DISCRETE APPROXIMATION OF STOCHASTIC MATHER MEASURES

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Abstract. In this paper, we construct measures which minimize a discrete version of the stochastic Mather problem associated to a Tonelli Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the flat d–dimensional torus. We show that the discrete variational problems approximate the stochastic Mather problem as the step of the discretisation goes to zero, in the sense that the minima of the discrete problems converge to the minimum of the stochastic Mather problem and the discrete minimizing measures converge to the unique stochastic Mather measure.

1. Introduction

For a given Tonelli Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, where $\mathbb{T}^d$ is the flat d–dimensional torus, the classical problem of Calculus of Variations amounts to minimize the action $\int_a^b L(\gamma, \dot\gamma)$ in a set of trajectories $\gamma : [a, b] \to \mathbb{T}^d$. These minimizers satisfy the Euler–Lagrange equations which are equivalent to Hamilton equations. Mather theory aims to study the dynamics of special solutions, those with global minimization properties. Weak KAM theory establishes a connection with viscosity solutions of Hamilton Jacobi equation. Solutions of the Hamilton Jacobi equation are given by the action of global minimizers.

When the trajectories considered in the minimization problem are no longer deterministic, the actions are defined by a Stochastic Optimal Control

$$u(x, t) = \inf_V \mathbb{E}_x \int_t^T L(X(s), V(s))\, ds + u(X(T), T)$$

with V varying in the set of bounded controls progressively measurable and

$$dX(s) = V(s)ds + \sqrt{2}dB_s, \quad X(t) = x$$

where B_s is a Brownian motion in $\mathbb{T}^d$. There is a feedback, u satisfies the viscous Hamilton–Jacobi equation

$$-u_t + H(x, Du) = \Delta u,$$

the optimal control is given by $H_p(x, Du)$ and so the corresponding process $X(s)$ possesses as probability distribution the solution m of the equation

$$m_t - \operatorname{div}(H_p(x, Du)m) = \Delta m.$$

It could be thought that the minimizing controls define the random perturbation

$$dP(s) = -H_x(X(s), P(s))ds, \quad dX(s) = H_p(X(s), P(s))ds + \sqrt{2}dB_s$$

of Hamilton equations.

The stationary viscous Hamilton–Jacobi corresponds to the ergodic problem

$$c_0 = \inf_V \limsup_{T\to\infty} \mathbb{E}_x \frac{1}{T}\int_0^T L(X(s), V(s))\, ds$$

with $V(s)$ as before. The probability distribution m of the stationary process is a solution of

$$\Delta m - \operatorname{div}(H_p(x, Du)m) = 0. \tag{1}$$

The stochastic Mather problem is the following variational problem

$$-\min_{\tilde\mu \in X} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde\mu, \tag{2}$$

where the minimization is performed over the set $X := \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^d)$ of *stochastic holonomic measures,* namely a family of probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ satisfying the relaxed version of (1)

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} (\Delta\varphi(x) + \langle D\varphi(x), q\rangle)\, d\tilde\mu(x, q) = 0 \qquad \text{for all } \varphi \in C^2(\mathbb{T}^d). \tag{3}$$

As proved in [11], there is a unique stochastic holonomic measure that solves the minimization problem (2), called *stochastic Mather measure*. Moreover, the value of the minimum in (2) thus obtained, hereafter denoted by α_0, turns out to be equal to the unique real constant c_0 for which the following viscous Hamilton–Jacobi equation

$$\Delta u + H(x, Du) = c_0 \qquad \text{in } \mathbb{T}^d \tag{4}$$

admits solutions in the viscosity sense, where $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian associated with L via the Fenchel transform. We refer the reader to [3] for definitions and basic results in viscosity solution theory.

In this paper we are interested in performing a discretization of the problems. The discretization depends on a parameter $\tau > 0$, which denotes the step of the interpolation.

The discrete counterpart of the continuous Stochastic Control problem is

$$\inf_V \mathbb{E}_x \sum_{i=0}^{N-1} \tau L(X_i, V_i) + u(X_N)$$

where the discrete process $V = (V_i)$ defines the dynamics

$$X_{i+1} = X_i + \tau V_i + \xi_i$$

with ξ_i gaussian random variables independient and identically distributed.

The stochastic Mather variational problem consists in solving the variational problem (2) by taking as X the set of *stochastic τ-holonomic measures* $\mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$, which is a family of probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ satisfying a sort of discretized version of (3), see Definition 2.4 for details. We prove that this variational problem admits minimizing measures, in general not unique. Furthermore, we show that the value of the minimum in (2) obtained in this way, that we will denote by α_τ, satisfies a property analogous to the one mentioned above for a discretized version of equation (4), see Theorem 3.1: it is equal to the unique real constant c_τ for which the following identity

$$u(x) = \mathcal{L}_\tau u(x) - \tau c_\tau \qquad \text{for all } x \in \mathbb{T}^d \tag{5}$$

holds true for some $u \in C(\mathbb{T}^d)$, where $\mathcal{L}_\tau : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$ is the discrete Lax–Oleinik operator, see Section 2.2.

Once the discrete problems are solved, a natural question arises: *can we recover the solution of the continuous model when the parameter τ goes to zero?* We give a positive answer to this question, at least under proper assumptions on the Lagrangian. Indeed, in Theorem 4.1 we show that the values of the minima α_τ of the discrete variational problems converge to the minimum α_0 of the stochastic Mather problem, and that the corresponding discrete minimizing measures converge to the unique stochastic Mather measure. As in the original Aubry–Mather theory, we can enhance the concepts and results by defining the rotation vector of a measure and by considering the varational problem restricted to stochastic τ-holonomic measures with a *fixed rotation vector*. We prove that this is equivalent to modify the Lagrangian in the variational problem, see Section 5. The problem of the convergence of the solutions of (5) to solutions of the viscous Hamilton–Jacobi equation (4) has been instead addressed in [9].

Discrete versions of Aubry and Mather theory have been already proposed and studied in literature, also in connection with different asymptotic problems, see for instance [4, 5, 8, 12, 15–17]. In particular, our approximation result is analogous to previous convergence results established in [4, 12, 15]. The main difference with respect to all the quoted references is that the models therein consideblue come from discretizations of "classical" Aubry–Mather theory, developed for first order Hamilton–Jacobi equation, see [7, 10]. The novelty of our work consists in proposing a discretization of the stochastic Aubry–Mather theory, as developed by Gomes in [11] in connection with viscous Hamilton–Jacobi equations.

