GLOBAL EXISTENCE FOR SOME SLIGHTLY SUPER-LINEAR PARABOLIC EQUATIONS WITH MEASURE DATA.

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Abstract. In this work we study the global existence of a solution to some parabolic problems whose model is

$$(1)\qquad \begin{cases} u_t - \Delta u &= g(u) + \mu\,, & (x,t) \in \Omega \times (0,\infty)\,;\\ u(x,t) &= 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,;\\ u(x,0) &= u_0(x)\,, & x \in \Omega\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $u_0 \in L^1(\Omega)$, μ is a finite Radon measure in $\Omega \times (0,\infty)$ and g is a real continuous function, *slightly superlinear at infinity* ("slightly" in the sense that $1/g$ is not integrable at ∞). One of the main tools is a new logarithmic Sobolev inequality.

We also prove some uniqueness results.

1. Introduction

In the present paper we deal with a class of parabolic problems whose basic model is the following.

$$(2)\qquad \begin{cases} u_t - \Delta u &= g(u) + \mu\,, & (x,t) \in Q = \Omega \times (0,\infty)\,;\\ u(x,t) &= 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,;\\ u(x,0) &= u_0(x)\,, & x \in \Omega\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $u_0 \in L^1(\Omega)$, μ is a finite Radon measure in $\Omega \times (0,\infty)$ (for the sake of simplicity, we may assume in this introduction that u_0 and μ are nonnegative) and $g : \mathbb{R} \to [0,\infty)$ is a slightly super-linear, even and continuous function. Parabolic problems with measure data have been studied, for instance, in [9], [8], [7], [23], [24], [1] and references therein.

Our model problem appears as the transformed by the Cole-Hopf change of unknown (see, for instance, [19]) of problem

$$(3)\qquad \begin{cases} w_t - \Delta w &= \beta(w)|\nabla w|^2 + 1\,, & (x,t) \in Q\,;\\ w(x,t) &= 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,;\\ w(x,0) &= w_0(x)\,, & x \in \Omega\,, \end{cases}$$

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where β is a positive, increasing, continuous function. Indeed by applying the change of unknown

$$u = \Psi(w) = \int_0^w \exp\Big(\int_0^s \beta(r)\,dr\Big)\,ds\,,$$

problem (3) transforms (formally) into problem (2) with $\mu = 0$, where

$$g(u) = \exp\Big(\int_0^w \beta(r)\,dr\Big)\,.$$

It can be checked that the function g verifies

$$(G)\quad \begin{cases} (g1) & g : [0,\infty) \to [0,\infty), \qquad \text{increasing, convex and } g(0) = 1, \\ (g2) & \displaystyle\int_0^\infty \frac{ds}{g(s)} = \infty\,, \\ (g3) & \displaystyle\lim_{s\to\infty} \frac{g(s)}{s} = \infty. \end{cases}$$

In [1] the Cole-Hopf change of unknown is studied in detail, and it is proved that if w is not regular enough, then a singular measure μ may occur in (2). Conversely, assuming μ is a positive, singular Radon measure (here "singular" means that it is concentrated on a set of zero parabolic capacity, see [1]), and that u is a solution of problem (2), then by performing the inverse change of variable $w = \Psi^{-1}(u)$ the measure "disappears", and w is a solution of problem (3).

The general relation between g and β can be seen, for instance, in [15]. Just to fix ideas, some examples about the behaviour of $g(s)$ at infinity are in order:

$$\begin{aligned} &\text{if} \quad \beta(s) = s^\lambda\,, && \text{then } g(s) \sim s\,(\log s)^{\frac{\lambda}{\lambda+1}}\,; \\ &\text{if} \quad \beta(s) = e^s\,, && \text{then } g(s) \sim s\,\log s\,; \\ &\text{if} \quad \beta(s) = e^{e^s}\,, && \text{then } g(s) \sim s\,(\log s)\,(\log\log s)\,. \end{aligned}$$

It is straightforward that in all cases, we might write $g(s)$ in the form

$$g(s) = 1 + s\,A(\log s)\,,$$

for large s. Indeed, our global existence result will be obtained under the following mild assumptions on A:

$$(H)\quad \begin{cases} (h1) & A \quad \text{is increasing}\,, \\ (h2) & \displaystyle\int_0^\infty \frac{ds}{A(s)} = \infty\,, \\ (h3) & \displaystyle\lim_{s\to\infty} A(s) = \infty\,, \\ (h4) & A \text{ satisfies the } \Delta_2\text{–condition: } A(2s) \le K A(s) \\ & \text{for all large } s \text{ and for some } K > 0\,. \end{cases}$$

Note that no convexity/concavity assumptions are assumed on g.

One of the main difficulties one encounters in looking for some a priori estimates for problem (2) is that solutions to parabolic problems with measure data are unbounded in general, so it would be difficult to find a priori estimates by means of supersolutions. We have to rely on some different method, for instance using test functions.

