# Nonlinear parabolic equations with natural growth in general domains

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**Abstract**

We prove an existence result for a class of parabolic problems whose principal part is the $p$-Laplace operator or a more general Leray-Lions type operator, and featuring an additional first order term which grows like $|\nabla u|^p$. Here the spatial domain can have infinite measure, and the data may be not regular enough to ensure the boundedness of solutions. As a consequence, solutions are obtained in a class of functions with exponential integrability. An existence result of bounded solutions is also given under additional hypotheses.

**Sunto**

In questo articolo si dimostra un risultato di esistenza per una classe di problemi parabolici la cui parte principale è l'operatore $p$-Laplaciano, oppure un operatore più generale del tipo di Leray-Lions, e in cui compare un termine aggiuntivo del primo ordine che cresce come $|\nabla u|^p$. Il dominio spaziale in cui si risolve il problema può avere misura infinita, e i dati possono non avere la regolarità necessaria per garantire la limitatezza delle soluzioni. Di conseguenza, si ottengono soluzioni in una classe di funzioni con integrabilità esponenziale. Sotto ipotesi più forti, si prova l'esistenza di soluzioni limitate.

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## 1 Introduction

In this paper we deal with existence results for nonlinear parabolic problems with first order terms having natural growth with respect to the gradient. More

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precisely, the model problem we refer to is

$$\begin{cases} u_t - \Delta_p u = d|\nabla u|^p + f(x,t) & \text{in } Q_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $1 < p < N$, $\Delta_p u$ is the $p$-Laplace operator and

$$f(x,t) \in L^r(0,T; L^q(\Omega)), \tag{2}$$

where $q, r > 1$ are such that

$$q\,\frac{r-1}{r} \geq \frac{N}{p}\,. \tag{3}$$

We point out that $\Omega$ is a general open set in $\mathbf{R}^N$ which may have infinite measure. No regularity requirement is assumed on $\partial\Omega$. Assuming for simplicity that the initial datum $u_0$ satisfies

$$\int_\Omega \left(e^{\lambda|u_0|} - 1\right)^2 dx < +\infty\,, \qquad \text{for every } \lambda \in \mathbf{R}, \tag{4}$$

we are able to prove the existence of at least one solution $u$ to problem (1), such that $|u|^\alpha(e^{\lambda|u|}-1) \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega))$ for every positive number $\lambda$, and for sufficiently large $\alpha$. We also prove that such a solution belongs to $L^p(0,T;W^{1,p}(\Omega^0))$ for every bounded open set $\Omega^0 \subset \Omega$. In fact the assumption on the initial datum can be weaker than (4) (see hypotheses (I1) and (I2) in the next section for the precise statement). For instance, one can assume that hypothesis (4) only holds for some fixed $\lambda$ (which has to be large enough), but in this case one obtains a correspondingly weaker integrability of the solution (see Theorem 1 below).

Moreover, if a strict inequality holds in (3) and $u_0$ is supposed to belong to $L^\infty(\Omega) \cap L^2(\Omega)$, we can prove that these solutions are also bounded in $Q_T$. We recall that, in the case where $\Omega$ is bounded and $d = 0$ (that is, if there is no first-order term in the right-hand side of (1)), Aronson and Serrin proved in [1] that the solutions are bounded if $q(r-1)/r > N/p$, and the result is sharp.

We use here a different technique to prove the same result in our more general framework.

If $\Omega$ is bounded, the previous solutions have finite energy, *i.e.,* they belong to $L^p(0,T;W_0^{1,p}(\Omega))$. If $\Omega$ is a general domain and $f(x,t)$ only satisfies hypothesis (2), we cannot say that the distributional solutions found by our method have finite energy, since we cannot obtain $L^p$-estimates on the gradient of $u$ outside a bounded domain and where $u$ is "small".

If one is interested in solutions having finite energy, one needs an additional hypothesis on $f(x,t)$, i.e., $f(x,t) \in L^\rho(0,T;L^\sigma(\Omega))$, where $\rho$ and $\sigma$ are the Hölder conjugate exponents of those given by the classical Gagliardo-Nirenberg interpolation inequality (see Remark 4 below). In this case, under hypothesis (4), we can prove the existence of solutions which satisfy

$$e^{\lambda|u|} - 1 \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) , \qquad \text{for every } \lambda > 0. \tag{5}$$

As before, the solutions we find are also bounded if the inequality in (3) is strict and the initial datum $u_0$ is bounded.

Let us recall some known results in the case where $\Omega$ is a bounded open set. Existence of weak bounded solution for a Cauchy problem like (1), whose principal part is a quasilinear parabolic operator, was proved by Boccardo, Murat and Puel in [4], if $p = 2$ (see also Orsina and Porzio [21], and Grenon [16] for more general principal part, and growth of order $p \neq 2$). In Landes, Mustonen [18] and Dall'Aglio, Orsina [9] the existence of unbounded solutions was proved under a sign assumption on the first-order term. In the present paper there is no such assumption. However, once this work was completed, we learned that a similar problem has been studied by Ferone, Posteraro and Rakotoson [14] and [15] in the case of unbounded solutions on a domain with finite measure. If we restrict our attention to bounded domains, the results concerning existence of unbounded solutions are analogous in our paper and theirs.

As far as the corresponding stationary problem is concerned, existence of bounded solutions satisfying Dirichlet boundary conditions was proved in several papers by Boccardo, Murat and Puel ([3], [6], see also Ferone, Posteraro and Rakotoson [13] and references therein). Unbounded solutions are found in [2], where a sign condition on the first order term is assumed, and in Ferone, Murat [12] with no sign assumption. For the stationary problem in domains having infinite measure, as far as we know the only references are Donato and Giachetti [11] and Dall'Aglio, Giachetti and Puel [8]. In all the mentioned papers the use of test functions of exponential type allows to get rid of the term $|\nabla u|^p$ and therefore is an essential tool in the proof. Here we make use of the same kind of test functions.

To obtain the existence result, since $\Omega$ can have infinite measure, we proceed by solving some approximate problems in bounded sets $\Omega_n$.

It is worth noticing that we have to prove uniform estimates on the solutions $u_n$, avoiding arguments which involve either the measure of $\Omega_n$ or any embedding result between Lebesgue spaces.

The plan of this article is the following. In Section 2 we state the hypotheses on problem (1), the approximation method and the main theorems. Section 3 is devoted to prove estimates on $u_n$, solution of the approximate problems, under various hypotheses on the source term $f(x,t)$. In Section 4, we prove local strong convergence of $\nabla u_n$, while Section 5 is devoted to the proof of the main theorems.

## 2 Assumptions and main result

Let $T$ be a positive number and $\Omega$ an open subset of $\mathbf{R}^N$, possibly of infinite measure. We denote by $\Gamma$ its boundary, by $Q_T$ the cilinder

$$Q_T = \Omega \times (0,T)$$

and by $\Sigma_T$ its lateral boundary

$$\Sigma_T = \Gamma \times (0,T)\,.$$

We are interested in proving the existence of solutions $u = u(x,t)$ for the following Cauchy problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,u,\nabla u) = H(x,t,u,\nabla u) + f(x,t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{$\tilde{\mathrm{P}}$}$$

The model problem we refer to is the following:

$$\begin{cases} u_t - \Delta_p u = d|\nabla u|^p + f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\Delta_p$ is the $p$-Laplace operator (i.e., $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$), with $1 < p < N$, and $d$ is a constant.

More generally we will assume the following hypotheses on the terms which appear in ($\tilde{\mathrm{P}}$).

**Assumptions on the data:**
$f(x,t):\ \Omega \times (0,T) \to \mathbf{R}$ is a measurable function such that:

(F1) $\qquad f(x,t) \in L^r(0,T;L^q(\Omega))\,, \quad \text{with } 1 < r, q < \infty\ , \quad \dfrac{q}{r'} \ge \dfrac{N}{p}\,.$

$u_0(x) : \Omega \to \mathbf{R}$ is a measurable function satisfying

(I1) $$\int_{\{|u_0|>1\}} e^{\bar{\lambda}|u_0|}\,dx < +\infty\,, \qquad \text{for some } \bar{\lambda} > \frac{p'd}{\Lambda_2}\ ,$$

(I2) $$\int_{\{|u_0|\le 1\}} |u_0|^{\bar{\alpha}+2}\,dx < +\infty\,, \qquad \text{for some } \bar{\alpha} \ge 0\ .$$

**Remark 1** Assumption (I2) is obviously satisfied for every $\alpha \geq 0$ if $\Omega$ has finite measure.

**Assumptions on** $a(x,t,s,\xi)$**:**
$a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory vector-valued function (*i.e.,* it is measurable with respect to $(x,t)$ for every $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$, and continuous with respect to $(s,\xi)$ for almost every $(x,t) \in Q_T$) such that

(A1) there exist constants $\Lambda_1, \gamma > 0$ such that

$$|a(x,t,s,\xi)| \leq \Lambda_1 \left(k_1(x,t) + |s|^\gamma + |\xi|^{p-1}\right)$$

for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$, where $k_1(x,t)$ is a positive function such that

$$k_1^{p'} \in L^{r_1}(0,T;L^{q_1}(\Omega)), \quad \text{with } 1 < r_1, q_1 < \infty, \quad \frac{q_1}{r_1'} \geq \frac{N}{p}, \tag{6}$$

(here $p'$ denotes Hölder's conjugate exponent of $p$, defined by $\frac{1}{p} + \frac{1}{p'} = 1$), and $\gamma$ is any positive number;

(A2) there exists a constant $\Lambda_2 > 0$ such that

$$a(x,t,s,\xi) \cdot \xi \geq \Lambda_2 |\xi|^p$$

for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$;

(A3)
$$[a(x,t,s,\xi) - a(x,t,s,\eta)] \cdot (\xi - \eta) > 0$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbf{R}$ and $\xi, \eta \in \mathbf{R}^N$, with $\xi \neq \eta$.

