Some Remarks on Elliptic Problems with Critical Growth in the Gradient

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Abstract

In this work we analyze existence, nonexistence, multiplicity and regularity of solution to problem

$$\begin{cases} -\Delta u = \beta(u)|\nabla u|^2 + \lambda f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where β is a continuous nondecreasing positive function and f belongs to some suitable Lebesgue spaces.

1 Introduction and preliminaries

This paper is devoted to some results concerning nonlinear elliptic equations of the form

$$\begin{cases} -\Delta u = \beta(u)|\nabla u|^2 + \lambda f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $\beta(s)$ is a positive continuous function, λ is a positive constant and $f(x)$ is a positive measurable function. We will assume that Ω has a smooth enough boundary, as an example, the interior sphere condition is sufficient to do all the arguments below. Equations of the form (2) have been widely studied in the literature. For instance, in the case where $\beta \equiv$ constant and $f \equiv 0$, this equation may be reckoned as the stationary part of the equation

$$u_t - \varepsilon\Delta u = |\nabla u|^2,$$

which may be viewed as the viscosity approximation as $\varepsilon \to 0^+$ of Hamilton-Jacobi type equations from stochastic control theory (see [36]). The same parabolic equation appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation

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(see [31]). Existence results for problem (2) start from the classic references [35] and [34]. Later on, many authors have been considering elliptic equations with first order terms having quadratic growth with respect to the gradients (see for instance [14], [15], [16], [17], [18], [19], [21], [22], [25], [26], [29], [32], [31], [40], [42], [43], [46], [50] and references therein). We will start from the study of the simpler case of equation (2), that is, the case where $\beta(s) \equiv$ constant. By rescaling, there is no loss in generality in assuming that $\beta(s) \equiv 1$. Therefore, let us consider problem

$$\begin{cases} -\Delta u &= |\nabla u|^2 + \lambda f(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

It is well known that in this case (see [32] and [25]) the change of variable $v = e^u - 1$ leads to the linear equation

$$\begin{cases} -\Delta v &= \lambda f(x)(v+1) \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

which admits a unique solution in $W_0^{1,2}(\Omega)$ provided $f \in L^{N/2}$ and λ is small enough. It is also known that the smallness condition on λ is necessary in order to have existence. This means that, for every $f(x) \geq 0$, with $f \not\equiv 0$, there is no solution of problem (4) for λ large. Therefore equation (2) has no solution in the space of functions u such that $e^u - 1 \in W_0^{1,2}(\Omega)$ (see also [25] for a detailed result in this direction). A first contribution of our paper is a non-existence result in the larger space $W_0^{1,2}(\Omega)$ when λ is large. More precisely, if $f(x)$ is a locally integrable function, verifying

(A) There exists $\phi_0 \in \mathcal{C}_0^\infty(\Omega)$ such that $\displaystyle\int_\Omega |\nabla \phi_0|^2 dx < \lambda \int_\Omega f \phi_0^2 dx < +\infty\,;$

then we show that problem (3) admits no solution in $W_0^{1,2}(\Omega)$ for such λ. We will also analyse the existence and nonexistence under regularity condition on f. It is well known that in general there is no uniqueness of solutions of (3). For instance, if $N > 2$, the functions

$$u_m(x) = \log\left(\frac{|x|^{2-N} - m}{1-m}\right) \in W_0^{1,2}(B_1),\ \ 0 \leq m < 1,$$

all solve the equation $-\Delta u = |\nabla u|^2$ in the unit ball $B_1 = \{x \in I\!R^N\ :\ |x| < 1\}$ (though only the zero function satisfies $e^u - 1 \in W_0^{1,2}(B_1)$). One of the main aims of this paper is to characterize this non-uniqueness phenomenon, and to show that *every* solution of problem (2) comes from a solution of a linear problem with measure data, after a suitable change of variable. The first step is to show that *all solutions* of equation (3), also satisfy some exponential integrability (independently on the regularity of $f(x)$, provided this function is nonnegative). More precisely they verify

$$e^{\delta u} - 1 \in W_0^{1,2}(B_1)\,, \quad \text{for every } \delta < \tfrac{1}{2}.$$

Note that the bound on δ is sharp by the previous example. The main novelty in the proof of this regularity result is the fact that the "regularizing" term is the right-hand side of the equation, rather than –as usually happens– the diffusion term. Using this regularity result we show that if we perform the change of variable $v = e^u - 1$, then the new function v is still in a (larger) Sobolev space, that is, $v \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. We show that v is a distributional solution of the problem

$$\begin{cases} -\Delta v &= \lambda f(x)(v+1) + \mu_s \quad \text{in } \Omega \\ \quad v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

where μ_s is a bounded positive Radon measure which is concentrated on a set of capacity zero. On the other hand, we can also prove a result in the opposite direction, that is, if f is a nonnegative function such that $f \in L^q(\Omega)$, with $q > \frac{N}{2}$, and if λ is small enough, and if μ_s is a bounded positive Radon measure, then problem (5) has a unique solution, a result which was proved for bounded f by Radulescu-Willem [45] (see also Orsina [41]). Then, if μ_s is concentrated on a set of zero capacity, we will show that $u = \log(1+v) \in W_0^{1,2}(\Omega)$ is a solution of problem (3). The remaining part of this paper is devoted to the study of equation (2) for a continuous nonnegative function β under some hypotheses that will be precised in each section. In this case, the change of unknown function

