L^∞ estimates for a class of nonlinear elliptic systems with nonstandard growth

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Riassunto

In questo articolo proviamo la locale limitatezza delle soluzioni di sistemi alle derivate parziali in forma di divergenza. I sistemi considerati comprendono le variazioni prime di funzionali dipendenti dalla variabile spaziale e con crescita non standard rispetto al gradiente, quali ad esempio il modello con crescita dipendente dal punto, cioè

$$\mathcal{F}(u) = \int_\Omega g(|\nabla u|)^{\alpha(x)}\, dx\,,$$

dove g è una N-funzione di classe Δ_2.

Abstract

We prove the local boundedness of solutions of partial differential systems in divergence form. The systems under consideration include the first variations of functionals depending on the space variable and having nonstandard growth with respect to the gradient, like for instance the model with growth depending on the point, that is,

$$\mathcal{F}(u) = \int_\Omega g(|\nabla u|)^{\alpha(x)}\, dx\,,$$

where g is an N-function of class Δ_2.

1 Introduction

In this paper we study the boundedness properties for local minimizers of the integral functional of the Calculus of Variations

$$\mathcal{F}(u) = \int_\Omega F(x, \nabla u)\, dx\,, \tag{1}$$

where Ω is an open set of $\mathbf{R}^n$ $(n \geq 2)$ and ∇u denotes the gradient of a vector-valued function $u : \Omega \to \mathbf{R}^N$. Such minimizers are also weak solutions of an elliptic system of the type

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha(x, \nabla u) = 0\,, \qquad \alpha = 1, \ldots, N,$$

where the vector field $a = (a_i^\alpha) : \Omega \times \mathbf{R}^{nN} \to \mathbf{R}^{nN}$ is the gradient with respect to ξ of the function $F(x, \xi)$.

We recall that $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$ is a local minimizer of $\mathcal{F}$ if

$$\int_{\Omega_0} F(x, \nabla u)\, dx < +\infty \qquad \text{for all } \Omega_0 \subset\subset \Omega$$

and

$$\int_{\operatorname{supp}\varphi} F(x, \nabla u)\, dx \leq \int_{\operatorname{supp}\varphi} F(x, \nabla u + \nabla \varphi)\, dx$$

for every $\varphi \in W^{1,1}_0(\Omega; \mathbf{R}^N)$ with $\operatorname{supp}\varphi \subset\subset \Omega$.

The regularity of minimizers of integral functionals has been widely studied in the scalar case ($N = 1$) under the so called natural growth conditions, *i.e.*, in the case where the integrand function F grows like a power of the modulus of the gradient, and under nonstandard growth conditions, *i.e.*, in the case where $F(x, \xi)$ is controlled from below and from above by two different powers $|\xi|^p$, $|\xi|^q$, with p and q not too far from each other (these are the so-called p, q-growth conditions), or by general convex functions.

In the vector case ($N > 1$), there are some well known counterexamples to the continuity of the minimizers (see De Giorgi [7], Giusti and Miranda [11], Necas [21]). However, in the case where $F(x, |\xi|) = |\xi|^p$, $p \geq 2$, Uhlenbeck proved in [23] that the minimizers are in $C^{1,\alpha}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$, a result which was later extended to more general integrands which grow like $|\xi|^p$ by Giaquinta and Modica [9] when $p \geq 2$ and by Acerbi and Fusco [1] in the case $1 < p < 2$. More recently Choe studied in [4] the regularity for minimizers of integral functionals not depending on x and with p, q-growth conditions. Marcellini [16] proved $C^{1,\alpha}$ regularity for minimizers of (1) in the case where $F(x, \xi) = g(|\xi|)$ and g satisfies a nonoscillating condition at infinity, and may also have exponential growth.

In this paper we consider the case

$$F(x, \xi) = g(x, |\xi|)\,,$$

where, for almost every $x \in \Omega$, $g(x, \cdot)$ is a N-function of class Δ_2 (see Definition 2.2) We recall that N-functions are convex functions which have been widely studied in connection with Orlicz spaces (see Krasnosel'skii and Rutickii [12], Adams [2], Rao and Ren [22]). We give some conditions on g which imply that every local minimizer of $\mathcal{F}$ is bounded on compact subsets of Ω, and we give an estimate of the supremum. Our results include the case of functionals with variable growth exponent, *i.e.*,

$$F(x, \xi) = h(|\xi|)^{\alpha(x)}\,, \tag{2}$$

where h is an N-function and $\alpha \in L^\infty(\Omega)$ satisfies some regularity assumptions. In the scalar case, the regularity of minimizers of functionals of this kind has been considered by Marcellini [14], [15], by Mascolo and Papi [19] and by Dall'Aglio, Mascolo and Papi [6]. The particular case $F(x, \xi) = |\xi|^{\alpha(x)}$ has been studied in detail by Chiadò Piat and Coscia [3] and, in the vector case, by Migliorini [20] and by Coscia and Mingione [5]. The last three papers, however, are obtained in the framework of p, q-growth conditions. Our result is a first step in the study of the regularity of vector-valued functionals of type (2) which cannot be treated by estimating the integrand with powers of the gradient, and seems to be new also in the case where F does not depend on x, *i.e.*, in the case where $F(x,\xi) = g(|\xi|)$, and g is an N-function.

The plan of the paper is the following. In Section 2 we state the boundedness theorem and give some examples of its applications. Section 3 is devoted to the proof of the result, which is based on a suitable version of Sobolev's embedding theorem and an iteration method.

2 Main result and applications

Let Ω be an open set of $\mathbf{R}^n$.

Definition 2.1 *We will say that a function $g : \Omega \times [0,+\infty) \to [0,+\infty)$ is a generalised N-function in Ω if*

(a) $g(\cdot,t)$ is a measurable function on Ω, for all $t \geq 0$;

(b) for almost every $x \in \Omega$, $g(x,\cdot)$ is a convex function on $[0,+\infty)$ such that $g(x,t) = 0$ if and only if $t = 0$, and

$$\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0\,, \qquad \lim_{t\to +\infty} \frac{g(x,t)}{t} = +\infty\,.$$

(c) there exist two constants Λ_1, Λ_2 such that

$$0 < \Lambda_1 \leq g(x,1) \leq \Lambda_2 \qquad \text{for almost every } x \in \Omega.$$

In the case where $g(x,t) = g(t)$, then the Definition above corresponds to the usual definition of N-functions (see [12], [22]).