Assuming for a moment that $\mu = 0$ and that the initial datum is positive, after multiplying problem

$$u_t - \Delta u = u\, A(\log^* u)$$

by u and integrating on Ω, one obtains

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx = \int_\Omega u^2\,A(\log^* u)\,dx\,.$$

To estimate the last integral, we need an inequality such as

$$\int_\Omega u^2 A(\log^* u)\,dx \le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + F\Big(\int_\Omega u^2\,dx\Big)\,,$$

for a suitable function F which does not grow too much: for instance, $F(s) = s\,A(\log^* s)$ would be fine, because the ordinary equation $y' = y\,A(\log^* y)$ has a global solution on $[0,\infty)$, so one could use a nonlinear version of Gronwall's lemma (see for instance [20]) to conclude. In other words, one needs a Sobolev inequality of logarithmic type (see [21], [2], [12], [15], [11]). Actually, the presence of the measure term worsens the situation, because in this case it is not possible to take u as a test, but only bounded functions of u are allowed (see, for instance, [9]). Taking one of these test functions, one obtains an equation similar to the preceding one, but with different powers of u. Therefore, an inequality such as

$$\int_\Omega u^q\,A(\log^* u)\,dx \le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + F\Big(\int_\Omega u^q\,dx\Big)$$

is necessary, with $2 < q < 2^*$. Since such a kind of inequalities is not available to us, we have to begin by proving a generalized logarithmic Sobolev's inequality. By applying such an inequality, we are able to prove a priori estimates (Proposition 1) and a global existence result (Theorem 1) for problem (2).

Under some stronger assumptions, we are also able to prove a uniqueness result (Theorem 3).

The rest of this paper is divided in three sections. In the next one we will give some notation and the precise assumptions for our problems, and we will state the main results and the logarithmic Sobolev inequality. We will prove the logarithmic inequality in Section 3. Finally, in Section 4 the proofs of the existence and uniqueness results are given.

2. Global existence of the Cauchy-Dirichlet problem

Let Ω be a bounded, open set in $\mathbb{R}^N$, $N \ge 1$. For $T > 0$, we write $Q_T = \Omega \times (0,T)$, and for $r, q \in [1,\infty]$, the symbols $L^q(\Omega)$, $L^r(0,T;L^q(\Omega))$, and so forth, denote the usual Lebesgue spaces, see for instance [19]. We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space of measurable functions having weak derivative in $L^q(\Omega)$ and zero trace on $\partial\Omega$. If $T > 0$, the spaces $L^r(0,T;L^q(\Omega))$ and $L^r(0,T;W_0^{1,q}(\Omega))$ have obvious meanings, see again [19].

Moreover, we will denote by q' Hölder's conjugate exponent of $q > 1$, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$. Finally, if $1 \le q < N$, we will denote by $q^* = Nq/(N-q)$ its Sobolev conjugate exponent.

For the sake of brevity, instead of writing "$u(x,t) \in L^r(0,\tau;W_0^{1,q}(\Omega))$ for every $\tau > 0$", we shall write $u(x,t) \in L^r_{\mathrm{loc}}([0,\infty);W_0^{1,q}(\Omega))$. Similarly, we shall write $u \in L^q_{\mathrm{loc}}(\overline{Q})$ instead of $u \in L^q(Q_\tau)$ for every $\tau > 0$.

Finally, throughout this paper, we will use the usual truncation at levels $\pm k$,

$$T_k\, s = \max\{-k, \min\{k, s\}\}\,.$$

We will consider the following parabolic problem

$$\left\{\begin{array}{rcll} u_t - \operatorname{div} a(x,t,u,\nabla u) &=& b(x,t,u) + \mu, & (x,t) \in Q = \Omega \times (0,\infty) \\ u(x,t) &=& 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) &=& u_0(x), & x \in \Omega, \end{array}\right. \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $u_0 \in L^1(\Omega)$, μ is a finite Radon measure in Q and the functions a and b verify the following hypotheses.

- $a(x,t,s,\xi) \,:\, \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, i.e., it is continuous with respect to (s,ξ) for a.e. $(x,t) \in Q$, and measurable with respect to (x,t) for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, such that there exist positive constants Λ_1, Λ_2 satisfying:

$$|a(x,t,s,\xi)| \le \Lambda_1 |\xi|\,, \tag{5}$$

$$a(x,t,s,\xi) \cdot \xi \ge \Lambda_2 |\xi|^2\,, \tag{6}$$

$$\big(a(x,t,s,\xi) - a(x,t,s,\eta)\big) \cdot (\xi - \eta) > 0\,, \tag{7}$$

  for a.e. $(x,t) \in Q$, for every $s \in \mathbb{R}$ and every $\xi, \eta \in \mathbb{R}^N$, with $\xi \ne \eta$.
- $b(x,t,s) \,:\, \Omega \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., it is continuous with respect to s for a.e. $(x,t) \in Q$, and measurable with respect to (x,t) for every $s \in \mathbb{R}$, such that there exists a positive constant Λ_3 satisfying:

$$|b(x,t,s)| \le \Lambda_3 \Big(1 + |s|\, A(\log^* |s|)\Big)\,, \tag{8}$$

  for a.e. $(x,t) \in Q$ and for every $s \in \mathbb{R}$, where

$$\log^* s = \max\{1, \log s\}$$

  The function $A(s) \,:\, [0,+\infty) \to [0,+\infty)$ which appears in (8) satisfies the following hypotheses:
- $A(s)$ is increasing and continuous, and $\lim\limits_{s\to+\infty} A(s) = +\infty$.
- A satisfies the so-called Δ_2–condition at infinity (see for instance [22] or [29]), that is, there exist positive constants t_0 and K such that

$$A(2t) \le K\, A(t) \qquad \text{for every } t \ge t_0.$$

- A satisfies the following "slow-growth" condition:

$$\int_1^{+\infty} \frac{ds}{A(s)} = +\infty\,. \tag{9}$$

  Let us remark that, by a simple change of variable, the previous condition is equivalent to

$$\int_1^{+\infty} \frac{ds}{s\, A(\log^* s)} = +\infty\,.$$

We set
$$t_1 = \exp(\max\{t_0, 1\})$$
and therefore $\log t = \log^* t$ for all $t \geq t_1$. Since A satisfies the Δ_2–condition then there exists $k > 0$ such that
$$A(s+t) \leq k(A(s) + A(t)), \text{ for all } r,\, s > t_0.$$
Since
$$\log^*(ab) \leq \log^* a + \log^* b\,, \qquad \text{for all positive } a \text{ and } b,$$
one obtains the subadditivity of $A(\log^* t)$:
$$A(\log^*(ab)) \leq c\big(A(\log^* a) + A(\log^* b)\big), \text{ for all } a,\, b > 0. \tag{10}$$

We wish to prove an existence result for problem (4). We first give a definition of weak solution for this problem.

Definition 1. *We will say that a function*
$$u \in L^\infty_{\text{loc}}([0,\infty); L^1(\Omega)) \cap L^r_{\text{loc}}([0,\infty); W_0^{1,r}(\Omega)) \cap L^\sigma_{\text{loc}}(\overline{Q})\,,$$
for every $r < 1 + \frac{1}{N+1}$ *and for every* $\sigma < 1 + \frac{2}{N}$*, is a weak solution of problem* (4) *if it verifies*

a) *For every* $\beta < \frac{1}{2}$, $\big((1+|u|)^\beta - 1\big) \in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$
b) *For all* $k > 0$, $T_k u \in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$
c) $b(x,t,u) \in L^1_{\text{loc}}(\overline{Q})$

and if for every $\xi \in \mathcal{C}^1_c([0,\infty) \times \Omega)$ *the following equality holds*
$$-\iint_Q u\,\xi_t\,dx\,dt - \int_\Omega u_0(x)\,\xi(x,0)\,dx + \iint_Q a(x,t,u,\nabla u)\cdot\nabla\xi\,dx\,dt = \iint_Q b(x,t,u)\,\xi\,dx\,dt + \iint_Q \xi\,d\mu\,.$$

Remark 1. It is easy to obtain that the solution u belongs to $L^r_{\text{loc}}([0,+\infty); W_0^{1,q}(\Omega))$ for all $r, q \geq 1$ such that
$$\frac{2}{r} + \frac{N}{q} > N+1$$
(as in [8]).

Remark 2. Every weak solution according to this definition has the following property: up to null sets in $(0,+\infty)$
$$\lim_{t\to 0^+} \int_\Omega u(x,t)\,\varphi(x)\,dx = \int_\Omega u_0(x)\,\varphi(x)\,dx$$
for every $\varphi \in C_0^\infty(\Omega)$. As a consequence, taking into account that $C_0^\infty(\Omega)$ is dense in $C_0(\Omega)$, one easily obtains that
$$\lim_{t\to 0^+} u(\cdot,t) = u_0(\cdot) \qquad \text{weakly-}* \text{ in the space of measures,}$$
still up to null sets. This gives further sense to the initial datum. Moreover, if μ is a function in $L^1_{\text{loc}}(\overline{Q})$, then the weak solution we obtain belongs to $C([0,+\infty); L^1(\Omega))$ because of Proposition 6.4 of [14]. See also Theorem 2, where more regularity is considered.