**Assumptions on** $H(x,t,s,\xi)$**:**
$H(x,t,s,\xi) : \Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a Carathéodory function such that:

(H) there exists a constant $d > 0$ such that for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$

$$|H(x,t,s,\xi)| \leq d|\xi|^p .$$

**Remark 2** It is obvious from the proofs of the results that the last assumption might be replaced by

$$|H(x,t,s,\xi)| \leq d|\xi|^p + f_1(x,t),$$

with $f_1$ satisfying the same hypotheses as $f$.

Let us point out that one can always suppose that a lower order term $\mu u$, with $\mu > 0$, appears in the left-hand side of the equation of problem $(\tilde{\mathrm{P}})$. Indeed, if we consider the function $v(x,t) = e^{-\mu t}u(x,t)$, it is easy to verify that it satisfies

$$\begin{cases} v_t - \operatorname{div} \tilde{a}(x,t,v,\nabla v) + \mu v = \tilde{H}(x,t,v,\nabla v) + \tilde{f}(x,t) & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\tilde{a}(x,t,s,\xi) = e^{-\mu t}a(x,t,e^{\mu t}s,e^{\mu t}\xi)$, $\tilde{H}(x,t,s,\xi) = e^{-\mu t}H(x,t,e^{\mu t}s,e^{\mu t}\xi)$, $\tilde{f}(x,t) = e^{-\mu t}f(x,t)$ satisfy the same kind of hypotheses as $a$, $H$ and $f$, respectively. For this reason in the sequel we will refer to the following problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,u,\nabla u) + \mu u = H(x,t,u,\nabla u) + f(x,t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{P}$$

with $\mu > 0$.

We denote by $\mathcal{D}_0^{1,p}(\Omega)$ the Banach space defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_{L^p(\Omega)}$. The first result we are going to prove is the following.

**Theorem 1** *Assume that* (F1), (I1), (I2), (A1)–(A3), (H) *are satisfied. Then there exists at least one solution $u$ of* (P) *in the sense of distributions such that, for every bounded open set $\Omega^0 \subset \Omega$,*

$$u \in L^p(0,T;W^{1,p}(\Omega^0)), \tag{7}$$

$$e^{\frac{\bar{\lambda}}{p}|u|} - 1 \in L^p(0,T;W^{1,p}(\Omega^0)) \cap L^\infty(0,T;L^p(\Omega^0)). \tag{8}$$

*Moreover $u$ satisfies:*

*1) (estimates for large values of $u$)*

$$\iint_{Q_T \cap \{|u|>1\}} e^{\bar{\lambda}|u|}|\nabla u|^p \,dx\,dt < \infty, \tag{9}$$

$$\sup_{t\in[0,T]} \int_{\Omega\cap\{|u|>1\}} e^{\bar{\lambda}|u(x,t)|}\,dx < \infty. \tag{10}$$

*2) (estimates for small values of $u$). There exists $\alpha \geq 0$ such that*

$$\iint_{Q_T} |u|^\alpha |\nabla u|^p \,dx\,dt < \infty, \tag{11}$$

$$\sup_{t\in[0,T]} \int_\Omega |u(x,t)|^{\alpha+2}\, dx < \infty\,. \tag{12}$$

**Remark 3** If $\Omega$ is a bounded set, we can take $\Omega^0 = \Omega$ in (7), (8) and therefore

$$e^{\frac{\bar\lambda}{p}|u|} - 1 \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^p(\Omega))\,.$$

If one assumes a slightly stronger hypothesis on $f(x,t)$ and $u_0$, i.e., (F1) is replaced by

(F1′) $\qquad f(x,t) \in L^r(0,T; L^q(\Omega))\,, \quad \text{with } 1 < r, q < \infty\,, \quad \dfrac{q}{r'} > \dfrac{N}{p}\,.$

and (I1) is replaced by

(I1′) $\qquad u_0 \in L^\infty(\Omega)\,,$

then one can prove the boundedness of the solutions found by Theorem 1. More precisely we have:

**Theorem 2** *Assume that* (F1′), (I1′), (I2), (A1)–(A3), (H) *are satisfied. Then the solution found by Theorem 1 satisfies*

$$u \in L^\infty(Q)\,. \tag{13}$$

We point out that in Theorems 1 and 2, for unbounded domains, we do not know that the solutions we find have finite energy, that is, $u \in L^p(0,T; W_0^{1,p}(\Omega))$. Indeed we cannot obtain estimates outside a bounded domain and where $u$ is "small". If we assume some additional hypotheses on the data $f$ and $u_0$, namely

(F2) $$\begin{cases} f(x,t) \in L^\rho(0,T; L^\sigma(\Omega))\,, \text{ with } \rho, \sigma \text{ such that} \\ 1 \le \rho \le p' = \dfrac{p}{p-1}\,, \quad \dfrac{2N}{\sigma} + \dfrac{Np - 2N + 2p}{\rho} = Np - N + 2p \end{cases}$$

(I3) $$\int_\Omega u_0^2\, dx < \infty\,,$$

then we can prove existence of solutions having finite energy. More precisely:

**Theorem 3** *Assume that* (A1)–(A3), (H) *are satisfied.*

*i) If $f(x,t)$ satisfies* (F1), (F2), *and $u_0$ satisfies* (I1), (I3), *then the solution provided by Theorem 1 satisfies in addition*

$$u \in L^p(0,T; \mathcal{D}_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))\,, \tag{14}$$

*ii) If $f(x,t)$ satisfies* (F1′), (F2), *and $u_0$ satisfies* (I1′), (I3), *then the solution provided by Theorem 1 satisfies in addition*

$$u \in L^\infty(Q) \cap L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;\mathcal{D}_0^{1,p}(\Omega)). \qquad (15)$$

**Remark 4** Let us comment hypotheses (F1) (or (F1′)) and (F2) on the source term $f(x,t)$. As far as the hypotheses (F1) and (F1′) are concerned, we recall that, in the case where $\Omega$ is bounded and $d = 0$, the curve (in the variables $p$, $q$) defined by $q(r-1)/r = N/p$ is the threshold above which (i.e., if (F1′) holds) the solution of the parabolic problem (P) is bounded, as proved in [1]. As far as (F2) is concerned, we remark that the numbers $\rho$, $\sigma$ given by (F2) are the Hölder conjugate exponents of those given (for $m = 2$, see below) by the classical Gagliardo-Nirenberg embedding theorem which we now recall (see for instance DiBenedetto [10]).

**Lemma 1** *Let $\Omega$ be a bounded open set of $\mathbf{R}^N$ and $T$ be a real positive number. Let $v(x,t)$ be a function such that*

$$v \in L^\infty(0,T;L^m(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)),$$

*with $1 < p < N$. Then $v \in L^{\rho_1}(0,T;L^{\sigma_1}(\Omega))$, where*

$$m \le \sigma_1 \le \frac{Np}{N-p} \quad \text{if } m \le \frac{Np}{N-p}, \qquad \frac{Np}{N-p} \le \sigma_1 \le m \quad \text{if } \frac{Np}{N-p} \le m,$$

$$p \le \rho_1 \le \infty$$

*and*

$$\frac{mN}{\sigma_1} + \frac{Np - m(N-p)}{\rho_1} = N, \qquad (16)$$

*and the following estimate holds*

$$\int_0^T \|v(t)\|^{\rho_1}_{L^{\sigma_1}(\Omega)}\, dt \le C(N,p,m)\, \|v\|^{\rho_1 - p}_{L^\infty(0,T;L^m(\Omega))} \int_0^T \|\nabla v(t)\|^p_{L^p(\Omega;\mathbf{R}^N)}\, dt. \qquad (17)$$

The proof of Theorems 1–3 will be obtained by approximation using the following problems on bounded domains $Q_{n,T} = \Omega_n \times (0,T)$, where $\Omega_n = \Omega \cap B(0,n)$ and $B(0,n)$ is the ball of center 0 and radius $n$ (we omit the dependence on $x$ and $t$ for the sake of brevity):

$$(\mathrm{P}_n) \qquad \begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div} a(u_n, \nabla u_n) + \mu u_n = H_n(u_n, \nabla u_n) + f_n & \text{in } Q_{n,T}, \\ u_n = 0 & \text{on } \partial\Omega_n \times (0,T), \\ u_n(0) = u_{0,n} & \text{in } \Omega_n, \end{cases}$$

with

$$H_n(x,t,s,\xi) = T_n(H(x,t,s,\xi)), \qquad f_n(x,t) = T_n(f(x,t)), \tag{18}$$

and $T_n(s)$ is the truncation defined by

$$T_n(s) = \begin{cases} s & \text{if } |s| < n, \\ n \text{ sign}(s) & \text{if } |s| \geq n. \end{cases} \tag{19}$$

Moreover $u_{0,n}$ is a sequence such that

$$u_{0,n} \in L^\infty(\Omega_n) \cap W_0^{1,p}(\Omega_n), \qquad u_{0,n} \to u_0 \quad \text{a.e. in } \Omega, \tag{20}$$

and such that $u_{0,n}$ is bounded in the same spaces as the initial datum $u_0$, that is,

$$\int\limits_{\Omega_n \cap \{|u_{0,n}|>1\}} e^{\bar{\lambda}|u_{0,n}|}\, dx \leq c \qquad \text{if (I1) holds,} \tag{21}$$

$$\int\limits_{\Omega_n \cap \{|u_{0,n}|\leq 1\}} |u_{0,n}|^{\bar{\alpha}+2}\, dx \leq c \qquad \text{if (I2) holds,} \tag{22}$$

$$\|u_{0,n}\|_{L^\infty(\Omega_n)} \leq c \qquad \text{if (I1}'\text{) holds,} \tag{23}$$

$$\int\limits_{\Omega_n} u_{0,n}^2\, dx \leq c \qquad \text{if (I3) holds} \tag{24}$$

(of course one could require more, for instance strong convergence in the respective spaces, but this will suffice). Such a regularization of the initial datum can be obtained by a standard technique of truncation and convolution. Moreover one can always assume that

$$\lim_{n\to\infty} \frac{1}{n} \|u_{0,n}\|_{W_0^{1,p}(\Omega_n)} = 0. \tag{25}$$

This condition will be used in the proof of the strong convergence of the gradients $\nabla u_n$ (see Section 4 below). It is well known (see [20]) that problems ($\mathrm{P}_n$) admit at least one distributional solution $u_n \in L^\infty(Q_{n,T}) \cap L^p(0,T; W_0^{1,p}(\Omega_n))$. In the following Section 3, we will find *a priori* estimates for the solutions $u_n$.