Let $\varphi(x,t) : \Omega \times (0,+\infty) \to (0,+\infty)$ be the left derivative of g with respect to t. Then $\varphi(x,\cdot)$ is positive, nondecreasing and left-continuous in $(0,+\infty)$ for almost every $x \in \Omega$. Moreover

$$\lim_{t\to 0^+} \varphi(x,t) = 0\,, \qquad \lim_{t\to +\infty} \varphi(x,t) = +\infty\,.$$

Definition 2.2 *We will say that a generalised N-function satisfies the Δ_2 condition, and we will write $g \in \Delta_2(\Omega)$, if there exists a constant $k > 1$ such that*

$$g(x,2t) \leq k\, g(x,t) \quad \text{for every } t \geq 0, \text{ for almost every } x \in \Omega.$$

By proceeding as in the proofs of Theorem 3, ch. 2, of [22], and Proposition 2.1 of [6], one can prove the following result:

Proposition 2.3 *Let g be a generalised N-function on Ω, and let φ be its left derivative with respect to t. Then the following properties are equivalent:*

- *$g \in \Delta_2(\Omega)$;*
- *there exists $m > 1$ such that*

$$\varphi(x,t)t \leq m g(x,t) \quad \text{for every } t > 0, \text{ for almost every } x \in \Omega; \tag{3}$$

- *there exists $m > 1$ such that*

$$g(x, \lambda t) \le \lambda^m g(x,t) \quad \textit{for every } t > 0 \textit{ and } \lambda > 1, \textit{ for almost every } x \in \Omega. \tag{4}$$

Moreover (3) implies (4) and viceversa with the same m.

For $m > 1$, we will say that g belongs to the class $\Delta_2^{(m)}(\Omega)$ if it satisfies (3) or (4). Assume that $g \in \Delta_2^{(m)}(\Omega)$. Then, for all $t_1, t_2 > 0$, since $\varphi(x, \cdot)$ is nondecreasing, we have:

$$\varphi(x,t_1)t_2 \le \varphi(x,t_1)t_1 + \varphi(x,t_2)t_2 \,. \tag{5}$$

Moreover, by the convexity of g with respect to t, we have:

$$g(x, t_1 + t_2) \le 2^{m-1}[g(x,t_1) + g(x,t_2)] \,. \tag{6}$$

Examples of generalised N-functions. The function

$$g_1(x,t) = a(x)h(t) \,,$$

where $a(x)$ is a measurable function in Ω, with $0 < \lambda \le a(x) \le \Lambda$, and h is an N-function, is a generalised N-function. Another example of generalised N-function is given by

$$g_2(x,t) = t^{\alpha(x)} \,,$$

where $\alpha(x)$ is a measurable bounded function with $\alpha(x) > 1$ for almost every $x \in \Omega$. More generally one can consider

$$g_3(x,t) = h(t)^{\alpha(x)} \,, \qquad g_4(x,t) = h(t)^{\alpha(x)} \log^\delta(e+t) \,,$$

where $\alpha(x) \ge 1$, δ is a positive constant and h is an N-function. Functions g_3 and g_4 are in $\Delta_2(\Omega)$ if and only if and only if $h \in \Delta_2$.

In the following we will consider the integral functional of the Calculus of Variations

$$\mathcal{F}(u) = \int_\Omega F(x, \nabla u)\, dx \,,$$

where $F(x, \xi) = g(x, |\xi|)$ and $g(x,t) : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfies the following assumptions:

(*i*) g is a generalised N-function such that $g \in \Delta_2^{(m)}(\Omega)$ for some $m > 1$;

(*ii*) for a.e. $x \in \Omega$, $g(x, \cdot)$ is differentiable in $[0, +\infty)$, and its derivative $g_t(x,t)$ is a Caratheodory function in $\Omega \times [0, +\infty)$;

(*iii*) for every $t \ge 0$ and for every $i = 1, \ldots, n$, $g(\cdot, t)$ has weak derivatives $g_{x_i}(\cdot, t)$ $(i = 1, \ldots, n)$ in $L^1_{\text{loc}}(\Omega)$. These derivatives $g_{x_i}(x,t)$ are Caratheodory functions in $\Omega \times [0, +\infty)$, and there exists a function $\gamma \in L^s_{\text{loc}}(\Omega)$, with $s > mn$ such that

$$|g_x(x,t)| \le \gamma(x) g(x,t) \log(e+t) \quad \text{for every } t \ge 0, \text{ for a.e. } x \in \Omega \tag{7}$$

where g_x denotes the weak gradient of g with respect to x. Of course it is not a restriction to assume that $\gamma(x) \ge 1$ almost everywhere.

Remark 2.4 Inequality (7) in assumption (*iii*) might look somewhat restrictive, although it is satisfied by the model function $g(x,t) = h(t)^{\alpha(x)}$. Actually hypothesis (7) could be replaced by

$$|g_x(x,t)| \le \gamma(x) g(x,t)(1+t^{\delta}) \quad \text{for every } t \ge 0, \text{ for a.e. } x \in \Omega\,, \tag{8}$$

where $\delta < \dfrac{s-mn}{ms(n-1)}$. It is not difficult to check that Theorem 2.7 below can be proved under this weaker hypothesis.

Remark 2.5 The functions g_1–g_4 considered above satisfy hypotheses (*i*)–(*iii*) if h is a C^1 N-function of class $\Delta_2^{(m)}$ for some $m > 1$, while $a(x)$ and $\alpha(x)$ are in $W^{1,s}_{\text{loc}}(\Omega)$ for $s > nm$.