$$v = \Psi(u) = \int_0^u e^{\int_0^s \beta(t)dt}\, ds$$

leads formally to the following semilinear problem,

$$\begin{cases} -\Delta v &= \lambda f(x)(1+g(v)) \quad \text{in } \Omega \\ \quad v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $g(s) : [0,+\infty) \to [0,+\infty)$ is a positive, increasing, convex function which is superlinear at infinity. The passage from problem (2) to problem (6) and viceversa is correct only if the function $v = \Psi(u)$ belongs to $W_0^{1,2}(\Omega)$. We emphasize the fact that, for every choice of increasing function $\beta(s)$, the function g is only very slightly superlinear, in the sense that

$$\int_0^{+\infty} \frac{ds}{1+g(s)} = +\infty\,. \tag{7}$$

However we do not know if, for any choice of $\beta(s)$, the transformed function $g(s)$ satisfies a pointwise condition of the form $g(s) \le cs^q$ for some $q > 1$, though for every choice of β among elementary functions (for instance $\beta(s) = (\log(1+s))^\alpha$, $\beta(s) = s^\alpha$, $\beta(s) = e^s$, $\beta(s) = e^{e^s}$, etc.) this condition is satisfied. We will always assume a condition of this kind (see assumption (**H**) below). We devote Section 3 to the study of slightly superlinear problems of the form (7). These problems are variational in nature, in the sense that their positive solutions are critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f(x) u_+ dx - \lambda \int_\Omega f(x) G(u_+)\, dx\ , \tag{8}$$

where $G(s)$ is the primitive of $g(s)$. For bounded f, and λ not too large, this functional has a concave-convex geometry, which suggests the existence of (at least) two distinct positive solutions. However, due to the slow growth of the nonlinearity, the usual Ambrosetti-Rabinowitz condition, which ensures that all the Palais-Smale sequences for the functional $J_\lambda(u)$ are bounded in $W_0^{1,2}(\Omega)$, does not hold. Therefore the proof of the existence of two solutions for problem (6) has to use more sophisticated tools, such as a more recent result by Jeanjean (see [30]). We show that there exist a positive number λ^* such that problem (6) has at least two positive solutions for $\lambda < \lambda^*$, at least one solution for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$. Therefore this means that for λ not too large problem (2) admits at least two solutions such that $\Psi(u) \in W_0^{1,2}(\Omega)$. The next natural question, by analogy with the case $\beta \equiv 1$, is whether there exist less regular solutions for which the change of variable $v = \Psi(u)$ generates a singular measure in the resulting semilinear problem. The last section of this paper is devoted to answering (positively) to this question. We show that all solutions to problem (2) satisfy some exponential integrability, and that the transformed function v satisfy some semilinear problem with measure data, of the form

$$\begin{cases} -\Delta v = \lambda f(x)(1+g(v)) + \mu_s & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where μ_s is a positive, bounded Radon measure, which is concentrated on a set of zero capacity. Viceversa, every solution of (9), with μ_s singular and $f(x)(1+g(v)) \in L^1(\Omega)$, generates a solution $u \in W_0^{1,2}(\Omega)$ of problem (2). Notice that, using some results by Baras-Pierre [7] and Adams-Pierre [3] (see also [9]) one can prove the existence of a solution of problem (9) for any measure μ_s, for every positive bounded function $f(x)$ and for every nonlinearity g satisfying condition (**H**) below for a small enough (which is true in all model cases). The existence of an infinite number of solutions in $W_0^{1,2}(\Omega)$ of problem (2) in the case of an increasing function $\beta(u)$ should be contrasted with the uniqueness result recently proven by Korkut, Pašić and Žubrinić in [33]. They show that if $\beta(s) \in L^\infty(I\!R) \cap L^1(I\!R)$ and if $f \equiv 0$, then the only solution $u \in W_0^{1,2}(\Omega)$ of problem (2) is the zero function. Remark that if $\beta \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ there exist solutions for all $f \in L^1(\Omega)$ and all $\lambda > 0$. See [43]. The non-uniqueness results are based on the following Picone type inequality (see [2]).

Theorem 1.1 *If $u \in W_0^{1,2}(\Omega)$, $u \geq 0$, $v \in W_0^{1,2}(\Omega)$, $-\Delta v \geq 0$ is a bounded Radon measure, $v|_{\partial\Omega} = 0$, $v \geq 0$ and not identically zero, then*

$$\int_\Omega |\nabla u|^2 \geq \int_\Omega \left(\frac{u^2}{v}\right)(-\Delta v).$$

The plan of the paper is the following: Section 2 is devoted to the study of $\beta \equiv 1$, while in Section 3 we study by variational methods, in the case of general β, the semi-linear problem obtained by change of variables with variational methods in the case of general β. Finally, Section 4 is devoted to study the existence and regularity of weaker solutions and its connection with semi-linear problems with measure data.

The parabolic case will be explained in a forthcoming paper.

2 The case $\beta(u) \equiv 1$: analysis of the solutions in $W_0^{1,2}(\Omega)$

Consider the problem

$$\begin{cases} -\Delta u = |\nabla u|^2 + \lambda f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $\lambda > 0$ and $f \in L^m(\Omega)$, $m \geq \dfrac{N}{2}$, $f(x) \geq 0$. Note that, in order to be a solution in the sense of distributions, a function u must be in $W_0^{1,2}(\Omega)$.