2. Preliminaries

2.1. **Stochastic Mather measures.** In this section we will recall the notion of stochastic Mather measure together with its main properties, as introduced and studied by Gomes in [11]. Let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2 Lagrangian, convex and superlinear in the velocity variable q, and $\mathbb{Z}^d$–periodic in the space variable x, that is to say, L can be thought as a Tonelli Lagrangian defined on $\mathbb{T}^d \times \mathbb{R}^d$. We associate to L the Hamiltonian H defined via the *Fenchel transform* as follows:

$$H(x,p) := \sup_{q\in\mathbb{R}^N} \{\langle p, q\rangle - L(x,q)\}, \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

The Hamiltonian H is of class C^2 and satisfies properties analogous to the ones fulfilled by L.

In the sequel, we will denote by $C(\mathbb{T}^d)$ the space of continuous functions on $\mathbb{T}^d$, or, equivalently, the family of $\mathbb{Z}^d$–periodic functions on $\mathbb{R}^d$. A similar remark applies to $C^k(\mathbb{T}^d)$ with $k \in \mathbb{N}$.

For any topological space X, we denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. Let C^0_ℓ be the set of continuous functions $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ having linear growth, i.e.

$$\|f\|_\ell := \sup_{(x,q)\in\mathbb{T}^d\times\mathbb{R}^d} \frac{|f(x,q)|}{1 + \|q\|} < +\infty.$$

We also denote by $\mathcal{P}_\ell$, the set of measures in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ such that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} \|q\| \; d\mu < +\infty.$$

The set of measures $\mathcal{P}_\ell$ is naturally embedded in $(C^0_\ell)'$, the dual of C^0_ℓ, and its topology coincides with that induced by the weak* topology on $(C^0_\ell)'$. Therefore, we will say that μ_n is converging to μ in $\mathcal{P}_\ell$, and we will write $\mu_n \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{n\to\infty} \int_{\mathbb{T}^d\times\mathbb{R}^d} f\, d\mu_n = \int_{\mathbb{T}^d\times\mathbb{R}^d} f\, d\mu \qquad \text{for all } f \in C^0_\ell.$$

The weak* topology is metrizable on $\mathcal{P}_\ell$. Indeed, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions with compact support on C^0_ℓ which is dense on C^0_ℓ in the topology of uniform convergence on compact sets of $\mathbb{T}^d \times \mathbb{R}^d$. The metric d on $\mathcal{P}_\ell$, defined as follows

$$d(\mu_1,\mu_2) = \Big| \int_{\mathbb{T}^d\times\mathbb{R}^d} |q|\, d\mu_1 - \int_{\mathbb{T}^d\times\mathbb{R}^d} |q|\, d\mu_2 \Big| + \sum_{n\geq 1} \frac{1}{2^n \|f_n\|_\infty} \Big| \int_{\mathbb{T}^d\times\mathbb{R}^d} f_n\, d\mu_1 - \int_{\mathbb{T}^d\times\mathbb{R}^d} f_n\, d\mu_2 \Big|,$$

provides the topology of $\mathcal{P}_\ell$ (see for instance Mañé [14]).

Remark 2.1. Let $\kappa \in \mathbb{R}$. It was proved in [14] that the set

$$A(\kappa) := \Big\{ \tilde{\mu} \in \mathcal{P}_\ell : \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu} \leq \kappa \Big\}$$

is compact in $\mathcal{P}_\ell$.

Following [11], we define stochastic holonomic measures as follows.

Definition 2.2. A measure $\tilde{\mu} \in \mathcal{P}_\ell$ satisfying

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} (\Delta\varphi(x) + \langle D\varphi(x), q\rangle)\, d\tilde{\mu}(x,q) = 0 \qquad \text{for all } \varphi \in C^2(\mathbb{T}^d) \tag{6}$$

is called *stochastic holonomic* measure. The subset of stochastic holonomic measures in $\mathcal{P}_\ell$ will be denoted by $\mathcal{C}(\mathbb{T}^d \times \mathbb{R}^d)$.

There is a very simple stochastic holonomic measure $\tilde{\nu}$ defined by

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f d\tilde{\nu} := \int_{\mathbb{T}^d} f(x,0)\, dx \qquad \text{for all } f \in C^0_\ell. \tag{7}$$

More generally, let $V : \mathbb{T}^d \to \mathbb{R}^d$ be a C^1–vector field and let μ be the unique element of $\mathcal{P}(\mathbb{T}^d)$ solving the following Fokker–Planck equation in the distributional sense:

$$\Delta\mu - \operatorname{div}(V(x)\mu) = 0 \quad \text{in } \mathbb{T}^d. \tag{8}$$

Then $\tilde{\mu} := G_{V\#}\mu$ is a stochastic holonomic measure, where $G_{V\#}\mu$ denotes the push–forward of the measure μ via the map $G_V : \mathbb{T}^d \ni x \mapsto (x, V(x)) \in \mathbb{T}^d \times \mathbb{R}^d$.

The stochastic Mather problem is the following variational problem:

$$-\alpha_0 := \min_{\tilde{\mu}\in\mathcal{C}(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}. \tag{9}$$

The stochastic holonomic measures solving (9) are called *stochastic Mather measures.*

We have the following result:

Theorem 2.3. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be the C^2 Tonelli Hamiltonian associated by duality with L. Suppose there exist constants $a_1, a_2 > 0$ such that $|\partial_x H| \leq a_1 H + a_2$ in $\mathbb{T}^d \times \mathbb{R}^d$.*

(i) There is a unique $c_0 \in \mathbb{R}$ such that

$$\Delta u + H(x, Du) = c_0 \quad \text{in } \mathbb{T}^d \tag{10}$$

has viscosity solutions. Furthermore, $c_0 = \alpha_0$.

(ii) If u is a continuous viscosity solution of (10), then u is of class $C^{2,\alpha}$, in particular u is a classical solution to (10).