Theorem 1. *Under the above hypotheses, for every $u_0 \in L^1(\Omega)$ and for every finite Radon measure μ, problem* (4) *admits a weak solution u in the sense of Definition 1.*

We also give an existence and regularity result in the case where $\mu = f(x,t)$ is a function such that

$$f(x,t) \in L^{\rho}_{\mathrm{loc}}([0,+\infty); L^{\sigma}(\Omega)), \qquad \text{with } \frac{N}{\sigma} + \frac{2}{\rho} = \frac{N+4}{2}. \tag{11}$$

Theorem 2. *Under the above hypotheses on the operator, if $\mu = f(x,t)$, with f satisfying* (11) *and $u_0 \in L^2(\Omega)$, there exists a weak solution to* (4) *such that*

$$u \in C^0([0,\infty); L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_0(\Omega)).$$

As far as uniqueness is concerned, we will give a result in the case of the heat equation (see also Remark 3 below), that is, when $a(x,t,s,\xi) = a(\xi) = \xi$. We will assume that the function $b(x,t,s)$ satisfies (8) and moreover

$$|b(x,t,s_1) - b(x,t,s_2)| \le \Lambda_4 \left(1 + |s_1|^{\delta} + |s_2|^{\delta}\right) |s_1 - s_2|, \qquad 0 < \delta < \frac{2}{N} \tag{12}$$

for a.e. $(x,t) \in Q$ and every $s_1, s_2 \in \mathbb{R}$, where $\Lambda_4 > 0$. Note that this condition is satisfied in all the model cases, for instance if $b(x,t,s) = g(x,t)\left(1 + |s|\,(\log^* |s|)^{\theta}\right)$, with $\theta \le 1$, or if $b(x,t,s) = g(x,t)\left(1 + |s|\,(\log^* |s|)\,(\log^* \log^* |s|)\right)$, and so on, with $g(x,t)$ bounded.

Theorem 3. *Assume that μ is a Radon measure on Q, that $u_0 \in L^1(\Omega)$ and that the function $b(x,t,s)$ satisfies assumptions* (8) *and* (12). *Then there exists a unique weak solution of problem*

$$\begin{cases} u_t - \Delta u = b(x,t,u) + \mu, & (x,t) \in Q \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{13}$$

Remark 3. In the last theorem we have considered the heat operator in order to avoid the need to introduce more complicated definitions of solution. It should be noted, indeed, that even in the linear case $a(x,t,s,\xi) = A(x,t)\,\xi$ and $b(x,t,u) \equiv 0$, problem (4) may admit multiple solutions in the sense of distributions (see the counterexample in [25], based on the elliptic result by Serrin [26]). Therefore, in order to obtain uniqueness one should consider the notion of *duality solutions* in the case of linear operators (see [23]), or else assume that the measure μ is a function in L^1 or more generally a *soft measure*, that is, a measure which does not charge sets of zero parabolic capacity (see [18]); then one has to use the notions of *entropy solutions* (see [3] and [25]) or *renormalized solutions* (see [6], [18] and [24]), or *approximate solutions* (see [13]). It should be pointed out that all these formulations have elliptic precedents, see [27], [5], [9], [16] and references therein. In anyone of these frameworks, if we assume that the vector-valued function $a(x,t,s,\xi)$ defined at the beginning of this Section does not depend on s (that is, on u), and that, instead of (7), it satisfies the stronger condition

$$\left(a(x,t,\xi) - a(x,t,\eta)\right) \cdot (\xi - \eta) \ge \Lambda_5 \, |\xi - \eta|^2,$$

for a.e. $(x,t) \in Q$ and every $\xi, \eta \in \mathbb{R}^N$, with $\Lambda_5 > 0$, then it is possible to prove a uniqueness result similar to Theorem 3.

As we explained in the Introduction, the main tool to obtain the a priori estimate necessary for the existence result is the following logarithmic Sobolev inequality, which we think may have an interest on its own:

Theorem 4. *Assume that the function A satisfies the hypotheses stated above. Let p, q be positive numbers such that $1 \le p \le q < p^*$ if $N > p$, and $p \le q$ if $N \le p$. Then there exists a positive constant C such that, for every $\varepsilon > 0$ and for every $v \in W_0^{1,2}(\Omega)$ the following inequality holds:*

$$\begin{aligned}(14)\qquad &\int_\Omega |v|^q A(\log^* |v|)\,dx \\ &\le C\left(\varepsilon \int_\Omega |\nabla v|^p\,dx + \|v\|_q^q A\Big(\log^* \frac{1}{\varepsilon}\Big) + \|v\|_q^q A(\log^* \|v\|_q^q) + \|v\|_q^q A(\log^* \|v\|_q^{-q})\right),\end{aligned}$$

and as a consequence it yields

$$\begin{aligned}(15)\qquad &\int_\Omega |v|^q A(\log^* |v|)\,dx \\ &\qquad \le C\left(\varepsilon \int_\Omega |\nabla v|^p\,dx + \|v\|_q^q A\Big(\log^* \frac{1}{\varepsilon}\Big) + \|v\|_q^q A(\log^* \|v\|_q^q) + 1\right).\end{aligned}$$

3. Proof of the logarithmic Sobolev inequality

In this Section we shall prove the logarithmic Sobolev inequality stated in Theorem 4. Throughout this Section, we will assume that the function A satisfies the hypotheses stated in the previous Section. To prove Theorem 4, we will need some lemmata:

Lemma 1. *There exists a positive constant c_1 such that*

$$\frac{A(\log s)}{s} \le c_1 \frac{A(\log t)}{t} \qquad \text{for every } s > t \ge t_1.$$

Proof. For s and t fixed, let $m \in \mathbb{N}$ be such that

$$t^{2^{m-1}} < s \le t^{2^m}.$$

Then, recalling the assumptions on A, one has

$$A(\log s) \le A(\log t^{2^m}) = A(2^m \log t) \le K^m A(\log t).$$

Therefore

$$\frac{A(\log s)}{s} \le \frac{K^m A(\log t)}{t^{2^{m-1}}} \le \frac{K^m}{t_1^{2^{m-1}-1}} \frac{A(\log t)}{t}.$$

To finish, it suffices to observe that the sequence $K^m/(t_1^{2^{m-1}-1})$ is bounded, since it tends to zero. □

Lemma 2. *Let ν be a nonnegative measure on Ω such that $\nu(\Omega) = 1$. Then there exists a positive constant c_2 such that*

$$\int_\Omega A(\log^* f)\,d\nu \le c_2 A\Big(\log^* \int_\Omega f\,d\nu\Big),$$

for all nonnegative $f \in L^1(\Omega, \nu)$.

Remark 4. The previous result, with $c_2 = 1$, would just be Jensen's inequality under the assumption that $A(\log^* s)$ is concave. We point out that this is not always true under our hypotheses.

Also note that $f \in L^1(\Omega, \nu)$ implies $A(\log^* f) \in L^1(\Omega, \nu)$, since A grows less than a power.

PROOF OF LEMMA 2. We set

$$B(s) = \begin{cases} A(\log s) & \text{if } s \geq t_1 \\ \dfrac{A(\log t_1)}{t_1}\, s & \text{if } 0 \leq s \leq t_1\,. \end{cases}$$

Then, applying Lemma 1, $B(t)$ satisfies

$$\frac{B(s)}{s} \leq c_3 \, \frac{B(t)}{t} \qquad \text{for every } s > t > 0 \text{ and for some constant } c_3 \geq 1.$$

Therefore it follows that

$$B(s) - c_3\, B(t) \leq c_3\, \frac{B(t)}{t}\,(s-t) \qquad \text{for every } s > t > 0.$$

We define

$$f_1(x) = f(x)\, \chi_{\{f(x) \geq \int f\, d\nu\}}\,.$$

For every $x \in \Omega$ such that $f(x) \geq \int f\, d\nu$ one has

$$B(f_1(x)) - c_3 B\Big(\int f\, d\nu\Big) \leq c_3\, \frac{B\Big(\displaystyle\int f\, d\nu\Big)}{\displaystyle\int f\, d\nu} \Big(f_1(x) - \int f\, d\nu\Big)\,.$$

It is trivial to check that this same inequality continues to hold for every x such that $f(x) < \int f\, d\nu$, i.e., such that $f_1(x) = 0$. Therefore, integrating the inequality over Ω and recalling that $\nu(\Omega) = 1$, one gets

$$\int B(f_1(x)) - c_3 B\Big(\int f\, d\nu\Big) \leq c_3\, \frac{B\Big(\displaystyle\int f\, d\nu\Big)}{\displaystyle\int f\, d\nu} \Big(\int f_1(x) - \int f\, d\nu\Big) \leq 0\,,$$

which means

$$\int B(f_1(x))\, d\nu \leq c_3 B\Big(\int f\, d\nu\Big)\,. \tag{16}$$

On the other hand, if we set

$$f_2(x) = f(x) - f_1(x) = f(x)\, \chi_{\{f(x) < \int f\, d\nu\}}\,,$$

then the monotonicity of B gives $B(f_2(x)) \leq B\Big(\int f\, d\nu\Big)$, from which, integrating,

$$\int B(f_2)\, d\nu \leq B\Big(\int f\, d\nu\Big)\,. \tag{17}$$

Adding up (16) and (17), one obtains

$$\int B(f)\, d\nu = \int B(f_1)\, d\nu + \int B(f_2)\, d\nu \leq (c_3 + 1)\, B\Big(\int f\, d\nu\Big)\,.$$

To obtain the result one only has to take into account that $A(\log^* t) - B(t)$ is a bounded function and that $A(\log^* t) \geq A(1) > 0$. $\square$

Lemma 3. *There exists a positive constant c_4 satisfying*

$$x\, A(\log^* y) \leq c_4 \Big(x\, A(\log^* x) + y \Big) ,$$

for all $x, y > 0$.

Remark 5. This result resembles Young's inequality for conjugate N-functions (see for instance [22]), but in this case we do not assume any hypothesis on concavity/convexity of $A(\log^* s)$.