2.1 Existence and nonexistence

Following Kazdan-Kramer [32] we perform the change of variable

$$v = e^u - 1$$

and then problem (10) becomes

$$\begin{cases} -\Delta v &= \lambda f(x)(v+1) \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $f \in L^m(\Omega)$, $m \geq \dfrac{N}{2}$, $f(x) \geq 0$.
It is well known that this problem admits a unique solution provided λ is small enough. As a straightforward consequence we obtain the following result.

Theorem 2.1 *If λ is small enough, there exists a unique solution to problem* (10) *such that $e^u - 1 \in W_0^{1,2}(\Omega)$.*

Next we will study a deeper existence and nonexistence result according with some hypothesis on f and the size of λ. Assume that f is a measurable, non-negative function such that f satisfies the following property:

(**A**) There exists $\phi_0 \in \mathcal{C}_0^\infty(\Omega)$ such that $\displaystyle\int_\Omega |\nabla \phi_0|^2 dx < \lambda \int_\Omega f \phi_0^2 dx < +\infty\,;$

then we have the following nonexistence result.

Theorem 2.2 *If λ, f verify the hypothesis* (**A**) *above, then problem* (10) *has no solution.*

Proof. By contradiction, assume that problem (10) has a solution u, then multiplying by ϕ_0^2 we obtain that

$$2\int_\Omega \phi_0 \nabla\phi_0 \nabla u\, dx = \int_\Omega \phi_0^2 |\nabla u|^2 dx + \lambda \int_\Omega f \phi_0^2 dx.$$

Hence we conclude that

$$\lambda \int_\Omega f\phi_0^2 dx = 2\int_\Omega \phi_0 \nabla \phi_0 \nabla u dx - \int_\Omega \phi_0^2 |\nabla u|^2 dx \leq \int_\Omega |\nabla \phi_0|^2 dx$$

a contradiction with the definition of ϕ_0. ∎

Assume that $f \in L^1(\Omega)$ is a non-negative function, and set

$$\lambda_1(f) = \inf_{\phi \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega f\phi^2 dx} \geq 0. \tag{12}$$

We consider the following hypothesis

(B) $$\lambda_1(f) > 0.$$

Notice that by hypothesis (**B**), $W_0^{1,2}(\Omega)$ is continuously imbedded in $L^2(\Omega, f(x)dx)$, moreover, using the Cauchy-Schwartz inequality for the measure $f(x)dx$ and hypothesis (**B**) we obtain

$$\int_\Omega vf(x)dx \leq \Big(\int_\Omega f(x)dx\Big)^{\frac{1}{2}}\Big(\int_\Omega v^2 f(x)dx\Big)^{\frac{1}{2}} \leq \lambda_1(f)^{-1/2}\Big(\int_\Omega f(x)dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla v|^2 dx\Big)^{\frac{1}{2}},$$

namely, hypothesis (**B**) implies that $f \in W^{-1,2}(\Omega)$.

Theorem 2.3 *Assume that* (**B**) *holds then problem* (10) *has no solution in* $W_0^{1,2}(\Omega)$ *for* $\lambda > \lambda_1(f)$ *and has a unique solution* u *such that* $e^u - 1 \in W_0^{1,2}(\Omega)$ *for* $\lambda < \lambda_1(f)$.

Proof. If $\lambda > \lambda_1(f)$, then by a density argument we can show that condition (**A**) holds. Therefore, by Theorem 2.2 we obtain that problem (10) has no solution. We prove now the existence result. Assume that $\lambda < \lambda_1(f)$ and consider the following problem

$$\begin{cases} -\Delta v &= \lambda f(v+1) & \text{in } \Omega \\ v > 0,\ v &= 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Since $0 < \lambda < \lambda_1(f)$ we have that the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda}{2}\int_\Omega f(x)v^2 - \lambda\int_\Omega fv$$

is well defined in $W_0^{1,2}(\Omega)$ and, moreover:

1) J is coercive, indeed,

$$J(v) \geq \Big(\frac{1}{2} - \frac{\lambda}{\lambda_1(f)}(\frac{1}{2}+\varepsilon)\Big)\int_\Omega |\nabla v|^2 - C(\varepsilon)\lambda\int_\Omega f$$

and if $0 < \varepsilon < \frac{1}{4}(\lambda_1(f) - \lambda)$, then $\delta = \Big(\frac{1}{2} - \frac{\lambda}{\lambda_1(f)}(\frac{1}{2}+\varepsilon)\Big) > 0$.

2) It is easy to see that J is Frechet-differentiable in $W_0^{1,2}(\Omega)$ and then by the *Ekeland Variational Principle* (see [24]), we obtain a sequence $\{v_k\}_{k\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$, $v_k > 0$, such that

$$i)\, v_k \rightharpoonup v \text{ weakly in } W_0^{1,2}(\Omega);\ ii) \quad J(v_k) \to c = \inf_{w\in W_0^{1,2}(\Omega)} J(w) \text{ and } \quad iii) \quad J'(v_k) \to 0$$