(iii) If u, v are classical solutions of (10), then $u - v$ is constant.

(iv) The measure $\tilde{\mu}$ is a solution to (9) if and only if $\tilde{\mu} := G_{V\#}\mu$ where $\mu \in \mathcal{P}(\mathbb{T}^d)$ is the solution of (8) with $V(x) := \partial_p H(x, Du(x))$ and u is any classical solution of (10)

We refer to [3] for the notion of viscosity (sub-, super-) solution.

Proof. Items (i), (iii) and (iv) are proved in [11]. Since we could not find an explicit reference for item (ii), we sketch a proof in the appendix, see Lemma A.1. In this regard, we remark that u is a viscosity solution to (10) if and only if $-u$ is a viscosity solution of $-\Delta v + H(x, -Dv) = c_0$ in $\mathbb{T}^d$. $\square$

Thus there is a unique $\tilde{\mu}_0 \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^d)$ that solves (9). In other words, there is only one stochastic Mather measure.

2.2. **Discretization.** For $\tau > 0$, the source of randomness comes from the following stochastic kernel η^τ on $\mathbb{T}^d$, which is defined below

$$\eta^\tau : \mathbb{T}^d \to \mathcal{P}(\mathbb{R}^d)$$
$$y \mapsto \eta^\tau_y,$$

where η^τ_y is given as follows:

$$\eta^\tau_y(A) := \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \int_A \mathrm{e}^{-\frac{|z-y|^2}{4\tau}}\, dz \quad \text{for all } A \in \mathscr{B}(\mathbb{R}^d).$$

Given $u \in C(\mathbb{T}^d)$, we observe

$$\int_{\mathbb{T}^d} u(z)\, d\eta^\tau_y(z) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(z)\ \mathrm{e}^{-\frac{|z-y|^2}{4\tau}}\, dz = (\eta_\tau * u)(y),$$

where $\eta_\tau(y) := (4\pi\tau)^{-\frac{d}{2}}\, \mathrm{e}^{-\frac{|y|^2}{4\tau}}$.

Next, we introduce the notion of stochastic τ-holonomic measure. It was inspiblue by the definition of τ-holonomic measure given in [15] and can be regarded as a discretization of the stochastic holonomic measure.

Definition 2.4. A measure $\mu \in \mathcal{P}_\ell$ satisfying

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \Big(\varphi(x) - (\eta_\tau * \varphi)(x + \tau q)\Big)\, d\tilde{\mu}(x, q) = 0 \qquad \text{for every } \varphi \in C(\mathbb{T}^d)$$

is called *stochastic τ-holonomic* measure. The subset of stochastic τ-holonomic measures in $\mathcal{P}_\ell$ will be denoted by $\mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$.

We note that the measure defined in (7) is also stochastic τ-holonomic.

The discrete analogous of the minimization problem (9) is the following

$$-\alpha_\tau := \min_{\tilde{\mu} \in \mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d)} \int_{\mathbb{T}^d \times \mathbb{R}^d} L\, d\tilde{\mu}. \tag{11}$$

We will show in the sequel that (11) admits minimizing measures, that we shall call τ–minimizing in the sequel, see Theorem 3.1. Furthermore, under the assumption that the Lagrangian $L(x,q)$ has at least quadratic growth in q, we will show that such τ–minimizing measures converge, as $\tau \to 0^+$, to the unique solution of the stochastic Mather problem (9), see Theorem 4.1.

Remark 2.5. Note that

$$\min_{\mathbb{T}^d\times\mathbb{R}^d} L \leq -\alpha_\tau \leq \max_{x\in\mathbb{T}^d} L(x,0),$$

where the second inequality is obtained by using the measure given in (7),

The discrete analogue of (10) is given as follows: define the discrete Lax-Oleinik operator $\mathcal{L}_\tau : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$ by

$$\mathcal{L}_\tau u(x) := \max_{q\in\mathbb{R}^d} \left(-\tau L(x,q) + (\eta_\tau * u)(x+\tau q)\right) \qquad \text{for every } x \in \mathbb{T}^d.$$

The following holds.

Theorem 2.6 ([9])**.** *There exists a unique $c_\tau \in \mathbb{R}$ such that*

$$u(z) = \mathcal{L}_\tau u(z) - \tau c_\tau \qquad z \in \mathbb{T}^d, \tag{12}$$

has a solution $u \in C(\mathbb{T}^d)$. Furthermore, the solution is unique up to additive constants.

3. Existence of τ–minimizing measures

In this section we will show the existence of minimizing measures for the variational problem (11) and we will prove that the minimum is equal to $-c_\tau$, in analogy with the continuous case. The precise statement is the following:

Theorem 3.1. *There exists a stochastic τ-holonomic measure $\tilde{\mu}_\tau$ such that*

$$-c_\tau = \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}_\tau = \min_{\tilde{\mu}\in\mathcal{C}_\tau(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}. \tag{13}$$

In particular, $\alpha_\tau = c_\tau$.

We will denote by $\mathcal{M}_\tau(L)$ the subset of stochastic τ-holonomic measures in $\mathcal{P}_\ell$ that realize the minimum in (13).

We start with some preliminary material. For a Borel measurable and bounded vector field $V : \mathbb{T}^d \to \mathbb{R}^d$ and a measure $\nu \in \mathcal{P}(\mathbb{T}^d)$ consider the Markov process $\{\xi_i : i \geq 0\}$ with initial distribution ν and transition kernels

$$\mathbb{P}_\nu^V(\xi_{i+1} \in B | \xi_0, \ldots, \xi_i) = \eta^\tau_{\xi_i+\tau V(\xi_i)}(B). \tag{14}$$

Here, the $\xi_i : \Omega \to \mathbb{R}^d$ are random variables defined on a measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a σ–algebra.

