mathbb{C}^n/\mathbb{C})$, $n = 2r \ge 6$. We denote by I respectively the identity matrix, if we consider $\mathfrak{g} = \mathrm{so}_n,\ U = S^2\mathbb{C}^n/\mathbb{C}$, or the matrix J, if we consider $\mathfrak{g} = \mathrm{sp}_n,\ U = \Lambda^2\mathbb{C}^n/\mathbb{C}$. It follows that $I^2 = i^2\mathbb{1}$, where $i^2 = \pm 1$, depending whether we consider $\mathfrak{g} = \mathrm{so}_n,\ U = S^2\mathbb{C}^n/\mathbb{C}$ or $\mathfrak{g} = \mathrm{sp}_n,\ U = \Lambda^2\mathbb{C}^n/\mathbb{C}$.*

(a) If Ω_Π denotes the eigenvalue of the Casimir operator $\Omega = J_\alpha J^\alpha \in U(\mathfrak{g})$ on the irreducible $\mathfrak{g}$–module Π, then

$$
\begin{array}{rcl}
\Omega_{\mathbb{C}^n} & = & \frac{1}{2}(n - i^2)\ , \\
\Omega_U & = & n\ , \\
\Omega_{\mathfrak{g}} & = & n - 2i^2\ .
\end{array}
$$

As usual we assume we sum over repeated indices.

(b) For every matrix $M \in Mat_n\mathbb{C}$ we have

$$
J_\alpha M J^\alpha \ = \ -\frac{1}{2} I M^T I + \frac{1}{2} Tr(M)\mathbb{1}\ .
$$

In particular

$$
\begin{array}{rcl}
J_\alpha X J^\alpha & = & \frac{1}{2} i^2 X\ , \quad \forall X \in \mathfrak{g}\ , \\
J_\alpha A J^\alpha & = & -\frac{1}{2} i^2 A\ , \quad \forall A \in U\ .
\end{array}
$$

(c) There is a unique, up to scalar multiplication, $\mathfrak{g}$–module homomorphism: $\Lambda^2 U \to \mathfrak{g}$, namely

$$
[A, B] \ = \ AB - BA\ , \quad \forall A, B \in U\ .
$$

(b) A basis for $Hom_{\mathfrak{g}}(S^2U, S^2\mathfrak{g})$ is given by the following three $\mathfrak{g}$–module homomorphisms $S^2U \to S^2\mathfrak{g}$

$$
\begin{array}{rcl}
P^{(1)}(A,B) & = & (A,B) J_\alpha \otimes J^\alpha\ , \\
P^{(2)}(A,B) & = & \frac{1}{2}\Big(\{\{A,B\}, J_\alpha\} \otimes J^\alpha + J_\alpha \otimes \{\{A,B\}, J^\alpha\}\Big) \\
 & = & J_\alpha \otimes \{\{A,B\}, J^\alpha\}\ , \\
P^{(3)}(A,B) & = & \frac{1}{2}\Big((A J_\alpha\ B + B J_\alpha A) \otimes J^\alpha + J_\alpha \otimes (A J^\alpha B + B J^\alpha A)\Big) \\
 & = & J_\alpha \otimes (A J^\alpha B + B J^\alpha A)\ .
\end{array}
$$

Here and further $\{A, B\}$ denotes the anti-commutator of A and B, namely $AB + BA$.

(e) In all cases except $\mathfrak{g} = \mathrm{sp}_6,\ U = \Lambda^2\mathbb{C}^6/\mathbb{C}$, the following expressions define linearly

independent $\mathfrak{g}$*–module homomorphisms* $S^3U \otimes \mathfrak{g} \to U$ *(*$X \in \mathfrak{g},\ A_i \in U,\ i = 1,2,3$*)*

$$\begin{aligned} M^{(1)}(X; A_1, A_2, A_3) &= \tfrac{1}{2}\textstyle\sum_{\sigma\in S_3}(A_{\sigma_1}, A_{\sigma_2})[X, A_{\sigma_3}] \ , \\ M^{(2)}(X; A_1, A_2, A_3) &= \textstyle\sum_{\sigma\in S_3}[\{A_{\sigma_1}A_{\sigma_2}, X\}, A_{\sigma_3}] \ , \\ M^{(3)}(X; A_1, A_2, A_3) &= \textstyle\sum_{\sigma\in S_3} A_{\sigma_1}[X, A_{\sigma_3}]A_{\sigma_2} \ . \end{aligned}$$

We denote by S_3 *the set of all permutations of* $(1,2,3)$*. For* $\mathfrak{g} = \mathrm{sp}_6,\ U = \Lambda^2\mathbb{C}^6/\mathbb{C}$ *the only relation of linear dependence among them is*

$$M^{(1)}(X; A_1, A_2, A_3) \ = \ 2M^{(2)}(X; A_1, A_2, A_3) \ + \ 2M^{(3)}(X; A_1, A_2, A_3) \ . \qquad (4.27)$$

(f) The following expressions define linearly independent $\mathfrak{g}$*–module homomorphisms:* $S^2U \otimes \mathfrak{g} \to \mathfrak{g}$ *(*$X \in \mathfrak{g},\ A, B \in U$*)*

$$\begin{aligned} R^{(1)}(X; A, B) &= (A, B)X \ , \\ R^{(2)}(X; A, B) &= \{\{A, B\}, X\} \ , \\ R^{(3)}(X; A, B) &= (AXB + BXA) \ . \end{aligned}$$

Proof. Parts (a) and (b) can be checked with a straightforward computation. Their proof will be omitted. (Note that we will only need them in the particular case $\mathfrak{g} = \mathrm{sp}_6$, $U = \Lambda^2\mathbb{C}^6/\mathbb{C}$). Part (c) follows immediately by the fact that $\dim \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 U, \mathfrak{g}) = 1$ (see [18]). Clearly all the maps $P^{(i)}:\ S^2U \to S^2\mathfrak{g}$, $M^{(i)}:\ S^3U\otimes\mathfrak{g} \to U$, $R^{(i)}:\ S^2U\otimes\mathfrak{g} \to \mathfrak{g}$, $i = 1,2,3$, are $\mathfrak{g}$–module homomorphisms. Since $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 3$, we only need to prove their linear independence (and relation (4.27) for $\mathfrak{g} = \mathrm{sp}_6$). This can be checked by direct inspection. ☐

It follows by (c) and (d) in Lemma 4.4.2 that the $\mathfrak{g}$–module homomorphisms K and P are

$$\begin{aligned} K(A, B) &= \sigma[A, B] \ , \\ P(A, B) &= \alpha_1 P^{(1)}(A, B) + \alpha_2 P^{(2)}(A, B) + \alpha_3 P^{(3)}(A, B) \ , \end{aligned}$$

for some values of $\sigma, \alpha_i \in \mathbb{C},\ i = 1,2,3$.

Let us impose now equation (3.42). In this setting, it takes the form

$$\sum_{i=1,2,3} \alpha_i \sum_{\sigma\in C_3} P_1^{(i)}(A_{\sigma_1}, A_{\sigma_2}) \otimes [P_2^{(i)}(A_{\sigma_1}, A_{\sigma_2}), A_{\sigma_3}] \ = \ 0 \ , \quad \forall A_i \in U \ , \qquad (4.28)$$

where C_3 is the set of cyclic permutations of $(1,2,3)$. After a straightforward computation we get the following identities

$$\sum_{\sigma\in C_3} P_1^{(i)}(A_{\sigma_1}, A_{\sigma_2}) \otimes [P_2^{(i)}(A_{\sigma_1}, A_{\sigma_2}), A_{\sigma_3}] \ = \ J_\alpha \otimes M^{(i)}(J^\alpha; A_1, A_2, A_3) \ , \quad \forall i = 1,2,3 \ ,$$

where $M^{(i)}(X; A_1, A_2, A_3)$ is defined in part (e) of Lemma 4.4.2. Thanks to the above equations, (4.28) can be rewritten as

$$\sum_{i=1,2,3} \alpha_i J_\alpha \otimes M^{(i)}(J^\alpha; A_1, A_2, A_3) \ = \ 0 \ . \tag{4.29}$$

By Part (e) in Lemma 4.4.2, the $M^{(i)}$'s are linearly independent in all cases except $\mathfrak{g} = \mathrm{sp}_6,\ U = \Lambda^2\mathbb{C}^6/\mathbb{C}$. Therefore equation (4.29) implies

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \ ,$$

unless $\mathfrak{g} = \mathrm{sp}_6,\ U = \Lambda^2\mathbb{C}^6/\mathbb{C}$, in which case (4.29) is equivalent to

$$\alpha_2 = \alpha_3 = -2\alpha_1 \ . \tag{4.30}$$

Let's now impose equation (3.39). In our setting it takes the form ($X \in \mathfrak{g},\ A, B \in U$)

$$\sum_{i=1}^{3} \alpha_i \Big(2k(X, P_1^{(i)}(A,B)) P_2^{(i)}(A,B) + [P_1^{(i)}(A,B), [P_2^{(i)}(A,B), X]] \Big)$$
$$= \ \sigma[A,[B,X]] + \sigma[B,[A,X]] - (A,B)X \ . \tag{4.31}$$

Notice that if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, equation (4.31) cannot be satisfied for all $A, B \in U$ and $X \in \mathfrak{g}$. This automatically rules out all the pairs $(\mathfrak{g}, U)$ except $\mathfrak{g} = \mathrm{sp}_6,\ U = \Lambda^2\mathbb{C}^6/\mathbb{C}$. Using Part (b) of Lemma (4.4.2), it is not hard to prove the following equations

$$\begin{aligned}
(X, P_1^{(i)}(A,B)) P_2^{(i)}(A,B) \ &= \ R^{(i)}(X; A, B) \ , \quad i = 1,2,3 \ , \\
[P_1^{(1)}(A,B), [P_2^{(1)}(A,B), X]] \ &= \ \Omega_{\mathfrak{g}} R^{(1)}(X; A, B) \ , \\
[P_1^{(2)}(A,B), [P_2^{(2)}(A,B), X]] \ &= \ \{J_\alpha\{A,B\}J^\alpha, X\} - \{\{A,B\}, J^\alpha X J^\alpha\} \\
&+ \ \Omega_{\mathbb{C}^n} \{\{A,B\}, X\} - J_\alpha\{\{A,B\},X\}J^\alpha \\
&= \ 2R^{(1)}(X; A, B) + (\Omega_{\mathbb{C}^n} - \tfrac{3}{2} i^2) R^{(2)}(X; A, B) \ , \\
[P_1^{(3)}(A,B), [P_2^{(3)}(A,B), X]] \ &= \ -\tfrac{1}{2} i^2 R^{(2)}(X; A, B) - i^2 R^{(3)}(X; A, B) \ , \\
[A,[B,X]] + [B,[A,X]] \ &= \ R^{(2)}(X; A, B) - 2R^{(3)}(X; A, B) \ ,
\end{aligned}$$

where $R^{(i)}(X; A, B),\ i = 1,2,3$, are defined in part (f) of Lemma 4.4.2. Substituting back in (4.31), we then get

$$\begin{aligned}(1+2k\alpha_1+\alpha_1\Omega_{\mathfrak{g}}+2\alpha_2)R^{(1)}(X;A,B)+(2k\alpha_2+\alpha_2\Omega_{\mathbb{C}^n}-\tfrac{3}{2}i^2\alpha_2-\tfrac{1}{2}i^2\alpha_3 \\ -\sigma)R^{(2)}(X;A,B)+(2k\alpha_3-i^2\alpha_3+2\sigma)R^{(3)}(X;A,B) \;=\; 0\ .\end{aligned} \tag{4.32}$$

By part (f) in Lemma 4.4.2, $R^{(1)}$, $R^{(2)}$ and $R^{(3)}$ are linearly independent $\mathfrak{g}$–module homomorphisms. Therefore equation (4.32) holds if and only if all the coefficients of $R^{(i)}(X;A,B)$ are zero. Since we are only interested in the case $\mathfrak{g}=\mathrm{sp}_6$, $U=\Lambda^2\mathbb{C}^6/\mathbb{C}$, we can replace $i^2=-1$, $\Omega_{\mathfrak{g}}=8$, $\Omega_{\mathbb{C}^n}=7/2$. Moreover, by (4.30) we have $\alpha_1=-\alpha$, $\alpha_2=\alpha_3=2\alpha$. It then follows from equation (4.32)

$$\begin{aligned}1-2k\alpha-4\alpha &= 0\ ,\\ 4k\alpha+11\alpha-\sigma &= 0\ ,\\ 2k\alpha+\alpha+\sigma &= 0\ .\end{aligned} \tag{4.33}$$

It is not hard to check that the system of equations (4.33) does not admit any solution. In conclusion, also the pair ($\mathfrak{g}=\mathrm{sp}_6$, $U=\Lambda^2\mathbb{C}^6/\mathbb{C}$) is ruled out.

4.5 Case $\dim \mathbf{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g})=2$

We are left to consider the case $\dim\mathrm{Hom}_{\mathfrak{g}}(S^2U,S^2\mathfrak{g})=2$. Suppose $u\otimes v$ is an element of the irreducible component $\Pi\subset S^2U$. Since the map $u\otimes v\mapsto(\{J^\alpha,J^\beta\}u,v)J^\alpha\otimes J^\beta$ is a $\mathfrak{g}$–module homomorphism: $S^2U\to S^2\mathfrak{g}$, we have

$$(\{J^\alpha,J^\beta\}u,v)J^\alpha\otimes J^\beta\ \begin{cases} =\ 0 & \text{if } \Pi\not\subset S^2\mathfrak{g}\ ,\\ =\ 2\frac{\Omega_U}{\dim\mathfrak{g}}(u,v)J^\alpha\otimes J^\alpha\ , & \text{if } \Pi\simeq\mathbb{C}\ .\end{cases}$$

In the second identity we used equation (4.9). It follows that the expression of $P(u,v)$ given by (4.7) and (4.8) becomes in this case

$$P(u,v)=\gamma(\{J^\alpha,J^\beta\}u,v)J^\alpha\otimes J^\beta+\beta(u,v)J^\alpha\otimes J^\alpha\ ,\quad \forall u,v\in U\ , \tag{4.34}$$

where β and γ are related to the α_Π's by

$$\alpha_{\mathbb{C}}=2\gamma\frac{\Omega_U}{\dim\mathfrak{g}}+\beta\ ,\qquad \alpha_{\tilde\Pi}=\gamma\ . \tag{4.35}$$

Here $\tilde\Pi$ denotes the (unique) non trivial irreducible component of S^2U which can be embedded in $S^2\mathfrak{g}$.

In order to solve Problem 3.5.3 we need to impose conditions (3.39), (3.40), (3.41) and (3.42). From equations (4.2), (4.10) and (4.35) we get that (3.40) is equivalent to

$$6k\sigma \;=\; c+4k\gamma\Omega_U+2\beta k\dim\mathfrak{g}\ , \tag{4.36}$$

and similarly, from equations (4.7), (4.8) and (4.35) we get that (3.39) is equivalent to

$$\begin{aligned}(2k+\Omega_{\mathfrak{g}})(2\gamma\Omega_U+\beta\dim\mathfrak{g}) &= 2\sigma\Omega_U-\dim\mathfrak{g}\ ,\\ (2k-\tfrac{1}{2}\Omega_{\tilde{\Pi}}+\Omega_{\mathfrak{g}})\gamma &= \sigma\ .\end{aligned} \tag{4.37}$$

We now want to use equation (3.42) to get a very strong restriction on the $\mathfrak{g}$–module U. This is stated in the following

Lemma 4.5.1. *(a) Suppose* $0\neq P:\ S^2U\to S^2\mathfrak{g}$ *is as in (4.34), and assume it satisfies equation (3.42). Then, for every irreducible component* $\Pi\subset S^2U$ *one has*

$$\Omega_\Pi(\gamma\Omega_\Pi-2\gamma\Omega_U+\beta) \;=\; 0\ . \tag{4.38}$$

(b) In particular, if $U=\Pi_\lambda$ *is the irreducible* $\mathfrak{g}$*–module with highest weight* λ*, then*

$$S^2U \;=\; \Pi_{2\lambda}\oplus\mathbb{C}\ . \tag{4.39}$$

Proof. We can rewrite equation (3.42) by using the expression (4.34) of $P(u,v)$. For every $u_1,u_2,u_3\in U$ we have

$$\gamma\sum_{\sigma\in C_3}(\{J^\alpha,J^\beta\}u_{\sigma_1},u_{\sigma_2})J^\alpha\otimes(J^\beta u_{\sigma_3})+\beta\sum_{\sigma\in C_3}(u_{\sigma_1},u_{\sigma_2})J^\alpha\otimes(J^\alpha u_{\sigma_3}) \;=\; 0\ . \tag{4.40}$$

We can replace u_3 by e^i in equation (4.40), take the tensor product of both sides by e^i, and sum over the repeated index $i=1,\ldots,\dim U$. The resulting equation is

$$\begin{aligned}&\gamma(\{J^\alpha,J^\beta\}u_1,u_2)J^\alpha\otimes(J^\beta e^i)\otimes e^i+\gamma J^\alpha\otimes(J^\beta u_1)\otimes(\{J^\alpha,J^\beta\}u_2)\\ &\quad+\gamma J^\alpha\otimes(J^\beta u_2)\otimes(\{J^\alpha,J^\beta\}u_1)+\beta(u_1,u_2)J^\alpha\otimes(J^\alpha e^i)\otimes e^i\\ &\quad+\beta J^\alpha\otimes(J^\alpha u_1)\otimes u_2+\beta J^\alpha\otimes(J^\alpha u_2)\otimes u_1 \;=\; 0\ .\end{aligned} \tag{4.41}$$

Notice that the expression in (4.41) is symmetric in u_1 and u_2. Namely we can think at the left hand side of (4.41) as a $\mathfrak{g}$–module homomorphism

$$\underline{\text{LHS}}\ :\ S^2U \longrightarrow \mathfrak{g}\otimes U\otimes U\ . \tag{4.42}$$

Let us assume that $u_1\otimes u_2$ (or rather a linear combination of such monomials) is an element of an irreducible component $\Pi\subset S^2U$. Notice that there is a surjective map $\pi:\ \mathfrak{g}\otimes U\twoheadrightarrow U$, given by $\pi(a\otimes u)=au$. We can then compose the homomorphism

(4.42) with π in two different ways: either

$$\underline{\mathrm{LHS}}_1 \ : \ S^2U \xrightarrow{\mathrm{LHS}} \mathfrak{g}\otimes U\otimes U \xrightarrow{\pi_{12}} U\otimes U$$
$$a\otimes u\otimes v \mapsto (au)\otimes v$$

or

$$\underline{\mathrm{LHS}}_2 \ : \ S^2U \xrightarrow{\mathrm{LHS}} \mathfrak{g}\otimes U\otimes U \xrightarrow{\pi_{13}} U\otimes U$$
$$a\otimes u\otimes v \mapsto u\otimes (av)$$

It is then clear that (4.41) implies the two equations

$$\underline{\mathrm{LHS}}_1 \ = \ 0 \quad , \qquad \underline{\mathrm{LHS}}_2 \ = \ 0 \ . \tag{4.43}$$

Equation (4.38) will follow by comparing these two conditions. Let us first compute $\underline{\mathrm{LHS}}_1$. By definition it is, for $u_1\otimes u_2 \in \Pi \subset S^2U$

$$\begin{aligned} \underline{\mathrm{LHS}}_1 \ &= \ \gamma(\{J^\alpha, J^\beta\}u_1, u_2)(J^\alpha J^\beta e^i)\otimes e^i + 2\gamma(J^\alpha J^\beta u_1)\otimes(\{J^\alpha, J^\beta\}u_2) \\ &+ \ \beta(u_1,u_2)(J^\alpha J^\alpha e^i)\otimes e^i + 2\beta(J^\alpha J^\alpha u_1)\otimes u_2 \ . \end{aligned} \tag{4.44}$$

Since the Casimir operator $\Omega = J^\alpha J^\alpha$ acts as scalar on the irreducible $\mathfrak{g}$–module U, we can replace in the last two terms of the right hand side of (4.44) $J^\alpha J^\alpha e^i = \Omega_U e^i$ and $J^\alpha J^\alpha u_1 = \Omega_U u_1$. Moreover, since by assumption $u_1\otimes u_2 \in \Pi \subset S^2U$, we have

$$\Omega_\Pi u_1\otimes u_2 \ = \ 2\Omega_U u_1\otimes u_2 + 2(J^\alpha u_1)\otimes(J^\alpha u_2) \ , \tag{4.45}$$

so that we can replace, in the second term of the right hand side of (4.44)

$$\begin{aligned} (J^\alpha J^\beta u_1)\otimes(\{J^\alpha, J^\beta\}u_2) \ &= \ 2(J^\alpha J^\beta u_1)\otimes(J^\alpha J^\beta u_2) - \tfrac{1}{2}([J^\alpha, J^\beta]u_1)\otimes([J^\alpha, J^\beta]u_2) \\ &= \ \Big(2(\tfrac{1}{2}\Omega_\Pi - \Omega_U)^2 + \tfrac{1}{2}\Omega_{\mathfrak{g}}(\tfrac{1}{2}\Omega_\Pi - \Omega_U)\Big)u_1\otimes u_2 \ . \end{aligned}$$

In the last equation we used the obvious identity

$$[J^\alpha, J^\beta]\otimes[J^\alpha, J^\beta] \ = \ -\Omega_{\mathfrak{g}} J^\alpha\otimes J^\alpha \ . \tag{4.46}$$

We can then use the above observations to rewrite (4.44) as

$$\begin{aligned}\underline{\mathrm{LHS}}_1 &= \gamma(\{J^\alpha, J^\beta\}u_1, u_2)(J^\alpha J^\beta e^i) \otimes e^i \\ &+ 2\gamma\Big(2(\tfrac{1}{2}\Omega_\Pi - \Omega_U)^2 + \tfrac{1}{2}\Omega_{\mathfrak{g}}(\tfrac{1}{2}\Omega_\Pi - \Omega_U)\Big)u_1 \otimes u_2 \\ &+ \beta\Omega_U(u_1, u_2)e^i \otimes e^i + 2\beta\Omega_U u_1 \otimes u_2 \ .\end{aligned} \tag{4.47}$$

Let us now compute $\underline{\mathrm{LHS}}_2$. By definition

$$\begin{aligned}\underline{\mathrm{LHS}}_2 &= \gamma(\{J^\alpha, J^\beta\}u_1, u_2)(J^\beta e^i) \otimes (J^\alpha e^i) + 2\gamma(J^\beta u_1) \otimes (J^\alpha\{J^\alpha, J^\beta\}u_2) \\ &+ \beta(u_1, u_2)(J^\alpha e^i) \otimes (J^\alpha e^i) + 2\beta(J^\alpha u_1) \otimes (J^\alpha u_2) \ .\end{aligned} \tag{4.48}$$

By (4.45) we can replace in the last two terms in the right hand side of (4.48)

$$(J^\alpha e^i) \otimes (J^\alpha e^i) \ = \ -\Omega_U e^i \otimes e^i \ ,$$

and

$$(J^\alpha u_1) \otimes (J^\alpha u_2) \ = \ (\frac{1}{2}\Omega_\Pi - \Omega_U)u_1 \otimes u_2 \ .$$

Moreover, by (4.5) and (4.46) we get

$$\begin{aligned}(J^\beta u_1) \otimes (J^\alpha\{J^\alpha, J^\beta\}u_2) &= 2(J^\beta u_1) \otimes (J^\alpha J^\alpha J^\beta u_2) + (J^\beta u_1) \otimes ([J^\alpha, J^\beta]J^\alpha u_2) \\ &= (2\Omega_U - \tfrac{1}{2}\Omega_{\mathfrak{g}})(\tfrac{1}{2}\Omega_\Pi - \Omega_U)u_1 \otimes u_2 \ .\end{aligned}$$

Substituting the above results back in (4.48) we thus get

$$\begin{aligned}\underline{\mathrm{LHS}}_2 &= \gamma(\{J^\alpha, J^\beta\}u_1, u_2)(J^\beta e^i) \otimes (J^\alpha e^i) \\ &+ 2\gamma(2\Omega_U - \tfrac{1}{2}\Omega_{\mathfrak{g}})(\tfrac{1}{2}\Omega_\Pi - \Omega_U)u_1 \otimes u_2 \\ &- \beta\Omega_U(u_1, u_2)e^i \otimes e^i + 2\beta(\tfrac{1}{2}\Omega_\Pi - \Omega_U)u_1 \otimes u_2 \ .\end{aligned} \tag{4.49}$$