Remark 2.6 Observe that assumptions (*i*)–(*iii*) ensure (see for instance [17]) that for every $v \in W^{1,1}_{\text{loc}}(\Omega; \mathbf{R}^N)$, $g(x, |v(x)|)$ belongs to $W^{1,1}_{\text{loc}}(\Omega)$ and the chain rule holds, i.e.,

$$D_{x_i} g(x, |v(x)|) = g_{x_i}(x, |v(x)|) + \frac{g_t(x,|v(x)|)}{|v(x)|} \sum_{\alpha=1}^{N} v^{\alpha}(x) v^{\alpha}_{x_i}(x)\,, \tag{9}$$

provided both $g(x,|v(x)|)$ and the right-hand side of (9) are locally integrable (the last term is defined to be zero in the set where $v = 0$).

We now state the main result of this paper, which is a local estimate of the supremum of $|u|$ in terms of the integral which defines functional $\mathcal{F}$.

Theorem 2.7 *Let $\mathcal{F}$ be as in (1), with $F(x,\xi) = g(x,|\xi|)$, where g is a generalised N-function satisfying (i), (ii) and (iii). Let $u \in W^{1,1}_{\text{loc}}(\Omega; \mathbf{R}^N)$ be a local minimizer of $\mathcal{F}$. Then u is locally bounded. Moreover, if Ω_0 is an open set compactly contained in Ω, then there exists R_0 (depending on n, m, s, Λ_1, $\|\gamma\|_{L^s(\Omega_0)}$ and $\|\nabla u\|_{L^1(\Omega_0)}$) such that, for every $x_0 \in \Omega_0$, for every ρ, R, with $0 < \rho < R \le \min\{\text{dist}(x_0, \partial\Omega_0), R_0\}$ and for every $\alpha > 1$, the following inequality holds*

$$\sup_{x \in B_\rho} g(x, |u(x) - u_R|) \le C \left\{ \frac{1}{(R-\rho)^q} \int_{B_R} \Big(1 + g(x, |\nabla u(x)|)\Big)\, dx \right\}^{\alpha}\,, \tag{10}$$

where $q = \frac{ms(n-1)}{s-mn}$, B_ρ and B_R are the balls with center x_0 and radii ρ and R respectively, $u_R = \fint_{B_R} u(x)\, dx$, and C depends on α, n, m, s, Λ_1 and $\|\gamma\|_{L^s(\Omega_0)}$.

In the case where g does not depend directly on x, the proof of the boundedness of u is heavily simplified, and it is possible to take $\alpha = 1$ (and of course $s = \infty$) in (10), as stated in the following theorem, which also provides a simple "global" estimate for $|u|$ on any $\Omega_0 \subset\subset \Omega$.

Theorem 2.8 *Assume that $g(t)$ is a C^1 generalised N-function in $\Delta_2^{(m)}$ for some $m > 1$. Let u be a local minimizer of $\mathcal{F}$. Then there exists $R_0 = R_0(n) > 0$ such that, for every $x_0 \in \Omega$ and for every $0 < \rho < R < \min\{R_0, \text{dist}(x_0, \partial\Omega)\}$,*

$$\sup_{x \in B_\rho} g(|u(x) - u_R|) \le \frac{C_1}{(R-\rho)^{m(n-1)}} \int_{B_R} g(|\nabla u(x)|)\, dx\,, \tag{11}$$

where C_1 depends on m and n. Moreover, if Ω_0 and Ω_1 are open sets such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, then the following estimate holds for the supremum of $|u|$ on the whole of Ω_0:

$$\sup_{x\in\Omega_0} g(|u(x)|) \leq C_2 \left[\int_{\Omega_1} g(|\nabla u(x)|)\, dx + \int_{\Omega_1} g(|u(x)|)\, dx\right], \tag{12}$$

where C_2 depends only on m, n and on $\mathrm{dist}(\Omega_0, \partial\Omega_1)$.

3 Proof of results

The core of the proof of Theorem 2.7 is given by the following result, whose proof will be obtained by a suitable iteration technique.

Proposition 3.1 *Under the same hypotheses of Theorem 2.7, let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbf{R}^N)$ be a local minimizer of $\mathcal{F}$. Then u is locally bounded. Moreover, for all $\Omega_0 \subset\subset \Omega$, $x_0 \in \Omega_0$, for all ρ, R such that $0 < \rho < R \leq \min\{1, \mathrm{dist}(x_0, \partial\Omega_0)\}$ and for every $\alpha > 1$, one has*

$$\sup_{x\in B_\rho} g(x, |u(x)|) \leq C \left\{ \frac{1}{(R-\rho)^{\frac{ms(n-1)}{s-mn}}} \left[\int_{B_R} \left(1 + g^{\frac{n}{n-1}}(x, |u(x)|)\right) dx\right]^{\frac{n-1}{n}} \right\}^{\alpha}, \tag{13}$$

where C depends on α, m, s, Λ_1 and $\|\gamma\|_{L^s(\Omega_0)}$.

Before proving Proposition 3.1, let us observe that the right hand side of (13) is finite, since the following version of the Sobolev-Poincaré embedding holds.

Proposition 3.2 *Assume that g satisfies (i)–(iii), and let u be a function in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbf{R}^N)$ such that $g(x, |\nabla u|) \in L^1_{\mathrm{loc}}(\Omega)$. Then for all open sets $\Omega_0 \subset\subset \Omega$ there exist $R_0 > 0$, depending on n, m, s, $\|\gamma\|_{L^s(\Omega_0)}$ and $\|\nabla u\|_{L^1(\Omega_0)}$, and $C = C(n, m) > 0$ such that for every $x_0 \in \Omega_0$, for every R such that $0 < R \leq \min\{\mathrm{dist}(x_0, \partial\Omega_0), R_0\}$, the following inequality holds:*

$$\left[\int_{B_R} g^{\frac{n}{n-1}}(x, |u - u_R|)\, dx\right]^{\frac{n-1}{n}} \leq C \int_{B_R} g(x, |\nabla u|)\, dx\,. \tag{14}$$