Lemma 3.2. *Let $V : \mathbb{T}^d \to \mathbb{R}^d$ be a bounded, Borel measurable vector field, and $\nu \in \mathcal{P}(\mathbb{T}^d)$. For every $f \in B_b(\mathbb{R}^d)$ we have*

$$\mathbb{E}_\nu^V\left[f(\xi_{i+1})\right] = \mathbb{E}_\nu^V\left[(\eta_\tau * f)(\xi_i + \tau V(\xi_i))\right] \qquad \text{for every } i \in \mathbb{N}.$$

Proof. Indeed, we have

$$\mathbb{E}_\nu^V\left[f(\xi_{i+1})\right] = \mathbb{E}_\nu^V\left[\mathbb{E}_\nu^V\left[f(\xi_{i+1}) \mid \xi_i\right]\right] = \mathbb{E}_\nu^V\left[\int_{\mathbb{T}^d} f(x)\, d\eta^\tau_{\xi_i+\tau V(\xi_i)}(x)\right]$$
$$= \mathbb{E}_\nu^V\left[\left(\eta_\tau * f\right)\left(\xi_i + \tau V(\xi_i)\right)\right].$$

□

We will also need the following lemma.

Lemma 3.3. *For each $\phi \in C(\mathbb{T}^d)$, let*

$$\mathbb{V}_\phi(x) := \{q \in \mathbb{R}^d : \mathcal{L}_\tau\phi(x) = (\eta_\tau * \phi)(x + \tau q) - \tau L(x, q)\} \qquad \text{for all } x \in \mathbb{T}^d.$$

Then there exists a bounded and measurable vector field $V_\phi : \mathbb{T}^d \to \mathbb{R}^d$ such that $V_\phi(x) \in \mathbb{V}_\phi(x)$ for every $x \in \mathbb{T}^d$.

Proof. From the continuity of ϕ and the growth conditions of L, it follows from [9] that the sets $\mathbb{V}_\phi(x)$, $x \in \mathbb{T}^d$, are nonempty, uniformly bounded and compact in $\mathbb{R}^d$. Moreover, the multi–function $x \mapsto \mathbb{V}_\phi(x)$ is upper–semicontinuous with respect to set inclusion, so we can apply Theorem III.8 in [6] to infer the existence of a bounded measurable map $V_\phi : \mathbb{T}^d \to \mathbb{R}^d$ such that $V_\phi(x) \in \mathbb{V}_\phi(x)$ for every $x \in \mathbb{T}^d$. □

The following will be a key tool to prove existence of τ–minimizing measures.

Theorem 3.4. *Let $\phi_\tau \in C(\mathbb{T}^d)$ be a solution of (12) and set $V = V_{\phi_\tau}$. For $\nu \in \mathcal{P}(\mathbb{T}^d)$ let $\{\xi_i : i \geq 0\}$ be the Markov process with initial distribution ν and transition kernels given by (14). Then, for any solution $u \in C(\mathbb{T}^d)$ of (12) and $n \in \mathbb{N}$, we have*

$$\mathbb{E}_\nu^V\left[\mathcal{L}_\tau^n u\left(\xi_0\right)\right] = \mathbb{E}_\nu^V\left[u\left(\xi_n\right) - \tau \sum_{i=0}^{n-1} L\left(\xi_i, V(\xi_i)\right)\right], \tag{15}$$

where $\mathbb{E}_\nu^V$ denotes the expectation with respect to the probability measure $\mathbb{P}_\nu^V$.

Proof. Let $u \in C(\mathbb{T}^d)$ be a solution of (12), then using that $u - \phi_\tau$ is constant and that the Lax operator $\mathcal{L}_\tau$ commutes with additive constants, we get

$$u(x) + \tau c_\tau = \mathcal{L}_\tau u(x) = (\eta_\tau * u)(x + \tau V(x)) - \tau L(x, V(x)) \quad \text{for all } x \in \mathbb{R}^d. \tag{16}$$

We will prove (15) by induction on $n \in \mathbb{N}$. Let us first show (15) for $n = 1$. By taking into account (16), we get, for every solution u of (12) and $\omega \in \Omega$,

$$\mathcal{L}_\tau u\left(\xi_0(\omega)\right) = \left(\eta_\tau * u\right)\left(\xi_0(\omega) + \tau V\left(\xi_0(\omega)\right)\right) - \tau L\left(\xi_0(\omega), V\left(\xi_0(\omega)\right)\right).$$

By integrating with respect to $\mathbb{P}_\nu^V$ and by recalling Lemma 3.2, we get

$$\mathbb{E}_\nu^V\left[\mathcal{L}_\tau u\left(\xi_0\right)\right] = \mathbb{E}_\nu^V\left[u(\xi_1) - \tau L\left(\xi_0, V\left(\xi_0\right)\right)\right],$$

which implies the first step of the induction. Let us now assume that (15) holds for n and for every solution u of (12). We first apply the induction hypothesis to $u := \mathcal{L}_\tau\phi_\tau$ and infer

$$\mathbb{E}_\nu^V\left[\left(\mathcal{L}_\tau^{n+1}\phi_\tau\right)\left(\xi_0\right)\right] = \mathbb{E}_\nu^V\left[\mathcal{L}_\tau\phi_\tau(\xi_n) - \tau \sum_{i=0}^{n-1} L\left(\xi_i, V(\xi_i)\right)\right]. \tag{17}$$

In view of (16), for every $\omega \in \Omega$, we get

$$(\mathcal{L}_\tau\phi_\tau)(\xi_n(\omega)) = (\eta_\tau * \phi_\tau)(\xi_n(\omega) + \tau V(\xi_n(\omega))) - \tau L(\xi_n(\omega), V(\xi_n(\omega))).$$

By integrating with respect to $\mathbb{P}_\nu^V$ and by recalling Lemma 3.2, we get

$$\begin{aligned}\mathbb{E}_\nu^V\left[\mathcal{L}_\tau\phi_\tau(\xi_n)\right] &= \mathbb{E}_\nu^V\left[(\eta_\tau*\phi_\tau)(\xi_n+\tau V(\xi_n))-\tau L(\xi_n,V(\xi_n))\right]\\ &= \mathbb{E}_\nu^V\left[\phi_\tau(\xi_{n+1}^V)-\tau L(\xi_n,V(\xi_n))\right].\end{aligned}$$

By using this relation in (17), we get (15) with $n+1$ instead of n and $u:=\phi_\tau$, and hence for every u solution of (12) since the Lax operator commutes with additive constants. □

We now construct stochastic τ-holonomic measures. Let $\{\xi_i\,:\,i\geq 0\}$ be the Markov process associated to a Borel measurable bounded vector field $V:\mathbb{T}^d\to\mathbb{R}^d$ and a measure $\nu\in\mathcal{P}(\mathbb{T}^d)$ via (14). For each $n\in\mathbb{N}$, let us define the measure $\tilde\mu_n\in\mathcal{P}_\ell$ by setting

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f(x,q)\,\mathrm{d}\tilde\mu_n(x,q):=\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}_\nu^V\left[f(\xi_i^V,V(\xi_i^V))\right]\qquad\text{for all } f\in C_\ell^0. \tag{18}$$

Since V is bounded, it is easily seen that the measures $\{\tilde\mu_n\,:\,n\in\mathbb{N}\}$ have equi–compact support, in particular they are relatively compact in $\mathcal{P}_\ell$.