Since $(ae^i) \otimes e^i = -e^i(ae^i)$, the sum of the first term in the right hand side of (4.47) and the first term in the right hand side of (4.49) is zero. Therefore, if we take the sum of the two equations (4.43), we get, after simple algebraic manipulations

$$0 = \underline{\mathrm{LHS}}_1 + \underline{\mathrm{LHS}}_2 = \Omega_\Pi(\gamma\Omega_\Pi - 2\gamma\Omega_U + \beta)u_1 \otimes u_2 \ , \quad \forall u_1 \otimes u_2 \in \Pi \subset S^2U \ .$$

This clearly implies part (a) of the lemma. Notice that (4.38) is an equation of degree 2 in $x = \Omega_\Pi$. One solution is obviously $x = 0$, which is obtained for $\Pi = \mathbb{C} \subset S^2U$. It follows that the Casimir operator Ω has the same eigenvalue on every non trivial irreducible component $\Pi \subset S^2U$. By Corollary 4.2.3 this is possible only if $S^2U/\mathbb{C} \simeq \Pi_{2\lambda}$. This concludes the proof of the lemma. □

It follows by part (b) of Lemma 4.5.1 that all cases in Table 4.5 with $d = 2$ and

$n \neq 0$ are ruled out. We are then left to consider the pairs $(\mathfrak{g}, U)$ in Table 4.5 for which $d = 2$, $n = 0$. It is clear from the isomorphisms: $A_1 \simeq \mathrm{so}_3$, $C_2 \simeq \mathrm{so}_5$, $A_3 \simeq \mathrm{so}_6$, $B_r \simeq \mathrm{so}_{2r+1}, r \geq 3$, $D_r \simeq \mathrm{so}_{2r}, r \geq 4$, that all such pairs are

$$
\begin{aligned}
&1. \quad \mathfrak{g} = \mathrm{so}_n \ , && U = \mathbb{C}^n \quad n \geq 3, \ n \neq 4 \ , \\
&2. \quad \mathfrak{g} = B_3 \ , && U = \pi_3 \ , \\
&3. \quad \mathfrak{g} = G_2 \ , && U = \pi_1 \ .
\end{aligned}
\tag{4.50}
$$

We now want to use Lemma 4.5.1 to study equations (3.41) and (3.42). We will denote for the rest of this section $\Pi_{2\lambda} = \Pi$. Every element $u_1 \otimes u_2 \in S^2U$ can be decomposed according to the decomposition (4.39) as follows

$$
u_1 \otimes u_2 = \left(u_1 \otimes u_2 - \frac{(u_1, u_2)}{\dim U} e^i \otimes e^i \right) + \frac{(u_1, u_2)}{\dim U} e^i \otimes e^i \ , \tag{4.51}
$$

so that the first expression in the right hand side belongs to $\Pi \subset S^2U$, while the last term belongs to $\mathbb{C} \subset S^2U$. The decomposition (4.51), together with (4.45), implies that for every element $u_1 \otimes u_2 \in S^2U$

$$
(J^\alpha u_1) \otimes (J^\alpha u_2) = \left(\frac{1}{2}\Omega_\Pi - \Omega_U \right) u_1 \otimes u_2 - \frac{1}{2}\Omega_\Pi \frac{(u_1, u_2)}{\dim U} e^i \otimes e^i \ . \tag{4.52}
$$

Equation (4.52) will be very useful to study equations (3.41) and (3.42). Let us consider first equation (3.41). Using the expressions (4.2) of $K(u, v)$ and (4.34) of $P(u, v)$, we get

$$
\begin{aligned}
\tfrac{1}{2}((u_1, u_3)u_2 + (u_2, u_3)u_1) - (u_1, u_2)u_3 &= \sigma((J^\alpha u_1, u_3)J^\alpha u_2 + (J^\alpha u_2, u_3)J^\alpha u_1) \\
&- \gamma\Big((\{J^\alpha, J^\beta\}u_1, u_3)J^\alpha J^\beta u_2 + (\{J^\alpha, J^\beta\}u_2, u_3)J^\alpha J^\beta u_1 \Big) \\
&-\beta\Big((u_1, u_3)J^\alpha J^\alpha u_2 + (u_2, u_3)J^\alpha J^\alpha u_1 \Big) \ .
\end{aligned}
\tag{4.53}
$$

To get (4.53) we replaced in the right hand side of (3.41) $P(u_1, u_2)u_3 = -P(u_1, u_3)u_2 - P(u_2, u_3)u_1$, which holds thanks to equation (3.42). Since (4.53) holds for every $u_3 \in U$, it is equivalent to the following equation

$$
\begin{aligned}
&2\gamma(\{J^\alpha, J^\beta\}u_1) \otimes (J^\alpha J^\beta u_2) + 2\beta\Omega_U u_1 \otimes u_2 - 2\sigma(J^\alpha u_1) \otimes (J^\alpha u_2) \\
&\quad + u_1 \otimes u_2 - (u_1, u_2)e^i \otimes e^i \ = \ 0 \ , \quad \forall u_1 \otimes u_2 \in S^2U \ .
\end{aligned}
\tag{4.54}
$$

We can rewrite the third term in the left hand side of (4.54) using (4.52). Moreover, for the first term in the left hand side of (4.54) we have the following sequence of identities, which can be proved using (4.46) and (4.52)

$$
\begin{aligned}
(\{J^\alpha, J^\beta\}u_1) \otimes (J^\alpha J^\beta u_2) &= 2(J^\alpha J^\beta u_1) \otimes (J^\alpha J^\beta u_2) - \tfrac{1}{2}([J^\alpha, J^\beta]u_1) \otimes ([J^\alpha, J^\beta]u_2) \\
&= 2(\tfrac{1}{2}\Omega_\Pi - \Omega_U)(\tfrac{1}{2}\Omega_\Pi - \Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}})u_1 \otimes u_2 \\
&- \Omega_\Pi(\tfrac{1}{2}\Omega_\Pi - 2\Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}})\tfrac{(u_1,u_2)}{\dim U} e^i \otimes e^i \ .
\end{aligned}
$$

Using the above results, we get that (4.54) is equivalent to the following equation, for every $u_1 \otimes u_2 \in S^2U$

$$
\begin{aligned}
&\Big(1 - 2\sigma(\tfrac{1}{2}\Omega_\Pi - \Omega_U) + 4\gamma(\tfrac{1}{2}\Omega_\Pi - \Omega_U)(\tfrac{1}{2}\Omega_\Pi - \Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}}) + 2\beta\Omega_U\Big)u_1 \otimes u_2 \\
&\quad - \Big(\dim U - \sigma\Omega_\Pi + 2\gamma\Omega_\Pi(\tfrac{1}{2}\Omega_\Pi - 2\Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}})\Big)\tfrac{(u_1,u_2)}{\dim U} e^i \otimes e^i \ = \ 0 \ .
\end{aligned} \tag{4.55}
$$

Clearly equation (4.55) holds if and only if both the coefficients of $u_1 \otimes u_2$ and $(u_1, u_2)e^i \otimes e^i$ are zero. We thus conclude that equation (3.41) is equivalent to the following two conditions

$$
\begin{aligned}
1 - 2\sigma(\tfrac{1}{2}\Omega_\Pi - \Omega_U) + 4\gamma(\tfrac{1}{2}\Omega_\Pi - \Omega_U)(\tfrac{1}{2}\Omega_\Pi - \Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}}) + 2\beta\Omega_U \ &= \ 0 \ , \\
\dim U - \sigma\Omega_\Pi + 2\gamma\Omega_\Pi(\tfrac{1}{2}\Omega_\Pi - 2\Omega_U + \tfrac{1}{4}\Omega_{\mathfrak{g}}) \ &= \ 0 \ .
\end{aligned} \tag{4.56}
$$

Finally, let us consider equation (3.42). By (4.34) we can rewrite it as

$$
\begin{aligned}
&\gamma \sum_{\sigma \in C_3} \Big((J^\alpha J^\beta u_{\sigma_1}, u_{\sigma_2})J^\alpha \otimes J^\beta u_{\sigma_3} + (J^\alpha J^\beta u_{\sigma_2}, u_{\sigma_1})J^\alpha \otimes J^\beta u_{\sigma_3}\Big) \\
&\qquad + \beta \sum_{\sigma \in C_3} (u_{\sigma_1}, u_{\sigma_2})J^\alpha \otimes J^\alpha u_{\sigma_3} \ = \ 0 \ , \quad \forall u_1, u_2, u_3 \in U \ .
\end{aligned} \tag{4.57}
$$

If S_3 denotes the group of all permutations of $(1, 2, 3)$, equation (4.57) can be rewritten as

$$
\gamma \sum_{\sigma \in S_3} (aJ^\alpha u_{\sigma_1}, u_{\sigma_2})J^\alpha u_{\sigma_3} + \frac{1}{2}\beta \sum_{\sigma \in S_3} (u_{\sigma_1}, u_{\sigma_2})au_{\sigma_3} \ = \ 0 \ , \quad \forall a \in \mathfrak{g}, \ u_1, u_2, u_3 \in U \ . \tag{4.58}
$$

Here we used the obvious fact that, if $\sum_\alpha J^\alpha \otimes u_\alpha = 0$ form some $u_\alpha \in U$, $\alpha = 1, \dots, \dim \mathfrak{g}$, then necessarily $u_\alpha = 0$, $\forall \alpha = 1, \dots, \dim \mathfrak{g}$. We can then use (4.52) to get from (4.58)

$$
\gamma\Big(\frac{1}{2}\Omega_\Pi - \Omega_U\Big) \sum_{\sigma \in S_3} (au_{\sigma_1}, u_{\sigma_2})u_{\sigma_3} + \frac{1}{2}\Big(\gamma\frac{\Omega_\Pi}{\dim U} + \beta\Big) \sum_{\sigma \in S_3} (u_{\sigma_1}, u_{\sigma_3})au_{\sigma_2} \ = \ 0 \ . \tag{4.59}
$$

Notice that the first term in the left hand side of (4.59) is identically zero, since it is both symmetric and skewsymmetric with respect to u_{σ_1} and u_{σ_2}. In conclusion equation (3.42) is equivalent to the following condition

$$\gamma \frac{\Omega_{\Pi}}{\dim U} + \beta \;=\; 0 \,. \tag{4.60}$$

So far we proved that the pairs $(\mathfrak{g}, U)$ in Table 4.5 with $d = 2$, $n = 0$ satisfy the assumptions of Problem 1.2.1, if and only if equations (4.36), (4.37), (4.56) and (4.60) are satisfied for some choice of the parameters c, σ, β, γ, k. Surprisingly enough, for every value of the Kac–Moody level k and in all examples listed in (4.50) one can find c, σ, β, γ such that all the equations (4.36), (4.37), (4.56) and (4.60) hold. The corresponding values of all the parameters are given in the following table.

Table 4.6:

	$(\mathrm{so}_n\ ,\ \mathbb{C}^n)$	$(B_3\ ,\ \pi_3)$	$(G_2\ ,\ \pi_1)$
$\dim U$	n	8	7
$\dim \mathfrak{g}$	$\frac{n(n-1)}{2}$	21	14
Ω_u	$n-1$	$\frac{21}{4}$	4
$\Omega_{\mathfrak{g}}$	$2(n-2)$	10	8
Ω_π	$2n$	12	$\frac{28}{3}$
c	$\frac{k(3k+n^2/2-5)}{k+n-3}$	$\frac{2k(2k+11)}{k+4}$	$\frac{k(9k+31)}{2(k+3)}$
σ/γ	$2k+n-4$	$2(k+2)$	$2k+\frac{10}{3}$
β/γ	-2	$-\frac{3}{2}$	$-\frac{4}{3}$
γ	$\frac{1}{4(k+n-3)}$	$\frac{1}{3(k+4)}$	$\frac{3}{8(k+3)}$

4.6 Final classification

We can summarize all the results of Chapter 4 in the following

Theorem 4.6.1. *A complete list of 7–tuples* $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ *satisfying all the assumptions of Problem 3.5.3, and such that* $\mathfrak{g}$ *is simple,* U *is irreducible and* $\varkappa$*,* Q *are not identically zero, is as follows. The pair* $(\mathfrak{g}, U)$ *is one of the following*

(i) $(\mathrm{so}_n,\ \mathbb{C}^n),\ n \geq 3,\ n \neq 4$

(ii) $(B_3,\ \pi_3 = Spin_7)$

(iii) $(G_2,\ \pi_1)$

The bilinear forms $\varkappa:\ S^2\mathfrak{g} \to \mathbb{C}$ *and* $Q:\ S^2U \to \mathbb{C}$ *are respectively* $\varkappa(a,b) = k(a,b)$ *and* $Q(u,v) = \varepsilon(u,v)$*, for* $a, b \in \mathfrak{g},\ u, v \in U$*, where* (a,b) *denotes the normalized killing form on* $\mathfrak{g}$*,* (u,v) *is the unique (up to scalar multiplication) invariant bilinear form on* U*, and* k, ε *are arbitrary constants. Without loss of generality we can assume* $\varepsilon = 1$*. The* $\mathfrak{g}$*–module homomorphisms* K*,* P *are given respectively by (4.2) and (4.34), and the constants* c, σ, β, γ *are given in Table 4.6.*

Thanks to Theorems 2.1.8, 3.2.6 and 3.4.2, the above result is equivalent to the following

Corollary 4.6.2. *(i) Let* V *be any vertex algebra strongly generated by a space* $R \subset V$ *as in (3.1). Assume that* V *is non degenerate according to Definition 3.2.4. Suppose moreover that the Lie algebra* $\mathfrak{g}$ *is simple, the* $\mathfrak{g}$*–module* U *is irreducible and the bilinear forms* $\varkappa:\ S^2\mathfrak{g} \to \mathbb{C}$*,* $Q:\ S^2U \to \mathbb{C}$ *(defined in Proposition 3.1.4) are not identically zero. Then the datum* $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ *defined by the* λ*–bracket structure in Table 3.1 is one of the 7–tuples listed in Theorem 4.6.1.*

(ii) Conversely, let $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ *be one of the 7–tuples listed in Theorem 4.6.1. Then there exists a vertex algebra* $U(R)$ *which is strongly generated by the space* R *as in (3.1) with* λ*–bracket structure on* $R \otimes R$ *given in Table 3.1.*

Chapter 5

Quasi–classical limit and transitivity of group action on quadrics

By looking at the list of vertex algebras in Corollary 4.6.2, we notice that each example in the list has the following two properties.

1) The Kac–Moody level k is an arbitrary complex number (except for some singular value). Namely for each pair $(\mathfrak{g}, U)$, there corresponds a family of vertex algebras $U_k(R)$, parametrized by the Kac–Moody level.

2) The complex connected algebraic group G associated to the Lie algebra $\mathfrak{g}$ acts transitively on the complex quadric $S^2 = \{u \in U \ , \text{ s.t. } (u, u) = 1\} \subset U$. This observation is related to a similar result in [13]. There it is proved that this property holds in the special case in which the space R in (3.1) is a Lie conformal superalgebra, namely it is closed under λ–bracket (or, equivalently, the g–module homomorphism $P: \ S^2U \to S^2\mathfrak{g}$ defined in Table 3.1.4 is identically zero). In [13] is also provided a complete classification of pairs $(\mathfrak{g}, U)$, where $\mathfrak{g}$ is a Lie algebra, U an orthogonal self contragradient $\mathfrak{g}$–module such that the connected complex algebraic group G associated to $\mathfrak{g}$ has transitive action on the quadric $S^2 \subset U$. It is then immediate to check that all pairs $(\mathfrak{g}, U)$ listed in Theorem 4.6.1 fall in this classification.

This chapter will be devoted to understand the intimate connection between properties 1) and 2) above. Loosely speaking, we will prove the following. Suppose $\{U_k(R), \ k \in \mathbb{C}\}$ is a family of vertex algebras strongly generated by the space R in (3.1), parametrized by the parameter $k \in \mathbb{C}$ (for example, the Kac–Moody level). Under certain regularity assumptions on the behavior at $k \sim \infty$, we will be able to define a "quasi–classical limit" of the vertex algebra structures of $U_k(R)$ for $k \to \infty$, which will be a Poisson vertex algebra structure on the symmetric algebra $S(R)$. We will then use this structure to prove that, in general, the connected algebraic group G associated to $\mathfrak{g}$ acts transitively on the quadric $S^2 \subset U$. This can be viewed as a generalization of the result in [13] to the context of vertex algebras. Using this result, we will find in Chapter 6 a complete classification of vertex algebras generated by a space R as in (3.1) which admit quasi–classical limit (see Definition 5.1.5), for generic reductive $\mathfrak{g}$ and (possibly reducible) U.

5.1 Definition of Poisson vertex algebras

A Poisson vertex algebra is the same as a vertex algebra which is associative (instead of quasi–associative), commutative (instead of skewsymmetric), and for which commutative (instead of non commutative) Wick formula holds. In other words, we replace Definition 1.1.2 with the following

Definition 5.1.1. A *Poisson vertex superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$–graded $\mathbb{C}[T]$–module V endowed with two operations: a λ–bracket $V \otimes V \to \mathbb{C}[\lambda] \otimes V$ which makes it a Lie conformal superalgebra; a product $V \otimes V \to V$, which makes it a super commutative, associative unital differential algebra, with derivative T, and unity denoted by $|0\rangle \in V$. They satisfy the following commutative Wick formula

$$[a\ _\lambda\ bc] \ = \ [a\ _\lambda\ b]c \ + \ p(a,b)\ b[a\ _\lambda\ c]\ . \tag{5.1}$$

Remark 5.1.2. It follows by sesquilinearity and skewsymmetry of the λ–bracket that the following (commutative) right Wick formula holds

$$[ab\ _\lambda\ c] \ = \ \left(e^{T\partial_\lambda}a\right)[b\ _\lambda\ c] \ + \ p(a,b)\ \left(e^{T\partial_\lambda}b\right)[a\ _\lambda\ c]\ . \tag{5.2}$$

Suppose R is a Lie conformal algebra. By Theorem 1.1.5 we can associate to R an enveloping vertex algebra $U(R)$. A similar (and in fact much easier) statement is true for Poisson vertex algebras.

Theorem 5.1.3. *Let R be a Lie conformal algebra, and let $S(R)$ be the symmetric algebra over the space R. The extension of the endomorphism T to $S(R)$ by derivation, and of the λ–bracket to $S(R) \otimes S(R)$ by left and right commutative Wick formulas (5.1) and (5.2), make $S(R)$ a Poisson vertex algebra.*

It is not hard to prove directly that $S(R)$ is indeed a Poisson vertex algebra. Instead, we will derive this result starting from the enveloping vertex algebra $U(R)$ via a limiting procedure known as *quasi–classical limit*. Given the Lie conformal algebra R and $\hbar \in \mathbb{C}/\{0\}$, we get a new Lie conformal algebra $R_\hbar$ (isomorphic to R) in the following way. As $\mathbb{C}[T]$–module, $R_\hbar = R$. The λ–bracket structure on $R_\hbar$, denoted by $[\ _\lambda\ ]_\hbar$, is given by

$$[a\ _\lambda\ b]_\hbar \ = \ \hbar[a\ _\lambda\ b]\ , \quad \forall a,b \in R\ .$$

It is obvious that $R_\hbar$ is again a Lie conformal algebra. Let then $R_\hbar^{\mathrm{Lie}}$ be the space R considered as a Lie algebra with respect to the Lie bracket

$$[a,b]_\hbar \ = \ \int_{-T}^{0} d\lambda\ [a\ _\lambda\ b]_\hbar\ .$$

By Theorem 1.1.5 the space $V_\hbar = U(R_\hbar^{\mathrm{Lie}})$ (the universal enveloping algebra over $R_\hbar^{\mathrm{Lie}}$) is endowed with a structure of vertex algebra, such that the normal order product restricted to $R_\hbar \otimes V_\hbar$ is compatible with the associative product of $U(R_\hbar^{\mathrm{Lie}})$, and the λ–bracket restricted to $R_\hbar \otimes R_\hbar$ is compatible with $[\ {}_\lambda\ ]_\hbar$. Theorem 5.1.3 follows immediately from the following

Theorem 5.1.4. *The space $S(R)$ is obtained as a limit for $\hbar \to 0$ of the space $V_\hbar$:*

$$V_\hbar \ \stackrel{\hbar\to 0}{\longrightarrow}\ S(R) \ .$$

Let us denote by $\pi_\hbar$ and π_0 the quotient maps $\pi_\hbar :\ \mathcal{T}(R) \twoheadrightarrow V_\hbar$, and $\pi_0 :\ \mathcal{T}(R) \twoheadrightarrow S(R)$. The associative commutative algebra structure of $S(R)$ is obtained as a limit of the normal order product structure of $V_\hbar$ (which is non commutative and non associative) in the following way

$$\pi_0(A)\pi_0(B) \ = \ \lim_{\hbar\to 0}\ :\pi_\hbar(A)\pi_\hbar(B):_\hbar \ , \quad \forall A,B \in \mathcal{T}(R) \ . \tag{5.3}$$

Here $:\ :_\hbar$ denotes the normal order product on $V_\hbar$. Finally, the following λ–bracket

$$[\pi_0(A)\ {}_\lambda\ \pi_0(B)]^{PB} \ = \ \lim_{\hbar\to 0}\ \frac{1}{\hbar}[\pi_\hbar(A)\ {}_\lambda\ \pi_\hbar(B)]_\hbar \ , \quad \forall A,B \in \mathcal{T}(R) \ , \tag{5.4}$$

makes $S(R)$ into a Poisson vertex algebra, compatible with the Lie conformal algebra structure of R.

Proof. Notice that, by definition of $[\ {}_\lambda\ ]_\hbar$, we have $\lim_{\hbar\to 0}[a\ {}_\lambda\ b]_\hbar \ = \ 0,\ \forall a,b \in R$. It follows that

$$V_\hbar \ = \ \mathcal{T}(R)/\langle a\otimes b - b\otimes a - [a,b]_\hbar \ ;\ a,b \in R\rangle \ \stackrel{\hbar\to 0}{\longrightarrow}\ S(R) \ .$$

Moreover, equation (5.3) follows immediately by the fact that, in the limit $\hbar \to 0$, quasi–associativity and skewsymmetry conditions in Definition 1.1.2 become respectively associativity and commutativity. Obviously $[\ {}_\lambda\ ]^{\mathrm{PB}}$ defined in (5.4) satisfies all axioms of Lie conformal algebra, and for $a,b \in R$ we simply have $[a\ {}_\lambda\ b]^{\mathrm{PB}} \ = \ [a\ {}_\lambda\ b]$, namely $[\ {}_\lambda\ ]^{\mathrm{PB}}$ restricted to $R\otimes R$ is compatible with the λ–bracket of R. We are left to show that $[\ {}_\lambda\ ]^{\mathrm{PB}}$ satisfies commutative Wick formula. For $A,B,C \in \mathcal{T}(R)$ we have

$$\begin{aligned}
\tfrac{1}{\hbar}[\pi_\hbar(A)\ {}_\lambda\ :\pi_\hbar(B)\pi_\hbar(C):_\hbar]_\hbar \ &= \ :\Big(\tfrac{1}{\hbar}[\pi_\hbar(A)\ {}_\lambda\ \pi_\hbar(B)]_\hbar\Big)\pi_\hbar(C):_\hbar \\
&+ \ p(A,B)\ :\pi_\hbar(B)\Big(\tfrac{1}{\hbar}[\pi_\hbar(A)\ {}_\lambda\ \pi_\hbar(C)]_\hbar\Big):_\hbar \\
&+ \ \tfrac{1}{\hbar}\int_0^\lambda d\mu\ [[\pi_\hbar(A)\ {}_\lambda\ \pi_\hbar(B)]_\hbar\ {}_\mu\ \pi_\hbar(C)]_\hbar \ .
\end{aligned} \tag{5.5}$$

In the limit $\hbar \to 0$, the left hand side of (5.5) converges to

$$[\pi_0(A)\ {}_\lambda\ \pi_0(B)\pi_0(C)]^{\mathrm{PB}} \ ,$$

and the first two terms in the right hand side of (5.5) converge to

$$[\pi_0(A)\ {}_\lambda\ \pi_0(B)]^{\text{PB}}\pi_0(C)\ +\ p(A,B)\ \pi_0(B)[\pi_0(A)\ {}_\lambda\ \pi_0(C)]^{\text{PB}}\ .$$

Finally, the last term in the right hand side of (5.5) has asymptotic behavior, for $\hbar \sim 0$

$$\hbar \int_0^\lambda d\mu\ [[\pi_0(A)\ {}_\lambda\ \pi_0(B)]^{\text{PB}}\ {}_\mu\ \pi_0(C)]^{\text{PB}}\ ,$$

so that in the limit $\hbar \to 0$ it converges to zero. This completes the proof of the theorem. ☐

In the next sections we will try to generalize the results of Theorems 5.1.3 and 5.1.4 to the case in which the λ–bracket structure on R admits "quadratic non linearities". In the next chapter we will then use these results to classify vertex algebras V strongly generated by primary fields of conformal weight 1 and 3/2, for which the symmetric space $S(R)$ over the generating set $R \subset V$ in (3.1) has a (compatible) Poisson vertex algebra structure. More precisely, we introduce the following

Definition 5.1.5. Suppose R is a super $\mathbb{C}[T]$–module endowed with a Lie λ–bracket of degree 2, and let V be a vertex algebra strongly generated by $R \subset V$, with compatible λ–bracket structure. We say that V *admits a quasi–classical limit* if $S(R)$ is a Poisson vertex algebra.