# 3 Estimates

## 3.1 Estimates under hypotheses (F1), (I1), (I2): unbounded solutions with infinite energy

This subsection is devoted to prove estimates on solutions $u_n$ of problem ($\mathrm{P}_n$), when $f(x,t)$ and $u_0(x)$ only satisfy (F1), (I1), (I2).

**Proposition 1** *Assume* (F1), (I1), (I2), (A2), (H) *are satisfied, and let* $u_n$ *be a solution of* $(\mathrm{P}_n)$*. Then there exists a constant* $C$*, depending on the data, such that:*

*1) (estimates for large values of* $u_n$*)*

$$\iint\limits_{Q_{n,T}\cap\{|u_n|>1\}} e^{\bar\lambda|u_n|}|\nabla u_n|^p\,dx\,dt \le C\,, \tag{26}$$

$$\sup_{t\in[0,T]} \int\limits_{\Omega_n\cap\{|u_n|>1\}} e^{\bar\lambda|u_n(x,t)|}\,dx \le C\,. \tag{27}$$

*2) (estimates for small values of* $u_n$*). There exists* $\alpha \ge 0$ *such that*

$$\iint\limits_{Q_{n,T}} |u_n|^\alpha|\nabla u_n|^p\,dx\,dt \le C\,, \tag{28}$$

$$\sup_{t\in[0,T]} \int\limits_{\Omega_n} |u_n(x,t)|^{\alpha+2}\,dx \le C\,. \tag{29}$$

*Moreover for every bounded open set* $\Omega^0 \subset \Omega$*,*

$$\iint\limits_{Q^0} |\nabla u_n|^p \le C(\Omega^0)\,, \tag{30}$$

*where* $Q^0_T = \Omega^0 \times (0,T)$*.*

**Proof.** Let $\alpha$ be a nonnegative number, and let $\eta(s) : \mathbf{R} \to [-1,1]$ be a smooth, increasing, odd function, such that

$$\eta(s) = |s|^\alpha s \quad \text{for } |s| \le \frac{1}{2}\,, \qquad \eta(s) = \text{sign } s \quad \text{for } |s| \ge 1\,. \tag{31}$$

Moreover we define the function

$$\varphi(s) = \eta(s)\,e^{\bar\lambda|s|}\,, \tag{32}$$

and its primitive

$$\Phi(s) = \int\limits_0^s \varphi(\sigma)\,d\sigma\,. \tag{33}$$

We take $\varphi(u_n)$ as test function in $(\mathrm{P}_n)$. Integrating on $Q_{n,\tau} = \Omega_n \times (0,\tau)$, we obtain (for simplicity of notation we omit the index $n$ from here on)

$$\int\limits_\Omega \Phi(u(\tau))\,dx - \int\limits_\Omega \Phi(u_0)\,dx + \Lambda_2 \iint\limits_{Q_\tau} |\nabla u|^p \varphi'(u)$$

$$\begin{aligned} &\le \; d \iint\limits_{Q_\tau} |\nabla u|^p |\varphi(u)| + \iint\limits_{Q_\tau} |f||\varphi(u)| \qquad (34)\\ &= \; d \iint\limits_{Q_\tau} |\nabla u|^p |\varphi(u)| + \iint\limits_{Q_\tau \cap \{|u|>1\}} |f||\varphi(u)| + \iint\limits_{Q_\tau \cap \{|u|\le 1\}} |f||\varphi(u)| \\ &= \; A + B + C\,. \end{aligned}$$

Using hypotheses (I1) and (I2), if $\alpha \ge \alpha_0$ one has

$$\int\limits_\Omega \Phi(u_0)\, dx \le c_1 \qquad (35)$$

It is immediate to check that (32) implies

$$\varphi'(s) \ge \bar\lambda |\varphi(s)| \qquad \text{for every } s \in \mathbf{R}, \qquad (36)$$

therefore

$$A \le \frac{d}{\bar\lambda} \iint\limits_{Q_\tau} |\nabla u|^p \varphi'(u). \qquad (37)$$

Before estimating the terms $B$ and $C$, let us observe that, using Sobolev's inequality,

$$\int\limits_\Omega |\nabla u(t)|^p \varphi'(u(t))\, dx = \int\limits_\Omega |\nabla(\Psi(u(t)))|^p\, dx \ge c_2(N,p) \left[ \int\limits_\Omega |\Psi(u(t))|^{p^*}\, dx \right]^{\frac{p}{p^*}}, \qquad (38)$$

where

$$\Psi(s) = \int\limits_0^{|s|} \varphi'(\sigma)^{1/p}\, d\sigma\,, \qquad (39)$$

while $p^* = Np/(N-p)$ denotes the Sobolev exponent of $p$. Let us observe that, for every $s$ such that $|s| \ge 1$,

$$\Psi(s)^p \ge c_3(\bar\lambda, p)|\varphi(s)|\,, \qquad \Phi(s) \ge c_4(\bar\lambda)|\varphi(s)|\,, \qquad (40)$$

where $c_3$ and $c_4$ are positive constants. Since, by assumption (F1) $1 < q' < \frac{p^*}{p}$, one has

$$\frac{1}{q'} = \frac{1-\theta}{1} + \frac{\theta p}{p^*}\,, \qquad \text{with } \theta = \frac{N}{pq} \in\ (0,1)\,.$$

Therefore, using interpolation and (40), we can estimate the term $B$ as follows:

$$B \;\le\; \int\limits_0^\tau \|f(t)\|_{L^q} \|\varphi(u(t))\|_{L^{q'}(\{|u|>1\})}\, dt$$

$$
\begin{aligned}
&\leq \int_0^\tau \|f(t)\|_{L^q} \|\varphi(u(t))\|^{1-\theta}_{L^1(\{|u|>1\})} \|\varphi(u(t))\|^{\theta}_{L^{p^*/p}(\{|u|>1\})}\, dt \\
&\leq c_5 \int_0^\tau \|f(t)\|_{L^q} \left[ \int_{\{|u|>1\}} \Phi(u(t))\, dx \right]^{1-\theta} \|\Psi(u(t))\|^{\theta p}_{L^{p^*}(\{|u|>1\})}\, dt \qquad (41) \\
&\leq \frac{c_2}{4}\left(\Lambda_2 - \frac{d}{\bar{\lambda}}\right) \int_0^\tau \left[ \int_\Omega \Psi(u(t))^{p^*}\, dx \right]^{\frac{p}{p^*}} dt \\
&\quad + c_6 \int_0^\tau \|f(t)\|^{\frac{1}{1-\theta}}_{L^q} \left[ \int_\Omega \Phi(u(t))\, dx \right] dt \,,
\end{aligned}
$$

where the constants depend only on the data. It is easy to check that

$$
\frac{1}{1-\theta} \leq r\,. \tag{42}
$$

As far as the term $C$ is concerned, we observe that

$$
|\varphi(s)| \leq c_7 |s|^{\alpha+1}\,, \qquad \text{for every } s \text{ such that } |s| \leq 1, \tag{43}
$$

$$
\Psi(s) \geq c_8 |s|^{\frac{\alpha+p}{p}}\,, \qquad \Phi(s) \geq c_9 |s|^{\alpha+2}\,, \qquad \text{for every } s \in \mathbf{R}, \tag{44}
$$

where $c_7$, $c_8$, $c_9$ depend on $\eta$ and $\bar{\lambda}$. Therefore

$$
C \leq c_{10} \int_0^\tau \|f(t)\|_{L^q} \left[ \int_{\{|u|\leq 1\}} |u(t)|^{(\alpha+1)q'}\, dx \right]^{\frac{1}{q'}} dt\,.
$$

It is easy to check that, if one chooses $\alpha$ large enough, one has

$$
\alpha + 2 < (\alpha+1)q' < \frac{(\alpha+p)p^*}{p} \tag{45}
$$

(in fact the second inequality holds for every nonnegative $\alpha$ by hypothesis (F1)). Therefore, using interpolation and inequalities (44),

$$
\begin{aligned}
C \;\leq\; & c_{11} \int_0^\tau \|f(t)\|_{L^q} \left[ \int_{\{|u|\leq 1\}} \Phi(u(t))\, dx \right]^{\frac{(\alpha+1)\tilde{\theta}}{\alpha+2}} \\
& \times \left[ \int_{\{|u|\leq 1\}} \Psi(u(t))^{p^*}\, dx \right]^{\frac{(\alpha+1)(1-\tilde{\theta})p}{(\alpha+p)p^*}} dt
\end{aligned}
$$

$$\leq \frac{c_2}{4}\left(\Lambda_2 - \frac{d}{\overline{\lambda}}\right)\int_0^\tau \left[\int_\Omega \Psi(u(t))^{p^*}\,dx\right]^{\frac{p}{p^*}} dt \tag{46}$$

$$+\, c_{12}\int_0^\tau \|f(t)\|_{L^q}^{\frac{\alpha+p}{\alpha+p-(\alpha+1)(1-\tilde{\theta})}} \left[\int_\Omega \Phi(u(t))\,dx\right]^{\frac{(\alpha+1)(\alpha+p)\tilde{\theta}}{(\alpha+2)[\alpha+p-(\alpha+1)(1-\tilde{\theta})]}} dt\,,$$

where $c_{11}$ and $c_{12}$ depend only on the data of the problem, while $\tilde{\theta} \in (0,1)$ is such that

$$\frac{1}{(\alpha+1)q'} = \frac{\tilde{\theta}}{\alpha+2} + \frac{(1-\tilde{\theta})p}{(\alpha+p)p^*}\,.$$

It is easy to check that

$$\frac{\alpha+p}{\alpha+p-(\alpha+1)(1-\tilde{\theta})} \leq r \tag{47}$$

and that

$$\frac{(\alpha+1)(\alpha+p)\tilde{\theta}}{(\alpha+2)[\alpha+p-(\alpha+1)(1-\tilde{\theta})]} < 1\,. \tag{48}$$