3) As a consequence we obtain that v is a weak solution, because for all test function ϕ,

$$0 = \lim_{k\to\infty}(-\int_\Omega v_k\Delta\phi - \int_\Omega f(v_k+1)\phi) = -\int_\Omega v\Delta\phi - \int_\Omega f(v+1)\phi = -\int_\Omega \Delta v\phi - \int_\Omega f(v+1)\phi.$$

then (13) has a unique positive solution $v \in W_0^{1,2}(\Omega)$. It is no too difficult to prove that in fact the convergence of the sequence is strong. Finally by setting $u = \log(v+1)$ we obtain that $u \in W_0^{1,2}(\Omega)$, $e^u - 1 \in W_0^{1,2}(\Omega)$ and

$$-\Delta u = |\nabla u|^2 + \lambda f \text{ in } \mathcal{D}'(\Omega).$$

■

Remark 2.4 Notice that the following examples verifies the assertion in Theorem 2.3:

a) If $f \in L^p(\Omega)$ with $p \geq \frac{N}{2}$, $\lambda_1(f)$ is attained by some eigenfunction $\phi_1 \in W_0^{1,2}(\Omega)$. Moreover, if $p > \frac{N}{2}$ the eigenfunction ϕ_1 is Hölder continuous and then even for $\lambda = \lambda_1(f)$ problem (10) has no solution. Indeed, by contradiction, if u is a solution, taking ϕ_1^2 as a test function in (10) we obtain that

$$2\int_\Omega \phi_1\nabla u\nabla\phi_1 = \int_\Omega |\nabla u|^2\phi_1^2 dx + \lambda_1(f)\int_\Omega f\phi_1^2 dx = \int_\Omega |\nabla u|^2\phi_1^2 dx + \int_\Omega |\nabla\phi_1|^2 dx.$$

Therefore we obtain that $\int_\Omega |\nabla\phi_1 - \phi_1\nabla u|^2 dx = 0$. Hence $\nabla u = \frac{\nabla\phi_1}{\phi_1} = \nabla(\log(\phi_1))$, a contradiction with the fact that $u \in W_0^{1,2}(\Omega)$.

b) If $f(x) = \dfrac{1}{|x|^2}$ we have the Hardy inequality. If we assume that $0 \in \Omega$ then it is well known that $\lambda_1(f) = \dfrac{(N-2)^2}{4}$ and is not attained. Then for $\lambda > \lambda_1(f)$ there is no solution and for $\lambda < \lambda_1(f)$ there exists solution. A nonexistence result of solutions in the class $e^u - 1 \in W_0^{1,2}(\Omega)$ for $\lambda > \lambda_1(f)$ has been obtained by L. Boccardo in [10] even for a general class of elliptic operators in divergence form. In the Laplacian case we prove that even in the larger class $W_0^{1,2}(\Omega)$ there is no solution if $\lambda > \lambda_1(f)$. See also the next remark.

In the case $\lambda = \lambda_1(f)$, using the *improved Hardy inequalities* (see [51] and [1]), it is possible to prove that problem (13) has a solution v in the space H obtained as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$||v||^2 = \int_\Omega |\nabla v|^2 dx - \lambda_1(f)\int_\Omega \frac{v^2}{|x|^2} dx.$$

As a consequence $u = \log(1+v) \in W_0^{1,2}(\Omega)$ and $e^u - 1 \in H$. Obviously the case where $0 \notin \overline{\Omega}$ is included in the previous case a).

c) If $f(x) = \dfrac{1}{\delta(x)}$, where $\delta(x)$ is the distance to the boundary, is not in L^1 but we have a Hardy inequality and that $f \in W^{-1,2}(\Omega)$ (see [20]). Then a slight modification of the argument in Theorem 2.3 allow us to conclude the same result.

Remark 2.5 The above nonexistence result can be easily extended to a large class of elliptic problems like

$$-\mathrm{div}(a(x,u,\nabla u)) = b(x,u,\nabla u) + \lambda f, \quad u \in W_0^{1,p}(\Omega) \tag{14}$$

where f and b are positive functions and

1. $|a(x,u,\xi)| \le c_1|\xi|^{p-1}$.
2. $\mu_1|\xi|^p \le \langle a(x,u,\xi),\xi\rangle \le \mu_2|\xi|^p$ for all $\xi \in I\!\!R^N$.
3. $b(x,u,\xi) \ge c_2|\xi|^p$.

Assume that $f \in L^1(\Omega)$ is a non-negative function, and consider

$$\Lambda(f) = \inf_{\phi \in W_0^{1,p}(\Omega)} \frac{\int\limits_\Omega |\nabla\phi|^p dx}{\int\limits_\Omega f|\phi|^p dx} \ge 0.$$

1. If $\Lambda(f) = 0$ then (14) has not solution.
2. If $\Lambda(f) > 0$ then there exists $\Lambda^* > 0$ such that problem (14) has no solution if $\lambda > \Lambda^*$.

Indeed, if $\phi \in \mathcal{C}_0^\infty(\Omega)$, $\phi \ge 0$ we consider ϕ^p as test function in (14) and by the structural hypotheses of the equation, 1), 2) and 3), we find

$$\lambda \int\limits_\Omega f\phi^p + c_2 \int\limits_\Omega |\nabla u|^p \phi^p \le pc_1 \int\limits_\Omega |\nabla u|^{p-1}\phi^{p-1}|\nabla\phi|,$$

then

$$\lambda \int\limits_\Omega f\phi^p + c_2 \int\limits_\Omega |\nabla u|^p \phi^p \le \varepsilon \int\limits_\Omega |\nabla u|^p \phi^p + C(\varepsilon,p) \int\limits_\Omega |\nabla\phi|^p$$

for ε small enough we obtain

$$\lambda \int\limits_\Omega f\phi^p \le C(\varepsilon,p) \int\limits_\Omega |\nabla\phi|^p,$$

and then for λ large we have a contradiction with the definition of $\Lambda(f)$.