5.2 Enveloping Poisson vertex algebra $S(R)$ over a conformal algebra R with "quadratic non linearities"

Suppose R is a super $\mathbb{C}[T]$–module endowed with a Lie λ–bracket of degree 2. Recall in Chapter 2 we constructed the enveloping vertex algebra $U(R)$, thus generalizing the statement of Theorem 1.1.5. In this section we want to prove an analogous result in the context of Poisson vertex algebras. Namely we want to generalize Theorem 5.1.3 to the situation in which the Lie λ–bracket structure on R admits "quadratic non linearities".

Throughout this section, we let R be a vector superspace $R = R_{\bar{0}} \oplus R_{\bar{1}}$, with an even endomorphism $T \in \text{End}R$. Let $S(R)$ denote the symmetric algebra over R, and we extend the action of T to $S(R)$ by derivation.

Definition 5.2.1. A *Poisson λ–bracket of degree 2* is a λ–bracket of degree 2, L_λ^{PB}, on R (see Definition 2.1.2) such that $L_\lambda^{\text{PB}}:\ R \otimes R \to \mathbb{C}[\lambda] \otimes S(R)$.

Lemma 5.2.2. *Let L_λ^{PB} be a Poisson λ–bracket of degree 2 on R. One can find uniquely a linear map*

$$L_\lambda^{PB}\ :\quad S(R)\otimes S(R)\ \longrightarrow\ \mathbb{C}[\lambda]\otimes S(R)\ ,$$

such that the restriction to $R \otimes R$ coincides with the given λ–bracket, 1 is a central element, namely $L_\lambda^{PB}(1,A) = L_\lambda^{PB}(A,1) = 0$, $\forall A \in S(R)$, and left and right commutative Wick formulas hold, namely for $A,B,C \in S(R)$

$$\begin{aligned} L_\lambda^{PB}(A,BC) &= L_\lambda^{PB}(A,B)C + p(A,B)BL_\lambda^{PB}(A,C) , \\ L_\lambda^{PB}(AB,C) &= \left(e^{T\partial_\lambda}A\right)L_\lambda^{PB}(B,C) + p(A,B)\left(e^{T\partial_\lambda}B\right)L_\lambda^{PB}(A,C) . \end{aligned}$$

Proof. The proof of this lemma is straightforward. □

Definition 5.2.3. A *Poisson Lie λ–bracket of degree 2* on R is a Poisson λ–bracket $L_\lambda^{\mathrm{PB}}:\ R \otimes R \to S(R)$ such that the following two conditions hold

(1) skewsymmetry $(a,b \in R)$

$$L_\lambda^{\mathrm{PB}}(a,b) = -p(a,b)L_{-\lambda-T}^{\mathrm{PB}}(b,a) , \tag{5.6}$$

(2) Jacobi identity $(a,b,c \in R)$

$$L_\lambda^{\mathrm{PB}}(a,L_\mu^{\mathrm{PB}}(b,c)) - p(a,b)L_\mu^{\mathrm{PB}}(b,L_\lambda^{\mathrm{PB}}(a,c)) = L_{\lambda+\mu}^{\mathrm{PB}}(L_\lambda^{\mathrm{PB}}(a,b)c) . \tag{5.7}$$

Here the triple λ–brackets are defined thanks to Lemma 5.2.2.

The following theorem is analogous to Theorem 2.1.8 in the context of Poisson vertex algebras.

Theorem 5.2.4. *Let L_λ^{PB} be a Poisson Lie λ–bracket on the space R. Then the map $L_\lambda^{PB}:\ S(R)\otimes S(R) \to \mathbb{C}[\lambda]\otimes S(R)$ defined in Lemma 5.2.2 is a Lie conformal algebra structure on $S(R)$. In particular $S(R)$ is a Poisson vertex algebra.*

Proof. We need to prove the L_λ^{PB} satisfies the axioms of conformal algebra. Notice that, by definition, for $A = a_1 \dots a_m,\ B = b_1 \cdots_n \in S(R)$, we have

$$L_\lambda^{\mathrm{PB}}(A,B) = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} p(i,j)\left(e^{T\partial_\lambda}a_1 \cdots \hat{i} \cdots a_m\right)(b_1 \cdots \hat{j} \cdots b_n)L_\lambda^{\mathrm{PB}}(a_i,b_j) ,$$

where $\hat{}$ denotes a missing element, and $p(i,j) = \pm 1$ is an appropriate sign. Sesquilinearity and skewsymmetry of L_λ^{PB} follow immediately by the above expression. We are left to prove Jacobi identity. Let us denote, for $A,B,C \in S(R)$

$$\begin{aligned} \mathrm{J}(A,B,C;\lambda,\mu) &= L_\lambda^{\mathrm{PB}}(A,L_\mu^{\mathrm{PB}}(B,C)) - p(A,B)L_\mu^{\mathrm{PB}}(B,L_\lambda^{\mathrm{PB}}(A,C)) \\ &- L_{\lambda+\mu}^{\mathrm{PB}}(L_\lambda^{\mathrm{PB}}(A,B),C) . \end{aligned}$$

We know by assumption that

$$\mathrm{J}(a,b,c;\lambda\mu) = 0 , \quad \forall a,b,c \in R ,$$

and we want to prove that

$$\mathrm{J}(A,B,C;\lambda,\mu) \;=\; 0 \;, \quad \forall A,B,C \in S(R) \;. \tag{5.8}$$

It is immediate to check the following identities. For $A,B,C,D \in S(R)$

$$\begin{aligned}
\mathrm{J}(A,B,CD;\lambda,\mu) &= \mathrm{J}(A,B,C;\lambda,\mu)D + p(A,C)p(B,C)C\,\mathrm{J}(A,B,D;\lambda,\mu) \;, \\
\mathrm{J}(A,BC,D;\lambda,\mu) &= p(A,B)\Big(e^{T\partial_\mu}B\Big)\mathrm{J}(A,C,D;\lambda,\mu) \\
&+ p(A,C)p(B,C)\Big(e^{T\partial_\mu}C\Big)\mathrm{J}(A,B,D;\lambda,\mu) \;, \\
\mathrm{J}(AB,C,D;\lambda,\mu) &= \Big(e^{T\partial_\lambda}A\Big)\mathrm{J}(B,C,D;\lambda,\mu) + p(A,B)\Big(e^{T\partial_\lambda}B\Big)\mathrm{J}(A,C,D;\lambda,\mu) \;.
\end{aligned}$$

Equation (5.8) follows by the above identities with an easy induction argument. □

Theorem 3.4.2 gives necessary and sufficient conditions for the space R in (3.1) to admit a Lie λ–bracket of degree 2. Similar computations can be used to prove an analogous statement about Poisson Lie λ–brackets of degree 2. The difference is that, since there are no integral terms in the left and right Wick formulas, in this case the computations are much easier. The result is described in the following

Theorem 5.2.5. *Consider the space $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T]\otimes\mathfrak{g}) \oplus (\mathbb{C}[T]\otimes U) \oplus (\mathbb{C}[T]L)$. Let $L^{PB}_\lambda : \; R\otimes R \to \mathbb{C}[T]\otimes S(R)$ be a linear map such that $|0\rangle$ is central, L is a Virasoro element of central charge $\bar{c}\in\mathbb{C}$, namely*

$$L^{PB}_\lambda(L,L) \;=\; (T+2\lambda)L + \frac{\bar{c}}{12}\lambda^3|0\rangle \;,$$

$\mathfrak{g}$ is a finite dimensional space of even primary elements of conformal weight 1, namely

$$L^{PB}_\lambda(L,a) \;=\; (T+\lambda)a \;, \quad \forall a\in\mathfrak{g} \;,$$

and U is a finite dimensional space of odd primary elements of conformal weight 3/2, namely

$$L^{PB}_\lambda(L,u) \;=\; (T+\frac{3}{2}\lambda)u \;, \quad \forall u\in U \;.$$

Then L^{PB}_λ is a Poisson Lie λ–bracket of degree 2 (and thus $S(R)$ has naturally the structure of Poisson vertex algebra) if and only if $\mathfrak{g}$ is a Lie algebra, U is a $\mathfrak{g}$–module, and there exist $\mathfrak{g}$-module homomorphisms $\bar{\varkappa} : \; S^2\mathfrak{g}\to\mathbb{C}$, $\bar{Q} : \; S^2U\to\mathbb{C}$, $\bar{K} : \; \Lambda^2U\to\mathfrak{g}$, $\bar{P} : \; S^2U\to S^2\mathfrak{g}$, such that the following equations hold ($a\in\mathfrak{g}$, $u,v,u_i\in U$, $i=$

1, 2, 3*)*

$$\bar{Q}(u,v)a + 2\bar{\varkappa}(a,\bar{P}_1(u,v))\bar{P}_2(u,v) \;=\; \bar{K}(au,v)+\bar{K}(av,u)\ , \tag{5.9}$$

$$6\bar{\varkappa}(a,\bar{K}(u,v)) = \bar{c}\bar{Q}(au,v)\ , \tag{5.10}$$

$$\frac{1}{2}\bar{Q}(u_2,u_3)u_1 + \frac{1}{2}\bar{Q}(u_1,u_3)u_2 \;=\; \bar{Q}(u_1,u_2)u_3 \;+\bar{K}(u_2,u_3)u_1 \tag{5.11}$$

$$+\bar{K}(u_1,u_3)u_2\ , \tag{5.12}$$

$$\sum_{\sigma\in C_3}\bar{P}_1(u_{\sigma_1},u_{\sigma_2})\cdot(\bar{P}_2(u_{\sigma_1},u_{\sigma_2})u_{\sigma_3}) \;=\; 0\ , \tag{5.13}$$

where C_3 *denotes the group of cyclic permutations of* $(1,2,3)$*, and* $\cdot$ *denotes the associative commutative product in* $S(R)$*. In this case the* λ*–bracket structure on* R *is given by the following table.*

Table 5.1:

	L	b	v
L	$(T+2\lambda)L + \frac{\bar{c}}{12}\lambda^3\lvert 0\rangle$	$(T+\lambda)b$	$(T+\frac{3}{2}\lambda)v$
a	λa	$[a,b]+\lambda\bar{\varkappa}(a,b)\lvert 0\rangle$	av
u	$(\frac{1}{2}T+\frac{3}{2}\lambda)u$	$-bu$	$\bar{Q}(u,v)\left(L+\frac{\bar{c}}{3}\lambda^{(2)}\lvert 0\rangle\right) + (T+2\lambda)\bar{K}(u,v)+\bar{P}(u,v)$

Proof. The proof is straightforward. □

5.3 Existence of the quasi–classical limit

If R is a Lie conformal algebra, Theorem 5.1.4 states that, starting from the enveloping vertex algebra $U(R)$, there is a canonical way to construct a Poisson vertex algebra structure on $S(R)$, which consists in taking a "quasi–classical" limit $\hbar \to 0$ of the family of vertex algebras $\{U(R_\hbar),\ \hbar\in\mathbb{C}\}$. It is natural to ask if a similar construction is possible in the more general situation in which the λ–bracket structure on R admits "quadratic non linearities".

Unfortunately in general such construction is not possible. To better understand what the problem is, suppose R is a space admitting a Lie λ–bracket of degree 2 (according to Definition 2.1.5) of kind $L_\lambda:\ R\otimes R\to\mathbb{C}[\lambda]\otimes S(R)$. Following the construction in Theorem 5.1.4, we would like to define a map $L^{\mathrm{PB}}_\lambda:\ S(R)\otimes S(R)\to$

$\mathbb{C}[\lambda] \otimes S(R)$ such that, when restricted to $R \otimes R$, it coincides with the original Lie λ–bracket on R,

$$L_\lambda^{\mathrm{PB}}(a,b) \;=\; L_\lambda(a,b)\ , \quad \forall a,b \in R\ .$$

On the other hand such a map cannot be a Poisson Lie λ–bracket on R (and therefore it does not define a structure of Poisson vertex algebra on $S(R)$). Indeed the Jacobi identity in the Definition 5.2.3 of a Poisson Lie λ–bracket of degree 2 is not the same as the Jacobi identity in the in the Definition 2.1.5 of a Lie λ–bracket of degree 2, for the following simple reason. While in equation (2.8) we compute the triple λ–brackets by using, when needed, the non commutative Wick formulas, in equation (5.7) the triple λ–brackets are defined by the commutative Wick formulas.

It turns out that, if the space R is endowed with a family of Lie λ–brackets of degree 2, $L_\lambda^{(k)}$ (parametrized, for example, by the Kac–Moody level k), then we can still define a limiting procedure, again called "quasi–classical limit", which will produce a Poisson Lie λ–bracket of degree 2 on R, and therefore a structure of Poisson vertex algebra on $S(R)$. The remaining of this section will be devoted to describe such construction.

Definition 5.3.1. Consider the space

$$R \;=\; \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)\ . \tag{5.14}$$

We say that R admits a *1 parameter family* of Lie λ–brackets of degree 2, if the following conditions hold.

(1) For arbitrarily large $k \in \mathbb{C}$ there is a Lie λ–bracket of degree 2, $L_\lambda^{(k)}$, such that $|0\rangle$ is central, L is a Virasoro element with central charge $c^{(k)}$, and $\mathfrak{g}$ (respectively U) is a finite dimensional space of even (resp. odd) primary elements of conformal weight 1 (resp. 3/2).

(2) The Lie algebra structure of $\mathfrak{g}$ and the $\mathfrak{g}$–module structure of U are independent of k, and the $\mathfrak{g}$–module homomorphisms $\varkappa^{(k)}:\ S^2\mathfrak{g} \to \mathbb{C}$, $Q^{(k)}:\ S^2U \to \mathbb{C}$, $K^{(k)}:$ $\Lambda^2 U \to \mathfrak{g}$, $P^{(k)}:\ S^2U \to S^2\mathfrak{g}$ defined in Theorem 3.4.2 satisfy the following limiting conditions ($a,b \in \mathfrak{g},\ u,v \in U$)

$$\varkappa^{(k)}(a,b) \sim k^\alpha \bar{\varkappa}(a,b)\ , \quad Q^{(k)}(u,v) \sim k^\beta \bar{Q}(u,v)\ , \quad K^{(k)}(u,v) \sim k^\gamma \bar{K}(u,v)\ , \tag{5.15}$$

for some constants $\alpha,\beta,\gamma \in \mathbb{R}$ such that $\alpha > 0$, and fixed $\mathfrak{g}$–module homomorphisms $\bar{\varkappa}$, $\bar{Q}$, $\bar{K}$ such that $\bar{\varkappa}$ and $\bar{Q}$ are non degenerate, and $\bar{K}$ is not identically zero. Here we are using the notation

$$f(k) \;\sim\; k^\alpha \bar{f} \quad \Longleftrightarrow \quad \lim_{k\to\infty} \frac{1}{k^\alpha} f(k) = \bar{f}\ .$$

(3) If $\{J_\alpha, J^\alpha;\ \alpha \in \mathcal{A}\}$ is a dual basis of $\mathfrak{g}$ with respect to the bilinear form $\bar{\varkappa}$, then the expression

$$\sum_{\alpha\in\mathcal{A}} \Big(J_\alpha u_1 \otimes J^\alpha u_2 \;+\; J_\alpha u_2 \otimes J^\alpha u_1 \Big)\ ,$$

is not zero for every pair $u_1, u_2 \in U$.

Remark 5.3.2. Notice that, up to a change of variable $k' = k^\alpha$, we can assume, without loss of generality, $\alpha = 1$. Moreover, the bilinear form $Q^{(k)}$ on U can be rescaled arbitrarily with a $\mathfrak{g}$–module isomorphism $U \stackrel{\sim}{\to} U,\ u \mapsto \delta u$, for $\delta \in \mathbb{C} - \{0\}$. We can thus assume, without loss of generality, $\beta = 0$.

Remark 5.3.3. If $\mathfrak{g}$ is any reductive Lie algebra, condition (3) in Definition 5.3.1 is automatically satisfied, unless U is a trivial $\mathfrak{g}$–module. For this, let $v_\lambda \in U$ be any singular vector of weight λ, let $\{J_\alpha, J^\alpha,\ \alpha \in \mathcal{A}\}$ be a dual basis of root vectors of $\mathfrak{g}$, and let $\{h_i,\ i = 1, \dots, r\} \subset \{J_\alpha,\ \alpha \in \mathcal{A}\}$ be an orthonormal basis of the Cartan subalgebra. We then have

$$\sum_{\alpha \in \mathcal{A}} J_\alpha v_\lambda \otimes J^\alpha v_\lambda \ = \ \sum_{i=1}^{r} h_i v_\lambda \otimes h_i v_\lambda \ = \ |\lambda|^2 v_\lambda \otimes v_\lambda \ \neq \ 0 \ .$$

Lemma 5.3.4. *Suppose the space R in (5.14) admits a 1–parameter family of Lie λ–brackets of degree 2. And assume, without loss of generality, that $\alpha = 1,\ \beta = 0$. Then there exists $\bar{c} \in \mathbb{C} - \{0\}$ such that, for $u, v \in U$*

$$\begin{aligned}
\lim_{k\to\infty} \tfrac{1}{k} c^{(k)} &= \bar{c} \ , \\
\lim_{k\to\infty} K^{(k)}(u,v) &= \bar{K}(u,v) = \tfrac{\bar{c}}{6} \bar{Q}(J_\alpha u, v) J^\alpha \ , \\
\lim_{k\to\infty} k P^{(k)}(u,v) &= \bar{P}(u,v) = \tfrac{\bar{c}}{12} \bar{Q}(\{J_\alpha, J_\beta\} u, v) J^\alpha \otimes J^\beta \\
&\qquad - \tfrac{1}{2} \bar{Q}(u,v) J_\alpha \otimes J^\alpha \ .
\end{aligned}$$

Proof. By Theorem 3.4.2, the $\mathfrak{g}$–module homomorphisms $\varkappa^{(k)},\ Q^{(k)},\ K^{(k)}$ and $P^{(k)}$ satisfy all equations (3.39)–(3.42). Consider first equation (3.39). In this setting, it takes the form

$$\begin{aligned}
-Q^{(k)}(u,v)a + K^{(k)}(au,v) + K^{(k)}(av,u) &= 2\varkappa^{(k)}(a, P_1^{(k)}(u,v)) P_2^{(k)}(u,v) \\
&\quad + [[a, P_1^{(k)}(u,v)], P_2^{(k)}(u,v)] \ .
\end{aligned} \tag{5.16}$$

We will consider separately the two situations $\gamma < 0$ and $\gamma \geq 0$.

Case $\gamma < 0$

In this case, by assumption (5.15) we have that $\mathcal{K}^{(k)}(u,v) \stackrel{k\to\infty}{\longrightarrow} 0,\ \forall u, v \in U$, so that if we take the limit for $k \to \infty$ of both sides of equation (5.16) we get

$$-\bar{Q}(u,v)a \ = \ \lim_{k\to\infty} \Big(2\varkappa^{(k)}(a, P_1^{(k)}(u,v)) P_2^{(k)}(u,v) + [[a, P_1^{(k)}(u,v)], P_2^{(k)}(u,v)] \Big) \ . \tag{5.17}$$

We can thus replace a by J_α in both sides of (5.17), take the tensor product by J^α

and sum over $\alpha = 1, \ldots, \dim \mathfrak{g}$. The resulting equation is

$$-\bar{Q}(u,v) J_\alpha \otimes J^\alpha = \lim_{k\to\infty} \Big(2k P^{(k)}(u,v) + 2k \Big(\tfrac{1}{k} \varkappa^{(k)}(J_\alpha, P_1^{(k)}(u,v)) - \bar{\varkappa}(J_\alpha, P_1^{(k)}(u,v)) \Big) P_2^{(k)}(u,v) \otimes J^\alpha + [[J_\alpha, P_1^{(k)}(u,v)], P_2^{(k)}(u,v)] \otimes J^\alpha \Big) . \qquad (5.18)$$

Note that the second and third term in parenthesis in the right hand side of (5.18) are negligible with respect to the first term, so that (5.18) implies

$$\lim_{k\to\infty} k P^{(k)}(u,v) \;=\; -\frac{1}{2} \bar{Q}(u,v) J_\alpha \otimes J^\alpha . \qquad (5.19)$$

Consider now equation (3.41), which takes the form

$$\tfrac{1}{2} Q^{(k)}(u_2,u_3)u_1 + \tfrac{1}{2} Q^{(k)}(u_1,u_3)u_2 - Q^{(k)}(u_1,u_2)u_3 \;=\; K^{(k)}(u_2,u_3)u_1 + K^{(k)}(u_1,u_3)u_2 + P_1^{(k)}(u_1,u_2)(P_2^{(k)}(u_1,u_2)u_3) . \qquad (5.20)$$

By the assumption $\gamma < 0$ we have $\lim_{k\to\infty} K^{(k)} = 0$, and by (5.19) we also have $\lim_{k\to\infty} P^{(k)} = 0$. Therefore, if we take the limit for $k \to \infty$ of both sides of equation (5.20) we get

$$\frac{1}{2} \bar{Q}(u_2,u_3)u_1 + \frac{1}{2} \bar{Q}(u_1,u_3)u_2 - \bar{Q}(u_1,u_2)u_3 \;=\; 0 . \qquad (5.21)$$

Since $\bar{Q}$ is non degenerate and U is not 1 dimensional, equation (5.21) cannot be satisfied for all choices of $u_1, u_2, u_3 \in U$. We thus conclude that the case $\gamma < 0$ is ruled out.