Proof of Proposition 3.2. For $k > 0$, let $T_k(s) = \min\{s, k\}$ be the truncation function at height k. For $x_0 \in \Omega_0$ let $0 < R \leq \min\{1, \mathrm{dist}(x_0, \partial\Omega_0)\}$. By Remark 2.6 the function $g\big(x, T_k(|u(x) - u_R|)\big)$ belongs to $W^{1,1}(B_R)$ for all $k > 0$, and the chain rule (9) holds for this function. By Sobolev's embedding, $g(x, T_k(|u - u_R|))$ belongs to $L^{n/(n-1)}(B_R)$, and one has

$$\begin{aligned}
&\left[\int_{B_R} g^{\frac{n}{n-1}}\big(x, T_k(|u - u_R|)\big)\, dx\right]^{\frac{n-1}{n}} \\
&\leq c_1(n) \left\{\int_{B_R} \left|\nabla\Big(g\big(x, T_k(|u - u_R|)\big)\Big)\right| dx + \int_{B_R} g\big(x, T_k(|u - u_R|)\big)\, dx\right\} \\
&\leq c_1(n) \Big\{\int_{B_R} g_t\big(x, T_k(|u - u_R|)\big) \left|\nabla T_k(|u - u_R|)\right| dx \\
&\qquad + \int_{B_R} \left|g_x\big(x, T_k(|u - u_R|)\big)\right| dx + \int_{B_R} g\big(x, T_k(|u - u_R|)\big)\, dx\Big\} \\
&= c_1(n)\, \{I_1 + I_2 + I_3\}\,.
\end{aligned}$$

On the other hand from (5), (3) and Hölder's inequality one obtains

$$\begin{aligned} I_1 &\leq m\left[\int_{B_R} g\Big(x, T_k(|u-u_R|)\Big)\,dx + \int_{B_R} g\Big(x, |\nabla T_k(|u-u_R|)|\Big)\,dx\right] \\ &\leq m\left[|B_R|^{1/n}\left(\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right)^{\frac{n-1}{n}} + \int_{B_R} g(x, |\nabla u|)\,dx\right], \end{aligned}$$

where $|B_R|$ denotes the n-dimensional Lebesgue measure of B_R. Similarly

$$I_3 \leq |B_R|^{1/n}\left(\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right)^{\frac{n-1}{n}}.$$

Finally, from hypothesis (iii) one obtains, for every $\delta > 0$,

$$|g_x(x,t)| \leq c(\delta)\gamma(x)g(x,t)(1+t)^{\delta} \quad \text{for every } t \geq 0, \text{ for a.e. } x \in \Omega \tag{15}$$

so that Hölder's inequality yields

$$\begin{aligned} I_2 &\leq c(\delta)\int_{B_R} \gamma(x) g\Big(x, T_k(|u-u_R|)\Big)\Big(1 + T_k(|u-u_R|)\Big)^{\delta}\,dx \\ &\leq c(\delta)\left[\int_{\Omega_0} \gamma^s(x)\,dx\right]^{\frac{1}{s}}\left[\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right]^{\frac{n-1}{n}} \times \\ &\qquad \times \left[\int_{B_R} (1+|u-u_R|)^{\frac{\delta ns}{s-n}}\,dx\right]^{\frac{s-n}{ns}}. \end{aligned}$$

If we choose (for instance) $\delta = \dfrac{s-n}{sn(n-1)}$, it follows that $\dfrac{\delta ns}{s-n} = \dfrac{1}{n-1}$, and therefore, using the usual Sobolev-Poincaré inequality, one has, for $R < 1$,

$$\begin{aligned} \left[\int_{B_R} (1+|u-u_R|)^{\frac{\delta ns}{s-n}}\,dx\right]^{\frac{s-n}{ns}} &\leq |B_R|^{\frac{s-n}{ns}} + \left[\int_{B_R} |u-u_R|^{\frac{n}{n-1}}\,dx\right]^{\frac{s-n}{n^2 s}} |B_R|^{\frac{(s-n)(n-1)}{n^2 s}} \\ &\leq |B_R|^{\frac{s-n}{ns}} + c_2(n,s)\left[\int_{\Omega_0} |\nabla u|\,dx\right]^{\frac{s-n}{sn(n-1)}} |B_R|^{\frac{(s-n)(n-1)}{n^2 s}} \\ &\leq c_3\left(n, s, \|\nabla u\|_{L^1(\Omega_0)}\right) |B_R|^{\frac{(s-n)(n-1)}{n^2 s}}, \end{aligned}$$

so that

$$I_2 \leq c_4\left[\int_{\Omega_0} \gamma^s(x)\,dx\right]^{\frac{1}{s}}\left[\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right]^{\frac{n-1}{n}} |B_R|^{\frac{(s-n)(n-1)}{n^2 s}},$$

where c_4 depends on n, s and $\|\nabla u\|_{L^1(\Omega_0)}$ Therefore we have proved that

$$\begin{aligned} &\left[\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right]^{\frac{n-1}{n}} \\ &\leq c_1(n)\left\{(m+1)|B_R|^{1/n} + c_4\left[\int_{\Omega_0} \gamma^s(x)\,dx\right]^{\frac{1}{s}} |B_R|^{\frac{(s-n)(n-1)}{n^2 s}}\right\} \times \\ &\qquad \times \left[\int_{B_R} g^{\frac{n}{n-1}}\Big(x, T_k(|u-u_R|)\Big)\,dx\right]^{\frac{n-1}{n}} + c_1(n)m\int_{B_R} g(x, |\nabla u|)\,dx\,. \end{aligned}$$

It is now possible to choose R_0 small enough (depending on n, m, s, $\|\nabla u\|_{L^1(\Omega_0)}$ and $\|\gamma\|_{L^s(\Omega_0)}$) so that for $R \le R_0$ the quantity enclosed in brackets is smaller than $1/2c_1(n)$. Therefore for every $R \le \min\{\mathrm{dist}(x_0, \partial\Omega_0), R_0\}$ one has

$$\left[\int_{B_R} g^{\frac{n}{n-1}}(x, T_k(|u-u_R|))\, dx\right]^{\frac{n-1}{n}} \le 2c_1(n)m \int_{B_R} g(x, |\nabla u|)\, dx\,.$$

Letting k tend to infinity, one obtains (14). ■

Remark 3.3 We point out that in the proof of Proposition 3.2 we have only used the assumption $s > n$. We also observe that, in the case where g does not depend on x, the term I_2 does not appear, and therefore the radius R_0 only depends on m and n.