Therefore, putting (34), (35), (37), (41), (46) together, taking (42), (47), (48) into account, and setting

$$h(\tau) = \int_\Omega \Phi(u(\tau))\,dx\,,$$

one can write

$$h(\tau)+\frac{1}{2}\left(\Lambda_2 - \frac{d}{\overline{\lambda}}\right)\iint_{Q_\tau} |\nabla u|^p \varphi'(u) \leq c_6\int_0^\tau g_1(\tau)h(\tau)\,dt + c_{12}\int_0^\tau g_2(\tau)h(\tau)^\nu\,dt + c_1\,,$$

where the functions

$$g_1(t) = \|f(t)\|_{L^q(\Omega)}^{\frac{1}{1-\theta}}\,, \qquad g_2(t) = \|f(t)\|_{L^q(\Omega)}^{\frac{\alpha+p}{\alpha+p-(\alpha+1)(1-\tilde{\theta})}}$$

belong to $L^1(0,T)$, while $\nu < 1$. An application of Gronwall's lemma yields that $h(\tau)$ is a bounded function, and that $\iint_{Q_\tau} |\nabla u|^p \varphi'(u)$ is also bounded. This implies (26)–(29), for every $\alpha$ such that (45) holds. Finally, we have to prove the "local" estimate (30). In view of (26), we only have to prove that

$$\iint_{Q_T^0} |T_1(u_n)|^p \leq C(\Omega^0)\,. \tag{49}$$

In order to do this, let us consider a cut-off function $\chi(x) \in C_0^1(\mathbf{R}^N)$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) \equiv 1$ on $\Omega^0$.

We use the test function $\chi(x)^p\varphi(T_1(u_n))$, where $\varphi(s) = \left(e^{\bar\lambda|s|} - 1\right)\operatorname{sign} s$. Note that (36) holds with this choice of $\varphi$. Defining

$$\Phi_1(s) = \int_0^s \varphi(T_1(\sigma))\, d\sigma\,, \tag{50}$$

we obtain, after integration on $Q_T$ (as before we omit the index $n$):

$$\begin{aligned}\iint_{Q_T} |\nabla T_1(u)|^p \varphi'(T_1(u))\chi^p \le p \iint_{Q_T} |a(x,t,u,\nabla u)|\, |\varphi(T_1(u))|\, |\nabla\chi|\, \chi^{p-1} \\ + d \iint_{Q_T} |\nabla T_1(u)|^p |\varphi(T_1(u))|\, \chi^p + d \iint_{Q_T} |\nabla G_1(u)|^p |\varphi(T_1(u))|\chi^p \\ + \iint_{Q_T} |f|\, |\varphi(T_1(u))|\chi^p + \int_\Omega \Phi_1(u_0)\, dx = I_1 + I_2 + I_3 + I_4 + I_5\,.\end{aligned} \tag{51}$$

The integrals $I_4$ and $I_5$ are bounded, and the same is true for $I_3$, by the estimate (26). Using (36), integral $I_2$ can be absorbed by the left-hand side. Finally, as far as $I_1$ is concerned, for every $\varepsilon > 0$ we have, by Young's inequality and (36):

$$\begin{aligned} I_1 &\le p\Lambda_1 \iint_{Q_T} \left(|k_1(x,t)| + |u|^\gamma + |\nabla T_1(u)|^{p-1} + |\nabla G_1(u)|^{p-1}\right) |\varphi(T_1(u))|\, |\nabla\chi|\, \chi^{p-1} \\ &\le \varepsilon \iint_{Q_T} |\nabla T_1(u)|^p \varphi'(T_1(u))\, \chi^p + c_{13}(\varepsilon) \iint_{Q_T} |\varphi(T_1(u))|\, |\nabla\chi|^p \\ &\quad + c_{14} \iint_{Q_T} \left(|k_1| + |u|^\gamma + |\nabla G_1(u)|^{p-1}\right)\,. \end{aligned}$$

By choosing $\varepsilon$ small enough, the first integral in the right-hand side can be absorbed by the left-hand side, while the two remaining integrals are bounded by (26) and (27). Therefore (49) is proved. ■

## 3.2 Estimates under hypotheses (F1), (F2), (I1), (I3): unbounded solutions with finite energy

In this subsection we look for an estimate on $u_n$ in the case where hypotheses (F1), (F2), (I1) and (I3) are assumed. The additional hypotheses (F2) and (I3) will allow us to obtain a better estimate in the region where $u_n$ is small, (i.e., $u_n = T_1(u_n)$), which in general has infinite measure. This improvement consists in taking $\alpha = 0$ in estimates (28) and (29) of Proposition 1.

**Proposition 2** *Assume that* (F1), (F2), (I1), (I3), (A2), (H) *are satisfied. Then the statement of Proposition 1 is true for* $\alpha = 0$.

**Proof.** We choose $\varphi(T_1(u_n))$ as test function, where $\varphi(s) = (e^{\bar\lambda|s|} - 1)\,\mathrm{sign}\, s$. We obtain, after integration on $Q_\tau = \Omega \times (0, \tau)$ (we again omit the index $n$)

$$\int_\Omega \Phi_1(u(\tau))\,dx - \int_\Omega \Phi_1(u_0)\,dx + \Lambda_2 \iint_{Q_\tau} |\nabla T_1(u)|^p \varphi'(T_1(u)) \tag{52}$$
$$\leq d \iint_{Q_\tau} |\nabla T_1(u)|^p |\varphi(T_1(u))| + d \iint_{Q_\tau} |\nabla G_1(u)|^p \varphi(1) + \iint_{Q_\tau} |f|\,|\varphi(T_1(u))|\,,$$

where, as in Proposition 1, $\Phi_1(s)$ is defined by (50). Using (52), (36) (which also holds for this choice of $\varphi$), (26), we obtain

$$\int_\Omega \Phi_1(u(\tau))\,dx + c_1 \iint_{Q_\tau} |\nabla T_1(u)|^p \leq \iint_{Q_\tau} |f|\,|\varphi(T_1(u))| + \int_\Omega \Phi_1(u_0)\,dx + c_2\,,$$

where $c_1$, $c_2$ depend on the data. Taking the supremum on $\tau$, and observing that

$$\Phi_1(s) \geq c_3(\bar\lambda)|T_1(s)|^2 \qquad \text{for every } s \in \mathbf{R},$$
$$|\varphi(s)| \leq c_4(\bar\lambda)|s| \qquad \text{for every } s \text{ such that } |s| \leq 1,$$

one obtains

$$\begin{aligned}
&c_3\|T_1(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + c_1 \iint_{Q_T} |\nabla T_1(u)|^p \\
&\leq 2c_4\|f\|_{L^\rho(0,T;L^\sigma(\Omega))}\|T_1(u)\|_{L^{\rho'}(0,T;L^{\sigma'}(\Omega))} + 2\int_\Omega \Phi_1(u_0)\,dx + 2c_2 \\
&\leq c_5\|f\|_{L^\rho(0,T;L^\sigma(\Omega))}\|T_1(u)\|^{1-\frac{p}{\rho'}}_{L^\infty(0,T;L^2(\Omega))}\left[\iint_{Q_T} |\nabla T_1(u)|^p\right]^{\frac{1}{\rho'}} \\
&\qquad + 2\int_\Omega \Phi_1(u_0)\,dx + 2c_2 \\
&\leq \frac{c_1}{2}\iint_Q |\nabla T_1(u)|^p + c_6\|f\|^\rho_{L^\rho(0,T;L^\sigma(\Omega))}\|T_1(u)\|^{\left(1-\frac{p}{\rho'}\right)\rho}_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad + 2\int_\Omega \Phi_1(u_0)\,dx + 2c_2\,.
\end{aligned} \tag{53}$$

Here we have used hypothesis (F2) on $f(x,t)$, the Gagliardo-Nirenberg inequality (17) (with $m = 2$) and Young's inequality. Since the exponent $\left(1 - \frac{p}{\rho'}\right)\rho$ is less than 2, one obtains, again by Young's inequality,

$$c_3\|T_1(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{c_1}{2}\iint_{Q_T} |\nabla T_1(u)|^p$$

$$\leq \frac{c_3}{2}\|T_1(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + c_7\|f\|^{\frac{2\rho}{2-\rho+p(\rho-1)}}_{L^\rho(0,T;L^\sigma(\Omega))} + 2\int_\Omega \Phi_1(u_0)\,dx + 2c_2\,.$$

The previous inequality implies (28), (29) with $\alpha = 0$. ∎

### 3.3 Bounded solutions

In this subsection we will prove that, if we replace (F1) by (F1′) (*i.e.,* if the exponents $r$ and $q$ satisfy a strict inequality) and (I1) by (I1′), the solutions $u_n$ are uniformly bounded in $L^\infty(Q_{n,T})$. To this aim we will adapt a technique introduced by Stampacchia, which is based on the following lemma (see [24]):

**Lemma 2** *Let $g$ be a nonnegative, nonincreasing function defined on the half line $[k_0,\infty)$. Suppose that there exist positive constants $A$, $\gamma$, $\beta$, with $\beta > 1$, such that*

$$g(h) \leq \frac{A}{(h-k)^\gamma} g(k)^\beta$$

*for every $h > k \geq k_0$. Then $g(k) = 0$ for every $k \geq k_1$, where*

$$k_1 = k_0 + A^{1/\gamma} 2^{\beta/(\beta-1)} g(k_0)^{(\beta-1)/\gamma}\,.$$

It will be useful, moreover, to define the real function

$$G_k(s) = s - T_k(s) = (|s|-k)_+ \operatorname{sign} s\,, \qquad k > 0\,, \tag{54}$$

where $T_k(s)$ is defined in (19), and the sets

$$A_{n,k}(t) = \{x : |u_n(x,t)| > k\}\,, \qquad A_{n,k} = \{(x,t) : |u_n(x,t)| > k\}\,.$$