Case $\gamma \geq 0$

If we divide both sides of equation (5.16) by k^γ and take the limit for $k \to \infty$, we obtain, with an argument similar to the one used to derive (5.19)

$$\lim_{k\to\infty} k^{1-\gamma} P^{(k)}(u,v) \;=\; \frac{1}{2} \Big(\bar{K}(J_\alpha u, v) + \bar{K}(J\alpha v, u) \Big) \otimes J^\alpha - \frac{1}{2} \delta_{\gamma,0} \bar{Q}(u,v) J_\alpha \otimes J^\alpha , \qquad (5.22)$$

where $\delta_{\gamma,0} = 0$ if $\gamma > 0$, and $\delta_{\gamma,0} = 1$ if $\gamma = 0$. Consider now equation (3.40). After dividing both sides by $k^{\gamma+1}$ we can write it as

$$\frac{6}{k^{\gamma+1}} \varkappa^{(k)}(a, K^{(k)}(u,v)) \;=\; \frac{c^{(k)}}{k^{\gamma+1}} Q^{(k)}(au,v) + \frac{2}{k^{\gamma+1}} \varkappa^{(k)}(P_1^{(k)}(au,v), P_2^{(k)}(au,v)) . \qquad (5.23)$$

In the limit for $k \to \infty$, the left hand side of (5.23) converges to

$$6 \bar{\varkappa}(a, \bar{K}(u,v)) ,$$

and the second term in the right hand side of (5.23) converges to zero, thanks to (5.22). We thus conclude, after replacing a by J_α, taking the tensor product by J^α, and summing over $\alpha = 1, \dots, \dim \mathfrak{g}$, that

$$\bar{K}(u,v) \;=\; \frac{\bar{c}}{6} \bar{Q}(J_\alpha u, v) J^\alpha \;, \tag{5.24}$$

where $\bar{c}$ is defined by

$$\bar{c} \;=\; \lim_{k\to\infty} \frac{c^{(k)}}{k^{\gamma+1}} \;.$$

Notice that, since by assumption $\bar{K}$ is not identically zero, it follows that $\bar{c} \neq 0$. Putting together (5.22) and (5.24) we get

$$\lim_{k\to\infty} k^{1-\gamma} P^{(k)}(u,v) \;=\; \frac{\bar{c}}{12} \bar{Q}(\{J_\alpha, J_\beta\}u, v) J^\alpha \otimes J^\beta - \frac{1}{2}\delta_{\gamma,0}\bar{Q}(u,v) J_\alpha \otimes J^\alpha \;. \tag{5.25}$$

In order to complete the proof of the lemma, we are left to show that $\gamma = 0$. Suppose then, by contradiction, that $\gamma > 0$, and divide both sides of equation (3.41) by k^γ to get

$$\begin{aligned} \tfrac{1}{k^\gamma}\Big(K^{(k)}(u_2,u_3)u_1 + K^{(k)}(u_1,u_3)u_2\Big) \;=\; & \tfrac{1}{2k^\gamma} Q^{(k)}(u_2,u_3)u_1 + \tfrac{1}{2k^\gamma} Q^{(k)}(u_1,u_3)u_2 \\ & - \tfrac{1}{k^\gamma} Q^{(k)}(u_1,u_2)u_3 + \tfrac{1}{k^\gamma} P_1^{(k)}(u_1,u_2)(P_2^{(k)}(u_1,u_2)u_3) \;. \end{aligned} \tag{5.26}$$

In the limit $k \to \infty$, every term in the right hand side of (5.26) converges to zero, so that equation (5.26) implies

$$\bar{K}(u_2,u_3)u_1 + \bar{K}(u_1,u_3)u_2 \;=\; 0 \;. \tag{5.27}$$

Since (5.27) holds for every $u_i \in U,\ i = 1,2,3$, and since $\bar{Q}$ is non degenerate, we get, after making the substitution (5.24)

$$J_\alpha u_1 \otimes J^\alpha u_2 + J_\alpha u_2 \otimes J^\alpha u_1 \;=\; 0 \;, \quad \forall u_1, u_2 \in U \;,$$

thus contradicting assumption (3) in Definition 5.3.1. This concludes the proof of the lemma. □

We can now state the main result of this section. It is a statement analogous to Theorem 5.1.4 in the situation in which the λ–bracket on R admits "quadratic non linearities".

Theorem 5.3.5. *Suppose the space R in (5.14) admits a 1 parameter family of Lie λ–brackets of degree 2.*

(1) Then $\bar{c} \in \mathbb{C} - \{0\}$ and the $\mathfrak{g}$–module homomorphisms $\bar{\varkappa} : S^2\mathfrak{g} \to \mathbb{C}$, $\bar{Q} : S^2U \to \mathbb{C}$, $\bar{K} : \Lambda^2 U \to \mathfrak{g}$, $\bar{P} : S^2U \to S^2\mathfrak{g}$ defined in Lemma 5.3.4 satisfy all equations (5.9)–(5.13).

(2) The map $L_\lambda^{PB} : R \otimes R \to \mathbb{C}[\lambda] \otimes S(R)$ defined in Table 5.1 is a Poisson Lie λ–bracket of degree 2 on R. Or, equivalently, L_λ^{PB} defines a structure of Poisson vertex algebra on $S(R)$.

(3) In particular, the enveloping vertex algebra $U_k(R)$ admits a quasi–classical limit (according to Definition 5.1.5).

Proof. Part (1) of the theorem follows by the following obvious considerations. If we take the limit for $k \to \infty$ of both sides of (3.39) we get equation (5.9). If we divide both sides of (3.40) by k and take the limit for $k \to \infty$, we get equation (5.10). If we take the limit for $k \to \infty$ of both sides of (3.41) we get equation (5.12). Finally, if we multiply both sides of (3.42) by k and take the limit for $k \to \infty$ we get equation (5.9). Part (2) of the theorem follows immediately by Theorems 5.2.5 and 5.2.4. □

The property that the vertex algebra $U_k(R)$ admits quasi–classical limit imposes very strong conditions on the structure of the space R. In particular, in the next two sections we will prove that if R admits a 1 parameter family of Lie λ–brackets of degree 2, or, more generally, if $S(R)$ admits a Poisson vertex algebra structure, then:

1. there exists a "little superalgebra" structure on the space $\hat{\mathfrak{g}} = \mathbb{C}L_{-1} \oplus U_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathbb{C}L_0 \oplus U_{1/2} \oplus \mathbb{C}L_1$.
2. the action of the connected complex algebraic group G on the quadric $S^2 = \{u \in U \text{ s.t. } \bar{Q}(u,u) = 1\}$ is transitive.

5.4 Little superalgebra

Let V be a vertex algebra. Recall in Section 1.1 we defined the n–th product of two elements $a, b \in V$ by the formulas

$$[a\ _\lambda\ b] = \sum_{n\geq 0} \lambda^{(n)} a_{(n)} b \ , \quad :(T^{(n)}a)b: = a_{(-n-1)}b \ , \ n \in \mathbb{Z}_+ \ .$$

In this section we will use a "shifted" notation. If a is eigenvalue of $L_{(1)}$ with conformal weight Δ_a, we define

$$a_n = a_{(n+\Delta_a-1)} \in \mathrm{End}V \ .$$

Notice that, if $\Delta_a = 3/2$, then a_n is defined for semi-integer values of n. By sesquilinearity we immediately get

$$(Ta)_m = -(m+\Delta_a)a_m \ , \tag{5.28}$$

and the super commutation relation (1.4) can be rewritten using this new notation as

$$[a_m, b_n] = \sum_{j\geq 0} \binom{m+\Delta_a-1}{j} (a_{(j)}b)_{m+n} \ . \tag{5.29}$$

Suppose now the space $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)$ admits a Lie λ–bracket of degree 2 (see Definition 2.1.5) such that $|0\rangle$ is central, L is a Virasoro element, $\mathfrak{g}$ (respectively U) is a finite dimensional space of even (resp. odd) primary elements of conformal weight 1 (resp. 3/2). The super commutation relation

(5.29), restricted to the operators L_n, $n = -1, 0, 1$, a_0 for $a \in \mathfrak{g}$ and u_n with $n = -1/2, 1/2$, $u \in U$, is described in the following table ($a, b \in \mathfrak{g}$, $u, v, x, y \in U$)

Table 5.2:

$[\ ,\ ]$	L_{-1}	$x_{-1/2}$	b_0	L_0	$y_{1/2}$	L_1
L_{-1}	0	0	0	$-L_{-1}$	$-y_{-1/2}$	$-2L_0$
$u_{-1/2}$	0	$Q(u,x)L_{-1}$ $+ :P(u,x):_{-1}$	$-(bu)_{-1/2}$	$-\frac{1}{2}u_{-1/2}$	$Q(u,y)L_0$ $- K(u,y)_0$ $+ :P(u,y):_0$	$-u_{1/2}$
a_0	0	$(ax)_{-1/2}$	$[a,b]_0$	0	$(ay)_{1/2}$	0
L_0	L_{-1}	$\frac{1}{2}x_{-1/2}$	0	0	$-\frac{1}{2}y_{1/2}$	$-L_1$
$v_{1/2}$	$v_{-1/2}$	$Q(v,x)L_0$ $+ K(v,x)_0$ $+ :P(v,x):_0$	$-(bv)_{1/2}$	$\frac{1}{2}v_{1/2}$	$Q(v,y)L_1$ $+ :P(v,y):_1$	0
L_1	$2L_0$	$x_{1/2}$	0	L_1	0	0

Notice that the elements in Table 5.2 are not closed under commutation, because of the occurrence of the elements $:P(u,v):_n$. On the other hand, if R admits a 1 parameter family of Lie λ–brackets of degree 2, then, by Lemma 5.3.4 we have $\lim_{k\to\infty} Q^{(k)}(u,v) = \bar{Q}(u,v)$, $\lim_{k\to\infty} K^{(k)}(u,v) = \bar{K}(u,v)$ and $\lim_{k\to\infty} P^{(k)}(u,v) = 0$. So that, in the limit $k \to \infty$, the elements $:P(u,v):_n$ disappear. We thus proved the following

Theorem 5.4.1. *Suppose the space R in (5.14) admits a 1 parameter family of Lie λ–brackets of degree 2. Let $\hat{\mathfrak{g}}$ be the vector space $\hat{\mathfrak{g}} = \mathbb{C}L_{-1} \oplus U_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathbb{C}L_0 \oplus U_{1/2} \oplus \mathbb{C}L_1$, where $\mathfrak{g}_0 \simeq \mathfrak{g}$ and $U_{-1/2} \simeq U_{1/2} \simeq U$. For $a \in \mathfrak{g}$, $u \in U$, denote by a_0 the corresponding element of $\mathfrak{g}_0$, and by $u_{-1/2}$ and $u_{1/2}$ the corresponding elements of $U_{-1/2}$ and $U_{1/2}$ respectively. Then $\hat{\mathfrak{g}}$ has the structure of a Lie superalgebra, known as the "little superalgebra" associated to R, with Lie bracket given by Table 5.2, with Q and K replaced by $\bar{Q}$ and $\bar{K}$ respectively, and with $P = 0$.*

It is natural to ask whether a converse statement to Theorem 5.4.1 is true. In fact this problem has been studied and solved in [14], [15]. More precisely, given any simple finite dimensional Lie superalgebra $\hat{\mathfrak{g}}$ with a $\frac{1}{2}\mathbb{Z}$–gradation $\hat{\mathfrak{g}} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$, with a non degenerate invariant bilinear form, and such that there are elements $f \in$

$\mathfrak{g}_{-1}$, $x \in \mathfrak{g}_0$, $e \in \mathfrak{g}_1$ forming an sl_2 triple, Kac and Wakimoto were able to reconstruct, with an operation "inverse" to quasi–classical limit known as *quantum reduction*, the corresponding family of vertex algebras $V_k(\hat{\mathfrak{g}})$, parametrized by the Kac–Moody level k.

5.5 Transitivity of group action on quadrics

Let U be a complex vector space with a non degenerate symmetric bilinear form $\bar{Q}: S^2U \to \mathbb{C}$. We denote by S^2 the complex quadric

$$S^2 \ = \ \{u \in U \ \text{ such that } \ \bar{Q}(u,u) = 1\} \ . \tag{5.30}$$

Let $SO(U)$ be the corresponding complex orthogonal group, namely the group of unimodular linear transformations of U preserving the bilinear form $\bar{Q}$.

Definition 5.5.1. Let G be an algebraic subgroup of $SO(U)$, and let $\mathfrak{g}$ be the corresponding Lie algebra. We say that the action of G on U is *infinitesimally transitive* on the quadric $S^2 \subset U$ if for every $u \in S^2$ one has

$$\mathfrak{g}u \ = \ u^{\perp} \ , \tag{5.31}$$

where $u^{\perp}$ denotes the orthogonal complement to u.

Remark 5.5.2. If $\dim U > 1$, then the action of G on U is infinitesimally transitive on the quadric $S^2 \subset U$ if and only if G acts transitively on S^2. Indeed, it follows from (5.31) that Gu is an open orbit on the quadric $S^2 \subset U$. Since this holds for every point on the quadric S^2, and, for $\dim U > 1$, S^2 is connected, we conclude that G acts transitively on $S^2 \subset U$.

Remark 5.5.3. For an arbitrary algebraic subgroup $G \subset SO(U)$, the inclusion $\mathfrak{g}u \subset u^{\perp}$ is always true, for every $u \in U$. Indeed by invariance of $\bar{Q}$, we have $\bar{Q}(au,u) = 0$, $\forall a \in \mathfrak{g}$, $u \in U$.

Theorem 5.5.4. *Suppose the space R in (5.14) admits a Poisson Lie λ–bracket of degree 2, and suppose the bilinear form $\bar{Q}$ defined in Theorem 5.2.5 is non degenerate. In particular, by Theorem 5.3.5, these assumptions are automatically true if R admits a 1 parameter family of Lie λ–brackets of degree 2.*

(1) Then for every $u \in S^2$ we have $\mathfrak{g}u = u^{\perp}$.

(2) Equivalently, the connected complex algebraic group G associated to $\mathfrak{g}$ has infinitesimally transitive action on the quadric $S^2 \subset U$ defined in (5.30).

Proof. By Theorem 5.2.5, there exist $\mathfrak{g}$–module homomorphisms $\bar{Q}: S^2U \to \mathbb{C}$, $\bar{K}: \Lambda^2U \to \mathfrak{g}$ and $\bar{P}: S^2U \to S^2\mathfrak{g}$ such that equations (5.9)–(5.13) are satisfied. In particular, if we choose $u_1 = u_2 = u$, $u_3 = v \in u^{\perp}$, we get, from equation (5.12)

$$-\bar{Q}(u,u)v \ = \ 2\bar{K}(u,v)u \ \in \ \mathfrak{g}u \ .$$

If $u \in S^2$, this implies that $u^\perp \subset \mathfrak{g}u$. The statement then follows from Remark 5.5.3. □

Remark 5.5.5. Theorem 5.5.4 generalizes a similar statement in [13], where the same result is proved under the assumption that R is a Lie conformal superalgebra.

In the particular case in which $\mathfrak{g}$ is a reductive Lie algebra, one can prove the following stronger result.

Theorem 5.5.6. *A) Suppose $\mathfrak{g}$ is a reductive Lie algebra, U is any $\mathfrak{g}$–module, and $\bar{\varkappa} : S^2\mathfrak{g} \to \mathbb{C}$, $\bar{Q} : S^2U \to \mathbb{C}$ are non degenerate symmetric invariant bilinear forms. Denote by $\{J_\alpha, J^\alpha;\ \alpha = 1, \dots, \dim\mathfrak{g}\}$ a dual basis of $\mathfrak{g}$ with respect to $\bar{\varkappa}$. The following are necessary and sufficient conditions for the space R in (5.14) to admit a Poisson Lie λ–bracket of degree 2.*

(i) The Casimir operator $\Omega = J_\alpha J^\alpha \in U(\mathfrak{g})$ acts as a scalar on U.

(ii) If we decompose $S^2U = \Pi \oplus \mathbb{C}$, where $\Pi = Ker(\bar{Q})$, then the Casimir operator Ω acts as a scalar on Π.

(iii) Let Ω_U and Ω_Π denote the eigenvalues of Ω on U and Π respectively. Then

$$\frac{1}{2}(\dim U - 1)\ \Omega_\Pi \ =\ \dim U\ \Omega_U\ . \tag{5.32}$$

B) Under these assumptions, we have the following expressions of $\bar{c}$, $\bar{K}$ and $\bar{P}$ defined in Theorem 5.2.5.

$$\begin{aligned} \bar{c} &= 3(\dim U - 1)/\Omega_U\ , & (5.33)\\ \bar{K}(u,v) &= \frac{\bar{c}}{6}\bar{Q}(J_\alpha u, v)J^\alpha\ , & (5.34)\\ \bar{P}(u,v) &= \frac{\bar{c}}{12}\bar{Q}(\{J_\alpha, J_\beta\}u, v)J^\alpha \cdot J^\beta - \frac{1}{2}\bar{Q}(u,v)J_\alpha \cdot J^\alpha\ , & (5.35) \end{aligned}$$

where $\cdot$ denotes the associative product in the symmetric algebra $S(R)$. As usual, we use the convention of summing over repeated indices.

Proof. From equation (5.10) we immediately get (5.34). Using this expression of $\bar{K}$, we then get from (5.12)

$$\begin{aligned} &\tfrac{1}{2}\bar{Q}(u_2,u_3)u_1 + \tfrac{1}{2}\bar{Q}(u_1,u_3)u_2 - \bar{Q}(u_1,u_2)u_3 \\ &= \tfrac{\bar{c}}{6}\bar{Q}(J_\alpha u_2, u_3)J^\alpha u_1 + \tfrac{\bar{c}}{6}\bar{Q}(J_\alpha u_1, u_3)J^\alpha u_2\ . \end{aligned} \tag{5.36}$$

Let $\{e_i, e^i;\ i = 1, \dots, \dim U\}$ be a basis of U dual with respect to $\bar{Q}$. If we replace $u_1 = e_i,\ u_2 = e^i$ in (5.36) and sum over the repeated index $i = 1, \dots, \dim U$, we get

$$(\dim U - 1)u_3\ =\ -\frac{\bar{c}}{3}\bar{Q}(J_\alpha e_i, u_3)J^\alpha e^i = \frac{\bar{c}}{3}\Omega(u_3)\ . \tag{5.37}$$

In the last equation we used the obvious identity $J_\alpha e_i \otimes J^\alpha e^i = -e_i \otimes \Omega(e^i)$. It follows from (5.37) that Ω acts on U as scalar multiplication by

$$\Omega_U \;=\; \frac{3}{\bar{c}}(\dim U - 1)\ . \tag{5.38}$$

If we replace $u_3 = e_i$ in (5.36), take the tensor product by e^i and sum over the repeated index $i = 1, \ldots, \dim U$, we get

$$\frac{1}{2}(u_1 \otimes u_2 + u_2 \otimes u_1) - \bar{Q}(u_1, u_2)e_i \otimes e^i \;=\; \frac{\bar{c}}{6}(J_\alpha u_1 \otimes J^\alpha u_2 + J_\alpha u_2 \otimes J^\alpha u_1)\ . \tag{5.39}$$

Since Ω acts as a scalar on U, we have

$$\Omega(u_1 \otimes u_2) \;=\; 2J_\alpha u_1 \otimes J^\alpha u_2 + 2\Omega_U\ u_1 \otimes u_2\ .$$

We thus get, from (5.39)

$$\Omega(u_1 \otimes u_2 + u_2 \otimes u_1) \;=\; \left(2\Omega_U + \frac{6}{\bar{c}}\right)(u_1 \otimes u_2 + u_2 \otimes u_1) - \frac{12}{\bar{c}}\bar{Q}(u_1, u_2)e_i \otimes e^i\ . \tag{5.40}$$

Let now $A \in \mathrm{Ker}(\bar{Q})$. Equation (5.40) implies

$$\Omega(A) \;=\; \left(2\Omega_U + \frac{6}{\bar{c}}\right)\ A\ .$$

In other words, Ω acts on Π as scalar multiplication by

$$\Omega_\Pi \;=\; 2\Omega_U + \frac{6}{\bar{c}}\ . \tag{5.41}$$

Equation (5.32) follows immediately from (5.38) and (5.41). So far we proved that conditions (1)–(3) in Theorem 5.5.6 are necessary conditions. Moreover we proved that, if we define $\bar{K}(u, v)$ by (5.34), then equations (5.10) and (5.12) are satisfied if and only if Ω acts as a scalar on U and $\Pi = S^2U/\mathbb{C}$, its eigenvalues satisfy equation (5.32) and $\bar{c}$ is given by (5.33). We are left to show that there exists $\bar{P}:\ S^2U \to S^2\mathfrak{g}$ such that equations (5.9) and (5.13) hold. From equation (5.9) we get the expression of $\bar{P}(u, v)$, namely

$$\bar{P}(u, v) = \frac{1}{2}\bar{K}(J_\beta u, v)\cdot J^\beta - \frac{1}{2}\bar{Q}(u, v)J_\alpha \cdot J^\alpha\ .$$

This, together with (5.34), gives (5.35). Consider now equation (5.13). We can rewrite it as

$$\frac{\bar{c}}{3}\sum_{\sigma \in S_3}\bar{Q}(J_\alpha J_\beta u_{\sigma_1}, u_{\sigma_3})J^\alpha \cdot (J^\beta u_{\sigma_2}) \;=\; \sum_{\sigma \in S_3}\bar{Q}(u_{\sigma_1}, u_{\sigma_2})J^\alpha \cdot (J^\alpha u_{\sigma_3})\ , \tag{5.42}$$

where S_3 denotes the group of all permutations of (1,2,3). Notice that, given any element $A = u \otimes v + v \otimes u \in S^2U$, we can decompose it as

$$A \;=\; \Big(u \otimes v + v \otimes u - \frac{2\bar{Q}(u,v)}{\dim U} e_i \otimes e^i\Big) + \frac{2\bar{Q}(u,v)}{\dim U} e_i \otimes e^i \ ,$$

so that the first expression is in $\Pi = \mathrm{Ker}(\bar{Q})$, and the last term is in $\mathbb{C} \subset S^2U$. If we apply the Casimir operator $\Omega = J_\alpha J^\alpha$, we then get

$$J_\alpha u \otimes J^\alpha v + J_\alpha v \otimes J^\alpha u \;=\; \Big(\frac{1}{2}\Omega_\Pi - \Omega_U\Big)(u \otimes v + v \otimes u) - \frac{\Omega_\Pi}{\dim U}\bar{Q}(u,v) e_i \otimes e^i \ . \quad (5.43)$$

We can now use (5.43) to rewrite equation (5.42) as

$$\frac{\bar{c}}{6}\Big(\Omega_\Pi - 2\Omega_U\Big)\sum_{\sigma \in S_3} \bar{Q}(u_{\sigma_1}, J_\alpha u_{\sigma_3}) J^\alpha \cdot u_{\sigma_2} = \Big(\frac{\bar{c}\ \Omega_\Pi}{6 \dim U} - 1\Big)\sum_{\sigma \in S_3} \bar{Q}(u_{\sigma_1}, u_{\sigma_2}) J^\alpha \cdot (J^\alpha u_{\sigma_3}) \ . \quad (5.44)$$

Notice that the left hand side of (5.44) is identically zero, by obvious symmetry considerations. Therefore (5.44) is equivalent to the single condition

$$\bar{c}\ \Omega_\Pi \;=\; 6 \dim U \ . \quad (5.45)$$

To conclude, we just notice that (5.45) is automatically satisfied as soon as (5.32) and (5.33) hold. □

Notice that the result of Theorem 5.5.6 is stronger than the result of Theorem 5.5.4. In fact we have the following

Lemma 5.5.7. *Let $\mathfrak{g}$ be any Lie algebra and let U be a $\mathfrak{g}$–module with a non degenerate symmetric invariant bilinear form $\bar{Q}: S^2U \to \mathbb{C}$. Consider the decomposition*

$$S^2U \;=\; \Pi \oplus \mathbb{C} \ ,$$

where $\Pi \subset S^2U$ is a complementary submodule to $\mathbb{C}$. If the Casimir operator acts as a scalar on U and on Π, then for every $u \in U$ such that $\bar{Q}(u,u) \neq 0$ we have $\mathfrak{g}u = u^\perp$. In particular the connected complex algebraic group G associated to $\mathfrak{g}$ has infinitesimally transitive action on the quadric S^2 defined in (5.30)

Proof. Let $u \in U$ be any element such that $\bar{Q}(u,u) \neq 0$, and let $w \in u^\perp$. It follows by (5.43) that

$$\begin{aligned} \bar{Q}(J_\alpha u, w) J^\alpha u \;&=\; \Big(\tfrac{1}{2}\Omega_\Pi - \Omega_U\Big)\bar{Q}(u,w) - \tfrac{\Omega_\Pi}{2\dim U}\bar{Q}(u,u)\bar{Q}(e_i,w)e^i \\ &=\; -\tfrac{\Omega_\Pi}{2\dim U}\bar{Q}(u,u) w \ . \end{aligned}$$

This of course implies $u^\perp \subset \mathfrak{g}u$. The lemma follows from Remark 5.5.3. □

In the next Chapter we will use Theorems 5.5.4 and 5.5.6 to classify all vertex algebras V strongly generated by a space R as in (3.1), with reductive $\mathfrak{g}$, which admit quasi–classical limit (see Definition 5.1.5).

Chapter 6

Classification under the assumption of existence of quasi-classical limit

In this Chapter we will consider vertex algebras V strongly generated by a Virasoro element and primary elements of conformal weight 1 and 3/2, which admit quasi–classical limit. In particular we will completely solve the problem of their classification, under the assumptions that the Lie algebra $\mathfrak{g}$ of elements of conformal weight 1 is reductive, and the bilinear forms $\bar{\varkappa}: S^2\mathfrak{g} \to \mathbb{C}$ and $\bar{Q}: S^2U \to \mathbb{C}$ defined in Theorem 5.2.5 are non degenerate.