Proof of Proposition 3.1 Since, for almost every $x \in \Omega$, $g(x, \cdot)$ is differentiable, convex and of class $\Delta_2^{(m)}$, it is possible to prove, proceeding as in Theorem 1.2 of [18], that every local minimizer u of functional (1) satisfies the weak form of the Euler system, that is,

$$\int_\Omega \sum_{i=1}^n F_{\xi_i^\alpha}(x, \nabla u)\, \Phi_{x_i}^\alpha\, dx = 0\,, \quad \alpha = 1, \ldots, N \tag{16}$$

for every $\Phi = (\Phi^\alpha)_{\alpha=1,\ldots,N} \in W^{1,1}(\Omega; \mathbf{R}^N)$ with compact support in Ω and such that $F(x, \nabla\Phi)$ belongs to $L^1(\Omega)$. Observe that, by Proposition 3.2, if u is a local minimizer, then $g(x, |\nabla u|)$ and $g^{\frac{n}{n-1}}(x, |u|)$ are locally integrable. For $0 < \rho < R \le \min\{1, \mathrm{dist}(x_0, \partial\Omega_0)\}$, we take

$$\Phi^\alpha = \eta^m \Psi(x, |u|) u^\alpha$$

as test function in (16), where η is a function such that

$$\eta \in C_0^1(B_R)\,, \quad 0 \le \eta \le 1 \text{ in } B_R\,, \quad \eta \equiv 1 \text{ in } B_\rho\,, \text{ and } |\nabla \eta| \le \frac{2}{R-\rho}\,,$$

while $\Psi(x, t)$ satisfies:

(Ψ_1) $\Psi : \Omega \times [0, +\infty) \to [0, +\infty)$ is a bounded Caratheodory function;

(Ψ_2) the derivative $\Psi_t(x, t)$ with respect to t is a nonnegative bounded Caratheodory function such that $\Psi_t(x, t)t$ is bounded;

(Ψ_3) for all $t \ge 0$, $\Psi(\cdot, t)$ has weak derivatives in $L^1_{\mathrm{loc}}(\Omega)$, which are Caratheodory function on $\Omega \times [0, +\infty)$, and are such that there exists $z \in L^s_{\mathrm{loc}}(\Omega)$, with $s > nm$, such that
$$|\Psi_x(x, t)| \le z(x) \quad \text{for every } t \ge 0\text{, for a.e. } x \in \Omega.$$

By Remark 2.6, $\Phi \in W_0^{1,1}(\Omega; \mathbf{R}^N)$ and the chain rule of differentiation gives

$$\begin{aligned} \Phi_{x_i}^\alpha &= m\eta^{m-1}\eta_{x_i}\Psi(x, |u|)u^\alpha + \eta^m \Psi(x, |u|)u_{x_i}^\alpha \\ &\quad + \eta^m u^\alpha \Psi_{x_i}(x, |u|) + \eta^m u^\alpha \frac{\Psi_t(x, |u|)}{|u|} \sum_{\beta=1}^N u^\beta u_{x_i}^\beta\,. \end{aligned} \tag{17}$$

We claim that $g(x, |\nabla\Phi|) \in L^1(\Omega)$. Indeed, using inequality (6),

$$\begin{aligned} g(x, |\nabla\Phi|) \;\; &\leq \;\; 2^{2(m-1)}\Big\{g\Big(x, m\eta^{m-1}|\nabla\eta|\,\Psi(x,|u|)\,|u|\Big) + g\Big(x, \eta^m\Psi(x,|u|)|\nabla u|\Big) \\ &\qquad + \; g\Big(x, \eta^m|u||\Psi_x(x,|u|)|\Big) + g\Big(x, \eta^m\Psi_t(x,|u|)|u||\nabla u|\Big)\Big\}\,. \end{aligned}$$

Since $\Psi(x,|u|)$ and $\Psi_t(x,|u|)|u|$ are bounded, one obtains

$$g(x, |\nabla\Phi|) \leq c\Big(g(x,|u|) + g(x,|\nabla u|) + g(x, z(x)|u|)\Big)$$

for some constant c. Since

$$g(x, z(x)|u|) \leq [1+z(x)]^m g(x,|u|)\,,$$

recalling that $g(x,|u|) \in L^{\frac{n}{n-1}}_{\mathrm{loc}}(\Omega)$ and that $[1+z(x)]^m \in L^{s/m}_{\mathrm{loc}}(\Omega)$, one obtains that $g(x,|\nabla\Phi|) \in L^1(\Omega)$, and therefore we can use Φ as test function in (16). Since

$$F_{\xi_i^\alpha}(x,\xi) = \frac{g_t(x,|\xi|)}{|\xi|}\,\xi_i^\alpha\,,$$

summing the Euler system (16) with respect to α gives

$$\begin{aligned} 0 \;\; &= \;\; \int\sum_{i,\alpha}\frac{g_t(x,|\nabla u|)}{|\nabla u|}\,m\eta^{m-1}\Psi(x,|u|)u^\alpha_{x_i}\eta_{x_i}u^\alpha\,dx \\ &\quad + \int\sum_{i,\alpha}\frac{g_t(x,|\nabla u|)}{|\nabla u|}\,\eta^m\Psi(x,|u|)u^\alpha_{x_i}u^\alpha_{x_i}\,dx \\ &\quad + \int\sum_{i,\alpha}\frac{g_t(x,|\nabla u|)}{|\nabla u|}\,\eta^m u^\alpha_{x_i}u^\alpha\Psi_{x_i}(x,|u|)\,dx \\ &\quad + \int\sum_{i,\alpha,\beta}\frac{g_t(x,|\nabla u|)}{|\nabla u|}\,\eta^m\frac{\Psi_t(x,|u|)}{|u|}u^\alpha u^\beta u^\alpha_{x_i}u^\beta_{x_i}\,dx \\ &= \;\; I_1 + I_2 + I_3 + I_4\,. \end{aligned} \tag{18}$$