We are interested in accumulation points of these measures.

Proposition 3.5. Let $\{\tilde\mu_n\,:\,n\in\mathbb{N}\}$ be the sequence of measures given by (18) and let $\tilde\mu\in\mathcal{P}_\ell$ be an accumulation point. The following holds:

(i) the measure $\tilde\mu$ is stochastic τ-holonomic;

(ii) if V is continuous, then $\tilde\mu=G_{V\#}\mu$, where $G_V\,:\,\mathbb{T}^d\ni x\mapsto(x,V(x))\in\mathbb{T}^d\times\mathbb{R}^d$ and $\mu:=\pi_{1\#}\tilde\mu$ is the push–forward of the measure $\tilde\mu$ via the standard projection map $\pi_1:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{T}^d$;

(iii) if $V=V_{\phi_\tau}$ with $\phi_\tau\in C(\mathbb{T}^d)$ solution of (12), then

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} L\,d\tilde\mu=-c_\tau.$$

Proof. Let us denote by $\left(\tilde\mu_{n_k}\right)_k$ a subsequence such that $\tilde\mu_{n_k}\stackrel{*}{\rightharpoonup}\tilde\mu$ in $\mathcal{P}_\ell$.

(i) Pick $f\in C(\mathbb{T}^d)$ and define $Tf\in C_\ell^0$ by

$$Tf(x,q)=(\eta_\tau*f)(x+\tau q)-f(x).$$

By an iterative application of Lemma 3.2 the sequence

$$M_n(f)=f(\xi_n)-\sum_{i=0}^{n-1}Tf(\xi_i,V(\xi_i)),\quad M_0(f)=f(\xi_0),$$

satisfies

$$\mathbb{E}_\nu^V\left[M_n(f)\right]=\mathbb{E}_\nu^V\left[M_0(f)\right],$$

i.e.

$$\mathbb{E}_\nu^V\left[f(\xi_n)\right]-n\int_{\mathbb{T}^d\times\mathbb{R}^d}Tf\,d\tilde\mu_n=\int_{\mathbb{T}^d}f\,d\nu. \tag{19}$$

By dividing equality (19) by n_k an letting $k\to\infty$, we get

$$\int_{\mathbb{T}^d\times\mathbb{R}^d}Tf(x,q)\,d\tilde\mu(x,q)=0,$$

i.e. $\tilde\mu$ is stochastic τ-holonomic .

(ii) We aim to show that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} g\,d\tilde\mu = \int_{\mathbb{T}^d} g(x,V(x))\,d\mu, \qquad \text{for all } g\in C^0_\ell.$$

Indeed, letting $\mu_n := \pi_{1\#}\tilde\mu_n$, for every $f\in C(\mathbb{T}^d)$ we have

$$\int_{\mathbb{T}^d} f\,d\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}^V_\nu\left[f(\xi_i)\right],$$

and then, for all $g\in C^0_\ell$,

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d} g\,d\tilde\mu &= \lim_{k\to+\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\mathbb{E}^V_\nu\left[g(\xi_i,V(\xi_i))\right]\\ &= \lim_{k\to+\infty}\int_{\mathbb{T}^d} g(x,V(x))\,d\mu_{n_k} = \int_{\mathbb{T}^d} g(x,V(x))\,d\mu(x).\end{aligned}$$

(iii) Being $\phi_\tau\in C(\mathbb{T}^d)$ a solution to (12), we have, for every $n\in\mathbb{N}$,

$$\mathcal{L}^n_\tau\phi_\tau = \phi_\tau + n\tau c_\tau \qquad \text{in} \quad \mathbb{T}^d.$$

According to (15) and to the definition of $\tilde\mu_n$, we infer

$$c_\tau = \frac{\mathbb{E}^V_\nu\left[\phi_\tau(\xi_n)-\phi_\tau(\xi_0)\right]}{n\tau} - \int_{\mathbb{T}^d\times\mathbb{R}^d} L(x,q)\,\mathrm{d}\tilde\mu_n.$$

By picking $n=n_k$ and by passing to the limit as $k\to+\infty$ we get (iii). □

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\phi_\tau\in C(\mathbb{T}^d)$ be a solution to (12). Then

$$\phi_\tau(x) = \mathcal{L}_\tau\phi_\tau(x) - \tau c_\tau \ge (\eta_\tau * \phi_\tau)(x+\tau q) - \tau L(x,q) - \tau c_\tau$$

for all $(x,q)\in\mathbb{T}^d\times\mathbb{R}^d$. By integrating this inequality with respect to any $\tilde\mu\in\mathcal{C}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ and by using

$$\int_{\mathbb{T}^d\times\mathbb{R}^d}\phi_\tau\,d\tilde\mu = \int_{\mathbb{T}^d\times\mathbb{R}^d}(\eta_\tau*\phi_\tau)(x+\tau q)\,d\tilde\mu,$$

we obtain

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} L\,d\tilde\mu \ge -c_\tau$$

The assertion follows in view of Proposition 3.5-(iii). □

4. From the discrete to the continuous model

In this section we discuss the asymptotics of stochastic τ-holonomic measures. The main result of this section is the following.