Proof of Lemma 3. We write $t = y/x$, and apply the inequality (10) to obtain

$$\begin{aligned} x\, A(\log^* y) &\leq c\, x\, A(\log^* x) + c\, x\, A(\log^* t) \leq c\, x\, A(\log^* x) + c'\, x\, (t+1) \\ &\leq c'' \Big(x\, A(\log^* x) + y + x \Big) \leq c''' \Big(x\, A(\log^* x) + y \Big) . \end{aligned}$$

$\square$

Proof of Theorem 4. Without loss of generality, we can assume that $\varepsilon < 1/e$, and that $v \geq 0$. We may also assume that $N > p$, since in the case $N \leq p$ one only has to replace p^* by any $r > q$.

Then, using Lemma 2 with $\nu = v^q / \|v\|_q^q$ and Sobolev's inequality, one has

$$\begin{aligned} \int_\Omega v^q\, A(\log^* v)\, dx &\leq c\, \|v\|_q^q \int_\Omega \frac{v^q}{\|v\|_q^q} A(\log^* v^{p^*-q})\, dx \leq c \|v\|_q^q\, A\Big(\log^* \int_\Omega \frac{v^{p^*}}{\|v\|_q^q}\, dx \Big) \\ &\leq c \|v\|_q^q \left[A\Big(\log^* \int_\Omega |\nabla v|^p\, dx \Big) + A\Big(\log^* \|v\|_q^{-q} \Big) \right] \\ &\leq c \|v\|_q^q \left[A\Big(\log^* \Big(\varepsilon \int_\Omega |\nabla v|^p\, dx \Big) \Big) + A\Big(\log^* \frac{1}{\varepsilon} \Big) \right] + c \|v\|_q^q\, A\Big(\log^* \|v\|_q^{-q} \Big) . \end{aligned}$$

We now apply Lemma 3 to obtain

$$\|v\|_q^q\, A\Big(\log^* \Big(\varepsilon \int_\Omega |\nabla v|^p\, dx \Big) \Big) \leq c \left(\|v\|_q^q\, A(\log^* \|v\|_q) + \varepsilon \int_\Omega |\nabla v|^p\, dx \right) ,$$

from which (14) follows; it implies (15) since

$$\|v\|_q^q\, A\Big(\log^* \|v\|_q^{-q} \Big) \leq c \Big(1 + \|v\|_q^q \Big) .$$

$\square$

4. Proof of the main results

To begin the proof of Theorem 1, we consider the truncated problems,

$$(18) \qquad \begin{cases} (u_n)_t - \operatorname{div} a(x, t, u_n, \nabla u_n) = b_n(x, t, u_n) + f_n, & \text{in } Q \\ u_n(x, t) = 0, & \text{on } \partial\Omega \times (0, \infty) \\ u_n(x, 0) = u_{0,n}(x), & x \in \Omega, \end{cases}$$

where $u_{0,n}$ is a sequence of bounded functions such that $u_{0,n} \to u_0$ strongly in $L^1(\Omega)$, $b_n(x, t, s) = T_n(b(x, t, s))$, while $f_n(x, t)$ is a sequence of bounded functions such that

$$\iint_Q \varphi(x, t)\, f_n(x, t)\, dx\, dt \to \iint_Q \varphi(x, t)\, d\mu$$

for every function $\varphi(x,t)$ continuous and with compact support in $\Omega\times[0,+\infty)$.

The existence of a bounded solution of problem (18) is well known. The first and most important step is to prove some a priori estimates.

Proposition 1. *Let $\{u_n\}$ be a sequence of solutions of the approximate problems (18) and let $T>0$. Then, for each $\beta<1/2$, the sequence $\big\{(1+|u_n|)^\beta-1\big\}_n$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega))$, and the sequence $\{u_n\}_n$ is bounded in*

$$L^\infty(0,T;L^1(\Omega))\cap L^r(0,T;W_0^{1,r}(\Omega))\cap L^\sigma(Q_T)$$

for all $1\le r<\frac{N+2}{N+1}$ and for all $1\le\sigma<\frac{N+2}{N}$.

Proof. Take $0<\alpha<\min\{1,2/N\}$ and set $p=2$ and $q=\frac{2}{1-\alpha}$, so that p,q satisfy the assumptions of Theorem 4. Consider $\phi(u_n)=\big(1-(1+|u_n|)^{-\alpha}\big)$ sign $u_n\chi_{(0,\tau)}$, with $0<\tau\le T$, as test function in the approximating problems. Then we reach the inequality

$$\begin{aligned}(19)\qquad &\int_\Omega\Phi(u_n(\tau))\,dx-\int_\Omega\Phi(u_{0,n})\,dx+\Lambda_2\iint_{Q_\tau}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\alpha+1}}\\ &\le\iint_{Q_\tau}b(x,t,u_n)\,\phi(u_n)+|\mu|(Q_\tau)\le\Lambda_3\iint_{Q_\tau}\big(1+|u_n|\,A(\log^*|u_n|)\big)+|\mu|(Q_\tau)\,,\end{aligned}$$

where

$$\Phi(s)=\int_0^{|s|}(1-(1+\sigma)^{-\alpha})\,d\sigma\equiv|s|+\frac{1}{1-\alpha}-\frac{1}{1-\alpha}(1+|s|)^{1-\alpha},$$