We estimate I_2 and I_4.

$$I_2 = \int\eta^m g_t(x,|\nabla u|)|\nabla u|\Psi(x,|u|)\,dx\,;$$

$$I_4 = \int\eta^m\frac{g_t(x,|\nabla u|)}{|\nabla u|}\,\frac{\Psi_t(x,|u|)}{|u|}\,\sum_i\Big(\sum_\alpha u^\alpha_{x_i}u^\alpha\Big)^2\,dx \geq 0\,.$$

Therefore we get:

$$\begin{aligned} &\int\eta^m g_t(x,|\nabla u|)|\nabla u|\Psi(x,|u|)\,dx \\ &\leq \;\; m\int\eta^{m-1}g_t(x,|\nabla u|)\Psi(x,|u|)|u|\,|\nabla\eta|\,dx + \int\eta^m g_t(x,|\nabla u|)\,|\Psi_x(x,|u|)|\,|u|\,dx\,. \end{aligned} \tag{19}$$

Let now q be a positive constant, and let $h_k(t) : [0,+\infty) \to [0,+\infty)$ be a sequence of bounded, smooth, increasing functions such that h_k is constant for large t, $h_k(t)$, $h_k'(t)$ are increasing with respect to k and tend to t^q, qt^{q-1} respectively, for every $t \geq 0$. Then, for every k, $\Psi(x,t) = h_k(g(x,t))$ satisfies assumptions (Ψ_1)–(Ψ_3), and can therefore be used in (19). We obtain

$$\begin{aligned}
&\int \eta^m g_t(x,|\nabla u|)|\nabla u| h_k(g(x,|u|))\, dx \\
&\qquad \le m \int \eta^{m-1} g_t(x,|\nabla u|) h_k(g(x,|u|))|u|\, |\nabla \eta|\, dx \\
&\qquad\qquad + \int \eta^m g_t(x,|\nabla u|) h_k'(g(x,|u|))\, |g_x(x,|u|)|\, |u|\, dx\,,
\end{aligned}$$

and therefore it is possible to pass to the limit for $k \to \infty$ by monotone convergence, obtaining, by inequality (15),

$$\begin{aligned}
&\int \eta^m g_t(x,|\nabla u|)|\nabla u| g^q(x,|u|)\, dx \\
&\le\ m \int \eta^{m-1} g_t(x,|\nabla u|) g^q(x,|u|)|u||\nabla \eta|\, dx \\
&\quad + c_1(\delta) q \int \eta^m g_t(x,|\nabla u|) g^q(x,|u|)\gamma(x)|u|(1+|u|)^\delta\, dx \\
&=\ A + B\,.
\end{aligned} \tag{20}$$

By (5) and (3), with $t_1 = |\nabla u|$ and $t_2 = 4m|u||\nabla\eta|/\eta$, for a.e. $x \in \Omega$ one has

$$g_t(x,|\nabla u|)\frac{m|u||\nabla \eta|}{\eta} \le \frac{1}{4} g_t(x,|\nabla u|)|\nabla u| + \frac{2^{3m-2} m^{m+1}}{\eta^m (R-\rho)^m} g(x,|u|)\,.$$

Therefore

$$A \le \frac{1}{4}\int g_t(x,|\nabla u|)|\nabla u|\eta^m g^q(x,|u|)\, dx + \frac{c_2}{(R-\rho)^m}\int_{B_R} g^{q+1}(x,|u|)\, dx\,, \tag{21}$$

where $c_2 = 2^{3m-2} m^{m+1}$. Moreover, using again (5) and (3), and recalling that $\gamma(x) \ge 1$ for a.e. $x \in \Omega$, we obtain

$$\begin{aligned}
&c_1 q \gamma(x) g_t(x,|\nabla u|)|u|(1+|u|)^\delta \\
&\le\ \frac{1}{4} g_t(x,|\nabla u|)|\nabla u| + \frac{m}{4} g\Big(x, 4c_1 q\gamma(x)|u|(1+|u|)^\delta\Big) \\
&\le\ \frac{1}{4} g_t(x,|\nabla u|)|\nabla u| + m4^{m-1} c_1^m (q+1)^m \gamma^m(x)(1+|u|)^{\delta m} g(x,|u|)\,.
\end{aligned}$$

Since

$$1 + t \le c_3(1 + g(x,t)) \tag{22}$$

for every $t \ge 0$ and almost every $x \in \Omega$, with $c_3 = c_3(\Lambda_1)$, we can conclude that

$$\begin{aligned}
B \le \frac{1}{4}&\int \eta^m g_t(x,|\nabla u|)|\nabla u| g^q(x,|u|)\, dx \\
&+ c_4 (q+1)^m \int_{B_R} \gamma^m(x)\Big(1 + g^{q+1+\delta m}(x,|u|)\Big)\, dx\,,
\end{aligned} \tag{23}$$

where c_4 depends on m, Λ_1 and δ. Then, putting (20), (21) and (23) together, we obtain

$$\int g_t(x,|\nabla u|)|\nabla u| g^q(x,|u|)\eta^m\, dx \le \frac{c_5 (q+1)^m}{(R-\rho)^m}\int_{B_R} \gamma^m(x)\Big(1 + g^{q+1+\delta m}(x,|u|)\Big)\, dx\,, \tag{24}$$