2.2 Regularity

We have found the solution such that $(e^{|u|}-1) \in W_0^{1,2}(\Omega)$. Following the examples in [21] and [25] we will discuss in the next subsection the existence of weaker solutions which still belong to $W_0^{1,2}(\Omega)$. In this subsection we will show that *every* solution $u \in W_0^{1,2}(\Omega)$ of problem (10), and not just the *regular* one given by Theorem 2.1, enjoy *some* exponential regularity. Precisely we have the following Theorem.

Theorem 2.6 *Assume that $u \in W_0^{1,2}(\Omega)$ is a solution of problem (10), where $f(x) \in L^1(\Omega)$ satisfies $f(x) \geq 0$ a.e. in Ω. Then*

$$e^{\delta|u|} - 1 \in W_0^{1,2}(\Omega) \ , \qquad \textit{for every } \delta < \frac{1}{2}. \tag{15}$$

Proof. Assume u is a weak solution to problem (11) and consider as test function

$$v_\varepsilon(x) = e^{\frac{2\delta u}{1+\varepsilon u}} - 1 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Then

$$\begin{aligned} \int_\Omega |\nabla u|^2 e^{\frac{2\delta u}{1+\varepsilon u}} \frac{2\delta}{(1+\varepsilon u)^2} dx &= \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta u}{1+\varepsilon u}} - 1\big) + \int_\Omega f\big(e^{\frac{2\delta u}{1+\varepsilon u}} - 1\big) \\ &\geq \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta u}{1+\varepsilon u}} - 1\big). \end{aligned}$$

Therefore

$$\int_\Omega |\nabla u|^2 \geq \int_\Omega e^{\frac{2\delta u}{1+\varepsilon u}} \big(1 - \frac{2\delta}{(1+\varepsilon u)^2}\big) |\nabla u|^2 dx$$

If $\delta < \frac{1}{2}$ then

$$1 - \frac{2\delta}{(1+\varepsilon u)^2} > 0$$

and by Fatou's Lemma we reach

$$\int_\Omega |\nabla u|^2 \geq \frac{(1-2\delta)}{\delta^2} \int_\Omega |\nabla (e^{\delta u} - 1)|^2 dx$$

■

Remark 2.7 Notice that the regularity given by the previous theorem is optimal. Indeed if we consider $f = 0$ and $\Omega = B_1(0)$, the unit ball, then the equation admits the following family of solutions (see [25])

$$u_m(x) = \log\left(\frac{|x|^{2-N} - m}{1-m}\right), \ 0 \leq m < 1,$$

which satisfies (15), but $e^{\frac{u}{2}} - 1 \notin W_0^{1,2}(\Omega)$.

In the case where f changes sign, we must require that its negative part is in $L^{\frac{N}{2}}(\Omega)$, and the regularity of u will depend on the norm of f_- in this space. More precisely we have the following result:

Theorem 2.8 *Assume that $u \in W_0^{1,2}(\Omega)$ is a solution of problem (10), where $f_+(x) \in L^1(\Omega)$ and $f_-(x) \in L^{\frac{N}{2}}(\Omega)$. Then*

$$e^{\delta|u|} - 1 \in W_0^{1,2}(\Omega) \ , \qquad \textit{for every } \delta \textit{ such that } 0 < \delta < \delta_0 = \frac{1}{1 + \sqrt{1 + S\|f_-\|_{\frac{N}{2}}}}, \tag{16}$$

where $S = S(N)$ is the best constant in the Sobolev inequality.

Proof. We take $v_\varepsilon(x) = e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1$ as test function. Then

$$\begin{aligned}
\int_\Omega |\nabla u|^2 e^{\frac{2\delta|u|}{1+\varepsilon|u|}} \frac{2\delta}{(1+\varepsilon|u|)^2} \operatorname{sign}(u)\, dx &= \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx + \int_\Omega f \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx \\
&\geq \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx - \int_\Omega f_- \ \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx.
\end{aligned}$$

Taking into account that for all $\eta > 0$

$$e^{2t} - 1 \leq (1+\eta)(e^t - 1)^2 + \frac{(1-\eta)^2}{\eta}$$

we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^2 e^{\frac{2\delta|u|}{1+\varepsilon|u|}} \frac{2\delta}{(1+\varepsilon|u|)^2} dx &\geq \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx - \int_\Omega f_- \ \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx \\
&\geq \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx - (1+\eta)||f_-||_{\frac{N}{2}} ||e^{\frac{\delta|u|}{1+\varepsilon|u|}} - 1||_{2^*}^2 - c_\eta ||f_-||_1 \\
&\geq \int_\Omega |\nabla u|^2 \big(e^{\frac{2\delta|u|}{1+\varepsilon|u|}} - 1\big)\, dx - (1+\eta)||f_-||_{\frac{N}{2}} S\delta^2 \int_\Omega |\nabla u|^2 \frac{e^{\frac{2\delta|u|}{1+\varepsilon|u|}}}{(1+\varepsilon|u|)^4} - c_\eta ||f_-||_1
\end{aligned}$$

therefore

$$\int_\Omega |\nabla u|^2 + c_\eta ||f_-||_1 \geq \int_\Omega |\nabla u|^2 e^{\frac{2\delta|u|}{1+\varepsilon|u|}} \Big(1 - 2\delta - (1+\eta)S\delta^2 ||f_-||_{\frac{N}{2}}\Big)\, dx$$

and taking limit in ε we reach the conclusion, provided $\Big(1 - 2\delta - (1+\eta)S\delta^2 ||f_-||_{\frac{N}{2}}\Big) > 0$ which gives the bound on δ. ∎

Remark 2.9 Note that δ_0 goes to $\frac{1}{2}$ when $||f_-||_{\frac{N}{2}} \to 0$ and $\delta_0 \to 0$ when $||f_-||_{\frac{N}{2}}$ increases.