Theorem 4.1. *Assume that there are $a_1,a_2>0$ such that*

$$|\partial_x H(x,p)| \le a_1 H(x,p) + a_2, \quad H(x,p)\le a_1|p|^2 + a_2 \qquad \textit{for any } (x,p)\in\mathbb{T}^d\times\mathbb{R}^d.$$

The following holds:

(i) $\lim_{\tau\to0^+}\alpha_\tau = \alpha_0$;

(ii) let $\tilde\mu_\tau\in\mathcal{M}_\tau(L)$ for every $\tau>0$. Then $\tilde\mu_\tau$ converges in $\mathcal{P}_\ell$ to the unique stochastic Mather measure $\tilde\mu_0$.

We show that the limit probability measure obtained from a sequence of stochastic τ_n-holonomic measures $\tilde{\mu}_{\tau_n}$ with $\tau_n \to 0$ is stochastic holonomic if either the measures $\tilde{\mu}_{\tau_n}$ have equi-compact supports or they are τ_n-minimizing and the Langragian has at least quadratic growth in q. More precisely, we have the following result.

Proposition 4.2. *Let $\tilde{\mu} \in \mathcal{P}_\ell$ be a measure obtained as a limit of a sequence of measures $\tilde{\mu}_{\tau_n} \in \mathcal{C}_{\tau_n}(\mathbb{T}^d \times \mathbb{R}^d)$ for some sequence $\{\tau_n : n \in \mathbb{N}\}$ converging to 0, such that either one of the following assumptions holds:*

(i) there is $\rho > 0$ such that

$$\operatorname{spt}(\tilde{\mu}_{\tau_n}) \subseteq \mathbb{T}^d \times B_\rho(0) \text{ for every } n \in \mathbb{N}, \tag{20}$$

(ii) there exist $c_1, c_2 > 0$ such that $|q|^2 \leq c_1 L(x,q) + c_2$ and $\tilde{\mu}_{\tau_n} \in \mathcal{M}_{\tau_n}(L)$.

Then $\tilde{\mu}$ is stochastic holonomic.

Proof. We observe that for $\varphi \in C(\mathbb{T}^d)$

$$(\eta_\tau * \varphi)(y) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} e^{-\frac{|z-y+k|^2}{4\tau}} \varphi(z)\, dz = \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} e^{-\tau|2\pi k|^2 + 2\pi i k \cdot z} \varphi(z)\, dz.$$

By approximation in C^2–norm, it will be enough to prove the condition (6) for $\varphi \in C^{4+s}(\mathbb{T}^d)$ with $2s > d$. For such a φ consider its Fourier expansion

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} \varphi_k e^{2\pi i k \cdot x}, \qquad x \in \mathbb{T}^d,$$

so that

$$(\eta_\tau * \varphi)(y) = \sum_{k \in \mathbb{Z}^d} \varphi_k e^{-\tau|2\pi k|^2 + 2\pi i k \cdot y},$$

and then

$$\begin{aligned} &\frac{(\eta_\tau * \varphi)(x + \tau q) - \varphi(x)}{\tau} - \langle D\varphi(x), q \rangle - \Delta\varphi(x) \\ &\qquad = \frac{1}{\tau} \sum_{k \in \mathbb{Z}^d} \varphi_k (e^{-\tau|2\pi k|^2 + 2\pi i \tau k \cdot q} - 1 + \tau|2\pi k|^2 - 2\pi i \tau k \cdot q) e^{2\pi i k \cdot x}. \end{aligned} \tag{21}$$

On the other hand, by straightforward computations, it is easy to show that

$$|e^z - 1 - z| \leq \frac{|z|^2}{2} \qquad \text{for } \Re z \leq 0.$$

Thus

$$|e^{-\tau|2\pi k|^2 + 2\pi i \tau k \cdot q} - 1 + \tau|2\pi k|^2 - 2\pi i \tau k \cdot q| \leq \frac{\tau^2}{2} ||2\pi k|^2 + 2\pi i k \cdot q|^2 \leq \frac{\tau^2}{2}(|2\pi k|^4 + |k|^2|2\pi q|^2),$$

and so

$$\begin{aligned} \Big|\sum_{k \in \mathbb{Z}^d} \varphi_k (e^{-\tau|2\pi k|^2 + 2\pi i \tau k \cdot q} - 1 + \tau|2\pi k|^2 - 2\pi i \tau k \cdot q) e^{2\pi i k \cdot x}\Big| &\leq \frac{\tau^2}{2} \sum_{k \in \mathbb{Z}^d} |\varphi_k|(|2\pi k|^4 + |k|^2|2\pi q|^2) \\ &\leq \frac{\tau^2}{2}\Big(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{-s}\Big)^{\frac{1}{2}} (|2\pi|^4 |\varphi|_{H^{4+s}} + |\varphi|_{H^{2+s}} |2\pi q|^2). \end{aligned}$$

where $|\ |_{H^r}$ is the Sobolev norm

$$|\varphi|^2_{H^r} = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^r |\varphi_k|^2.$$

Integrating (21) with respect to $\tilde\mu_\tau \in \mathcal{C}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ we deduce

$$\left| \int_{\mathbb{T}^d\times\mathbb{R}^d} \langle D\varphi(x), q\rangle + \Delta\varphi(x) \quad d\tilde\mu_\tau(x,q) \right| \le C\tau \left(1 + \int_{\mathbb{T}^d\times\mathbb{R}^d} |q|^2 \, d\tilde\mu_\tau(x,q) \right). \tag{22}$$

Under any of our assumptions the sequence

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} |q|^2 \, d\tilde\mu_{\tau_n}(x,q) < \infty, \qquad n\in\mathbb{N},$$

is uniformly bounded. Indeed, under assumption (i), this follows at once from condition (20), while under assumption (ii) this holds because for $\tilde\mu_\tau \in \mathcal{M}_\tau(L)$,

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} L(x,q)\, d\tilde\mu_\tau(x,q) = -\alpha_\tau \in \left[\min_{\mathbb{T}^d\times\mathbb{R}^d} L, \max_{y\in\mathbb{T}^d} L(y,0) \right].$$

Hence by taking $\tau\to 0$ in (22), we obtain

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} \langle D\varphi(x), q\rangle + \Delta\varphi(x) \;\; d\tilde\mu(x,q) = 0,$$

as requiblue. The proof is now complete. □

Now, we are ready to prove the main result of this section.