**Proposition 3** *We assume* (F1′), (I1′), (A2), (H). *Then there exists a constant $C$ depending on the data such that*

$$\|u_n\|_{L^\infty(Q_{n,T})} \leq C\,. \tag{55}$$

**Proof.**

We take $\varphi(G_k(u_n))\,\chi_{(0,\tau)}(t)$ as test function in $(\mathrm{P}_n)$, where $\varphi$ is defined by (31), (32), $k$ is greater than some positive $k_0$ to be chosen hereafter, and the exponent $\alpha$ in (31) is given by

$$\alpha = r(p-1) - p > -1\,. \tag{56}$$

In the case $-1 < \alpha < 0$ we cannot use this function directly, since $\varphi$ is not smooth near zero, and we will have to take approximations. We will examine this point below. For every $H > 0$, choosing (see (23))

$$k \geq k_0 = \max\left\{\sup_n \|u_{0,n}\|_{L^\infty(\Omega_n)}\,,\ \frac{H}{\mu}\right\}\,,$$

and using (36), we obtain (once again, we omit the dependence on $n$)

$$\int_\Omega \Phi\left(G_k(u(\tau))\right)\,dx + \left(\Lambda_2 - \frac{d}{\overline{\lambda}}\right)\iint_{Q_\tau} |\nabla G_k(u)|^p \varphi'(G_k(u))\,dx$$

$$+\,\mu \iint_{Q_\tau} |u|\,|\varphi(G_k(u))|$$

$$\le \iint_{\{|f(t)|\le H\}} |f|\,|\varphi(G_k(u))| + \iint_{\{|f(t)|>H\}} |f|\,|\varphi(G_k(u))|\,,$$

where the function $\Phi(s)$ is defined by (33). Since $\mu k_0 \ge H$, the first integral in the right-hand side is smaller than the last integral of the left-hand side. Therefore

$$\begin{aligned} &\sup_{\tau\in[0,T]} \int_\Omega \Phi\left(G_k(u(\tau))\right)\,dx + c_1 \int_0^T \left[\int_\Omega \Psi(u(t))^{p^*}\,dx\right]^{\frac{p}{p^*}} dt \\ &\le\ 2\left(\iint_{A_{k+1}\cap\{|f(t)|>H\}} |f|\,|\varphi(G_k(u))| + \iint_{A_k\setminus A_{k+1}} |f|\,|\varphi(G_k(u))|\right), \qquad (57)\\ &=\ 2(I+J)\,, \end{aligned}$$

where $\Psi$ is defined as in (39). If $\alpha < 0$, then the function $\varphi$ is not Lipschitz continuous near zero. For this reason we take, instead of $\eta(s)$ defined in (31), its smooth approximation

$$\eta_\delta(s) = \Big(\eta(|s|+\delta) - \eta(\delta)\Big)\operatorname{sign} s\,,$$

where $\delta > 0$. We also define

$$\varphi_\delta(s) = \eta_\delta(s)e^{\overline{\lambda}|s|}\,, \qquad \Psi_\delta(s) = \int_0^{|s|} (\varphi'_\delta(\sigma))^{1/p}\,d\sigma\,, \qquad \Phi_\delta(s) = \int_0^s \varphi_\delta(\sigma)\,d\sigma\,.$$

Since we have again

$$|\varphi_\delta(s)| \le \frac{1}{\overline{\lambda}}\,\varphi'_\delta(s) \qquad \text{for every } s \in \mathbf{R},$$

we obtain (57) for the approximate functions, and we can now pass to the limit for $\delta \to 0$. Therefore (57) is proved in any case. By Hölder's inequality,

$$I \le \|f\,\chi_{\{|f|>H\}}\|_{L^r(0,T;L^q(\Omega))}\ \|\varphi(G_k(u))\,\chi_{A_{k+1}}\|_{L^{r'}(0,T;L^{q'}(\Omega))}\,.$$

As in the proof of Proposition 1, using (40) one shows that

$$\|\varphi(G_k(u))\,\chi_{A_{k+1}}\|_{L^{r'}(0,T;L^{q'}(\Omega))}$$

$$\leq \quad c_2 \left[ \sup_{\tau\in[0,T]} \int_\Omega \Phi\left(G_k(u(\tau))\right)\, dx + \int_0^T \left[ \int_\Omega \Psi(u(t))^{p^*}\, dx \right]^{\frac{p}{p^*}} dt \right],$$

where $c_2$ depends only on $\bar\lambda$, $\alpha$ and the data. Therefore, by choosing $H$ large enough, one can assume that $\|f\,\chi_{\{|f|>H\}}\|_{L^r(0,T;L^q(\Omega))}$ is very small, so that the term $I$ can be absorbed by the left-hand side. As far as the term $J$ is concerned, using (43) and Hölder's inequality, one obtains

$$J \leq c_3 \iint_{Q_T} |G_k(u)|^{\alpha+1}|f| \;\leq\; c_4\, \||G_k(u)|^{\alpha+1}\|_{L^{r'}(0,T;L^{q'}(\Omega))}$$

With the choice (56) of $\alpha$, using Hölder's inequality and the assumption (F1′), and recalling (44), we can write, for every $\varepsilon>0$,

$$\begin{aligned} J \quad &\leq \quad c_5 \left\{ \int_0^T \left[ \int_\Omega \Psi(u(t))^{p^*}\, dx \right]^{\frac{p}{p^*}} dt \right\}^{\frac{1}{r'}} \sup_{\tau\in[0,T]} (\text{meas } A_k(\tau))^{\frac{1}{q'}-\frac{p}{p^* r'}} \\ &\leq \quad \varepsilon \int_0^T \left[ \int_\Omega \Psi(u(t))^{p^*}\, dx \right]^{\frac{p}{p^*}} dt + c_6(\varepsilon) \sup_{\tau\in[0,T]} (\text{meas } A_k(\tau))^{\frac{r}{q'}-\frac{p(r-1)}{p^*}} . \end{aligned}$$

Therefore, choosing $\varepsilon$ small enough, we obtain

$$\sup_{\tau\in[0,T]} \int_\Omega \Phi(G_k(u(\tau)))\, dx \leq c_7 \sup_{\tau\in[0,T]} (\text{meas}\, A_k(\tau))^{\frac{r}{q'}-\frac{p(r-1)}{p^*}} .$$

On the other hand, using the last inequality of (44), one has, for $h>k\geq k_0$,

$$\int_\Omega \Phi(G_k(u(\tau)))\, dx \geq \frac{1}{\alpha+2} \int_\Omega |G_k(u(\tau))|^{\alpha+2}\, dx \geq \frac{(h-k)^{\alpha+2}}{\alpha+2} \text{meas}\, A_h(\tau)\,.$$

Therefore we have proved the following inequality

$$\sup_{\tau\in[0,T]} \text{meas}\, A_k(\tau) \leq \frac{c_8}{(h-k)^{\alpha+2}} \sup_{\tau\in[0,T]} (\text{meas}\, A_k(\tau))^{\frac{r}{q'}-\frac{p(r-1)}{p^*}} .$$

It is easy to check that, under hypothesis (F1′), the last exponent is greater than 1, therefore one can apply Lemma 2 with

$$g(k) = \sup_{\tau\in[0,T]} \text{meas}\, A_k(\tau)\,.$$

Let us remark that, by (29), the function $g(k)$ is bounded for every $k\geq k_0$.

■

## 4 Strong convergence of $\nabla u_n$

This section will be essentially devoted to the strong convergence of the approximate solutions $u_n$ on bounded sets. We first extend $u_n$ to zero in $\Omega \setminus \Omega_n$. By Proposition 1 there exist a subsequence (which we will still denote by $u_n$) and a function $u$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W^{1,p}(\Omega^0)) \text{ and } *\text{-weakly in } L^\infty(0,T;L^q(\Omega^0)), \tag{58}$$

$$e^{\frac{\bar\lambda}{p}u_n} \rightharpoonup e^{\frac{\bar\lambda}{p}u} \quad \text{weakly in } L^p(0,T;W^{1,p}(\Omega^0)), \tag{59}$$

for every bounded open subset $\Omega^0$ of $\Omega$ and for every $q < \infty$. We will first prove that, still extracting a subsequence,

$$u_n \to u \qquad \text{a.e. in } Q_T^0 \text{ and strongly in } L^q(Q_T^0), \text{ for every } q < \infty, \tag{60}$$

where $Q_T^0 = \Omega^0 \times (0,T)$. Indeed, let us consider a function $\eta(x)$ such that

$$\eta \in C_0^\infty(B_R)\,, \quad 0 \le \eta \le 1\,, \quad \eta \equiv 1 \text{ in } \Omega^0, \tag{61}$$

where $B_R$ is a ball containing $\Omega^0$. For simplicity we denote $B_R \cap \Omega$ again by $B_R$. By (30) and the equation satisfied by $u_n$, the sequence $\left\{\frac{\partial}{\partial t}(\eta u_n)\right\}_{n\in\mathbf{N}}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(B_R)) + L^1((0,T)\times B_R)$. Then, by a well-known compactness result (see for instance [23]), the sequence $\eta u_n$ is relatively compact in $L^p((0,T)\times B_R)$. Since, by Proposition 1, $u_n$ is bounded in $L^q(Q_T^0)$, for every $q < \infty$, (60) holds.

As usually happens in nonlinear problems, the crucial point is the strong convergence of the gradients $\nabla u_n$ in $Q_T^0$. The remaining part of this section will be devoted to this aim. We refer to [4], [9] for similar results. We confine ourselves to the case of unbounded solutions (i.e., if (F1) and (I) hold). Indeed, since we only need convergence on bounded sets, we can refer to [21] in case the stronger assumptions (F1′), (I′) are satisfied. In that paper the authors prove strong convergence of $\nabla u_n$ under a uniform $L^\infty$-estimate on $u_n$, in the case where $\Omega$ is a bounded set.