2.3 Existence of weaker solutions: Connection with elliptic problems with measure data

In this subsection we will show a tight relation between problems with first order quadratic terms and linear equations with measure data. This relation will imply a very strong form of non-uniqueness for distributional solutions of problem (10).
We recall that, given a Radon measure μ on Ω and a Borel set $E \subset \Omega$, then μ is said to be concentrated on E if $\mu(B) = \mu(B \cap E)$ for every Borel set B.
Moreover we define by $\operatorname{cap}(E) = \operatorname{cap}_{1,2}(E)$ the capacity of subsets of Ω, which is induced by the norm $\|u\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega |\nabla u|^2\, dx$ (we refer to [38] for an introduction to capacity).

Theorem 2.10 *Let $u \in W_0^{1,2}(\Omega)$ be a solution to problem (10), where $f \in L^1(\Omega)$ is a positive function. Consider $v = e^u - 1$, then there exists a measure μ_s, which is concentrated on a set of zero capacity, such that*

$$\left\{\begin{array}{rcl} -\Delta v &=& \lambda f(x)(v+1) + \mu_s \ \textit{in } \mathcal{D}'(\Omega) \\ v &\in& W_0^{1,q}(\Omega) \ \textit{for all } q < \frac{N}{N-1} \\ T_k(v) &\in& W_0^{1,2}(\Omega), \qquad \log(1+v) \in W_0^{1,2}(\Omega). \end{array}\right. \tag{17}$$

Moreover μ_s can be characterized as a weak limit in the space of bounded Radon measures, as follows:

$$\mu_s = \lim_{\varepsilon \to 0} |\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right). \tag{18}$$

Proof. Since λ does not play any role, we will take $\lambda = 1$. We set $v = e^u - 1$, then by the regularity result of Theorem 2.6 and Hölder's inequality we obtain that $v \in W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$. For $\varepsilon > 0$, take $e^{\frac{u}{1+\varepsilon u}} - 1 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ as test function in (10) Then integrating by parts,

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) dx + \int_\Omega f\,(e^{\frac{u}{1+\varepsilon u}} - 1)\, dx$$

Hence

$$\int_\Omega f\,(e^{\frac{u}{1+\varepsilon u}} - 1)\, dx \le \int_\Omega |\nabla u|^2 dx$$

and then by monotone convergence we conclude that

$$\int_\Omega f\,(e^{\frac{u}{1+\varepsilon u}} - 1) \to \int_\Omega f v dx \le \int_\Omega |\nabla u|^2 dx < +\infty. \tag{19}$$

On the other hand again by the same argument

$$\int_\Omega |\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) dx \le \int_\Omega |\nabla u|^2 dx \tag{20}$$

then, up to a subsequence,

$$|\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) \rightharpoonup \mu_s$$

a positive Radon measure. Notice that μ_s is concentrated on the set $A \equiv \{x \in \Omega : u(x) = +\infty\}$. This follows from the fact that

$$\int\limits_{u \le k} |\nabla u|^2 e^{\frac{u}{1+\varepsilon u}} \left(1 - \frac{1}{(1+\varepsilon u)^2}\right) dx \to 0 \text{ as } \varepsilon \to 0.$$

Since $u \in W_0^{1,2}(\Omega)$, we conclude that $\text{cap}(A) = 0$.
We now define

$$v_\varepsilon(x) = \int_0^{u(x)} e^{\frac{s}{1+\varepsilon s}} ds \in W_0^{1,2}(\Omega).$$

It is easy to check that v_ε solves

$$\begin{aligned} -\Delta v_\varepsilon &= e^{\frac{u}{1+\varepsilon u}}(-\Delta u) - e^{\frac{u}{1+\varepsilon u}} \frac{|\nabla u|^2}{(1+\varepsilon u)^2} \\ &= e^{\frac{u}{1+\varepsilon u}} |\nabla u|^2 (1 - \frac{1}{(1+\varepsilon u)^2}) + \lambda f(x) e^{\frac{u}{1+\varepsilon u}} \end{aligned} \tag{21}$$

in the sense of distributions. The last term converges in $L^1(\Omega)$ by (19), while the remaining one converges to μ_s. Since $v_\varepsilon \to v$ in $L^1(\Omega)$, we obtain that v solves the equation (17) in the sense of distribution. Therefore μ_s is uniquely determined and the convergence in (18) holds for the whole sequence. ■

Remark 2.11 Notice that in the case where $e^{|u|/2} - 1 \in W_0^{1,2}(\Omega)$, that is, the regular solution, the limit in (18) is zero, by Lebesgue's convergence theorem.

Remark 2.12 We emphasize the fact that, given the special elliptic operator under consideration (the Laplace operator), then for measure data the notions of solution in the sense of distributions, in the sense of duality (see [48]) and of renormalized solutions (see [39] and [23]) all coincide (see also [44]).