where $c_5 = 4\max\{c_2, c_4\}$. If we now define

$$w = \eta^m \left(1 + g^{q+1}(x,|u|)\right) ,$$

by Remark 2.6 this function belongs to $W_0^{1,1}(\Omega)$, with

$$\begin{aligned} w_{x_i} &= m\eta^{m-1}\eta_{x_i}(1+g^{q+1}(x,|u|)) + (q+1)\eta^m g^q(x,|u|)g_{x_i}(x,|u|) \\ &\quad + (q+1)\eta^m g^q(x,|u|)\frac{g_t(x,|u|)}{|u|}\sum_\alpha u^\alpha u^\alpha_{x_i} , \end{aligned} \tag{25}$$

as soon as the right hand side of (25) belongs to $L^1(\Omega)$. Since (5) and (3) imply that

$$g_t(x,|u|)|\nabla u| \le g_t(x,|\nabla u|)|\nabla u| + mg(x,|u|) ,$$

taking (25), (15), (22) and (24) into account, since $R-\rho<1$, we deduce that

$$\begin{aligned} \int |\nabla w|\,dx &\le \frac{2m}{R-\rho}\int_{B_R}\left(1+g^{q+1}(x,|u|)\right)dx \\ &\quad + c(\delta,\Lambda_1)(q+1)\int_{B_R}\gamma(x)\left(1+g^{q+1+\delta}(x,|u|)\right)dx \\ &\quad + \frac{c_5(q+1)^{m+1}}{(R-\rho)^m}\int_{B_R}\gamma^m(x)\left(1+g^{q+1+\delta m}(x,|u|)\right)dx \\ &\quad + m(q+1)\int_{B_R} g^{q+1}(x,|u|)\,dx \\ &\le \frac{c_6\theta^{m+1}}{(R-\rho)^m}\int_{B_R}\gamma^m(x)\left(1+g^{\theta+\delta m}(x,|u|)\right)dx , \end{aligned}$$

where $\theta = q+1$, and c_6 depends on m, Λ_1, δ and R_0. By Sobolev's embedding theorem applied to w, we get

$$\left[\int_{B_\rho}\left(1+g^\theta(x,|u|)\right)^{1^*}dx\right]^{\frac{1}{1^*}} \le \frac{c_6\theta^{m+1}}{(R-\rho)^m}\int_{B_R}\gamma^m(x)\left(1+g^{\theta+\delta m}(x,|u|)\right)dx , \tag{26}$$

where, following the standard notation, we have set $1^* = \dfrac{n}{n-1}$. We now estimate the right-hand side of (26). Using Hölder's inequality with exponents $\dfrac{s}{m}$ and $\dfrac{s}{s-m}$, we obtain

$$\begin{aligned} &\int_{B_R}\gamma^m(x)\left(1+g^{\theta+\delta m}(x,|u|)\right)dx \\ &\quad\le \left[\int_{B_R}\gamma^s(x)\,dx\right]^{\frac{m}{s}}\left[\int_{B_R}\left(1+g^{\theta+\delta m}(x,|u|)\right)^{\frac{s}{s-m}}dx\right]^{\frac{s-m}{s}} . \end{aligned}$$

Let σ be a number such that $\dfrac{s}{s-m} < \sigma < 1^*$; again by Hölder's inequality, since

$$1+g^{\theta+\delta m}(x,|u|) \le \left(1+g^\theta(x,|u|)\right)\left(1+g^{\delta m}(x,|u|)\right) ,$$

we obtain

$$\begin{aligned} &\int_{B_R}\left(1+g^{\theta+\delta m}(x,|u|)\right)^{\frac{s}{s-m}}dx \\ &\quad\le \left[\int_{B_R}\left(1+g^\theta(x,|u|)\right)^\sigma dx\right]^{\frac{s}{\sigma(s-m)}}\left[\int_{B_R}\left(1+g^{\delta m}(x,|u|)\right)^{\frac{s\sigma}{\sigma(s-m)-s}}dx\right]^{1-\frac{s}{\sigma(s-m)}} , \end{aligned}$$

so that, choosing δ such that

$$\frac{\delta m s \sigma}{\sigma(s-m)-s} = 1^*\,, \tag{27}$$

we get

$$\begin{aligned}
&\int_{B_R} \gamma^m(x)\ \left(1+g^{\theta+\delta m}(x,|u|)\right)\ dx \\
&\le\ c_7 \left[\int_{\Omega_0} \gamma^s(x)\,dx\right]^{\frac{m}{s}} \left[\int_{B_R}\left(1+g^{1^*}(x,|u|)\right)dx\right]^{\frac{s-m}{s}-\frac{1}{\sigma}} \left[\int_{B_R}\left(1+g^{\theta}(x,|u|)\right)^{\sigma}\ dx\right]^{\frac{1}{\sigma}}\,,
\end{aligned}$$

where $c_7 = c_7(m,s,\sigma)$. Since $1+t^{1^*} \le (1+t)^{1^*}$ and $(1+t)^\sigma \le 2^{\sigma-1}(1+t^\sigma)$, (26) becomes

$$\begin{aligned}
&\left[\int_{B_\rho}\left(1+g^{\theta 1^*}(x,|u|)\right)dx\right]^{\frac{1}{1^*}} \\
&\le\ \frac{c_8\theta^{m+1}}{(R-\rho)^m}\left[\int_{B_R}\left(1+g^{1^*}(x,|u|)\right)dx\right]^{\frac{s-m}{s}-\frac{1}{\sigma}} \left[\int_{B_R}\left(1+g^{\theta\sigma}(x,|u|)\right)\ dx\right]^{\frac{1}{\sigma}}\,,
\end{aligned} \tag{28}$$

where c_8 depends on m, s, σ, Λ_1 and $\|\gamma\|_{L^s(\Omega_0)}$. This holds under the only requirement that $g(x,|u|) \in L^{\theta\sigma}(B_R)$. We now fix $\bar{R}$, $\bar{\rho}$ such that $0<\bar{\rho}<\bar{R}\le \min\{1, \operatorname{dist}(x_0,\partial\Omega_0)\}$. For all $j = 1, 2, \ldots$, we set

$$\rho_j = \bar{\rho} + \frac{\bar{R}-\bar{\rho}}{2^{j-1}}\,, \qquad \theta_j = \left(\frac{1^*}{\sigma}\right)^j\,, \qquad A_j = \left[\int_{B_{\rho_j}}\left(1+g^{\theta_j\sigma}(x,|u|)\right)dx\right]^{\frac{1}{\theta_j\sigma}}\,.$$