Proof of Theorem 4.1. In view of Remark 2.5, we know that any sequence $(\alpha_{\tau_n})_n$ with $\tau_n \to 0^+$ admits a converging subsequence. Let us denote by

$$A := \left\{ \alpha\in\mathbb{R} \,:\, \alpha = \lim_{n\to+\infty} \alpha_{\tau_n} \quad \text{for some } \tau_n\to 0^+ \right\},$$

the set of accumulation points of the α_τ as $\tau\to 0^+$. We want to show that $A = \{\alpha_0\}$. To this aim, we pick $\alpha\in A$ and let $\{\tau_n \,:\, n\in\mathbb{N}\}$ be an infinitesimal sequence such that $\alpha_{\tau_n} \to \alpha$ as $n\to+\infty$.

We first show that $\alpha \ge \alpha_0$. Let $\tilde\mu_0 \in \mathcal{P}(\mathbb{T}^d\times\mathbb{R}^d)$ be the unique stochastic holonomic measure solving

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde\mu_0 = -\alpha_0.$$

According to Theorem 2.3, $\tilde\mu_0 = G_{V\#}\mu_0$ where $\mu_0\in\mathcal{P}(\mathbb{T}^d)$ is the solution of the Fokker–Planck equation (8) with $V(x) := \partial_p H(x, Du(x))$ and u any solution of (10). For every $n\in\mathbb{N}$, let us denote by $\tilde\mu^V_{\tau_n}$ a measure in $\mathcal{C}_{\tau_n}(\mathbb{T}^d\times\mathbb{R}^d)$ defined according to the construction provided by Proposition 3.5 with the vector field $V(x) = \partial_p H(x, Du(x))$. We know that $\tilde\mu^V_{\tau_n} = G_{V\#}\mu^V_{\tau_n}$. Up to subsequences, we may assume that $\mu^V_{\tau_n} \overset{*}{\rightharpoonup} \mu^V$ in $\mathcal{P}(\mathbb{T}^d)$, therefore $\tilde\mu^V_{\tau_n}$ converges to $\tilde\mu^V := G_{V\#}\mu^V$ in $\mathcal{P}_\ell$. By Proposition 4.2–(i), we know that $\tilde\mu^V$ is stochastic holonomic. Otherwise stated, μ^V is the solution of the Fokker–Planck equation (8) with $V(x) = \partial_p H(x, Du(x))$. By uniqueness, we deduce that $\mu^V = \mu_0$ and hence $\tilde\mu^V = \tilde\mu_0$. Since the measures $\{\mu^V_{\tau_n} \,:\, n\in\mathbb{N}\}$ have equi–compact support, we finally get

$$-\alpha_0 = \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde\mu_0 = \lim_{n\to+\infty} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde\mu^V_{\tau_n} \ge \lim_{n\to+\infty} -\alpha_{\tau_n} = -\alpha,$$

as it was to be shown.

Next, we show that $\alpha \leq \alpha_0$. Let $\tilde{\mu}_{\tau_n} \in \mathcal{M}_{\tau_n}(L)$, then

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}_{\tau_n} = -\alpha_{\tau_n} \leq \max_{x \in \mathbb{T}^d} L(x, 0) =: \kappa.$$

By Remark 2.1, up to subsequences we can assume that $\tilde{\mu}_{\tau_n}$ converges to $\tilde{\mu}$ in $\mathcal{P}_\ell$. In view of Proposition 4.2–(ii), we deduce that $\tilde{\mu}$ is stochastic holonomic. We get

$$-\alpha = \lim_{n \to +\infty} -\alpha_{\tau_n} = \lim_{n \to +\infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}_{\tau_n} \geq \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} \geq -\alpha_0 \tag{23}$$

according to Theorem 2.3. This finally shows that $\alpha = \alpha_0$ and that the stochastic holonomic measure $\tilde{\mu}$ in (23) minimizes (9). Hence $\tilde{\mu} = \tilde{\mu}_0$, by uniqueness of the stochastic Mather measure. The last assertion in the statement of the Theorem follows from this fact. □

5. Rotation vectors

Given a measure $\tilde{\mu}$ in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, we define its rotation vector, $\rho(\tilde{\mu})$ as

$$\rho(\tilde{\mu}) = \int_{\mathbb{T}^d \times \mathbb{R}^d} q \, d\tilde{\mu}(x, q).$$

Given $h \in \mathbb{R}^d$ we can plug the constant vector field $V(x) = h$ in the construction of Proposition 3.5 to obtain a stochastic τ-holonomic measure with rotation vector h.

Indeed, if $\{\xi_i : i \geq 0\}$ is the Markov process given in (14) and $\{\tilde{\mu}_n : n \in \mathbb{N}\}$ are the measures defined in the proof Proposition 3.5–(ii), we have

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} q \, d\tilde{\mu}_n = \frac{1}{n} \sum_{i=0}^{n} \mathbb{E}_\nu^V(V(\xi_i)) = h.$$

Hence, it follows that the rotation vector of the limit $\tilde{\mu}$ is also h.

We define $\beta_\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$\beta_\tau(h) := \inf \left\{ \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} : \tilde{\mu} \in \mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d), \rho(\tilde{\mu}) = h \right\}. \tag{24}$$

The above infimum is actually a minimum. In fact, we have

Theorem 5.1. *There exists $\tilde{\mu} \in \mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ with $\rho(\tilde{\mu}) = h$ such that*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} = \beta_\tau(h). \tag{25}$$

Proof. Let $\{\tilde{\mu}_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{C}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ with $\rho(\tilde{\mu}_n) = h$ for every $n \in \mathbb{N}$ such that

$$\beta_\tau(h) = \lim_{n \to \infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}_n.$$

Hence, there exists $\kappa \in \mathbb{R}$ such that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}_n \leq \kappa \qquad \text{for all } n \in \mathbb{N}.$$

By Remark 2.1, there exists a subsequence $\{\tilde{\mu}_{n_k} : k \geq 1\}$ converging to $\tilde{\mu}$ in $\mathcal{P}_\ell$. Since L is continuous and it is bounded from below, we have

$$\beta_\tau(h) = \lim_{k\to\infty} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}_{n_k} \geq \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}. \tag{26}$$