**Proposition 4** *If* (A1), (A2), (A3), (H), (F1), (I1), (I2) *hold true, then there exist a subsequence (still denoted by $u_n$) and a function $u$ such that, for every bounded open set $\Omega^0 \subset \Omega$,*

$$\nabla(e^{\frac{\lambda}{p}|u_n|}) \to \nabla(e^{\frac{\lambda}{p}|u|}) \qquad \textit{strongly in } L^p(Q_T^0), \textit{ for every } \lambda < \bar\lambda, \tag{62}$$

*(and of course weakly for $\lambda = \bar\lambda$) where $Q_T^0 = \Omega^0 \times (0,T)$.*

Note that (62) implies

$$\nabla u_n \to \nabla u \qquad \text{strongly in } L^p(Q_T^0)\,. \tag{63}$$

The most delicate part in proving this result is the strong convergence of the truncated functions $\nabla T_k(u_n)$. In order to prove this convergence, we need a technical result to deal with the derivative with respect to time. We start by introducing a suitable regularization with respect to time (see [18]).

Let $\{u_n\}_{n\in\mathbf{N}}$ be a sequence of solutions of $(\mathrm{P}_n)$ and $u_{0,\nu}$ defined as in (20)–(25). For every $k>0$ and $\nu>0$, we define $T_k(u)_\nu$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{\nu}\ [T_k(u)_\nu]' + T_k(u)_\nu = T_k(u) \\ \\ T_k(u)_\nu(0) = T_k(u_{0,\nu})\,. \end{cases} \tag{64}$$

This means that the following representation formula holds:

$$T_k(u)_\nu(t) = e^{-\nu t} T_k(u_{0,\nu}) + \nu \int_0^t e^{-\nu(t-s)} T_k(u)(s)\, ds\,.$$

We observe that, by $(\mathrm{P}_n)$,

$$\frac{\partial u_n}{\partial t} = \rho_n \in L^{p'}(0,T;W^{-1,p'}(\Omega))$$

in the sense of distributions, where

$$\rho_n = \operatorname{div} a(x,t,u_n,\nabla u_n) - \mu u_n + H_n(x,t,u_n,\nabla u_n) + f_n(x,t)\,.$$

In the sequel we will denote by $\omega^\gamma(h)$ a quantity which goes to zero as $h$ goes to infinity, for every $\gamma$ fixed. Let $\eta(x)$ be a function in $C_0^\infty(\mathbf{R}^N)$ and, as before, $\varphi(s) = \big(e^{\lambda|s|}-1\big)\operatorname{sign} s,\ s\in\mathbf{R}$.

**Lemma 3** *The following inequality holds for every $\delta$ such that $0<\delta<\bar\lambda$:*

$$\langle\langle \rho_n, \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu) e^{\delta|G_k(u_n)|}\rangle\rangle \geq \omega^\nu\,(n) + \omega\,(\nu)\ .$$

*where $\langle\langle\cdot\,,\cdot\rangle\rangle$ denotes the duality between $L^{p'}(0,T;W^{-1,p'}(\Omega_n))$ and $L^p(0,T;W_0^{1,p}(\Omega_n))$.*

**Remark 5** Note that if $n$ is large enough, one has $\operatorname{supp}\eta\cap\Omega\subset\Omega_n$, and therefore the test function $\eta(x)\varphi(T_k(u_n)-T_k(u)_\nu)e^{\delta|G_k(u_n)|}$ belongs to $L^p(0,T;W_0^{1,p}(\Omega_n))$.

**Proof of Lemma 3.** For $\sigma>0$, we define, as before, $u_{n,\sigma}$ as the solution of

$$\begin{cases} \dfrac{1}{\sigma}\ u'_{n,\sigma} + u_{n,\sigma} = u_n \\ \\ u_{n,\sigma}(0) = u_{0,n}\,. \end{cases}$$

We have

$$u_{n,\sigma}\in L^p(0,T;W_0^{1,p}(\Omega_n))\,, \qquad u'_{n,\sigma}\in L^p(0,T;W_0^{1,p}(\Omega_n))\,,$$

$$\|u_{n,\sigma}\|_{L^\infty(\Omega_n\times(0,T))} \leq \|u_n\|_{L^\infty(\Omega_n\times(0,T))}\,,$$

$$u_{n,\sigma} \xrightarrow{\sigma} u_n \qquad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega_n))\,, \tag{65}$$

and

$$u'_{n,\sigma} \xrightarrow{\sigma} \rho_n \qquad \text{strongly in } L^{p'}(0,T;W^{-1,p'}(\Omega_n))\,.$$

(see [4], [16]). Let us call, for simplicity

$$w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\delta|G_k(u_n)|}\,,$$

$$w_\sigma(x,t) = \eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu)e^{\delta|G_k(u_{n,\sigma})|}\,.$$

Then

$$\begin{aligned}
\langle\langle \rho_n, w\rangle\rangle &= \lim_{\sigma\to\infty} \iint_{Q_T} \frac{\partial u_{n,\sigma}}{\partial t}\, w_\sigma = \lim_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t}\,[T_k(u_{n,\sigma}) + G_k(u_{n,\sigma})]\; w_\sigma \\
&= \lim_{\sigma\to\infty} \Bigg( \iint_{Q_T} \frac{\partial}{\partial t}\,[T_k(u_{n,\sigma})]\; \eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu) \\
&\qquad\qquad + \iint_{Q_T} \frac{\partial}{\partial t}\,[G_k(u_{n,\sigma})]\; w_\sigma \Bigg) \\
&= \lim_{\sigma\to\infty} \Bigg( \iint_{Q_T} \frac{\partial}{\partial t}\,[T_k(u_{n,\sigma}) - T_k(u)_\nu]\; \eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu) \\
&\qquad\qquad + \iint_{Q_T} \frac{\partial}{\partial t}\,[T_k(u)_\nu]\; \eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu) \\
&\qquad\qquad + \iint_{Q_T} \frac{\partial}{\partial t}\,[G_k(u_{n,\sigma})]\; w_\sigma \Bigg) \\
&= \lim_{\sigma\to\infty} (I_\sigma^{(1)} + I_\sigma^{(2)} + I_\sigma^{(3)})
\end{aligned}$$

(here we have used the fact that the term $\frac{\partial}{\partial t}T_k(u_{n,\sigma})$ is zero a.e. on the set where $G_k(u_{n,\sigma})$ is different from zero). If we set $\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma$, we get

$$\begin{aligned}
I_\sigma^{(1)} &= \int_\Omega \Phi(T_k(u_{n,\sigma}(T)) - T_k(u)_\nu(T))\,\eta(x)\,dx - \int_\Omega \Phi(T_k(u_{0,n}) - T_k(u_{0,\nu}))\,\eta(x)\,dx \\
&\geq -\int_\Omega \Phi(T_k(u_{0,n}) - T_k(u_{0,\nu}))\,\eta(x)\,dx = \omega^\nu(n) + \omega(\nu)
\end{aligned}$$

(using (20)).

$$\begin{aligned}
I_\sigma^{(2)} &= \nu \iint_{Q_T} [T_k(u) - T_k(u)_\nu]\; \eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu) \\
&= \nu \iint_{Q_T} [T_k(u) - T_k(u)_\nu]\; \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu) + \omega^{\nu,n}(\sigma)
\end{aligned}$$

$$
\begin{aligned}
&= \nu \iint\limits_{Q_T} [T_k(u) - T_k(u)_\nu]\ \eta(x)\varphi(T_k(u) - T_k(u)_\nu) + \omega^{\nu,n}(\sigma) + \omega^\nu(n) \\
&\geq \omega^{\nu,n}(\sigma) + \omega^\nu(n)\ .
\end{aligned}
$$

Here we have used (64) and the convergences (65) and 36a.

$$
\begin{aligned}
I_\sigma^{(3)} &= \iint\limits_{Q_T} \frac{\partial}{\partial t}\left[\frac{e^{\delta|G_k(u_{n,\sigma})|}-1}{\delta}\operatorname{sign} u_{n,\sigma}\right]\eta(x)\varphi(T_k(u_{n,\sigma}) - T_k(u)_\nu) \\
&= \int\limits_\Omega \frac{e^{\delta|G_k(u_{n,\sigma}(T))|}-1}{\delta}\operatorname{sign} u_{n,\sigma}(T)\,\eta(x)\varphi(T_k(u_{n,\sigma}(T)) - T_k(u)_\nu(T))\,dx \\
&\quad - \int\limits_\Omega \frac{e^{\delta|G_k(u_{0,n})|}-1}{\delta}\operatorname{sign} u_{0,n}\,\eta(x)\varphi(T_k(u_{0,n}) - T_k(u_{0,\nu}))\,dx \\
&\quad - \iint\limits_{Q_T} Q_T \frac{e^{\delta|G_k(u_{n,\sigma})|}-1}{\delta}\operatorname{sign} u_{n,\sigma}\,\eta(x)\varphi'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\frac{\partial}{\partial t}[T_k(u_{n,\sigma}) - T_k(u)_\nu] \\
&= I_{3,1} + I_{3,2} + I_{3,3}\ .
\end{aligned}
$$

As far as $I_{3,1}$ is concerned, we observe that $I_{3,1} \geq 0$, since, on the set where the term $G_k(u_{n,\sigma}(T))$ is different from zero, that is, where $|u_{n,\sigma}(T)| > k$, the function $\varphi(T_k(u_{n,\sigma})(T) - T_k(u)_\nu(T))$ has the same sign as $u_{n,\sigma}(T)$ (note that $|T_k(u)_\nu| \leq k$). Moreover, by(20), (21), since $\delta < \bar{\lambda}$,

$$
\begin{aligned}
I_{3,2} &= -\int\limits_\Omega \frac{e^{\delta|G_k(u_0)|}-1}{\delta}\operatorname{sign} u_0\,\eta(x)\varphi(T_k(u_0) - T_k(u_{0,\nu}))\,dx + \omega^\nu(n) \\
&= \omega^\nu(n) + \omega(\nu)\ .
\end{aligned}
$$

$$
I_{3,3} = \iint\limits_{Q_T} \frac{e^{\delta|G_k(u_{n,\sigma})|}-1}{\delta}\operatorname{sign} u_{n,\sigma}\,\eta(x)\varphi'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\frac{\partial}{\partial t}[T_k(u)_\nu]
$$