According to Definition 5.1.5, to say that the vertex algebra V admits a quasi–classical limit is equivalent to say that the generating set $R \subset V$ admits a Poisson Lie λ–bracket of degree 2. Or, equivalently, that the symmetric space $S(R)$ has a Poisson vertex algebra structure. Therefore our first task will be to classify all pairs $(\mathfrak{g}, U)$, where $\mathfrak{g}$ is any reductive Lie algebra and U is any non zero $\mathfrak{g}$–module, for which the corresponding $\mathbb{C}[T]$–module $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)$ admits a Poisson Lie λ–bracket of degree 2. For this we will use the results of Theorem 5.5.4 and 5.5.6. Thanks to Theorem 5.5.4 we can restrict ourselves to pairs $(\mathfrak{g}, U)$ for which the connected complex algebraic group G associated to $\mathfrak{g}$ acts transitively on the quadric $S^2 = \{u \in U \text{ such that } \bar{Q}(u,u) = 1\} \subset U$. A list of such pairs is provided in [13, Theorem 3.1], and is reproduced in Table 6.1. We will then go through the list in Table 6.1 and check, case by case, whether all conditions in Theorem 5.5.6 are satisfied. We will thus have, in Theorem 6.1.2, a complete list of pairs $(\mathfrak{g}, U)$ for which the corresponding space R admits a Poisson Lie λ–bracket of degree 2. To complete our classification, we will be left to see whether for every pair $(\mathfrak{g}, U)$ listed in Table 6.2, there exists a vertex algebra V which is strongly generated by the space $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes U) \oplus (\mathbb{C}[T]L)$. This will be done in Section 6.2. In particular we will prove that for each pair $(\mathfrak{g}, U)$ in Table 6.2 the corresponding space R admits a 1 parameter family of Lie λ–brackets of degree 2. So that, thanks to Theorem 2.1.8, there is a family of enveloping vertex algebras $U_k(R)$, strongly generated by R, with compatible λ–bracket structure.

6.1 Classification of Poisson vertex algebras

The following lemma follows by Theorem 3.1 in [13].

Lemma 6.1.1. *Let $SO(U)$ be the complex orthogonal group of linear transformations of the vector space U preserving a non degenerate symmetric bilinear form $\bar{Q}: S^2U \to \mathbb{C}$. Consider connected closed subgroups $G \subset SO(U)$, such that the corresponding Lie algebra $\mathfrak{g}$ is reductive. A complete list of pairs $(\mathfrak{g}, U)$ for which the group G has infinitesimally transitive action on the quadric $S^2 = \{u \in U \text{ such that } \bar{Q}(u,u) = 1\}$ is given in the following table.*

Table 6.1:

$\mathfrak{g}$	U
so_n	$\mathbb{C}^n$, $n \geq 1$
sl_n	$\mathbb{C}^n \oplus \mathbb{C}^{n,*}$, $n \geq 2$
gl_n	$\mathbb{C}^n \oplus \mathbb{C}^{n,*}$, $n \geq 2$
sp_n	$\mathbb{C}^n \oplus \mathbb{C}^n$, $n \geq 4$
$\mathrm{sp}_n \oplus \mathbb{C}$	$\mathbb{C}^n \oplus \mathbb{C}^{n,*}$, $n \geq 4$
$\mathrm{sp}_n \oplus \mathrm{sp}_2$	$\mathbb{C}^n \otimes \mathbb{C}^2$, $n \geq 2$
$B_3 = \mathrm{so}_7$	$\mathbb{C}^8 = \mathrm{Spin}_7 = V_{\pi_3}$
$B_4 = \mathrm{so}_9$	$\mathbb{C}^{16} = \mathrm{Spin}_9 = V_{\pi_4}$
G_2	$\mathbb{C}^7 = V_{\pi_1}$

Here and further, for simple $\mathfrak{g}$ we denote by V_λ the irreducible $\mathfrak{g}$–module with highest weight λ, and by $\pi_1, \ldots, \pi_r$ the fundamental weights of $\mathfrak{g}$. In particular, for classical Lie algebras, V_{π_1} is the fundamental representation.

We want to find which of the pairs $(\mathfrak{g}, U)$ listed in Table 6.1 satisfy all conditions in Theorem 5.5.6, namely there is a non degenerate invariant bilinear form $\bar{\varkappa}: S^2\mathfrak{g} \to \mathbb{C}$ for which the corresponding Casimir operator $\Omega = J_\alpha J^\alpha$ is such that $\Omega|_U = \Omega_U 1\!\!1_U$, $\Omega|_{S^2U/\mathbb{C}} = \Omega_\Pi 1\!\!1_{S^2U/\mathbb{C}}$, for some constants Ω_U and Ω_Π, and equation (5.32) holds.

For the pairs $(\mathrm{so}_n, \mathbb{C}^n;\ n \geq 1)$, $(B_3,\ V_{\pi_3})$ and (G_2, V_{π_1}), the values of Ω_U and Ω_Π are provided in Table 4.6. It is immediate to check that, in all three cases, equation (5.32) is satisfied. On the contrary, the pair (B_4, V_{π_4}) is easily ruled out. Indeed, from Table 4.5 we have $S^2V_{\pi_4} = V_{2\pi_4} \oplus V_{\pi_1} \oplus \mathbb{C}$, and, by Lemma 4.2.2, $\Omega|_{V_{2\pi_4}} > \Omega|_{V_{\pi_1}}$. In particular Ω does not act as a scalar on $S^2U/\mathbb{C}$. We will consider each of the remaining cases separately.

Case: $\mathfrak{g} = \mathrm{sl}_n,\ U = \mathbb{C}^n \oplus \mathbb{C}^{n,*};\ n \geq 2$

By definition $\mathbb{C}^n = V_{\pi_1}$ and $\mathbb{C}^{n,*} = V_{\pi_r}$, where $r = n-1$ is the rank of $\mathfrak{g}$. Recall that the space S^2U decomposes as

$$S^2U \ = \ V_{2\pi_1} \oplus V_{2\pi_r} \oplus V_{\pi_1+\pi_r} \oplus \mathbb{C}\ .$$

The unique (up to scalar multiplication) invariant bilinear form on $\mathfrak{g}$ is the trace form $\bar{\varkappa}(a,b) = \mathrm{Tr}_{\mathbb{C}^n}(ab)$. Let $\{J_\alpha, J^\alpha,\ \alpha = 1,\dots,\dim\mathfrak{g}\}$ be a dual basis of $\mathfrak{g}$ with respect to $\bar{\varkappa}$, and let $\Omega = J_\alpha J^\alpha \in U(\mathfrak{g})$ be the corresponding Casimir operator. The eigenvalues of Ω on U and S^2U are the following

$$\Omega_{\pi_1} = \Omega_{\pi_r} = n - \frac{1}{n}\ , \quad \Omega_{2\pi_1} = \Omega_{2\pi_r} = 2n + 2 - \frac{4}{n}\ , \quad \Omega_{\pi_1+\pi_r} = 2n\ .$$

Since we ask that $\Omega_{2\pi_1} = \Omega_{\pi_1+\pi_r}$, all values of n except $n = 2$ are ruled out. For $n = 2$ we then have $\Omega_U = 3/2,\ \Omega_\Pi = 4$, and it is immediate to check that condition (5.32) holds.

Case: $\mathfrak{g} = \mathrm{gl}_n,\ U = \mathbb{C}^n \oplus \mathbb{C}^{n,*};\ n \geq 2$

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathrm{sl}_n \oplus \mathbb{C}$. Irreducible finite dimensional $\mathfrak{g}$–modules are $V_{(\lambda,\mu)}$, where $\lambda = \sum_{i=1}^r k_i\pi_i,\ k_i \geq 0$, is a dominant weight for sl_n and $\mu \in \mathbb{C}$. With this notation $\mathbb{C}^n = V_{(\pi_1,1)},\ \mathbb{C}^{n,*} = V_{\pi_r,-1}$, and the space S^2U decomposes as

$$S^2U \ = \ V_{(2\pi_1,2)} \oplus V_{(2\pi_r,-2)} \oplus V_{(\pi_1+\pi_r,0)} \oplus \mathbb{C}\ .$$

There are two independent invariant bilinear forms on $\mathfrak{g}$, namely $\varkappa_1(a,b) = \mathrm{Tr}_{\mathbb{C}^n}(ab)$ and $\varkappa_2(a,b) = \frac{1}{n^2}\mathrm{Tr}_{\mathbb{C}^n}(a)\mathrm{Tr}_{\mathbb{C}^n}(b)$, and accordingly there are two linearly independent central elements in $U(\mathfrak{g})$ of degree 2, namely $\Omega^1 = \sum_{1\leq i,j\leq n} E_{ij}E_{ji}$ and $\Omega^2 = 1\!\!1 1\!\!1$. It is immediate to find the eigenvalues of Ω^1 and Ω^2 on the irreducible components of U and S^2U. They are as follows

$$\begin{aligned}
&\Omega^1_{(\pi_1,1)} = \Omega^1_{(\pi_r,-1)} = n\ , && \Omega^2_{(\pi_1,1)} = \Omega^2_{(\pi_r,-1)} = 1\ ,\\
&\Omega^1_{(2\pi_1,2)} = \Omega^1_{(2\pi_r,-2)} = 2(n+1)\ , && \Omega^2_{(2\pi_1,2)} = \Omega^2_{(2\pi_r,-2)} = 4\ ,\\
&\Omega^1_{(\pi_1+\pi_r,0)} = 2n\ , && \Omega^2_{(\pi_1+\pi_r,0)} = 0\ .
\end{aligned}$$

Since $\Omega \in U(\mathfrak{g})$ is a central element of degree 2, we have $\Omega = \alpha\Omega^1 + \beta\Omega^2$, and the condition that $\Omega_{(2\pi_1,2)} = \Omega_{(\pi_1+\pi_r,0)}$ imposes $\beta = -\frac{1}{2}\alpha$. In other words it must be (up to scalar multiplication)

$$\Omega = \sum_{1\leq i,j\leq n} E_{ij}E_{ji} - \frac{1}{2} 1\!\!1 1\!\!1 \ \in\ U(\mathfrak{g})\ .$$

It is not hard to check that the corresponding bilinear form is, for $n \neq 2$,

$$\bar{\varkappa}(a,b) = \mathrm{Tr}_{\mathbb{C}^n}(ab) - \frac{1}{n-2}\mathrm{Tr}_{\mathbb{C}^n}(a)\mathrm{Tr}_{\mathbb{C}^n}(b) \ .$$

Notice that, for $n = 2$, $\Omega \in U(\mathrm{sl}_2) \subset U(\mathfrak{g})$, so that the corresponding bilinear form is degenerate. Therefore the case $n = 2$ has to be ruled out. With the above choice of Ω we have $\Omega_U = n - \frac{1}{2}$ and $\Omega_\Pi = 2n$. It is immediate to check that condition (5.32) is satisfied for every n.

Case: $\mathfrak{g} = \mathrm{sp}_n,\ U = \mathbb{C}^n \oplus \mathbb{C}^n;\ n = 2r \geq 4$

The symmetric and wedge square power of the fundamental representation of sp_n decompose as $S^2 V_{\pi_1} = V_{2\pi_1}$ and $\Lambda^2 V_{\pi_1} = V_{\pi_2} \oplus \mathbb{C}$. It follows that

$$S^2 U \ = \ (V_{2\pi_1} \otimes \mathbb{C}^3) \oplus V_{\pi_2} \oplus \mathbb{C} \ .$$

Since $\mathfrak{g}$ is a simple Lie algebra, there is a unique (up to scalar multiplication) Casimir element $\Omega \in U(\mathfrak{g})$ of degree 2. By Lemma 4.2.2 we have $\Omega_{2\pi_1} > \Omega_{\pi_2}$, so that Ω does not act as a scalar on $S^2U/\mathbb{C}$. This case is therefore ruled out.

Case: $\mathfrak{g} = \mathrm{sp}_n \oplus \mathbb{C},\ U = \mathbb{C}^n \oplus \mathbb{C}^{n,*};\ n = 2r \geq 4$

As for gl_n, we denote by $V_{(\lambda,\mu)}$ the irreducible finite dimensional $\mathfrak{g}$–modules, with $\lambda = \sum_{i=1}^r k_i \pi_i$ dominant weight for sp_n and $\mu \in \mathbb{C}$. With this notation $U = V_{(\pi_1,1)} \oplus V_{(\pi_1,-1)}$, and its symmetric square decomposes as

$$S^2 U \ = \ V_{(2\pi_1,2)} \oplus V_{(2\pi_1,-2)} \oplus V_{(2\pi_1,0)} \oplus V_{(\pi_2,0)} \oplus \mathbb{C} \ .$$

There are two independent central elements of $U(\mathfrak{g})$ of degree 2, $\Omega^{\mathrm{sp}_n} \in U(\mathrm{sp}_n) \subset U(\mathfrak{g})$, namely the Casimir element of $\mathrm{sp}_n \subset \mathfrak{g}$, and $\Omega^{\mathbb{C}} = 1\!\!1 1\!\!1 \in U(\mathfrak{g})$. Obviously $\Omega^{\mathbb{C}}|_{V_{(2\pi_1,0)}} = \Omega^{\mathbb{C}}|_{V_{(\pi_2,0)}} = 0$, and, by Lemma 4.2.2, $\Omega^{\mathrm{sp}_n}|_{V_{(2\pi_1,0)}} > \Omega^{\mathrm{sp}_n}|_{V_{(\pi_2,0)}}$. It immediately follows that there is no Casimir element $\Omega \in U(\mathfrak{g})$ which acts as a scalar on $S^2U/\mathbb{C}$. In conclusion this case has to be ruled out.

Case: $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2,\ U = \mathbb{C}^n \otimes \mathbb{C}^2;\ n \geq 2$

We are left to consider the case $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2,\ U = \mathbb{C}^n \otimes \mathbb{C}^2$ for $n \geq 2$. With the notation introduced above, we have $U = V_{(\pi_1,\pi_1)}$, and its symmetric square decomposes, for $n \geq 3$, as

$$S^2 U \ = \ V_{(2\pi_1,2\pi_1)} \oplus V_{(\pi_2,0)} \oplus \mathbb{C} \ ,$$

and for $n = 2$ as $S^2U \ = \ V_{(2\pi_1,2\pi_1)} \oplus \mathbb{C}$. Again, there are two independent Casimir elements, namely $\Omega^{\mathrm{sp}_n} \in U(\mathrm{sp}_n) \subset U(\mathfrak{g})$ and $\Omega^{\mathrm{sp}_2} \in U(\mathrm{sp}_2) \subset U(\mathfrak{g})$. For example we can choose $\Omega^{\mathrm{sp}_n} = \sum_{\alpha \in \mathcal{A}} J_\alpha J^\alpha$, where $\{J_\alpha, J^\alpha,\ \alpha \in \mathcal{A}\}$ is the dual basis described in Remark 4.4.1, associated to the bilinear form $\varkappa^{\mathrm{sp}_n}(a,b) = \mathrm{Tr}_{\mathbb{C}^n}(ab)$. By Lemma 4.4.2 we have that the eigenvalues of Ω^{sp_n} and Ω^{sp_2} in the irreducible components of U and

S^2U are as follows

$$\Omega^{\mathrm{sp}_n}_{(\pi_1,\pi_1)} = \tfrac{n+1}{2} \ , \quad \Omega^{\mathrm{sp}_n}_{(2\pi_1,2\pi_1)} = n+2 \ , \quad \Omega^{\mathrm{sp}_n}_{(\pi_2,0)} = n \ ,$$
$$\Omega^{\mathrm{sp}_2}_{(\pi_1,\pi_1)} = \tfrac{3}{2} \ , \quad \Omega^{\mathrm{sp}_2}_{(2\pi_1,2\pi_1)} = 4 \ , \quad \Omega^{\mathrm{sp}_2}_{(\pi_2,0)} = 0 \ .$$

Every central element $\Omega \in U(\mathfrak{g})$ of degree 2 can be written as $\Omega = \beta\Omega^{\mathrm{sp}_n} + \alpha\Omega^{\mathrm{sp}_2}$. With an appropriate normalization we can fix $\beta = 1$, and the condition that $\Omega_{(2\pi_1,2\pi_1)} = \Omega_{(\pi_2,0)}$ imposes $\alpha = -\frac{1}{2}$. Notice that, since for $n = 2$ there is no component $V_{(\pi_2,0)}$ in S^2U, in this case the value of α is arbitrary. The corresponding bilinear form is

$$\bar{\varkappa}(a,b) = \varkappa^{\mathrm{sp}_n}(a,b) - \frac{1}{2}\varkappa^{\mathrm{sp}_2}(a,b) \ ,$$

unless $n = 2$, in which case

$$\bar{\varkappa}(a,b) = \varkappa^{\mathrm{sp}_n}(a,b) + \alpha\varkappa^{\mathrm{sp}_2}(a,b) \ ,$$

for arbitrary value of α. The request that $\bar{\varkappa}$ is non degenerate just imposes $\alpha \neq 0$. With the above choice of Ω, we have $\Omega_U = \frac{2n-1}{4}$ and $\Omega_\Pi = n$ for $n \geq 3$, and $\Omega_U = \frac{3}{2}(\alpha+1)$ and $\Omega_\Pi = 4(\alpha+1)$ for $n = 2$. It is immediate to check that, in both cases, equation (5.32) is satisfied.

All the above results, together with Theorem 5.5.6, give a complete classification of Poisson vertex algebras generated by R as in (3.1), with reductive $\mathfrak{g}$ and non degenerate bilinear forms $\bar{\varkappa}$ and $\bar{Q}$. This is stated in the following

Theorem 6.1.2. *Let $\mathfrak{g}$ be a non zero reductive Lie algebra and let U be a non zero $\mathfrak{g}$–module. Suppose the space $R = \mathbb{C}|0\rangle \oplus (\mathbb{C}[T]\otimes\mathfrak{g}) \oplus (\mathbb{C}[T]\otimes U) \oplus (\mathbb{C}[T]L)$ admits a Poisson Lie λ–bracket of degree 2, L^{PB}_λ. Moreover assume that the invariant bilinear forms $\bar{\varkappa} :\ S^2\mathfrak{g} \to \mathbb{C}$ and $\bar{Q} :\ S^2U \to \mathbb{C}$ defined in Table 5.1 are non degenerate. Then the pair $(\mathfrak{g}, U)$ is one of the examples listed in the following Table 6.2.*

Remark 6.1.3. (i) In the third column of Table 6.2 there is the expression of the bilinear form $\bar{\varkappa} :\ S^2\mathfrak{g} \to \mathbb{C}$. For $a \in \mathrm{sp}_n \oplus \mathrm{sp}_2$ we use the notation $a = a_1 + a_2$, with $a_1 \in \mathrm{sp}_n$ and $a_2 \in \mathrm{sp}_2$. If $\{J_\alpha, J^\alpha,\ \alpha \in \mathcal{A}\}$ is a dual basis of $\mathfrak{g}$ with respect to $\bar{\varkappa}$, the corresponding Casimir element is by definition $\Omega = J_\alpha J^\alpha \in U(\mathfrak{g})$.

(ii) The value of $\bar{c}$ in the fourth column is obtained by equation (5.33).

(iii) Notice that the data provided in Table 6.2 describe completely the corresponding Poisson Lie λ–bracket of degree 2, $L^{\mathrm{PB}}_\lambda :\ R\otimes R \to \mathbb{C}[\lambda]\otimes S(R)$. Indeed the structure of L^{PB}_λ is as in Table 5.1, and the $\mathfrak{g}$–module homomorphisms $\bar{K} :\ \Lambda^2U \to \mathfrak{g}$ and $\bar{P} :\ S^2U \to S^2\mathfrak{g}$ are given by equations (5.34) and (5.35).

(iv) In the last column of Table 6.2 there is the corresponding Little superalgebras $\hat{\mathfrak{g}}$, defined in Section 5.4. For a detailed account of simple Lie superalgebras, see [11].

Table 6.2:

$\mathfrak{g}$	U	$\bar{\varkappa}(a,b)$	$\bar{c}$	$\hat{\mathfrak{g}}$
$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}^*$	ab	3	$\mathrm{sl}(2\vert 1)$
so_n	$\mathbb{C}^n,\ n \geq 3, n \neq 4$	$\frac{1}{2}\mathrm{Tr}_{\mathbb{C}^n}(ab)$	3	$\mathrm{spo}(2\vert m)$
sl_2	$\mathbb{C}^2 \oplus \mathbb{C}^2$	$\mathrm{Tr}_{\mathbb{C}^2}(ab)$	6	$\mathrm{sl}(2\vert 2)/\mathbb{C}$
gl_n	$\mathbb{C}^n \oplus \mathbb{C}^{n,*},\ n \geq 3$	$\mathrm{Tr}_{\mathbb{C}^n}(ab) - \frac{1}{n-2}\mathrm{Tr}_{\mathbb{C}^n}(a)\mathrm{Tr}_{\mathbb{C}^n}(b)$	6	$\mathrm{sl}(2\vert n)$
$\mathrm{sp}_2 \oplus \mathrm{sp}_2$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	$\mathrm{Tr}_{\mathbb{C}^2}(a_1 b_1) + \alpha \mathrm{Tr}_{\mathbb{C}^2}(a_2 b_2)$	12	$D(2,1;\alpha)$
$\mathrm{sp}_n \oplus \mathrm{sp}_2$	$\mathbb{C}^n \otimes \mathbb{C}^2,\ n \geq 4$	$\mathrm{Tr}_{\mathbb{C}^n}(a_1 b_1) - \frac{1}{2}\mathrm{Tr}_{\mathbb{C}^2}(a_2 b_2)$	12	$\mathrm{osp}(4\vert n)$
B_3	V_{π_3}	$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\, a\ \ \mathrm{ad}\, b)$	4	$F(4)$
G_2	V_{π_1}	$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\, a\ \ \mathrm{ad}\, b)$	$\frac{9}{2}$	$G(3)$

Remark 6.1.4. In Theorem 6.1.2 we did not consider the situation in which either $\mathfrak{g}$ or U is zero. In this case we have the following possibilities. If $\mathfrak{g} = U = 0$, then the space R in (3.1) is the Virasoro Lie conformal algebra. If $U = 0$ but $\mathfrak{g} \neq 0$, then the space R is the current Lie conformal algebra. Finally, if $\mathfrak{g} = 0$ but $U \neq 0$, it follows by Theorem 5.5.4 that $U = \mathbb{C}$, and the space R in (3.1) is the Neveu–Schwarz Lie conformal superalgebra, [17].

6.2 Quantization

In this section we will "quantize" the Poisson vertex algebras described in Theorem 6.1.2. More precisely, for each pair $(\mathfrak{g}, U)$ in Table 6.2 we will classify all 7–ples $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ satisfying the assumptions of Problem 3.5.3, so that the corresponding product defined by Table 3.1 is a Lie λ–bracket of degree 2 on the space R defined by (3.1), and then, by Theorem 2.1.8, there exists an enveloping vertex algebra $U(R)$ strongly generated by R. In fact we will see that, for each pair $(\mathfrak{g}, U)$, the space R admits a 1–parameter family of Lie λ–brackets of degree 2 (see Definition 5.3.1), $\{L_\lambda^{(k)},\ k \in \mathbb{C}\}$, with arbitrarily large value of the Kac Moody level k, and the corresponding "quasi–classical limit", described in Theorem 5.3.5, coincides with the Poisson vertex algebra structure on $S(R)$ defined by the data in Table 6.2 and by equations (5.34) and (5.35), as explained in Remark 6.1.3.

6.2.1 Case: $\mathfrak{g}$ simple, U irreducible

Consider the pairs $(\mathrm{so}_n, \mathbb{C}^n,\ n \geq 3, n \neq 4)$, (B_3, V_{π_3}) and (G_2, V_{π_1}). We proved in Corollary 4.6.2 that, for an arbitrary value of the Kac–Moody level k (except at most some singular value), there is a corresponding Lie λ–bracket of degree 2, $L_\lambda^{(k)} : R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$, given by Table 3.1, where $\varkappa^{(k)}$, $Q^{(k)}$, $K^{(k)}$ and $P^{(k)}$ are defined in Theorem 4.6.1. Furthermore, by looking at the expressions of the parameters c, σ, β, γ in Table (4.6), it follows that $\{L_\lambda^{(k)},\ k \in \mathbb{C}\}$ is in fact a 1 parameter family of Lie λ–brackets of degree 2 on R. We can summarize these observations in the following

Proposition 6.2.1. *A) Let $(\mathfrak{g}, U)$ be one of the pairs $(\mathrm{so}_n, \mathbb{C}^n,\ n \geq 3, n \neq 4)$, (B_3, V_{π_3}), (G_2, V_{π_1}). Then the space R in (3.1) admits a 1 parameter family of Lie λ–brackets of degree 2, $\{L_\lambda^{(k)},\ k \in \mathbb{C}\}$, as in Table 3.1 with $\varkappa^{(k)}$, $Q^{(k)}$, $K^{(k)}$, $P^{(k)}$ defined in Theorem 4.6.1. The corresponding Poisson Lie λ–bracket of degree 2, L_λ^{PB}, defined in Theorem 5.3.5, coincides with the Poisson Lie λ–bracket described in Remark 6.1.3.*

B) R does not admit any other Lie λ–bracket of degree 2.