Theorem 2.13 *Let $f(x)$ be a positive function in $L^r(\Omega)$, with $r > N/2$, and set*

$$\lambda_1(f) = \inf_{\phi \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla \phi|^2 dx}{\displaystyle\int_\Omega f \phi^2 dx}. \tag{22}$$

Let μ be a positive Radon measure with bounded total variation. Then, for all $\lambda < \lambda_1(f)$, problem

$$\left\{ \begin{array}{rcl} -\Delta v &=& \lambda f(x)(v+1) + \mu \text{ in } \mathcal{D}'(\Omega) \\ v &\in& W_0^{1,q}(\Omega) \text{ for all } q < \frac{N}{N-1} \\ T_k(v) &\in& W_0^{1,2}(\Omega), \qquad \log(1+v) \in W_0^{1,2}(\Omega). \end{array} \right. \tag{23}$$

has a unique positive solution v.

Proof. We follow an approximation argument, as in [41]. Let $\{g_n\}$ a sequence of a positive bounded functions such that $g_n \to \mu$ in $\mathcal{M}_0(\Omega)$ and consider the problem

$$\begin{cases} -\Delta v_n &= \lambda f(x)(v_n+1)+g_n & \text{in } \Omega \\ \quad v_n &= 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Since $\lambda < \lambda_1(f)$, then problem (24) has a unique positive solution $v_n \in W_0^{1,2}(\Omega)$.
We claim that v_n is bounded in $L^{r'}(\Omega)$, where $r' = \dfrac{r}{r-1}$. If not, we can extract a subsequence (still denoted by $\{v_n\}$) such that $\|v_n\|_{r'} \to +\infty$. Then we set

$$w_n = \frac{v_n}{\|v_n\|_{r'}}\,.$$

Then w_n solves the equation

$$-\Delta w_n = \lambda f\, w_n + \frac{\lambda f + g_n}{\|v_n\|_{r'}}\,. \tag{25}$$

Since the right-hand side of (25) is bounded in $L^1(\Omega)$, it follows (see [48]) that w_n is bounded in $W^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, and in $L^s(\Omega)$ for every $s < \frac{N}{N-2}$. Then one can extract a subsequence which converges weakly in the same spaces to w. Passing to the limit in (25), one sees that w solves

$$-\Delta w = \lambda f\, w\,. \tag{26}$$

Moreover, by Rellich's compactness theorem, $w_n \to w$ strongly in $L^{r'}(\Omega)$, therefore $w \neq 0$. Moreover, by a bootstrap argument applied to problem (26), one can check that $w \in W_0^{1,2}(\Omega)$. Therefore λ must be an eigenvalue of problem (26), which contradicts the assumption on λ. This proves that v_n is bounded in $L^{r'}(\Omega)$.
Therefore, by applying the same arguments with the sequence $\{w_n\}$ replaced by $\{v_n\}$, one can extract a subsequence which converges weakly to a solution v of (23). Since $\lambda < \lambda_1(f)$, it is easy to prove that $v(x) > 0$ in Ω. Notice that $v \in W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$ and (see again [48]) $T_k(v) \in W_0^{1,2}(\Omega)$ for all $k > 0$. We prove now that $\log(v+1) \in W_0^{1,2}(\Omega)$. By using $z_n = \dfrac{v_n}{v_n+1}$ as a test function in (24) we obtain that

$$\int_\Omega \frac{|\nabla v_n|^2}{(v_n+1)^2}dx = \int_\Omega f\, v_n\, dx + \int_\Omega g_n \frac{v_n}{v_n+1}dx.$$

Hence we conclude that

$$\int_\Omega \frac{|\nabla v_n|^2}{(v_n+1)^2}dx \le C.$$

Therefore by Fatou lemma we conclude that

$$\int_\Omega |\nabla \log(v+1)|^2 dx = \int_\Omega \frac{|\nabla v|^2}{(v+1)^2}dx \le C.$$

The uniqueness follows by a standard bootstrap argument. Note that the operator is the Laplacean, so the well-known counterexamples to uniqueness in the space $W_0^{1,q}(\Omega)$ (see [47]) do not apply. ■

Remark 2.14 The previous proof shows that problem (24) admits a unique solution v for every λ which is not an eigenvalue of problem (26). However, $v > 0$ only for $\lambda < \lambda_1(f)$.

As a consequence we obtain the next result.

Theorem 2.15 *Let μ_s be a bounded positive measure which is concentrated on a set of zero capacity and f is in the hypothesis of Theorem 2.13. For $\lambda < \lambda_1(f)$, let v be the solution to problem*

$$\begin{cases} -\Delta v &= \lambda f(x)(v+1) + \mu_s \ \textit{in } \mathcal{D}'(\Omega) \\ v &\in W_0^{1,q}(\Omega) \ \textit{for all } q < \frac{N}{N-1} \\ T_k(v) &\in W_0^{1,2}(\Omega), \qquad \log(1+v) \in W_0^{1,2}(\Omega). \end{cases} \tag{27}$$

We set $u = \log(v+1)$, then u verifies

$$\begin{cases} -\Delta u &= |\nabla u|^2 + \lambda f(x) \ \textit{in } \mathcal{D}'(\Omega) \\ u &\in W_0^{1,2}(\Omega). \end{cases} \tag{28}$$