(since $\frac{\partial}{\partial t}[T_k(u_{n,\sigma})] = 0$ on the set where $G_k(u_{n,\sigma}) \neq 0$). Therefore, by (64),

$$
\begin{aligned}
I_{3,3} &= \nu \iint\limits_{Q_T} \frac{e^{\delta|G_k(u_{n,\sigma})|}-1}{\delta}\operatorname{sign} u_{n,\sigma}\,\eta(x)\varphi'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\ [T_k(u) - T_k(u)_\nu] \\
&= \nu \iint\limits_{Q_T} \frac{e^{\delta|G_k(u_n)|}-1}{\delta}\operatorname{sign} u_n\,\eta(x)\varphi'(T_k(u_n) - T_k(u)_\nu)\ [T_k(u) - T_k(u)_\nu] \\
&\qquad + \omega^{\nu,n}(\sigma) \\
&= \nu \iint\limits_{Q_T} \frac{e^{\delta|G_k(u)|}-1}{\delta}\operatorname{sign} u\ \eta(x)\varphi'(T_k(u) - T_k(u)_\nu)\ [T_k(u) - T_k(u)_\nu] \\
&\qquad + \omega^{\nu,n}(\sigma) + \omega^\nu(n) \\
&\geq \omega^{\nu,n}(\sigma) + \omega^\nu(n)\ .
\end{aligned}
$$

Putting all these estimates together, we obtain the desired result. ■

**Proof of Proposition 4.** By estimate (30), we can say that, up to a subsequence,

$$\nabla u_n \rightharpoonup \nabla u \qquad \text{weakly in } L^p(Q_T^0; \mathbf{R}^N).$$

*Step 1.* We begin by proving that

$$\nabla T_k(u_n) \to \nabla T_k(u) \qquad \text{strongly in } L^p(Q_T^0; \mathbf{R}^N), \text{ for every } k > 0. \tag{66}$$

It will be enough to prove that (from now on we omit the explicit dependence on $x$ and $t$ of the functions)

$$\lim_{n\to\infty} \iint_{Q_T} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \cdot (\nabla T_k(u_n) - \nabla T_k(u))\, \eta = 0\,. \tag{67}$$

where $\eta(x)$ is a cut-off function satisfying (61). Indeed this implies the result by standard arguments (see [19], [20]).

We take

$$w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu) e^{\delta |G_k(u_n)|}$$

as test function in ($\mathrm{P}_n$), where

$$\delta = \frac{d}{\Lambda_2}\,,$$

$T_k(u)_\nu$ is defined by (64), $\varphi(s) = (e^{\lambda |s|} - 1) \operatorname{sign} s$, with $d/\Lambda_2 < \lambda < \bar{\lambda}$, and $n$ is large enough to ensure that $w \in L^p(0,T; W_0^{1,p}(\Omega_n))$. Using Lemma 3 one obtains

$$\begin{aligned}
A + B \;&=\; \iint_{Q_T} a(u_n, \nabla u_n) \cdot \nabla (T_k(u_n) - T_k(u)_\nu) \varphi'(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta \\
&\quad + \mu \iint_{Q_T} u_n \varphi(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta \\
&\le\; \iint_{Q_T} f_n \varphi(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta \\
&\quad + d \iint_{Q_T} |\nabla u_n|^p \varphi(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta \\
&\quad - \delta \iint_{Q_T} a(u_n, \nabla u_n) \cdot \nabla G_k(u_n) \operatorname{sign} u_n\, e^{\delta |G_k(u_n)|}\, \varphi(T_k(u_n) - T_k(u)_\nu)\, \eta \\
&\quad - \iint_{Q_T} a(u_n, \nabla u_n) \cdot \nabla \eta\; e^{\delta |G_k(u_n)|}\, \varphi(T_k(u_n) - T_k(u)_\nu) + \omega(\nu, n) \\
&=\; C + D + E + F + \omega(\nu, n)\,,
\end{aligned}$$

where, using the notation introduced at the beginning of this section,

$$\omega(\nu,n)=\omega^{\nu}(n)+\omega(\nu)\ .$$

As far as the term $C$ is concerned,

$$\begin{aligned} C &= \iint_{Q_T} f_n\varphi(T_k(u_n)-T_k(u)_\nu)\, e^{\delta|G_k(u_n)|}\,\eta \\ &= \iint_{Q_T} f\varphi(T_k(u)-T_k(u)_\nu)\, e^{\delta|G_k(u)|}\,\eta+\omega^{\nu}(n) \\ &= \omega(\nu,n)\,. \end{aligned}$$

Indeed, by (F1) and (18), $f_n$ converges strongly to $f$ in $L^r(0,T;L^q(\Omega))$, $\varphi(T_k(u_n)-T_k(u)_\nu)$ is bounded and converges almost everywhere. Moreover, by Proposition 1, $e^{\frac{\delta}{p}|G_k(u_n)|}\,\eta$ is bounded in $L^\infty(0,T;L^p(B_R))\cap L^p(0,T;W_0^{1,p}(B_R))$, since $\delta\le\bar\lambda$; therefore by Lemma 1 (applied with $m=p$),

$$e^{\frac{\delta}{p}|G_k(u_n)|}\eta \quad \text{is bounded in } L^{\rho_1}(0,T;L^{\sigma_1}(\Omega)), \tag{68}$$

for every $\rho_1$, $\sigma_1$ satisfying (16) with $m=p$, hence $e^{\delta|G_k(u_n)|}\eta$ converges weakly in $L^{r'}(0,T;L^{q'}(\Omega))$.

We next deal with the term $F$. Using the growth assumption (A1) on $a$, one has

$$\begin{aligned} F \ \le\ & \Lambda_1\iint_{Q_T}(k_1(x,t)+|u_n|^\gamma)\ |\nabla\eta|\, e^{\delta|G_k(u_n)|}\,|\varphi(T_k(u_n)-T_k(u)_\nu)| \\ & +\Lambda_1\iint_{Q_T}|\nabla u_n|^{p-1}\,|\nabla\eta|\, e^{\delta|G_k(u_n)|}\,|\varphi(T_k(u_n)-T_k(u)_\nu)| = F_1+F_2\,. \end{aligned}$$

For the integral $F_1$, we observe that the term $(k_1(x,t)+|u_n|^\gamma)$ is bounded in $L^{r_1}(0,T;L^{q_1}(B_R))$ (in fact in $L^{p'r_1}(0,T;L^{p'q_1}(B_R))$), with $r_1$, $q_1$ satisfying (6), so using (68) (with $\eta$ replaced by $\nabla\eta$) one obtains

$$F_1\le\omega(\nu,n)\,.$$

Moreover, by Hölder's inequality,

$$F_2\le\left(\ \iint_{B_R\times(0,T)}|\nabla u_n|^p\, e^{\delta|G_k(u_n)|}\right)^{1/p'}\left(\iint_{Q_T}|\varphi(T_k(u_n)-T_k(u)_\nu)|^p|\nabla\eta|^p\, e^{\delta|G_k(u_n)|}\right)^{1/p}.$$

The first integral is bounded by (26) and (30), since $\delta\le\bar\lambda$, and the second one converges to zero as $n$ and $\nu$ go to infinity. Therefore we have proved that

$$F\le\omega(\nu,n)\,.$$

Similarly, one easily shows that

$$B = \mu \iint_{Q_T} u_n \varphi(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta = \omega(\nu, n)\,.$$

Let us examine the term $A$.

$$\begin{aligned}
A \;=\; & \iint_{\{|u_n|\le k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\
& \qquad\qquad \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'(T_k(u_n) - T_k(u)_\nu)\, \eta \\
& + \iint_{\{|u_n|\le k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\
& \qquad\qquad \cdot \nabla(T_k(u) - T_k(u)_\nu) \varphi'(T_k(u_n) - T_k(u)_\nu)\, \eta \\
& + \iint_{\{|u_n|\le k\}} a(T_k(u_n), \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u)_\nu) \varphi'(T_k(u_n) - T_k(u)_\nu)\, \eta \\
& - \iint_{\{|u_n|> k\}} a(u_n, \nabla u_n) \cdot \nabla T_k(u)_\nu \varphi'(T_k(u_n) - T_k(u)_\nu)\, e^{\delta |G_k(u_n)|}\, \eta \\
=\; & A_1 + A_2 + A_3 + A_4\,.
\end{aligned}$$

Now,

$$\begin{aligned}
|A_2| \;\le\; & \varphi'(2k)\, \| [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\, \eta \|_{L^{p'}(Q_T;\mathbf{R}^N)} \\
& \times \|\nabla(T_k(u) - T_k(u)_\nu)\|_{L^p(Q_T;\mathbf{R}^N)}\,.
\end{aligned}$$

Since the first norm is bounded by the assumption (A1) and (30), while the second goes to zero as $\nu \to \infty$, we conclude that

$$A_2 = \omega(\nu, n)\,.$$

It is easy to check that the same holds for the term $A_3$. As far as the term $A_4$ is concerned, one can use assumption (A1) to obtain

$$\begin{aligned}
A_4 \;\le\; & c_1 \varphi'(2k) \left( \iint_{B_R \times (0,T)} \left( k_1(x,t)^{p'} + |u_n|^{\gamma p'} + |\nabla u_n|^p \right) e^{\delta p' |G_k(u_n)|} \right)^{1/p'} \\
& \times \left( \iint_{B_R \times (0,T)} |\nabla T_k(u)_\nu|^p\, \chi_{\{|u_n|>k\}} \right)^{1/p}.
\end{aligned}$$