6.2.2 Case: $\mathfrak{g} = \mathbb{C},\ U = \mathbb{C} \oplus \mathbb{C}^*$

Let $\mathfrak{g} = \mathbb{C}h$, $U = \mathbb{C}v_+ \oplus \mathbb{C}v_-$, with action of $\mathfrak{g}$ on U given by $hv_\pm = \pm v_\pm$. The $\mathfrak{g}$–module homomorphisms $\varkappa : S^2\mathfrak{g} \to \mathbb{C}$, $Q : S^2U \to \mathbb{C}$, $K : \Lambda^2 U \to \mathfrak{g}$ and $P : S^2U \to S^2\mathfrak{g}$ are necessarily of the form

$$\begin{aligned}
\varkappa(h,h) &= k\ , \\
Q(v_\pm, v_\mp) &= 1\ , & Q(v_\pm, v_\pm) &= 0\ , \\
K(v_\pm, v_\mp) &= \pm\sigma\ h\ , & K(v_\pm, v_\pm) &= 0\ , \\
P(v_\pm, v_\mp) &= \gamma\ h \otimes h\ , & P(v_\pm, v_\pm) &= 0\ ,
\end{aligned} \tag{6.1}$$

for some constants $k, \sigma, \gamma \in \mathbb{C}$. It is not hard to check that equation (3.42) implies $\gamma = 0$, and equations (3.39), (3.40), (3.41) are equivalent to the following conditions

$$\sigma = \frac{1}{2}\ , \quad c = 3k\ . \tag{6.2}$$

Since $P = 0$, the space R in (3.1) is closed under the λ–bracket in Table 3.1. In other words, R is the $N = 2$ Lie conformal superalgebra, [12]. We thus proved the following

Proposition 6.2.2. *A) Let $\mathfrak{g} = \mathbb{C},\ U = \mathbb{C} \oplus \mathbb{C}^*$. The space R in (3.1) admits a structure of Lie conformal superalgebra. The λ–bracket is given in Table 3.1, with $P = 0$ and $c, \varkappa, Q, K$ defined by (6.1) and (6.2).*

B) R does not admit any other Lie λ–bracket of degree 2.

6.2.3 Case: $\mathfrak{g} = \mathrm{sp}_2,\ U = \mathbb{C}^2 \oplus \mathbb{C}^2$

Recall that $\mathrm{sl}_2 = \mathrm{sp}_2 = \{a \in \mathrm{Mat}_2\mathbb{C} \mid aJ + Ja^t = 0\}$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We will denote an element of U as $u_1 + v_2$, with u_1 in the first component of U and v_2 in the second component. The following facts follow immediately from representation theory of sl_2,

$$\mathfrak{g} = V_{2\pi_1} \ , \qquad S^2\mathfrak{g} = V_{4\pi_1} \oplus \mathbb{C} \ ,$$
$$U = V_{\pi_1} \oplus V_{\pi_1} \ , \quad S^2U = (V_{2\pi_1} \otimes \mathbb{C}^3) \oplus \mathbb{C} \ , \quad \Lambda^2 U = V_{2\pi_1} \oplus \mathbb{C}^3 \ .$$

In particular the $\mathfrak{g}$–module homomorphisms $\varkappa : S^2\mathfrak{g} \to \mathbb{C}$, $Q : S^2U \to \mathbb{C}$, $K : \Lambda^2U \to \mathfrak{g}$, $P : S^2U \to S^2\mathfrak{g}$ are uniquely defined (up to scalar multiplication) and they are given by

$$\begin{aligned}
\varkappa(a,b) &= k\mathrm{Tr}_{\mathbb{C}^2}(ab) \ , \\
Q(u_1,v_2) &= Q(v_2,u_1) = (v^tJu) \ , \qquad & Q(u_i,v_i) &= 0 \ , \\
K(u_1,v_2) &= -K(v_2,u_1) = \sigma(uv^t+vu^t)J \ , & K(u_i,v_i) &= 0 \ , \\
P(u_1,v_2) &= P(v_2,u_1) = \gamma(v^tJu)J_\alpha \otimes J^\alpha \ , & P(u_i,v_i) &= 0 \ ,
\end{aligned} \tag{6.3}$$

for some constants $k, \sigma, \gamma \in \mathbb{C}$. Here $\{J_\alpha, J^\alpha, \ \alpha \in \mathcal{A}\}$ denotes a dual basis of sl_2 with respect to the trace form, namely $\mathrm{Tr}_{\mathbb{C}^2}(J_\alpha J^\beta) = \delta_{\alpha\beta}$. As before, it is not hard to check that equations (3.39), (3.40), (3.41) and (3.42) are equivalent to the following conditions

$$\sigma = \frac{1}{2} \ , \quad c = 3k \ , \quad \gamma = 0 \ . \tag{6.4}$$

We thus proved the following

Proposition 6.2.3. *A) Let $\mathfrak{g} = \mathrm{sl}_2$, $U = \mathbb{C}^2 \oplus \mathbb{C}^2$. The space R in (3.1) admits a structure of Lie conformal superalgebra (known as $N = 4$ conformal algebra, [12]). The λ–bracket is given in Table 3.1, with $P = 0$ and $c, \varkappa, Q, K$ defined by (6.3) and (6.4).*

B) R does not admit any other Lie λ–bracket of degree 2.

6.2.4 Case: $\mathfrak{g} = \mathrm{gl}_n$, $U = \mathbb{C}^n \oplus \mathbb{C}^{n,*}$, $n = r+1 \geq 3$

We identify $\mathrm{gl}_n = \mathrm{Mat}_n\mathbb{C}$, $\mathbb{C}^n$ with the space of column n–vectors, and $\mathbb{C}^{n,*}$ with the space of row n–vectors. Namely $\mathbb{C}^{n,*} = \{u^t, \ u \in \mathbb{C}^n\}$. The action of gl_n on $\mathbb{C}^n$ is given by (left) matrix multiplication, and on $\mathbb{C}^{n,*}$ it is given by $a(u^t) = -u^ta$. We have $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2\mathfrak{g}, \mathbb{C}) = 2$, and every symmetric invariant bilinear form on $\mathfrak{g}$ is of kind

$$\varkappa(a,b) = k_1\mathrm{Tr}_{\mathbb{C}^n}(ab) + k_2\mathrm{Tr}_{\mathbb{C}^n}(a)\mathrm{Tr}_{\mathbb{C}^n}(b) \ , \tag{6.5}$$

for arbitrary $k_1, k_2 \in \mathbb{C}$. Moreover $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, \mathbb{C}) = 1$, and the unique (up to scalar multiplication) symmetric invariant bilinear form on U is

$$Q(u,v^t) = Q(v^t,u) = v^tu \ , \quad Q(u,v) = Q(v^t,u^t) = 0 \ . \tag{6.6}$$

Furthermore, $\dim \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 U, \mathfrak{g}) = 2$, and every $\mathfrak{g}$–module homomorphism $K:\ \Lambda^2 U \to \mathfrak{g}$ is of the form

$$K(u, v^t) = -K(v^t, u) = \sigma_1 u v^t + \sigma_2 (v^t u)\mathbb{1}\ , \quad K(u, v) = K(v^t, u^t) = 0\ . \tag{6.7}$$

Finally, $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2 U, S^2 \mathfrak{g}) = 4$, and every $\mathfrak{g}$–module homomorphism $P:\ S^2 U \to S^2 \mathfrak{g}$ is of the form

$$\begin{aligned} P(u, v^t) &= \alpha(v^t u)\mathbb{1}\otimes\mathbb{1} + \beta(v^t u) J_\alpha \otimes J^\alpha + \tfrac{1}{2}\gamma(uv^t\otimes\mathbb{1} + \mathbb{1}\otimes uv^t) \\ &+ \tfrac{1}{2}\delta(J_\alpha u v^t \otimes J^\alpha + J_\alpha \otimes J^\alpha u v^t)\ , \\ P(u, v) &= P(v^t, u^t)\ =\ 0\ . \end{aligned} \tag{6.8}$$

Here we are denoting by $\{J_\alpha, J^\alpha,\ \alpha\in\mathcal{A}\}$ a dual basis of $\mathfrak{g}$ with respect to the trace form, namely such that $\mathrm{Tr}_{\mathbb{C}^n}(J_\alpha J^\beta) = \delta_{\alpha\beta}$, and we are using the convention of summing over repeated indices. For example we can choose $J_\alpha = E_{ij},\ J^\alpha = E_{ji},\ 1\le i, j\le n$. The following lemma is an easy fact from linear algebra

Lemma 6.2.4. *For every $u, v\in\mathbb{C}^n,\ A\in Mat_n\mathbb{C}$, we have*

$$Tr_{\mathbb{C}^n}(J_\alpha A)J^\alpha = A\ , \quad Tr_{\mathbb{C}^n}(uv^t) = v^t u\ , \quad J_\alpha A J^\alpha = Tr_{\mathbb{C}^n}(A)\mathbb{1}\ .$$

Let us first impose equation (3.39) with the above expressions of $\varkappa, Q, K, P$. After simple algebraic manipulations based on Lemma 6.2.4, we can rewrite it as

$$\begin{gathered} (\delta k_1 + \tfrac{1}{2}n\delta - \sigma_1)\{a, uv^t\} + (2\delta k_2 + \gamma k_1 + n\gamma k_2 - \delta)\mathrm{Tr}_{\mathbb{C}^n}(a) uv^t \\ + (1 + 2\beta k_1 + \delta + 2n\beta)(v^t u)a + (\gamma k_1 - \delta - 2\sigma_2)(v^t a u)\mathbb{1} \\ +(\gamma k_2 + 2\beta k_2 + 2\alpha k_1 + 2n\alpha k_2 - 2\beta)(v^t u)\mathrm{Tr}_{\mathbb{C}^n}(a)\mathbb{1}\ =\ 0\ . \end{gathered}$$

Notice that the expressions $\{a, uv^t\},\ \mathrm{Tr}_{\mathbb{C}^n}(a)uv^t,\ (v^t u)a,\ (v^t a u)\mathbb{1},\ (v^t u)\mathrm{Tr}_{\mathbb{C}^n}(a)\mathbb{1}$ define linearly independent elements of $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}^n\otimes\mathbb{C}^{n,*}\otimes\mathfrak{g}, \mathfrak{g})$, so that equation (3.39) is equivalent to the following conditions

$$\begin{aligned} &\delta k_1 + \tfrac{1}{2}n\delta - \sigma_1\ =\ 0\ , \\ &2\delta k_2 + \gamma k_1 + n\gamma k_2 - \delta\ =\ 0\ , \\ &1 + 2\beta k_1 + \delta + 2n\beta\ =\ 0\ , \\ &\gamma k_1 - \delta - 2\sigma_2\ =\ 0\ , \\ &\gamma k_2 + 2\beta k_2 + 2\alpha k_1 + 2n\alpha k_2 - 2\beta\ =\ 0\ . \end{aligned} \tag{6.9}$$

In a similar way, it is not hard to prove that equation (3.40) is equivalent to the following conditions

$$
\begin{aligned}
& c + 2(\delta + n\beta)(nk_1 + k_2) + 2(\gamma + \mathfrak{n}\alpha)(k_1 + nk_2) \ = \ 6\sigma_1 k_1 \ , \\
& \sigma_1 k_2 + \sigma_2(k_1 + nk_2) \ = \ 0 \ .
\end{aligned} \tag{6.10}
$$

Consider now equation (3.41). We have to consider separately the two situations $u_1, u_2 \in \mathbb{C}^n,\ u_3^t \in \mathbb{C}^{n,*}$ and $u_1, u_3 \in \mathbb{C}^n,\ u_2^t \in \mathbb{C}^{n,*}$. In the first case equation (3.41) takes the form

$$
(\sigma_1 + \sigma_2 - \frac{1}{2})((u_3^t u_2)u_1 + (u_3^t u_1)u_2) \ = \ 0 \ ,
$$

and in the second case it takes the form

$$
(\frac{1}{2}\delta + n\beta + \alpha + 1 - \sigma_1)(u_2^t u_1)u_3 + (\frac{1}{2}n\delta + \gamma - \frac{1}{2} - \sigma_2)(u_2^t u_3)u_1 \ = \ 0 \ .
$$

In conclusion equation (3.41) is equivalent to the following three conditions

$$
\begin{aligned}
& \sigma_1 + \sigma_2 - \tfrac{1}{2} \ = \ 0 \ , \\
& \tfrac{1}{2}\delta + n\beta + \alpha + 1 - \sigma_1 \ = \ 0 \ , \\
& \tfrac{1}{2}n\delta + \gamma - \tfrac{1}{2} - \sigma_2 \ = \ 0 \ .
\end{aligned} \tag{6.11}
$$

We are left to impose equation (3.42). By symmetry considerations, it is enough to consider the situation $u_1, u_2 \in \mathbb{C}^n,\ u_3 = w^t \in \mathbb{C}^{n,*}$. In this case equation (3.42) takes the form

$$
\sum_{\sigma \in S_2} \Big(\tfrac{1}{2}\delta(J_\alpha u_{\sigma_1} w^t \otimes J^\alpha u_{\sigma_2} + (w^t u_{\sigma_2}) J_\alpha \otimes J^\alpha u_{\sigma_1}) + \tfrac{1}{2}\gamma(u_{\sigma_1} w^t \otimes u_{\sigma_2} + (w^t u_{\sigma_2}) \mathbb{1} \otimes u_{\sigma_1})
$$
$$
+ \beta(w^t u_{\sigma_1}) J_\alpha \otimes J^\alpha u_{\sigma_2} + \alpha(w^t u_{\sigma_1}) \mathbb{1} \otimes u_{\sigma_2} \Big) \ = \ 0 \ .
$$

In the first term of the left hand side of the above equation we can replace

$$
J_\alpha u_{\sigma_1} w^t \otimes J^\alpha u_{\sigma_2} \ = \ u_{\sigma_2} w^t \otimes u_{\sigma_1} \ .
$$

Using this fact, we conclude that equation (3.42) is equivalent to the following conditions

$$
\begin{aligned}
\delta + \gamma \ &= \ 0 \ , \\
\tfrac{1}{2}\delta + \beta \ &= \ 0 \ , \\
\tfrac{1}{2}\gamma + \alpha \ &= \ 0 \ .
\end{aligned} \tag{6.12}
$$

Surprisingly enough, all equations (6.9), (6.10), (6.11) and (6.12) admit a family of solutions, with arbitrary value of $k_1 \neq 1 - n$. For given k_1, the corresponding values of all other parameters are as follows

$$
\begin{aligned}
k_2 &= -\tfrac{k_1+1}{n-2} \,, \\
\sigma_1 &= \tfrac{2k_1+n}{2(k_1+n-1)} \,, \quad \sigma_2 \;=\; -\tfrac{k_1+1}{2(k_1+n-1)} \,, \\
\delta &= \tfrac{1}{k_1+n-1} \;=\; 2\alpha \;=\; -2\beta \;=\; -\gamma \,, \\
c &= 6k_1+(n^2+5n-2)-\tfrac{(n-1)(n^2+5n-3)}{k_1+n-1} \,.
\end{aligned}
\tag{6.13}
$$

The following proposition is an easy consequence of the above results.

Proposition 6.2.5. *A) Let $\mathfrak{g} = \mathrm{gl}_n$, $U = \mathbb{C}^n \oplus \mathbb{C}^{n,*}$, $n \geq 3$. A complete list of 7-ples $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ satisfying all assumptions of Problem 3.5.3 is given by (6.5), (6.6), (6.7) and (6.8), with $k_1 \neq 1-n$ arbitrary, and all other parameters given by (6.13).*

B) The corresponding Lie λ–brackets $L_\lambda^{(k_1)}$, defined by Table 3.1, form a 1 parameter family of Lie λ–brackets of degree 2 on the space R in (3.1).

C) The Poisson Lie λ bracket of degree 2, L_λ^{PB}, defined in Theorem 5.3.5, coincides with the Poisson Lie λ–bracket described in Remark 6.1.3.

6.2.5 Case: $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2,\ U = \mathbb{C}^n \otimes \mathbb{C}^2,\ n = 2r \geq 2$

We are left to consider the case $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2$, $U = \mathbb{C}^n \otimes \mathbb{C}^2$, $n \geq 2$. The generic element in $\mathfrak{g}$ is of kind $A+a$, with $A \in \mathrm{sp}_n$ and $a \in \mathrm{sp}_2$. Moreover every element of U is linear combination of monomials $u \otimes x$, with $u \in \mathbb{C}^n$ and $x \in \mathbb{C}^2$. Irreducible finite dimensional $\mathfrak{g}$–modules are denoted by $V_{(\lambda,\mu)}$, where $\lambda = \sum_{i=1}^r k_i \pi_i$ is a dominant weight of sp_n and $\mu = k\pi_1$ is a dominant weight of sp_2. With this notation we have

$$
\begin{aligned}
\mathfrak{g} &= V_{(2\pi_1,0)} \oplus V_{(0,2\pi_1)} \,, \quad U \;=\; V_{(\pi_1,\pi_1)} \,, \\
S^2\mathfrak{g} &= V_{(4\pi_1,0)} \oplus \delta_{n\geq 3} V_{(2\pi_2,0)} \oplus \delta_{n\geq 3} V_{(\pi_2,0)} \oplus V_{(0,4\pi_1)} \oplus V_{(2\pi_1,2\pi_1)} \oplus \mathbb{C}^2 \,, \\
S^2U &= V_{(2\pi_1,2\pi_1)} \oplus \delta_{n\geq 3} V_{(\pi_2,0)} \oplus \mathbb{C} \,, \quad \Lambda^2 U \;=\; \delta_{n\geq 3} V_{(\pi_2,2\pi_1)} \oplus V_{(2\pi_1,0)} \oplus V_{(0,2\pi_1)} \,.
\end{aligned}
$$

It follows that $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2\mathfrak{g}, \mathbb{C}) = 2$ and the generic symmetric invariant bilinear form on $\mathfrak{g}$ is of kind

$$
\varkappa(A+a, B+b) = k_1 \mathrm{Tr}_{\mathbb{C}^n}(AB) + k_2 \mathrm{Tr}_{\mathbb{C}^2}(ab) \,. \tag{6.14}
$$

The unique (up to scalar multiplication) symmetric invariant bilinear form on U is given by

$$
Q(u \otimes x, v \otimes y) = (v^t J u)(y^t j x) \,. \tag{6.15}
$$

In the first factor of the right hand side J denotes the nxn matrix (4.26), and in the second factor j denotes the corresponding 2x2 matrix, $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since

$\dim \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 U, \mathfrak{g}) = 2$, every $\mathfrak{g}$–module homomorphism $K:\ \Lambda^2 U \to \mathfrak{g}$ can be written as

$$K(u\otimes x, v\otimes y) = \frac{1}{2}\sigma_1 (y^t jx)(uv^t J + vu^t J) + \frac{1}{2}\sigma_2 (vJu^t)(xy^t j + yx^t j)\ , \tag{6.16}$$

for some value of $\sigma_1, \sigma_2 \in \mathbb{C}$. Notice that the first term in the right hand side is an element of sp_n, while the second term is in sp_2. For $n=2$ we have $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 3$, and for $n \geq 3$ we have $\dim \mathrm{Hom}_{\mathfrak{g}}(S^2U, S^2\mathfrak{g}) = 4$. It follows that every $\mathfrak{g}$–module homomorphism $P:\ S^2U \to S^2\mathfrak{g}$ can be written as

$$\begin{aligned} P(u\otimes x, v\otimes y) \ &= \ \alpha(v^t Ju)(y^t jx) j_\alpha \otimes j^\alpha + \beta (v^t Ju)(y^t jx) J_\alpha \otimes J^\alpha \\ &+ \tfrac{1}{8}\gamma(y^t jx)\Big(\{J_\alpha, uv^t J - vu^t J\}\otimes J^\alpha + J_\alpha \otimes \{J^\alpha, uv^t J - vu^t J\}\Big) \\ +\tfrac{1}{8}\delta\Big(&(uv^t J + vu^t J)\otimes(xy^t j + yx^t j) + (xy^t j + yx^t j)\otimes(uv^t J + vu^t J)\Big)\ . \end{aligned} \tag{6.17}$$

Here $\{J_\alpha, J^\alpha,\ \alpha \in \mathcal{A}\}$ denotes a dual basis of sp_n with respect to the trace form. For example we can take $J_\alpha,\ J^\alpha$ as in Remark 4.4.1(b). $\{j_\alpha, j^\alpha,\ \alpha\in\mathcal{A}\}$ denotes the corresponding dual basis of sp_2. Notice that for $n=2$ we have

$$uv^t J - vu^t J \ = \ (v^t Ju)\mathbb{1}\ , \tag{6.18}$$

so that the third term in the right hand side of (6.17) can be written as linear combination of the other three. In particular, for $n=2$ we can assume $\gamma = 0$.

We can now use the above expressions of $\varkappa,\ Q,\ K,\ P$ to impose equations (3.39), (3.40), (3.41), (3.42). For this it is convenient to use Lemma 4.4.2 and equation (6.18). After a straightforward though rather lengthy computation, similar to the one done in the previous section, we find the following results. For $n=2$, equation (3.39) is equivalent to

$$\begin{aligned} &\sigma_1 \ = \ \tfrac{1}{2}\delta k_2\ , \quad \sigma_2 \ = \ \tfrac{1}{2}\delta k_1\ , \\ &1 + 2\beta k_1 + 4\beta - \sigma_1 \ = \ 0\ , \\ &1 + 2\alpha k_2 + 4\alpha - \sigma_2 \ = \ 0\ , \end{aligned} \tag{6.19}$$

equation (3.40) is equivalent to

$$6\sigma_1 k_1 \ = \ 6\sigma_2 k_2 \ = \ c + 6\beta k_1 + 6\alpha k_2\ , \tag{6.20}$$

equation (3.41) is equivalent to

$$\begin{aligned} &1 + \tfrac{1}{2}\delta - \sigma_1 - \sigma_2 + 3\alpha + 3\beta \ = \ 0\ , \\ &1 - \tfrac{1}{4}\delta - \sigma_1 - \sigma_2 + \tfrac{3}{2}\alpha + \tfrac{3}{2}\beta \ = \ 0\ , \end{aligned} \tag{6.21}$$

and equation (3.42) is equivalent to

$$\alpha = \beta = -\frac{1}{4}\delta \ . \tag{6.22}$$

For $n \geq 3$, equation (3.39) is equivalent to

$$\begin{aligned} \sigma_1 &= \tfrac{1}{2}\delta k_2 \ , \quad \sigma_2 = \tfrac{1}{2}\delta k_1 \ , \\ 1 + 2\alpha k_2 + 4\alpha - \sigma_2 &= 0 \ , \\ 1 + 2\beta k_1 + \tfrac{1}{2}\gamma + (n+2)\beta &= 0 \ , \\ \gamma k_1 + \tfrac{n+4}{4}\gamma - \sigma_1 &= 0 \ , \end{aligned} \tag{6.23}$$

equation (3.40) is equivalent to

$$6\sigma_1 k_1 = 6\sigma_2 k_2 = c + (n+1)\gamma k_1 + n(n+1)\beta k_1 + 6\alpha k_2 \ , \tag{6.24}$$

equation (3.41) is equivalent to

$$\begin{aligned} 1 + \tfrac{1}{4}\gamma + \tfrac{n+1}{2}\beta + \tfrac{3}{2}\alpha - \tfrac{1}{2}\sigma_1 &= 0 \ , \\ \tfrac{1}{2}\delta - 1 - \sigma_2 + \tfrac{n+2}{4}\gamma &= 0 \ , \\ \sigma_1 + 2\sigma_2 + \tfrac{1}{2}\delta - \tfrac{n+2}{4}\gamma &= 0 \ , \end{aligned} \tag{6.25}$$

and equation (3.42) is equivalent to

$$\frac{1}{4}\delta + \alpha = \frac{1}{4}\gamma + \beta = \delta + \gamma = 0 \ . \tag{6.26}$$

It is not hard to check that, for arbitrary values of k_1 and k_2 such that $k_1 + k_2 \neq -2$, all equations (6.19), (6.20), (6.21) and (6.22) admit a common solution. The corresponding values of all other parameters are as follows

$$\begin{aligned} \sigma_1 &= \tfrac{k_2}{k_1+k_2+2} \ , \quad \sigma_2 = \tfrac{k_1}{k_1+k_2+2} \ , \\ \delta &= \tfrac{2}{k_1+k_2+2} = -4\alpha = -4\beta \ , \\ c &= \tfrac{3(2k_1k_2+k_1+k_2)}{k_1+k_2+2} \ . \end{aligned} \tag{6.27}$$

Similarly, for $n \geq 3$, all equations (6.23), (6.24), (6.25) and (6.26) admit a family of solutions parametrized by $k_1 \in \mathbb{C} - \{-n/2\}$. For given k_1, the corresponding values of all other parameters are as follows

$$\begin{aligned}
k_2 &= -(2k_1 + \tfrac{1}{2}n + 2) \ , \\
\sigma_1 &= \tfrac{4k_1+n+4}{2k_1+n} \ , \quad \sigma_2 \ = \ -\tfrac{2k_1}{2k_1+n} \ , \\
\alpha &= \tfrac{1}{2k_1+n} \ = \ -\beta \ = \ \tfrac{1}{4}\gamma \ = \ -\tfrac{1}{4}\delta \ , \\
c &= 12k_1 + \tfrac{1}{2}(n^2 - 9n + 32) - \tfrac{(n-2)(n-3)(n-4)}{2(2k_1+n)} \ .
\end{aligned} \tag{6.28}$$

The following propositions follow immediately by the above results.