For $\rho=\rho_{j+1}$, $R=\rho_j$ and $\theta=\theta_j$, (28) gives

$$A_{j+1} \le \left(\frac{c_8 M}{(\bar{R}-\bar{\rho})^m}\right)^{\frac{1}{\theta_j}} \theta_j^{\frac{m+1}{\theta_j}}\, 2^{\frac{jm}{\theta_j}}\, A_j\,, \quad \text{with } M = \left[\int_{B_{\bar{R}}}\left(1+g^{1^*}(x,|u|)\right)dx\right]^{\frac{s-m}{s}-\frac{1}{\sigma}}. \tag{29}$$

By iterating (29), we obtain

$$A_{j+1} \le \left(\frac{c_8 M}{(\bar{R}-\bar{\rho})^m}\right)^{\sum_{k=1}^j \frac{1}{\theta_k}} \left(\prod_{k=1}^j \theta_k^{\frac{1}{\theta_k}}\right)^{m+1} 2^{m\sum_{k=1}^j \frac{k}{\theta_k}}\, A_1$$

(note that, by Proposition 3.2, M, the term A_1, and therefore every A_j, are finite). Since

$$\sum_{k=1}^{+\infty}\frac{1}{\theta_k} = \frac{\sigma}{1^*-\sigma}\,, \quad \sum_{k=1}^{+\infty}\frac{k}{\theta_k} = \sum_{k=1}^{+\infty} k\left(\frac{\sigma}{1^*}\right)^k < +\infty\,, \quad \prod_{k=1}^{+\infty}\theta_k^{\frac{1}{\theta_k}} = \exp\left(\sum_{k=1}^{+\infty}\frac{\ln\theta_k}{\theta_k}\right) < +\infty\,,$$

we conclude that, for every $j \in \mathbf{N}$,

$$A_j \le c_9\left(\frac{c_8 M}{(\bar{R}-\bar{\rho})^m}\right)^{\frac{\sigma}{1^*-\sigma}} \left[\int_{B_{\bar{R}}}\left(1+g^{1^*}(x,|u|)\right)\ dx\right]^{\frac{1}{1^*}}\,,$$

with $c_9 = c_9(n,\sigma)$; recalling the meaning of M, the last inequality can be rewritten as

$$A_j \le C\left\{\frac{1}{(\bar{R}-\bar{\rho})^{\frac{ms(n-1)}{s-mn}}}\left[\int_{B_{\bar{R}}}\left(1+g^{\frac{n}{n-1}}(x,|u(x)|)\right)dx\right]^{\frac{n-1}{n}}\right\}^{\alpha}\,, \tag{30}$$

where $\alpha = \alpha(\sigma) = \dfrac{\sigma(s - mn)}{s(n - \sigma(n-1))}$, and C depends on α (or, which is equivalent, on σ), n, m, s, Λ_1 and $\|\gamma\|_{L^s(\Omega_0)}$. Note that, depending on the choice of σ, the exponent α can assume any value greater than 1. Finally

$$\sup_{B_{\bar\rho}} g(x, |u|) = \lim_{j \to +\infty} \left[\int_{B_{\bar\rho}} g^{\theta_j \sigma}(x, |u|)\, dx \right]^{\frac{1}{\theta_j \sigma}} \leq \limsup_{j \to +\infty} A_j\,. \tag{31}$$

Replacing $\bar R$ and $\bar\rho$ with R and ρ respectively, (13) follows from (30) and (31). ■

Remark 3.4 If one assumes hypothesis (8) instead of (7), then one should use equality (27) to define σ, and then observe that (8) is equivalent to $\sigma < 1^*$.

Proof of Theorem 2.7. The proof follows easily by putting Propositions 3.1 and 3.2 together, and observing that, if u is a local minimum, then so is $u - u_R$. ■

Proof of Theorem 2.8. By proceeding as in the previous proof, it is not difficult to check that in this case inequality (26) can be replaced by

$$\left[\int_{B_\rho} g^{\theta 1^*}(|u|)\, dx \right]^{\frac{1}{1^*}} \leq \frac{c_7 \theta}{(R-\rho)^m} \int_{B_R} g^{\theta}(|u|)\, dx\,,$$

where c_7 depends on m and n. Therefore, by adapting the iteration to this case, estimate (11) follows easily. To prove the "global" estimate (12), let us choose $R < \min\{R_0, \mathrm{dist}(\Omega_0, \partial\Omega_1)\}$. Let $\{B_{R/2}(x_k)\}$, $k = 1, 2, \ldots, M$, be a covering of Ω_0 with balls of radius $R/2$. then, if we set $u_R^k = \fint_{B_R(x_k)} u(x)\, dx$, we can write

$$\sup_{x \in \Omega_0} g(|u(x)|) \leq 2^{m-1} \left[\max_k \sup_{x \in B_{\frac{R}{2}}(x_k)} g(|u(x) - u_R^k|) + \max_k g(|u_R^k|) \right].$$

By the estimate (11) we obtain

$$\sup_{x \in B_{\frac{R}{2}}(x_k)} g(|u(x) - u_R^k|) \leq c(m,n,R) \int_{B_R(x_k)} g(|\nabla u|)\, dx \leq c(m,n,R) \int_{\Omega_1} g(|\nabla u|)\, dx\,.$$

On the other hand, by Jensen's inequality, one has

$$g(|u_R^k|) = g\left(\left| \fint_{B_R(x_k)} u\, dx \right| \right) \leq \fint_{B_R(x_k)} g(|u|)\, dx \leq \frac{1}{R^n} \int_{\Omega_1} g(|u|)\, dx\,.$$

Putting the last three estimates together, one obtains (12). ■

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