hence, there exist positive constants c_1, c_2 such that

$$i)\quad\Phi(s)\ge c_1|s|-c_2,\qquad ii)\quad\Phi(s)\le|s|.$$

Therefore

$$\begin{aligned}(20)\qquad &\int_\Omega|u_n(\tau)|dx+c(\alpha,\Lambda_2)\iint_{Q_\tau}|\nabla[(1+|u_n|)^{\frac{1-\alpha}{2}}-1]|^2\\ &\qquad\le c\Big(\iint_{Q_\tau}|u_n|\,A(\log^*|u_n|)+\int_\Omega|u_{0,n}|\,dx+1\Big).\end{aligned}$$

Calling $v=(1+|u_n|)^{\frac{1-\alpha}{2}}-1$ we find that $|u_n|=(v+1)^q-1$ and, moreover, in terms of v, (20) becomes

$$\int_\Omega v^q(t)dx+\iint_{Q_t}|\nabla v|^2\le c\Big(1+\iint_{Q_t}v^qA(\log^*v)\Big).$$

Next, using Theorem 4 with a suitable choice of $\epsilon>0$, we get

$$\|v(t)\|_q^q+\iint_{Q_t}|\nabla v|^2\le c\Big(1+\int_0^t\|v(\tau)\|_q^qA(\log^*\|v(\tau)\|_q)\,d\tau\Big),\quad t\in[0,T]\,.$$

By condition (9), we can apply a nonlinear version of Gronwall's lemma (see, for instance, Lemma 3.11 in [20]) to obtain the desired estimate on v in

$$L^\infty(0,T;L^q(\Omega))\cap L^2(0,T,W_0^{1,2}(\Omega))\,,$$

and consequently the right-hand side of (20) is bounded.

Therefore, the sequences $\{u_n\}_n$ and $\big\{(1+|u_n|)^{\frac{1-\alpha}{2}}-1\big\}_n$ are bounded in the spaces $L^\infty(0,T;L^1(\Omega))$ and $L^2(0;T;W_0^{1,2}(\Omega))$, respectively. Moreover, going back to (19) we know that

$$\iint_{Q_T}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\alpha+1}}\le C(\alpha)\,,\quad \alpha>0\,;$$

thus, by the Boccardo-Gallouët estimates (see [9]), the sequence $\{u_n\}_n$ is bounded in $L^r(0,T;W^{1,r}(\Omega))$ for $1\le r<\frac{N+2}{N+1}$. Finally, by applying the Gagliardo-Nirenberg interpolation Theorem (see, for instance, [17]), we obtain that the sequence $\{u_n\}_n$ is also bounded in $L^\sigma(Q_T)$ for $1\le\sigma<\frac{N+2}{N}$. □

Proof of Theorem 1. Let $\{u_n\}_n$ be a sequence of solutions to the approximating problems (18). For each $T>0$, taking Proposition 1 into account, the sequence $\{(u_n)_t\}_n$ is bounded in $L^1(Q_T)+L^{r'}(0,T,W^{-1,r'}(\Omega))$, $1\le r<\frac{N+2}{N+1}$, so that the Aubin result (see [4] or [28]) implies that, up to subsequences, there exists a measurable function u_T such that $u_n\to u_T$ a.e. in Q_T and strongly in $L^1(Q_T)$. Actually, since the sequence $\{u_n\}_n$ is bounded in $L^\sigma(Q_T)$ for all $1\le\sigma<\frac{N+2}{N}$, it follows that we may assume $u_n\to u_T$ strongly in $L^\sigma(Q_T)$ for all such σ. Thus, a diagonal argument allows us to find a limit u that does not depend on T, that is:

$$u_n\to u\quad\text{a.e. in } Q\quad\text{and in}\qquad (21)$$
$$L^\sigma_{\rm loc}(\overline{Q})\ \text{ for all } 1\le\sigma<\frac{N+2}{N}\,.$$

Moreover, Fatou's Lemma yields $\big((1+|u|)^\beta-1\big)\in L^2_{\rm loc}([0,+\infty);W_0^{1,2}(\Omega))$, for every $0<\beta<\frac12$, and $u\in L^\infty_{\rm loc}([0,+\infty);L^1(\Omega))\cap L^r_{\rm loc}([0,+\infty);W_0^{1,r}(\Omega))\cap L^\sigma_{\rm loc}(\overline{Q})$ for all $1\le r<\frac{N+2}{N+1}$ and for all $1\le\sigma<\frac{N+2}{N}$.

Furthermore, the right-hand side of the equation converges in $L^1_{\rm loc}(\overline{Q})$. Indeed, on the one hand

$$b(x,t,u_n)\to b(x,t,u)\quad\text{a.e. in } Q\,,$$

because of (21) and the Carathéodory condition. On the other hand,

$$|b(x,t,u_n)|\le\Lambda_3(1+|u_n|A(\log^*|u_n|))\le\Lambda_3(1+|u_n|^\sigma)\,,\quad 1<\sigma<\frac{N+2}{N}\,,$$

since A grows less than a power.