Proposition 6.2.6. *A) Let* $\mathfrak{g} = \mathrm{sp}_2 \oplus \mathrm{sp}_2$, $U = \mathbb{C}^2 \otimes \mathbb{C}^2$. *A complete solution of Problem 3.5.3 is as follows. The maps* $\varkappa, Q, K, P$ *are given by the equations (6.14), (6.15), (6.16) and (6.17),* k_1 *and* k_2 *are arbitrary parameters such that* $k_1 + k_2 \neq -2$, $\gamma = 0$, *and all other parameters are given by (6.27).*

B) For a given value of the ratio k_2/k_1 *we have a 1 parameter family of Lie* λ*–brackets of degree 2,* $L_\lambda^{(k_1,k_2)}$, *defined on the space* R *in (3.1) by Table 3.1.*

C) The corresponding Poisson Lie λ *bracket,* $L_\lambda^{PB,k_2/k_1}$, *defined in Theorem 5.3.5, coincides with the Poisson Lie* λ*–bracket described in Remark 6.1.3, with* $\alpha = k_2/k_1$.

Proposition 6.2.7. *A) Let* $\mathfrak{g} = \mathrm{sp}_n \oplus \mathrm{sp}_2$, $U = \mathbb{C}^n \otimes \mathbb{C}^2$, *for* $n \geq 4$. *A complete list of 7-ples* $(\mathfrak{g}, U, c, \varkappa, Q, K, P)$ *satisfying all assumptions of Problem 3.5.3 is given by (6.14), (6.15), (6.16) and (6.17), with arbitrary* $k_1 \neq -n/2$ *and all other parameters given by (6.28).*

B) The corresponding maps $L_\lambda^{(k_1)} : \ R \otimes R \to \mathbb{C}[\lambda]\mathcal{T}(R)$ *defined in Table 3.1 form a 1 parameter family of Lie* λ*–brackets of degree 2.*

C) The Poisson Lie λ *bracket of degree 2,* L_λ^{PB}, *defined in Theorem 5.3.5, coincides with the Poisson Lie* λ*–bracket described in Remark 6.1.3.*

6.3 Final classification

We can summarize all the results of Chapter 6 in the following

Theorem 6.3.1. *Let* $\mathfrak{g}$ *be a reductive Lie algebra and* U *a* $\mathfrak{g}$*–module, with non degenerate symmetric invariant bilinear forms* $\bar{\varkappa} : \ S^2\mathfrak{g} \to \mathbb{C}$ *and* $Q : \ S^2U \to \mathbb{C}$. *Denote by* R *the* $\mathbb{C}[T]$*–module in (3.1). The space* R *admits a Poisson Lie* λ*–bracket of degree 2,* L_λ^{PB}, *as in Table 5.1 (namely* $S(R)$ *is a Poisson vertex algebra), if and only if* R *admits a 1 parameter family of Lie* λ*–brackets of degree 2,* $L_\lambda^{(k)}$, *whose "quasi–classical limit" (described in Theorem 5.3.5) coincides with* L_λ^{PB}.

Corollary 6.3.2. *Let* V *be any vertex algebra strongly generated by a space* $R \subset V$ *as in (3.1), which admits quasi–classical limit, namely such that* $S(R)$ *has a compatible structure of Poisson vertex algebra. Suppose* $\mathfrak{g}$ *is a reductive Lie algebra,* U *is a finite dimensional* $\mathfrak{g}$*–module, and the bilinear forms* $\bar{\varkappa}, \bar{Q}$ *defined in Table 5.1 are non* *degenerate. Then $(\mathfrak{g}, U)$ is one of the examples listed in Table 6.2. Moreover, for each such pair $(\mathfrak{g}, U)$, the λ–bracket structure of V restricted to R is given in Table 3.1, with $c, \varkappa, Q, K$ and P defined in the corresponding Proposition 6.2.1, 6.2.2, 6.2.3, 6.2.5, 6.2.6 or 6.2.7.*

Remark 6.3.3. So far we were able to classify vertex algebras strongly generated by a space R as in (3.1) which admit quasi–classical limit. As we have seen, this is equivalent to classify all the spaces R which admit a 1 parameter family of Lie λ–brackets of degree 2, with arbitrarily large value of the Kac–Moody level k. It is natural to ask whether there are vertex algebras V, strongly generated by R, which do not admit quasi–classical limit, and for which the Kac–Moody level k is bounded. At the conjectural level, the answer to this question is yes. In fact, already for $\mathfrak{g}$ simple and U irreducible such vertex algebras appear, which violate the "non degeneracy" condition in Definition 3.2.4 (namely, for which PBW Theorem fails).

Appendix A

A.1 Proof of Lemma 2.2.4

Proof. of equation (2.19). By definition, the left hand side of (2.19) is

$$\mathrm{skl}(a, b \otimes c; \lambda) = L_\lambda(a, b \otimes c) + p(a,b)p(a,c)L_{-\lambda-T}(b \otimes c, a) \ . \tag{A.1}$$

By the left Wick formula (2.3), the first term in the right hand side of (A.1) is

$$N(L_\lambda(a,b),c) + p(a,b)b \otimes L_\lambda(a,c) + \int_0^\lambda d\mu \ L_\mu(L_\lambda(a,b),c) \ , \tag{A.2}$$

and, by the right Wick formula (2.4), the second term in the right hand side of (A.1) is

$$\begin{aligned} &p(a,b)p(a,c)b \otimes L_{-\lambda-T}(c,a) + p(a,b)p(a,c)p(b,c)c \otimes L_{-\lambda-T}(b,a) \\ &\quad + p(a,b)p(a,c)p(b,c) \textstyle\int_0^\mu d\nu \ L_\nu(c, L_{\mu-\nu}(b,a))\big|_{\mu=-\lambda-T} \ . \end{aligned} \tag{A.3}$$

By skew–symmetry of the λ–bracket L_λ, the sum of the second term of (A.2) and the first term of (A.3) is zero. Moreover, adding the first term of (A.2) and the second term of (A.3) we get

$$\mathrm{skn}(L_\lambda(a,b), c, 1) + \int_{-T}^0 d\mu \ L_\mu(L_\lambda(a,b),c) \ . \tag{A.4}$$

Finally, by summing the last term of (A.2), (A.3) and (A.4), we get, after a change of variable of integration

$$\int_{-T}^\lambda d\mu \ \mathrm{skl}(L_\lambda(a,b), c; \mu) \ .$$

Putting together the above results we thus get equation (2.19). □

Proof. of equation (2.21). The left hand side of (2.21) is ($b, c \in R_{\bar{0}}$)

$$\begin{aligned} \mathrm{skn}(a, b \otimes c, D) &= a \otimes N(b \otimes c, D) - N(b \otimes c, a \otimes D) \\ &\quad - N\Big(\Big(\textstyle\int_{-T}^0 d\lambda \ L_\lambda(a, b \otimes c)\Big), D\Big) \ . \end{aligned} \tag{A.5}$$

The first two terms in the right hand side of (A.5) are, by quasi–associativity (2.6), respectively

$$a \otimes b \otimes c \otimes D + a \otimes \Big(\int_0^T d\lambda \ b \Big) \otimes L_\lambda(c, D) + a \otimes \Big(\int_0^T d\lambda \ c \Big) \otimes L_\lambda(b, D) \ , \quad \text{(A.6)}$$

and

$$-b \otimes c \otimes a \otimes D - \Big(\int_0^T d\lambda \ b \Big) \otimes L_\lambda(c, a \otimes D) - \Big(\int_0^T d\lambda \ c \Big) \otimes L_\lambda(b, a \otimes D) \ . \quad \text{(A.7)}$$

We can then use the left Wick formula (2.3) to rewrite the second and third terms of (A.7) respectively as

$$\begin{aligned} -\Big(\textstyle\int_0^T d\lambda \ b \Big) \otimes L_\lambda(c,a) \otimes D - \Big(\textstyle\int_0^T d\lambda \ b \Big) \otimes a \otimes L_\lambda(c,D) \\ - \Big(\textstyle\int_0^T d\lambda \ b \Big) \otimes \textstyle\int_0^\lambda d\mu \ L_\mu(L_\lambda(c,a), D) \ , \end{aligned} \quad \text{(A.8)}$$

and

$$\begin{aligned} -\Big(\textstyle\int_0^T d\lambda \ c \Big) \otimes L_\lambda(b,a) \otimes D - \Big(\textstyle\int_0^T d\lambda \ c \Big) \otimes a \otimes L_\lambda(b,D) \\ - \Big(\textstyle\int_0^T d\lambda \ c \Big) \otimes \textstyle\int_0^\lambda d\mu \ L_\mu(L_\lambda(b,a), D) \ . \end{aligned} \quad \text{(A.9)}$$

The last term in the right hand side of (A.5) is, by left Wick formula (2.3)

$$\begin{aligned} -N(\Big(\textstyle\int_{-T}^0 d\lambda \ L_\lambda(a,b) \otimes c \Big), D) - N(\Big(\textstyle\int_{-T}^0 d\lambda \ b \otimes L_\lambda(a,c) \Big), D) \\ - \Big(\textstyle\int_{-T}^0 d\lambda \textstyle\int_0^\lambda d\mu \ L_\mu(L_\lambda(a,b), c) \Big) \otimes D \ . \end{aligned} \quad \text{(A.10)}$$

Notice that, since by assumption, $b, c \in R_{\bar{0}}$, every term in (A.10) is well defined. We can then use quasi–associativity (2.6) to rewrite the first two terms of (A.10) respectively as

$$\begin{aligned} -\Big(\textstyle\int_{-T}^0 d\lambda \ L_\lambda(a,b) \otimes c \Big) \otimes D - \Big(\textstyle\int_0^T d\mu \textstyle\int_{\mu-T}^0 d\lambda \ L_\lambda(a,b) \Big) \otimes L_\mu(c,D) \\ - \Big(\textstyle\int_0^T d\mu \textstyle\int_{\mu-T}^0 d\lambda \ c \Big) \otimes L_\mu(L_\lambda(a,b), D) \ , \end{aligned} \quad \text{(A.11)}$$

and

$$\begin{aligned} -\Big(\textstyle\int_{-T}^0 d\lambda \ b \otimes L_\lambda(a,c) \Big) \otimes D - \Big(\textstyle\int_0^T d\mu \textstyle\int_{\mu-T}^0 d\lambda \ b \Big) \otimes L_\mu(L_\lambda(a,c), D) \\ - \Big(\textstyle\int_0^T d\mu \textstyle\int_{\mu-T}^0 d\lambda \ L_\lambda(a,c) \Big) \otimes L_\mu(b,D) \ . \end{aligned} \quad \text{(A.12)}$$

The sum of the first term of (A.6) and (A.7) can be written as

$$\begin{aligned} \mathrm{skn}(a,b,c \otimes D) + \Big(\textstyle\int_{-T}^0 d\lambda \ L_\lambda(a,b) \Big) \otimes c \otimes D \\ + b \otimes \mathrm{skn}(a,c,D) + b \otimes \Big(\textstyle\int_{-T}^0 d\lambda \ L_\lambda(a,c) \Big) \otimes D \ . \end{aligned} \quad \text{(A.13)}$$

Using skew–commutativity of the λ–bracket, it is not hard to check that the sum of the second term of (A.6), the second term of (A.8) and the second term of (A.11) gives

$$\mathrm{skn}(a, \Big(\int_0^T d\lambda\ b\Big), L_\lambda(c, D)) \ ,$$

likewise, the sum of the third term of (A.6), the second term of (A.9) and the third term of (A.12) gives

$$\mathrm{skn}(a, \Big(\int_0^T d\lambda\ c\Big), L_\lambda(b, D)) \ .$$

By skewsymmetry of the λ–bracket it also follows that the sum of the first term of (A.8), the first term of (A.12) and the last term of (A.13) is zero. And with a similar computation, we can write the sum of the first term of (A.9), the first term of (A.11) and the second term of (A.13) as

$$\int_0^{T_c} d\lambda\ \mathrm{skn}(L_\lambda(b, a), c, D) + \Big(\int_{-T}^0 d\lambda \int_0^{\lambda+T} d\mu\ L_\lambda(L_\mu(b, a), c)\Big) \otimes D \ . \qquad \text{(A.14)}$$

Again, using skewsymmetry of the λ–bracket, it is not hard to check that the third term of (A.8) simplifies with the second term of (A.12), the third term of (A.9) simplifies with the third term of (A.11) and the third term of (A.10) simplifies with the second term of (A.14) This concludes the proof of equation (2.21). □

A.2 Proof of Lemma 2.2.6

Proof. of equation (2.22). The left hand side of (2.22) is, by definition

$$\begin{aligned} \mathrm{J}(a, b, c \otimes D; \lambda, \mu) \ &= \ L_\lambda(a, L_\mu(b, c \otimes D)) \ - \ p(a, b) L_\mu(b, L_\lambda(a, c \otimes D)) \\ &- \ L_{\lambda+\mu}(L_\lambda(a, b), c \otimes D) \ . \end{aligned} \qquad \text{(A.15)}$$

We can use (2.3) to rewrite the first term in the right hand side of (A.15) as

$$\begin{aligned} & L_\lambda(a, N(L_\mu(b, c), D)) \ + \ p(b, c) L_\lambda(a, c \otimes L_\mu(b, D)) \\ & \quad + \ \textstyle\int_0^\mu d\nu\ L_\lambda(a, L_\nu(L_\mu(b, c), D)) \ . \end{aligned} \qquad \text{(A.16)}$$

Similarly the second term in the right hand side of (A.15) is

$$\begin{aligned} & -p(a, b) L_\mu(b, N(L_\lambda(a, c), D)) \ - \ p(a, b) p(a, c) L_\mu(b, c \otimes L_\lambda(a, D)) \\ & \quad - \ p(a, b) \textstyle\int_0^\lambda d\nu\ L_\mu(b, L_\nu(L_\lambda(a, c), D)) \ . \end{aligned} \qquad \text{(A.17)}$$

Moreover the third term in the right hand side of (A.15) can be written as

$$\begin{aligned} & -\mathrm{lW}(L_\lambda(a,b),c,D;\lambda+\mu) - N(L_{\lambda+\mu}(L_\lambda(a,b),c),D) \\ & - p(a,c)p(b,c)c\otimes L_{\lambda+\mu}(L_\lambda(a,b),D) - \textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(L_{\lambda+\mu}(L_\lambda(a,b),c),D)\ . \end{aligned} \tag{A.18}$$

We can then rewrite the first term in (A.16) and (A.17) respectively as

$$\begin{aligned} & \mathrm{lW}(a,L_\mu(b,c),D;\lambda) + N(L_\lambda(a,L_\mu(b,c)),D) \\ & + p(a,b)p(a,c)N(L_\mu(b,c),L_\lambda(a,D)) + \textstyle\int_0^\lambda d\nu\ L_\nu(L_\lambda(a,L_\mu(b,c)),D)\ , \end{aligned} \tag{A.19}$$

and

$$\begin{aligned} & -p(a,b)\mathrm{lW}(b,L_\lambda(a,c),D;\mu) - p(a,b)N(L_\mu(b,L_\lambda(a,c)),D) \\ & - p(b,c)N(L_\lambda(a,c),L_\mu(b,D)) - p(a,b)\textstyle\int_0^\mu d\nu\ L_\nu(L_\mu(b,L_\lambda(a,c)),D)\ . \end{aligned} \tag{A.20}$$

By (2.3) we can also rewrite the second term in (A.16) and (A.17) respectively as

$$\begin{aligned} & p(b,c)N(L_\lambda(a,c),L_\mu(b,D)) + p(a,c)p(b,c)c\otimes L_\lambda(a,L_\mu(b,D)) \\ & + p(b,c)\textstyle\int_0^\lambda d\nu\ L_\nu(L_\lambda(a,c),L_\mu(b,D))\ , \end{aligned} \tag{A.21}$$

and

$$\begin{aligned} & -p(a,b)p(a,c)N(L_\mu(b,c),L_\lambda(a,D)) - p(a,b)p(a,c)p(b,c)c\otimes L_\mu(b,L_\lambda(a,D)) \\ & - p(a,b)p(a,c)\textstyle\int_0^\mu d\nu\ L_\nu(L_\mu(b,c),L_\lambda(a,D))\ . \end{aligned} \tag{A.22}$$

Notice that the third term of (A.20) simplifies with the first term of (A.21) and the third term of (A.19) simplifies with the first term of (A.22). If we put together the second term of (A.18), (A.19) and (A.20) we get

$$N(\mathrm{J}(a,b,c;\lambda,\mu),D)\ .$$

Moreover, if we add the third term of (A.18), the second term of (A.21) and the second term of (A.22) we get

$$p(a,b)p(a,c)c\otimes \mathrm{J}(a,b,D;\lambda,\mu)\ .$$

By Jacobi identity, the last term of (A.18) becomes

$$\begin{aligned} & -\textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(L_\lambda(a,L_\mu(b,c)),D) + p(a,b)\textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(L_\mu(b,L_\lambda(a,c)),D) \\ & + \textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(\mathrm{J}(a,b,c;\lambda,\mu),D)\ . \end{aligned} \tag{A.23}$$

If we put together the last term of (A.16), the last term of (A.19), the last term of

(A.22) and the first term of (A.23), we get

$$\int_0^\mu d\nu \ \mathrm{J}(a, L_\mu(b,c), D; \lambda, \nu) \ .$$

Similarly, if we put together the last term of (A.17), the last term of (A.21), the last term of (A.20) and the second term of (A.23), we get

$$-p(a,b) \int_0^\lambda d\nu \ \mathrm{J}(b, L_\lambda(a,c), D; \mu, \nu) \ .$$

Equation (2.22) follows immediately by the above results. □

Proof. of equation (2.23). The left hand side of (2.23) is $(b, c \in R_{\bar{0}})$

$$\begin{aligned} \mathrm{J}(a, b\otimes c, D; \lambda, \mu) &= L_\lambda(a, L_\mu(b\otimes c, D)) - L_\mu(b\otimes c, L_\lambda(a,D)) \\ &- L_{\lambda+\mu}(L_\lambda(a, b\otimes c), D) \ . \end{aligned} \tag{A.24}$$

By (2.4) the first term in the right hand side of (A.24) is

$$\begin{aligned} &L_\lambda(a, \left(e^{T\partial_\mu} b\right) \otimes L_\mu(c,D)) + L_\lambda(a, \left(e^{T\partial_\mu} c\right) \otimes L_\mu(b,D)) \\ &+ \textstyle\int_0^\mu d\nu \ L_\lambda(a, L_\nu(c, L_{\mu-\nu}(b,D))) \ , \end{aligned} \tag{A.25}$$

and the first and second terms of (A.25) can be rewritten, using (2.3) and sesquilinearity, respectively as

$$\begin{aligned} &\left(e^{(\lambda+T)\partial_\mu} L_\lambda(a,b)\right) \otimes L_\mu(c,D) + \left(e^{T\partial_\mu} b\right) \otimes L_\lambda(a, L_\mu(c,D)) \\ &+ \textstyle\int_0^\lambda d\nu \ L_\nu(L_\lambda(a,b), L_{\lambda+\mu-\nu}(c,D)) \ , \end{aligned} \tag{A.26}$$

and

$$\begin{aligned} &\left(e^{(\lambda+T)\partial_\mu} L_\lambda(a,c)\right) \otimes L_\mu(b,D) + \left(e^{T\partial_\mu} c\right) \otimes L_\lambda(a, L_\mu(b,D)) \\ &+ \textstyle\int_0^\lambda d\nu \ L_\nu(L_\lambda(a,c), L_{\lambda+\mu-\nu}(b,D)) \ . \end{aligned} \tag{A.27}$$

We can rewrite the second term in the right hand side of (A.24) by using the right commutative Wick formula (2.4)

$$\begin{aligned} &-\left(e^{T\partial_\mu} b\right) \otimes L_\mu(c, L_\lambda(a,D)) - \left(e^{T\partial_\mu} c\right) \otimes L_\mu(b, L_\lambda(a,D)) \\ &- \textstyle\int_0^\mu d\nu \ L_\nu(c, L_{\mu-\nu}(b, L_\lambda(a,D))) \ . \end{aligned} \tag{A.28}$$

Finally, the third term in the right hand side of (A.24) is, by the left Wick formula (2.3)

$$\begin{aligned} &-L_{\lambda+\mu}(L_\lambda(a,b)\otimes c, D) - L_{\lambda+\mu}(b\otimes L_\lambda(a,c), D) \\ &- \textstyle\int_0^\lambda d\nu \ L_{\lambda+\mu}(L_\nu(L_\lambda(a,b),c), D) \ . \end{aligned} \tag{A.29}$$

Notice that each term in (A.29) is well defined since, by assumption, $b, c \in R_{\bar{0}}$. Since $L_\lambda(a,b)$ and $L_\lambda(a,c)$ are in $R[\lambda]$, we can use the right Wick formula (2.4) to write the first and second term of (A.29) respectively as

$$\begin{aligned} -\Big(e^{T\partial_\mu} L_\lambda(a,b)\Big) \otimes L_{\lambda+\mu}(c,D) - \Big(e^{T\partial_\mu} c\Big) \otimes L_{\lambda+\mu}(L_\lambda(a,b),D) \\ - \textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(c, L_{\lambda+\mu-\nu}(L_\lambda(a,b),D)) \ , \end{aligned} \tag{A.30}$$

and

$$\begin{aligned} -\Big(e^{T\partial_\mu} b\Big) \otimes L_{\lambda+\mu}(L_\lambda(a,c),D) - \Big(e^{T\partial_\mu} L_\lambda(a,c)\Big) \otimes L_{\lambda+\mu}(b,D) \\ - \textstyle\int_0^{\lambda+\mu} d\nu\ L_\nu(L_\lambda(a,c), L_{\lambda+\mu-\nu}(b,D)) \ . \end{aligned} \tag{A.31}$$

Adding the second term of (A.26), the first term of (A.28) and the first term of (A.31) we get

$$\Big(e^{T\partial_\mu} b\Big) \otimes \mathrm{J}(a,c,D;\lambda,\mu) \ .$$

Similarly, the second term of (A.27), the second term of (A.28) and the second term of (A.30) give

$$\Big(e^{T\partial_\mu} c\Big) \otimes \mathrm{J}(a,b,D;\lambda,\mu) \ .$$

Moreover the first term of (A.26) simplifies with the first term of (A.30) and the first term of (A.27) simplifies with the second term of (A.31). Adding the last term of expressions (A.25), (A.27) and (A.31) we get

$$\int_0^\mu d\nu\ \mathrm{J}(a,c,L_{\mu-\nu}(b,D);\lambda,\nu) + \int_0^\mu d\nu\ L_\nu(c, L_\lambda(a, L_{\mu-\nu}(b,D))) \ . \tag{A.32}$$