The function $|\nabla T_k(u)_\nu|^p\, \chi_{\{|u_n|>k\}}$ converges strongly in $L^1$ (as $n$ and then $\nu$ go to $\infty$) to $\chi_{\{|u|>k\}} \nabla T_k(u) \equiv 0$. On the other hand the first integral of the last formula is bounded by the hypothesis in (A1) on $k_1$, and by the estimates (26), (27) and (30), which by Lemma 1 imply easily, since $\delta p' < \bar{\lambda}$, that

$$e^{\frac{\delta}{p-1}|G_k(u_n)|} \quad \text{is bounded in } L^{\rho_1}(0,T;L^{\sigma_1}(B_R)), \tag{69}$$

for every $\rho_1$, $\sigma_1$ satisfying (16) with $m = p$. Note that this is the point where the full assumptions on $k_1$ and on $\bar{\lambda}$ are used. Therefore we have shown that

$$A = A_1 + \omega(\nu, n)\,.$$

We now deal with the terms $D$ and $E$, which will be estimated together. Using assumption (A2), one has

$$\begin{aligned} D + E \quad \le \quad & \frac{d}{\Lambda_2} \iint_{Q_T} a(u_n, \nabla u_n) \cdot \nabla u_n \, |\varphi(T_k(u_n) - T_k(u)_\nu)| \, e^{\delta |G_k(u_n)|} \, \eta \\ & - \delta \iint_{\{|u_n|>k\}} a(u_n, \nabla u_n) \cdot \nabla u_n \, |\varphi(T_k(u_n) - T_k(u)_\nu)| \, e^{\delta |G_k(u_n)|} \, \eta \end{aligned}$$

(since $\operatorname{sign} u_n \, \varphi(T_k(u_n) - T_k(u)_\nu) = |\varphi(T_k(u_n) - T_k(u)_\nu)|$ where $|u_n| > k$), and therefore, since $\delta = d/\Lambda_2$,

$$\begin{aligned} D + E \quad \le \quad & \frac{d}{\Lambda_2} \iint_{\{|u_n| \le k\}} a(T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) |\varphi(T_k(u_n) - T_k(u)_\nu)| \, \eta \\ = \quad & \frac{d}{\Lambda_2} \iint_{\{|u_n| \le k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\ & \qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)] |\varphi(T_k(u_n) - T_k(u)_\nu)| \, \eta \\ & + \frac{d}{\Lambda_2} \iint_{\{|u_n| \le k\}} a(T_k(u_n), \nabla T_k(u)) \\ & \qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)] |\varphi(T_k(u_n) - T_k(u)_\nu)| \, \eta \\ & + \frac{d}{\Lambda_2} \iint_{\{|u_n| \le k\}} a(T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u) \, |\varphi(T_k(u_n) - T_k(u)_\nu)| \, \eta\,. \end{aligned}$$

It is easy to check that the two last integrals go to zero as $n$ and $\nu$ go to infinity. Therefore we have proved that

$$\begin{aligned} D + E \quad \le \quad & \frac{d}{\Lambda_2} \iint_{\{|u_n| \le k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \\ & \qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)] |\varphi(T_k(u_n) - T_k(u)_\nu)| \, \eta + \omega(\nu, n)\,. \end{aligned}$$

Since $\lambda \ge d/\Lambda_2$, the last integral is less than a fraction of the term $A_1$, and can be cancelled. This shows that $A_1 \le \omega(\nu, n)$, which implies also (since $\varphi'(s) \ge \lambda$)

that

$$\iint\limits_{\{|u_n|\le k\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))\,\eta \le \omega(\nu,n)\,.$$

Therefore

$$\iint\limits_{Q_T} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))\,\eta$$

$$\le \iint\limits_{\{|u_n|>k\}} a(T_k(u_n),\nabla T_k(u))\cdot\nabla T_k(u)\,\eta + \omega(\nu,n) = \omega(\nu,n)\,,$$

which proves (67) and therefore (66).

*Step 2.* We will now prove convergence (62). From (66), since $k$ is arbitrary, it follows that (passing to a subsequence)

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } Q_T^0,$$

Therefore, using (26), (30) and (60), we deduce that, for every $\lambda \le \bar\lambda$,

$$\nabla(e^{\frac{\lambda}{p}|u_n|}) \rightharpoonup \nabla(e^{\frac{\lambda}{p}|u|}) \qquad \text{weakly in } L^p(Q_T^0;\mathbf{R}^N).$$

In order to prove (62), we only have to show that

$$\lim_{n\to\infty}\iint\limits_{Q_T^0} e^{\lambda|u_n|}|\nabla u_n|^p = \iint\limits_{Q_T^0} e^{\lambda|u|}|\nabla u|^p\,,$$

for every $\lambda < \bar\lambda$. For these values of $\bar\lambda$, one has

$$\iint\limits_{Q_T^0} e^{\lambda|u_n|}|\nabla u_n|^p = \iint\limits_{Q_T^0\cap\{|u_n|<k\}} e^{\lambda|u_n|}|\nabla u_n|^p + \iint\limits_{Q_T^0\cap\{|u_n|>k\}} e^{\lambda|u_n|}|\nabla u_n|^p = I_1+I_2\,.$$

By (66), it is easy to verify that, for every positive $k$,

$$I_1 \to \iint\limits_{Q_T^0\cap\{|u|<k\}} e^{\lambda|u|}|\nabla u|^p \qquad \text{as } n\to\infty.$$

Moreover, by (26),

$$I_2 \le e^{(\lambda-\bar\lambda)k}\iint\limits_{\{|u_n|>k\}} e^{\bar\lambda|u_n|}|\nabla u_n|^p \le C(\lambda)e^{(\lambda-\bar\lambda)k}\,.$$

Since $\lambda<\bar\lambda$, taking $k\to\infty$ we obtain the result. ∎

## 5 Proof of the main theorems

As a consequence of the results of the previous section, we can give the proof of the main theorems in a very short way.

By Proposition 4, (A1), (H) and (60), we get (for a subsequence)

$$a(u_n, \nabla u_n) \to a(u, \nabla u) \qquad \text{strongly in } L^{p'}(Q_T^0; \mathbf{R}^N),$$

$$H_n(u_n, \nabla u_n) \to H(u, \nabla u) \qquad \text{strongly in } L^1(Q_T^0; \mathbf{R}^N).$$

In particular, for every bounded open set $\Omega^0 \subset \Omega$,

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \qquad \text{strongly in } L^1(0, T; H^{-m}(\Omega^0)),$$

for $m$ sufficiently large. This implies that

$$u_n \to u \qquad \text{strongly in } C([0, T]; H^{-m}(\Omega^0)),$$

and therefore, since, by the assumptions on the sequence $u_{0,n}$, the initial condition is satisfied by $u$. We can now pass to the limit in ($\mathrm{P}_n$), obtaining that $u$ is a distributional solution of (P). The regularity stated in Theorem 1 for the solution $u$ follows from Proposition 1.

The regularity stated in Theorem 2 follows from Proposition 3, while, if we use Propositions 1 and 2, we get *i)* of Theorem 3. As far as *ii)* is concerned, it is sufficient to use again Proposition 3.

**Remark 6** In the previous theorems we have proved existence of solutions in the sense of distributions. In fact it is easy to verify, by approximation arguments, that one can also take test functions $\phi \in C^\infty(Q_T^0)$ such that $\phi = 0$ on $\Sigma_T = \partial\Omega \times ]0, T[$.

Moreover, we can also use in problem (P) test functions of the form $\psi(x, t) = \varphi(u)\eta(x)$, where $\varphi(s) : \mathbf{R} \to \mathbf{R}$ is any locally Lipschitz function satisfying $\varphi(0) = 0$, $|\varphi'(s)| \le c e^{\lambda|s|}$ for some $c, \lambda > 0$, with $\lambda < \bar{\lambda}$, and $\eta(x)$ is a cut-off function in $C_0^\infty(\Omega)$. For example, one can take $\varphi(s) = s$ or $\varphi(s) = (e^{\lambda|s|} - 1)\mathrm{sign}\, s$.

Indeed, let us take $\varphi(u_n)\eta(x)$ as test function in ($\mathrm{P}_n$). For $t \in (0, T]$ we obtain:

$$\int_\Omega \Phi(u_n(t))\eta\, dx = \int_\Omega \Phi(u_{0,n})\eta\, dx - \iint_{Q_t} a(u_n, \nabla u_n) \cdot \nabla(\varphi(u_n)\eta)$$
$$- \mu \iint_{Q_t} u_n \varphi(u_n)\eta + \iint_{Q_t} H_n(u_n, \nabla u_n)\, \varphi(u_n)\eta + \iint_{Q_t} f_n\, \varphi(u_n)\eta\,.$$

Here, as usual, $\Phi(s) = \int_0^s \varphi(\sigma)\, d\sigma$. Using convergences (60)–(63), the hypotheses on $u_{0,n}$ and the assumptions on the terms of the equation, and using the same techniques as in the proof of Proposition 4, one can easily pass to the limit in all the terms in the right-hand side. Let us now study the term

$$z_n(t) = \int_\Omega \Phi(u_n(t))\eta\, dx\,.$$

We are going to prove that $z_n(t)$ converges in $C([0,T])$ to $z(t) = \int_\Omega \Phi(u(t))\eta\,dx$, using the Ascoli-Arzelà theorem. Indeed, for every pair $0 \le t_1 < t_2 \le T$,

$$
\begin{aligned}
|z_n(t_2) - z_n(t_1)| \quad \le \quad & \iint\limits_{\Omega\times(t_1,t_2)} |a(u_n, \nabla u_n)|\,|\nabla(\varphi(u_n)\eta)| + \mu \iint\limits_{\Omega\times(t_1,t_2)} |u_n \varphi(u_n)\eta| \\
& + \iint\limits_{\Omega\times(t_1,t_2)} |H_n(u_n, \nabla u_n)\,\varphi(u_n)\eta| + \iint\limits_{\Omega\times(t_1,t_2)} |f_n\,\varphi(u_n)\eta|\,.
\end{aligned}
$$

the integrals in the right-hand side are small (uniformly with respect to $n$) if $t_2 - t_1$ is sufficiently small. This is due to the strong convergence in $L^1(Q_T)$ of each of the integrand functions.

Since $z_n(0)$ converges to $z(0)$, the Ascoli-Arzelà theorem implies that, up to subsequences, $z_n(t)$ converges in $C([0,T])$ to some function, which is necessarily $z$ because $z_n$ converges to $z$ strongly in $L^1(0,T)$. Therefore we have proved our assertion.

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