Since the operator is nonlinear, to pass to the limit in the weak formulation of (18), one has to prove that the gradients ∇u_n converge to ∇u almost everywhere in Q. To prove this, one can use the techniques introduced in [10], [8], [7] and, more recently, [24] and [23] in our general setting. This shows that u solves (4) in the sense of distributions. □

Proof of Theorem 2. The proof is very similar to the one of the previous theorem. One has to use u_n as a test function in the approximate problems (18), use again the logarithmic Sobolev inequality (15) and Gagliardo-Nirenberg inequality to get rid of the term with f_n. □

Proof of Theorem 3. Assume that u and v are two weak solutions of problem (13). Then, if we set

$$w=u-v\,,$$

then w satisfies the equation

$$\begin{cases} w_t - \Delta w &= b(x,t,u) - b(x,t,v) \quad \text{in } \mathcal{D}'(Q) \\ w(x,0) &= 0, \qquad\qquad\qquad\qquad x \in \Omega. \end{cases}$$

We would like to multiply this equation by $|w|^{\alpha-1}w$, for some α such that $0 < \alpha < \frac{2}{N}$, but we cannot do it directly because this function is not regular when $w = 0$ and moreover is not bounded. Therefore we multiply by $T_k\big((\varepsilon+|w|)^\alpha - \varepsilon^\alpha\big)\,\mathrm{sign}\,w$, and we integrate on Q_τ, obtaining

$$\begin{aligned}(22)\qquad \int_\Omega S_{k,\varepsilon}(w(x,\tau))\,dx + \alpha \iint_{Q_\tau} |\nabla w|^2\,(\varepsilon+|w|)^{\alpha-1} \\ \le \iint_{Q_\tau} |b(x,t,u)-b(x,t,v)|\; T_k\big((\varepsilon+|w|)^\alpha - \varepsilon^\alpha\big)\,,\end{aligned}$$

where

$$S_{k,\varepsilon}(s) = \int_0^{|s|} T_k\big((\varepsilon+\sigma)^\alpha - \varepsilon^\alpha\big)\,d\sigma\,.$$

Since we know that u and v belong to $L^\sigma(Q_\tau)$ for every $\sigma < \frac{N+2}{N}$ and that $|b(x,t,s)|$ grows less than any power $|s|^{1+\nu}$ at infinity, we obtain that $|b(x,t,u)-b(x,t,v)|$ also belongs to $L^\sigma(Q_\tau)$ for every $\sigma < \frac{N+2}{N}$. Therefore we can first let k go to infinity, and then ε go to zero in the right-hand side of (22). As far as the left-hand side is concerned, the passages to the limit are justified by monotone convergence. In the end we obtain

$$\begin{aligned}&\int_\Omega |w(x,\tau)|^{\alpha+1}\,dx + \iint_{Q_\tau} |\nabla w|^2\,|w|^{\alpha-1} \\ &\le c \iint_{Q_\tau} |b(x,t,u)-b(x,t,v)|\,|w|^\alpha \le c \iint_{Q_\tau} \big(1+|u|^\delta+|v|^\delta\big)\,|w|^{\alpha+1}\,.\end{aligned}$$

Therefore, if we set

$$\eta(x,t) = |w(x,t)|^{(\alpha+1)/2}\,, \qquad f(x,t) = 1+|u(x,t)|^\delta + |v(x,t)|^\delta\,,$$

then we obtain that $f \in L^q(Q_T)$ for every $T>0$ and for some $q \ge \frac{N+2}{2}$. Moreover, using Hölder's inequality,

$$\begin{aligned}\|\eta\|^2_{L^\infty(0,T;L^2(\Omega))} &+ \|\eta\|^2_{L^2(0,T;W_0^{1,2}(\Omega))} \\ &\le c \iint_{Q_T} |f|\,\eta^2 \le c\|f\|_{L^{(N+2)/2}(Q_T)}\,\|\eta\|^2_{L^{2(N+2)/N}(Q_T)}\,.\end{aligned}$$

By the Gagliardo-Nirenberg inequality (see for instance [17]), one also has

$$\|\eta\|^2_{L^{2(N+2)/N}(Q_T)} \le c(N)\left(\|\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\eta\|^2_{L^2(0,T;W_0^{1,2}(\Omega))}\right),$$

therefore, if we take T small enough, we can assume that $\|f\|_{L^{(N+2)/2}(Q_T)}$ is small, and therefore the last two formulas imply that $\eta \equiv 0$ in Q_T. Since f is a fixed function in $L^{(N+2)/2}(Q_T)$ for every $T>0$, we can divide the time interval $]0,T[$ in a finite number of intervals such that the previous argument can be carried out in each of them. This shows that $u \equiv v$. $\square$

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