Adding the last term of expressions (A.26), (A.29) and (A.30) we get

$$\int_0^\lambda d\nu\ \mathrm{J}(L_\lambda(a,b),c,D;\nu,\lambda+\mu-\nu) - \int_0^\mu d\nu\ L_\nu(c, L_{\lambda+\mu-\nu}(L_\lambda(a,b),D)) \ . \tag{A.33}$$

Finally, the second term of (A.32) and (A.33), together with the last term of (A.28), gives

$$\int_0^\mu d\nu\ L_\nu(c, \mathrm{J}(a,b,D;\lambda,\mu-\nu)) \ .$$

Putting together all the above results, we get equation (2.23). □

Proof. of equation (2.24). The left hand side of (2.24) is $(b, c \in R_{\bar{0}})$

$$\begin{aligned} \mathrm{lW}(a, b\otimes c, D;\lambda) = L_\lambda(a, N(b\otimes c, D)) - N(L_\lambda(a, b\otimes c), D) \\ - N(b\otimes c, L_\lambda(a,D)) - \textstyle\int_0^\lambda d\mu\ L_\mu(L_\lambda(a,b\otimes c),D) \ . \end{aligned} \tag{A.34}$$

Using quasi–associativity (2.6) we can rewrite the first term in the right hand side of (A.34) as

$$L_\lambda(a, b\otimes c\otimes D)+L_\lambda\Big(a, \Big(\int_0^T d\mu\, b\Big)\otimes L_\mu(c, D)\Big)+L_\lambda\Big(a, \Big(\int_0^T d\mu\, c\Big)\otimes L_\mu(b, D)\Big)\ . \quad \text{(A.35)}$$

Notice that, by assumption, $b, c \in R_{\bar{0}}$, and therefore $L_\lambda(a, b),\ L_\lambda(a, c) \in R[\lambda]$ and $L_\mu(L_\lambda(a, b), c) \in R[\lambda, \mu]$. We can thus apply twice the left Wick formula (2.3) to rewrite the first term of (A.35) as

$$\begin{aligned} L_\lambda(a, b) \otimes c \otimes D + b \otimes L_\lambda(a, c) \otimes D + b \otimes c \otimes L_\lambda(a, D) \\ + \textstyle\int_0^\lambda d\mu\ L_\mu(L_\lambda(a, b), c) \otimes D + \int_0^\lambda d\mu\ b \otimes L_\mu(L_\lambda(a, c), D) \\ + \textstyle\int_0^\lambda d\mu\ c \otimes L_\mu(L_\lambda(a, b), D) + \int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\mu(L_\lambda(a, b), c), D)\ . \end{aligned} \quad \text{(A.36)}$$

Moreover, by (2.3) the second and third terms of (A.35) are respectively

$$\begin{aligned} \Big(\textstyle\int_0^{\lambda+T} d\mu\ L_\lambda(a, b)\Big) \otimes L_\mu(c, D) + \Big(\int_0^T d\mu\ b\Big) \otimes L_\lambda(a, L_\mu(c, D)) \\ + \textstyle\int_0^\lambda d\mu \int_0^{\lambda-\mu} d\nu\ L_\mu(L_\lambda(a, b), L_\nu(c, D))\ , \end{aligned} \quad \text{(A.37)}$$

and

$$\begin{aligned} \Big(\textstyle\int_0^{\lambda+T} d\mu L_\lambda(a, c)\Big) \otimes L_\mu(b, D) + \Big(\int_0^T d\mu\ c\Big) \otimes L_\lambda(a, L_\mu(b, D)) \\ + \textstyle\int_0^\lambda d\mu \int_0^{\lambda-\mu} d\nu\ L_\mu(L_\lambda(a, c), L_\nu(b, D))\ . \end{aligned} \quad \text{(A.38)}$$

Let us now consider the second term in the right hand side of (A.34). By (2.3) it is equal to

$$-N(L_\lambda(a, b) \otimes c, D) - N(b \otimes L_\lambda(a, c), D) - \int_0^\lambda d\mu\ L_\mu(L_\lambda(a, b), c) \otimes D\ . \quad \text{(A.39)}$$

Notice that, since $b, c \in R_{\bar{0}}$, every term above is well defined. We can then use quasi–associativity (2.6) to rewrite the first and second term of (A.39) respectively as

$$\begin{aligned} -L_\lambda(a, b) \otimes c \otimes D - \Big(\textstyle\int_0^T d\mu\ L_\lambda(a, b)\Big) \otimes L_\mu(c, D) \\ - \Big(\textstyle\int_0^T d\mu\ c\Big) \otimes L_\mu(L_\lambda(a, b), D)\ , \end{aligned} \quad \text{(A.40)}$$

and

$$\begin{aligned} -b \otimes L_\lambda(a, c) \otimes D - \Big(\textstyle\int_0^T d\mu\ b\Big) \otimes L_\mu(L_\lambda(a, c), D) \\ - \Big(\textstyle\int_0^T d\mu\ L_\lambda(a, c)\Big) \otimes L_\mu(b, D)\ . \end{aligned} \quad \text{(A.41)}$$

For the third term in the right hand side of (A.34) we can just use quasi–associativity

(2.6) to get

$$-b \otimes c \otimes L_\lambda(a,D) - \Big(\textstyle\int_0^T d\mu\ b \Big) \otimes L_\mu(c, L_\lambda(a,D)) \\ - \Big(\textstyle\int_0^T d\mu\ c \Big) \otimes L_\mu(b, L_\lambda(a,D)) \ . \tag{A.42}$$

We are left to study the last term in the right hand side of (A.34). By (2.3) we can rewrite it as

$$- \textstyle\int_0^\lambda d\mu\ L_\mu(L_\lambda(a,b) \otimes c, D) - \int_0^\lambda d\mu\ L_\mu(b \otimes L_\lambda(a,c), D) \\ - \textstyle\int_0^\lambda d\mu \int_0^\lambda d\nu\ L_\nu(L_\mu(L_\lambda(a,b),c),D) \ . \tag{A.43}$$

We can then use the right Wick formula (2.4) to rewrite the first two terms of (A.43) respectively as

$$- \textstyle\int_0^\lambda d\mu\ \Big(e^{T\partial_\mu} L_\lambda(a,b) \Big) \otimes L_\mu(c,D) - \int_0^\lambda d\mu\ \Big(e^{T\partial_\mu} c \Big) \otimes L_\mu(L_\lambda(a,b),D) \\ - \textstyle\int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(c, L_{\mu-\nu}(L_\lambda(a,b),D)) \ , \tag{A.44}$$

and

$$- \textstyle\int_0^\lambda d\mu\ \Big(e^{T\partial_\mu} b \Big) \otimes L_\mu(L_\lambda(a,c),D) - \int_0^\lambda d\mu\ \Big(e^{T\partial_\mu} L_\lambda(a,c) \Big) \otimes L_\mu(b,D) \\ - \textstyle\int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\lambda(a,c), L_{\mu-\nu}(b,D)) \ . \tag{A.45}$$

The first three terms of (A.36) simplify with the first term of expressions (A.40), (A.41) and (A.42) respectively. Moreover, the fourth term of (A.36) simplifies with the third term of (A.39). Adding the fifth term of (A.36), the second term of (A.37), the second term of (A.41) and the first term of (A.45), we get

$$\Big(\int_0^T d\mu\ b \Big) \otimes \mathrm{J}(a,c,D;\lambda,\mu) \ ,$$

and similarly, adding the sixth term of (A.36), the second term of (A.38), the third term of (A.40), the second term of (A.42) and the second term of (A.44), we get

$$\Big(\int_0^T d\mu\ c \Big) \otimes \mathrm{J}(a,b,D;\lambda,\mu) \ .$$

Moreover, the sum of first term of (A.37), the second term of (A.40) and the first term of (A.44) is zero. Likewise, the sum of the first term of (A.38), the third term of (A.41) and the second term of (A.45) is zero. We finally notice that the last term of (A.38) simplifies with the last term of (A.45), and by adding the last term of expressions (A.36), (A.37), (A.43) and (A.44) we get

$$\int_0^\lambda d\mu \int_0^{\lambda-\mu} d\nu\ \mathrm{J}(L_\lambda(a,b),c,D;\mu,\nu) \ .$$

Putting together all the above results we thus get equation (2.24). □

Proof. of equation (2.25). The left hand side of (2.25) is defined as $(a, b \in R_{\bar{0}})$

$$\begin{aligned} \mathrm{lW}(a \otimes b, c, D; \lambda) = L_\lambda(a \otimes b, c \otimes D) - N(L_\lambda(a \otimes b, c), D) \\ - c \otimes L_\lambda(a \otimes b, D) - \textstyle\int_0^\lambda d\mu\ L_\mu(L_\lambda(a \otimes b, c), D)\ . \end{aligned} \tag{A.46}$$

The first term in the right hand side of (A.46) can be rewritten, by the right Wick formula (2.4), as

$$\begin{aligned} \left(e^{T\partial_\lambda} a\right) \otimes L_\lambda(b, c \otimes D) + \left(e^{T\partial_\lambda} b\right) \otimes L_\lambda(a, c \otimes D) \\ + \textstyle\int_0^\lambda d\mu\ L_\mu(b, L_{\lambda-\mu}(a, c \otimes D))\ . \end{aligned} \tag{A.47}$$

The first two terms of (A.47) are respectively equal, by (2.3), to

$$\begin{aligned} \left(e^{T\partial_\lambda} a\right) \otimes L_\lambda(b, c) \otimes D + \left(e^{T\partial_\lambda} a\right) \otimes c \otimes L_\lambda(b, D) \\ + \left(\textstyle\int_0^{\lambda+T} d\mu\ e^{T\partial_\lambda} a\right) \otimes L_\mu(L_\lambda(b, c), D)\ , \end{aligned} \tag{A.48}$$

and

$$\begin{aligned} \left(e^{T\partial_\lambda} b\right) \otimes L_\lambda(a, c) \otimes D + \left(e^{T\partial_\lambda} b\right) \otimes c \otimes L_\lambda(a, D) \\ + \left(\textstyle\int_0^{\lambda+T} d\mu\ e^{T\partial_\lambda} b\right) \otimes L_\mu(L_\lambda(a, c), D)\ , \end{aligned} \tag{A.49}$$

and the last term of (A.47) can be rewritten, by using twice the left Wick formula (2.3), as

$$\begin{aligned} \textstyle\int_0^\lambda d\mu\ L_\mu(b, L_{\lambda-\mu}(a, c)) \otimes D + \int_0^\lambda d\mu\ L_{\lambda-\mu}(a, c) \otimes L_\mu(b, D) \\ + \textstyle\int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\mu(b, L_{\lambda-\mu}(a, c)), D) + \int_0^\lambda d\mu\ L_\mu(b, c) \otimes L_{\lambda-\mu}(a, D) \\ + \textstyle\int_0^\lambda d\mu\ c \otimes L_\mu(b, L_{\lambda-\mu}(a, D)) + \int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\mu(b, c), L_{\lambda-\mu}(a, D)) \\ + \textstyle\int_0^\lambda d\mu \int_0^{\lambda-\mu} d\nu\ L_\mu(b, L_\nu(L_{\lambda-\mu}(a, c), D))\ . \end{aligned} \tag{A.50}$$

In order to compute the second term in the right hand side of (A.46) we need to use first the right Wick formula (2.4) and then quasi–associativity (2.6). The result is

$$\begin{aligned} -\left(e^{T\partial_\lambda} a\right) \otimes L_\lambda(b, c) \otimes D - \left(\textstyle\int_0^T d\mu\ e^{T\partial_\lambda} a\right) \otimes L_\mu(L_\lambda(b, c), D) \\ - \left(\textstyle\int_0^T d\mu\ L_{\lambda-\mu}(b, c)\right) \otimes L_\mu(a, D) - \left(e^{T\partial_\lambda} b\right) \otimes L_\lambda(a, c) \otimes D \\ -\left(\textstyle\int_0^T d\mu\ e^{T\partial_\lambda} b\right) \otimes L_\mu(L_\lambda(a, c), D) - \left(\int_0^T d\mu\ L_{\lambda-\mu}(a, c)\right) \otimes L_\mu(b, D) \\ - \textstyle\int_0^\lambda d\mu\ L_\mu(b, L_{\lambda-\mu}(a, c)) \otimes D\ . \end{aligned} \tag{A.51}$$

The third term in the right hand side of (A.46) can be easily rewritten by using the right Wick formula (2.4)

$$\begin{aligned} -c\otimes\Big(e^{T\partial_\lambda}a\Big)\otimes L_\lambda(b,D) - c\otimes\Big(e^{T\partial_\lambda}b\Big)\otimes L_\lambda(a,D) \\ -\textstyle\int_0^\lambda d\mu\ c\otimes L_\mu(b,L_{\lambda-\mu}(a,D))\ . \end{aligned} \tag{A.52}$$

Finally, to compute the last term in the right hand side of (A.46) we need to use twice the right Wick formula (2.4). The result is

$$\begin{aligned} &-\textstyle\int_0^\lambda d\mu\ \Big(\Big(e^{T(\partial_\lambda+\partial_\mu)}a\Big)\otimes L_\mu(L_\lambda(b,c),D)+\Big(e^{T\partial_\mu}L_{\lambda-\mu}(b,c)\Big)\otimes L_\mu(a,D)\Big) \\ &\qquad -\textstyle\int_0^\lambda d\mu\int_0^\mu d\nu\ L_\nu(L_{\lambda-\mu+\nu}(b,c),L_{\mu-\nu}(a,D)) \\ &-\textstyle\int_0^\lambda d\mu\ \Big(\Big(e^{T(\partial_\lambda+\partial_\mu)}b\Big)\otimes L_\mu(L_\lambda(a,c),D)+\Big(e^{T\partial_\mu}L_{\lambda-\mu}(a,c)\Big)\otimes L_\mu(b,D)\Big) \\ &\qquad -\textstyle\int_0^\lambda d\mu\int_0^\mu d\nu\ L_\nu(L_{\lambda-\mu+\nu}(a,c),L_{\mu-\nu}(b,D)) \\ &\qquad -\textstyle\int_0^\lambda d\mu\int_0^\lambda d\nu\ L_\nu(L_\mu(b,L_{\lambda-\mu}(a,c)),D)\ . \end{aligned} \tag{A.53}$$

The first term in (A.48) simplifies with the first term in (A.51), likewise the first term in (A.49) simplifies with the fourth term in (A.51). Moreover, adding the second term of (A.48) and the first term of (A.52) we get

$$\mathrm{skn}(\Big(e^{T\partial_\lambda}a\Big),c,L_\lambda(b,D))+\Big(\int_{-T}^0 d\mu\ L_\mu(a,c)\Big)\otimes L_{\lambda-\mu}(b,D)\ , \tag{A.54}$$

and similarly, adding the second term of (A.49) and the second term of (A.52) we get

$$\mathrm{skn}(\Big(e^{T\partial_\lambda}b\Big),c,L_\lambda(a,D))+\Big(\int_{-T}^0 d\mu\ L_\mu(b,c)\Big)\otimes L_{\lambda-\mu}(a,D)\ . \tag{A.55}$$

The sum of the last term of (A.48), the second term of (A.51) and the first term of (A.53) is zero, and similarly the sum of the last term of (A.49), the fifth term of (A.51) and the fourth term of (A.53) is zero. The first term of (A.50) simplifies with the last term of (A.51), and similarly the fifth term of (A.50) simplifies with the last term of (A.52). The sum of the second term of (A.50), the sixth term of (A.51), the fifth term of (A.53) and the second term of (A.54) is zero. Likewise, the sum of the fourth term of (A.50), the third term of (A.51), the second term of (A.53) and the second term of (A.55) is zero. The sixth term of (A.50) simplifies with the third term of (A.53). Finally, the sum of the third and last terms of (A.50) and the last two terms of (A.53) gives

$$\int_0^\lambda d\mu\int_0^{\lambda-\mu} d\nu\ \mathrm{J}(a,L_{\lambda-\mu}(a,c),D;\mu,\nu)\ .$$

This concludes the proof of equation (2.25). □

A.3 Proof of Equation (2.28)

By definition, the left hand side of equation (2.28) is

$$\begin{aligned} L_\lambda(a, \mathrm{skn}(b,c,D)) &= L_\lambda(a, b\otimes c\otimes D) - p(b,c)L_\lambda(a, c\otimes b\otimes D) \\ &\quad - L_\lambda(a, N(\Big(\textstyle\int_{-T}^0 d\mu\ L_\mu(b,c)\Big), D))\ . \end{aligned} \tag{A.56}$$

The first term in the right hand side of (A.56) is, by (2.3)

$$N(L_\lambda(a,b), c\otimes D) + p(a,b)b\otimes L_\lambda(a, c\otimes D) + \int_0^\lambda d\mu\ L_\mu(L_\lambda(a,b), c\otimes D)\ . \tag{A.57}$$

The second term of (A.57) is

$$\begin{aligned} &p(a,b)b\otimes N(L_\lambda(a,c), D) + p(a,b)p(a,c)b\otimes c\otimes L_\lambda(a,D) \\ &\quad + p(a,b)\textstyle\int_0^\lambda d\mu\ b\otimes L_\mu((L_\lambda(a,c), D)\ , \end{aligned} \tag{A.58}$$

and the last term of (A.57) can be written as

$$\begin{aligned} &\textstyle\int_0^\lambda d\mu\ \mathrm{lW}((L_\lambda(a,b), c, D;\mu) + \int_0^\lambda d\mu\ N(L_\mu(L_\lambda(a,b),c), D) \\ &\quad + p(a,c)p(b,c)\textstyle\int_0^\lambda d\mu\ c\otimes L_\mu(L_\lambda(a,b), D) \\ &\quad + \textstyle\int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\mu(L_\lambda(a,b),c), D)\ . \end{aligned} \tag{A.59}$$

Similarly, the second term in the right hand side of (A.56) gives

$$\begin{aligned} &-p(b,c)N(L_\lambda(a,c), b\otimes D) - p(a,c)p(b,c)c\otimes N(L_\lambda(a,b), D) \\ &\quad - p(a,b)p(a,c)p(b,c)c\otimes b\otimes L_\lambda(a,D) \\ &-p(a,c)p(b,c)\textstyle\int_0^\lambda d\mu\ c\otimes L_\mu(L_\lambda(a,b), D) - p(b,c)\int_0^\lambda d\mu \mathrm{lW}(L_\lambda(a,c), b, D;\mu) \\ &\quad - p(b,c)\textstyle\int_0^\lambda d\mu\ N(L_\mu(L_\lambda(a,c),b), D) - p(a,b)\int_0^\lambda d\mu\ b\otimes L_\mu(L_\lambda(a,c), D) \\ &\quad -p(b,c)\textstyle\int_0^\lambda d\mu \int_0^\mu d\nu\ L_\nu(L_\mu(L_\lambda(a,c),b), D)\ . \end{aligned} \tag{A.60}$$

The last term in the right hand side of (A.56) can be written as

$$\begin{aligned} &-\mathrm{lW}(a, \Big(\textstyle\int_{-T}^0 d\mu\ L_\mu(b,c)\Big), D;\lambda) - N(L_\lambda(a, \Big(\int_{-T}^0 d\mu\ L_\mu(b,c)\Big)), D) \\ &\quad - p(a,b)p(a,c)N(\Big(\textstyle\int_{-T}^0 d\mu\ L_\mu(b,c)\Big), L_\lambda(a,D)) \\ &\quad - \textstyle\int_0^\lambda d\mu\ L_\mu(L_\lambda(a, \Big(\int_{-T}^0 d\nu\ L_\nu(b,c)\Big)), D)\ . \end{aligned} \tag{A.61}$$

Notice that the last term of (A.58) simplifies with the seventh term of (A.60), and similarly the third term of (A.59) simplifies with the fourth term of (A.60). Moreover, adding the first term of (A.57) and the second term of (A.60), we get

$$-p(a,c)p(b,c)\mathrm{skn}(c,L_\lambda(a,b),D)-p(a,c)p(b,c)N(\Big(\int_{-T}^{0}d\mu\ L_\mu(c,L_\lambda(a,b))\Big),D)\ , \tag{A.62}$$

likewise, adding the first term of (A.58) and the first term of (A.60), we get

$$p(a,b)\mathrm{skn}(b,L_\lambda(a,c),D)+p(a,b)N(\Big(\int_{-T}^{0}d\mu\ L_\mu(b,L_\lambda(a,c))\Big),D)\ . \tag{A.63}$$

Similarly, adding the second term of (A.58), the third term of (A.60) and the third term of (A.61), we get

$$p(a,b)p(a,c)\mathrm{skn}(b,c,L_\lambda(a,D))\ .$$

The second term of (A.59) can be rewritten as

$$\int_0^\lambda d\mu\ N(\mathrm{skl}(L_\lambda(a,b),c;\mu),D)-p(a,c)p(b,c)N(\Big(\int_{-\lambda-T}^{-T}d\mu\ L_\mu(c,L_\lambda(a,b))\Big),D)\ , \tag{A.64}$$

likewise, the sixth term of (A.60) can be rewritten as

$$-p(b,c)\int_0^\lambda d\mu\ N(\mathrm{skl}(L_\lambda(a,c),b;\mu),D)+p(a,b)N(\Big(\int_{-\lambda-T}^{-T}d\mu\ L_\mu(b,L_\lambda(a,c))\Big),D)\ , \tag{A.65}$$

and the second term of (A.61) can be rewritten as

$$\begin{aligned}&-N(\Big(\textstyle\int_{-\lambda-T}^{0}d\mu\ \mathrm{skl}(a,L_\mu(b,c);\lambda)\Big),D)\\&+p(a,b)p(a,c)N(\Big(\textstyle\int_{-\lambda-T}^{0}d\mu\ L_{-\lambda-T}(L_\mu(b,c),a)\Big),D)\ .\end{aligned} \tag{A.66}$$

By summing all the last terms of (A.62), (A.63), (A.64), (A.65) and (A.66) we then get, after simple algebraic manipulations

$$-p(a,b)p(a,c)N(\Big(\int_{-\lambda-T}^{0}d\mu\ \mathrm{J}(b,c,a;\mu,-\lambda-\mu-T)\Big),D)\ .$$

After changing order of integration and using skewsymmetry of the λ–bracket, the last term of (A.59) becomes

$$\begin{aligned}&-p(a,b)\textstyle\int_0^\lambda d\nu\int_\nu^\lambda d\mu\ L_\nu(\mathrm{skl}(L_{\mu-\lambda}(b,a),c;\mu),D)\\&+p(a,b)p(a,c)p(b,c)\textstyle\int_0^\lambda d\nu\int_\nu^\lambda d\mu\ L_\nu(L_{\nu-\mu}(c,L_{\mu-\lambda}(b,a)),D)\ ,\end{aligned} \tag{A.67}$$

similarly the last term of (A.60) becomes

$$
\begin{aligned}
&p(a,c)p(b,c) \textstyle\int_0^\lambda d\nu \int_\nu^\lambda d\mu \; L_\nu(\mathrm{skl}(L_{\mu-\lambda}(c,a),b;\mu),D) \\
&- p(a,b)p(a,c) \textstyle\int_0^\lambda d\nu \int_\nu^\lambda d\mu \; L_\nu(L_{\nu-\mu}(b,L_{\mu-\lambda}(c,a)),D) \ ,
\end{aligned} \tag{A.68}
$$

and the last term of (A.61) becomes

$$
\begin{aligned}
&- \textstyle\int_0^\lambda d\nu \int_\nu^\lambda d\mu \; L_\nu(\mathrm{skl}(a,L_{\mu-\lambda}(b,c);\lambda),D) \\
&+ p(a,b)p(a,c) \textstyle\int_0^\lambda d\nu \int_\nu^\lambda d\mu \; L_\nu(L_{\nu-\lambda}(L_{\mu-\lambda}(b,c),a),D) \ .
\end{aligned} \tag{A.69}
$$

To conclude, we notice that the sum of all the second terms of (A.67), (A.68) and (A.69) is

$$
-p(a,b)p(a,c) \int_0^\lambda d\nu \int_\nu^\lambda d\mu \; L_\nu(\mathrm{J}(b,c,a;\nu-\mu,\mu-\lambda),D) \ .
$$

Putting together all the above results we get, as we wanted, equation (2.28).

□

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