The opposite inequality comes by the very definition of $\beta_\tau(h)$ since $\tilde{\mu}$ is stochastic τ-holonomic and satisfies $\rho(\tilde{\mu}) = h$, as it is easily seen. □

The function β_τ is convex, indeed, if $h_1, h_2 \in \mathbb{R}^d$ consider a convex combination $h = \lambda h_1 + (1-\lambda)h_2$. Let $\tilde{\mu}_1, \tilde{\mu}_2$ be minimizing measures with rotation vectors h_1 and h_2, and let $\tilde{\mu}$ be the convex combination $\lambda\tilde{\mu}_1 + (1-\lambda)\tilde{\mu}_2$. Then $\tilde{\mu}$ is a stochastic τ-holonomic probability measure with rotation vector h and by definition

$$\begin{aligned}\beta_\tau(h) \leq \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu} &= \lambda \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}_1 + (1-\lambda)\int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu}_2 \\ &= \lambda\beta_\tau(h_1) + (1-\lambda)\beta_\tau(h_2).\end{aligned}$$

The convex dual $\alpha_\tau : \mathbb{R}^d \to \mathbb{R}$ of the function β_τ is our original problem for a modified Lagrangian. For a vector w consider the one form ω defined by $\omega(q) = w\cdot q$. By definition of the convex dual

$$\begin{aligned}\alpha_\tau(w) &= \sup_h (w\cdot h - \beta_\tau(h)) \\ &= -\inf_h \min_{\rho(\tilde{\mu})=h} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu} - w\cdot h \\ &= -\min_{\tilde{\mu}\in\mathcal{C}_\tau(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} L\, d\tilde{\mu} - w\cdot \int_{\mathbb{T}^d\times\mathbb{R}^d} q\, d\tilde{\mu}(x,q) \\ &= -\min_{\tilde{\mu}\in\mathcal{C}_\tau(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} (L-\omega)\, d\tilde{\mu}.\end{aligned}$$

In [13] we defined the stochastic Mather function $\alpha_0 : \mathbb{R}^d \to \mathbb{R}$ by

$$\alpha_0(w) = -\min_{\tilde{\mu}\in\mathcal{C}(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} (L-\omega)\, d\tilde{\mu}.$$

When the Lagrangian L has at least quadratic growth in q, we infer from Theorem 4.1 that

$$\lim_{\tau\to 0} \alpha_\tau(w) = \alpha_0(w) \qquad \text{for any} \quad w \in \mathbb{R}^d.$$

Appendix A

Lemma A.1. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^1 Tonelli Hamiltonian and assume that $|\partial_x H| \leq a_1 H + a_2$ in $\mathbb{T}^d \times \mathbb{R}^d$ for some constants $a_1, a_2 > 0$. Then any continuous viscosity solution of $-\Delta u + H(x, Du(x)) = c_0$ in $\mathbb{T}^d$ is of class $C^{2,\alpha}$.*

Proof. Define $F(x,p,M) = -\operatorname{Tr} M + H(x,p) - c_0$ so that the viscous HJ equation is $F(x, Du, D^2u) = 0$. We first show that the solution u is actually Lipschitz continuous on $\mathbb{T}^d$. We want to apply [2, Theorem 1]. Let us check that condition (H1) in [2, Theorem 1] is satisfied, namely that there exist a constant $\alpha > 0$ such that

$$\partial_x H \cdot p + g_0 \operatorname{Tr} M^2 \geq \alpha$$

in a neighborhood $W(L)$ of the set $\{(x,p,M) : |p| \geq L, Mp = 0, F = 0\}$ for a sufficiently large $L > 0$, where $g_0 = \sqrt{2}(2-\sqrt{2})(\sqrt{2}+1)^{-1}$. Indeed, in the set $F = 0$ we have $d \operatorname{Tr} M^2 \geq (\operatorname{Tr} M)^2 = (H(x,p) - c_0)^2$ and so

$$\partial_x H \cdot p + g_0 \operatorname{Tr} M^2 \geq -|p|(a_1 H + a_2) + \frac{g_0}{d}(H - c_0)^2 = |p|^2 \left(-\frac{a_1 H + a_2}{|p|} + \frac{g_0 (H - c_0)^2}{d|p|^2} \right).$$

The superlinearity of H implies that there is $L > 0$ such that

$$\partial_x H \cdot p + g_0 \operatorname{Tr} M^2 > 1$$

when $F(x,p,M) = 0$ and $|p| \geq L$. From [2, Theorem 1] we get that u is Lipschitz continuous.

As u is Lipschitz, $H(x, Du)$ is at least essentially bounded. Now, plug this into the viscous Hamilton–Jacobi equation to yield that u is in $W^{2,p}$ for any $p > 1$, and in particular, it means that u is $C^{1,\alpha}$. Since u is $C^{1,\alpha}$ and H is C^1, we have that $H(x, Du)$ is in $C^{0,\alpha}$, and therefore, by Schauder's estimates, u is in $C^{2,\alpha}$. □

Proposition A.2. The following conditions are equivalent:

(i) $H(x,p) \leq b_1 |p|^2 + b_2$ for some constants $b_1, b_2 > 0$.

(ii) $|q|^2 \leq c_1 L(x,q) + c_2$ for some constants $c_1, c_2 > 0$.

Proof. Suppose (i) holds. Let $x \in \mathbb{T}^d$, $q \in \mathbb{R}^d$, for $|p| \leq \frac{|q|}{2b_1}$ we have

$$p \cdot q - H(x,p) \geq p \cdot q - \frac{|q|^2}{4b_1} - b_2,$$

thus

$$\begin{aligned} L(x,q) &\geq \max\left\{ p \cdot q - H(x,p) : |p| \leq \frac{|q|}{2b_1} \right\} \geq \max\left\{ p \cdot q - \frac{|q|^2}{4b_1} - b_2 : |p| \leq \frac{|q|}{2b_1} \right\} \\ &= \frac{|q|^2}{4b_1} - b_2 \end{aligned}$$

Suppose (ii) holds. Let $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$, then

$$H(x,p) = \max_q p \cdot q - L(x,q) \leq \max_q p \cdot q - \frac{|q|^2}{c_1} + \frac{c_2}{c_1} = \frac{c_1 p^2}{4} + \frac{c_2}{c_1}.$$

□

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