Proof. The existence of v is obtained in Theorem 2.13, where it is also proved that $u = \log(v+1) \in W_0^{1,2}(\Omega)$. Let $\{g_n\}$ be a sequence of a bounded positive function such that $\|g_n\|_1 \le c$ and $g_n \to \mu_s$ in $\mathcal{M}_0(\Omega)$. Let v_n be the unique solution to problem

$$\begin{cases} -\Delta v_n &= \lambda T_n(f(v+1)) + g_n(x) \text{ in } \Omega \\ v_n &\in W_0^{1,2}(\Omega). \end{cases} \tag{29}$$

Notice that $v_n \to v$ in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$. We set $u_n = \log(1+v_n)$, then by a direct computation one can obtain that

$$-\Delta u_n = |\nabla u_n|^2 + \lambda \frac{T_n(f(v+1))}{v_n+1} + \frac{g_n}{v_n+1} \text{ in } \mathcal{D}'(\Omega). \tag{30}$$

We will show that the right-hand side of (30) converges to $|\nabla u|^2 + \lambda f$ in $\mathcal{D}'(\Omega)$. This will suffice to prove that u solves (28). It is easy to check that $\dfrac{T_n(f(v+1))}{v_n+1} \to f(x)$ in $L^1(\Omega)$. We now claim that $\dfrac{g_n}{v_n+1} \to 0$ in $\mathcal{D}'(\Omega)$. To prove the claim, let $A \subset \Omega$ be such that $\text{cap}(A) = 0$ and μ is concentrated on A, then for all $\varepsilon > 0$ we get the existence of an open set U_ε such that $A \subset U_\varepsilon$ and $\text{cap}(U_\varepsilon) \le \varepsilon$. Namely for all $\varepsilon > 0$ we get the existence of $\phi \in \mathcal{C}_0^\infty(\Omega)$ such that $\phi \ge 0$, $\phi \equiv 1$ in U_ε and $\int\limits_\Omega |\nabla \phi|^2 dx \le 2\varepsilon$. By using Picone type inequality, see [2], we have

$$\int\limits_\Omega |\nabla \phi|^2 dx \ge \int\limits_\Omega \frac{-\Delta(v_n+1)}{v_n+1} \phi^2 dx \ge \int\limits_{U_\varepsilon} \frac{g_n}{v_n+1}\, dx\,.$$

Hence we conclude that

$$\int\limits_{U_\varepsilon} \frac{g_n}{v_n+1}\, dx \le 2\varepsilon$$

for every n. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$; we wish to show that

$$\lim_{n\to\infty} \int_\Omega \phi \frac{g_n}{v_n+1}\, dx = 0.$$

we can write

$$\int_\Omega \phi \frac{g_n}{v_n+1}\, dx = \int_{U_\varepsilon} \phi \frac{g_n}{v_n+1}\, dx + \int_{\Omega\setminus U_\varepsilon} \phi \frac{g_n}{v_n+1}\, dx.$$

Hence

$$\Big|\int_\Omega \phi \frac{g_n}{v_n+1}\, dx\Big| \le ||\phi||_\infty \int_{U_\varepsilon} \frac{g_n}{v_n+1}\, dx + \int_{\Omega\setminus U_\varepsilon} |\phi|\, g_n\, dx \le 2\varepsilon ||\phi||_\infty + \int_{\Omega\setminus U_\varepsilon} |\phi|\, g_n\, dx.$$

Now since $g_n \to \mu_s$ in $\mathcal{M}_0(\Omega)$ and μ is concentrated on $A \subset U_\varepsilon$, we conclude that

$$\int_{\Omega\setminus U_\varepsilon} |\phi|\, g_n\, dx \to 0 \text{ as } n \to \infty,$$

hence the claim follows. To conclude the proof, let us show that

$$|\nabla u_n|^2 \to |\nabla u|^2 \quad \text{strongly in } L^1(\Omega),$$

that is,

$$\frac{|\nabla v_n|^2}{(1+v_n)^2} \to \frac{|\nabla v|^2}{(1+v)^2} \quad \text{strongly in } L^1(\Omega).$$

Since the sequence converges a.e. in Ω, by Vitali's theorem we only have to show that it is equi-integrable. Let $E \subset \Omega$ be a measurable set. Then, for every $\delta \in (0,1)$ and $k > 0$,

$$\begin{aligned}\int_E \frac{|\nabla v_n|^2}{(1+v_n)^2}\, dx &= \int_{E\cap\{v_n\le k\}} \frac{|\nabla v_n|^2}{(1+v_n)^2}\, dx + \int_{E\cap\{v_n> k\}} \frac{|\nabla v_n|^2}{(1+v_n)^2}\, dx \\ &\le \int_E |\nabla T_k(v_n)|^2\, dx + \frac{1}{(1+k)^{1-\delta}} \int_\Omega \frac{|\nabla v_n|^2}{(1+v_n)^{1+\delta}}\, dx\,.\end{aligned}$$

The last integral is uniformly bounded with respect to n (see, for instance [12]), therefore the corresponding term can be made small by choosing k large enough. Moreover, for every $k > 0$, one has that $T_k(v_n) \to T_k(v)$ strongly on $W_0^{1,2}(\Omega)$ (see [23]), therefore the integral $\int_E |\nabla T_k(v_n)|^2\, dx$ is uniformly small if meas (E) is small enough. The equi-integrability of $|\nabla u_n|^2$ follows immediately, and the proof is completed. ∎

Remark 2.16 If one takes the solution v to problem (27), and makes the change of variable $u = \log(1+v)$, then it is easy to check that u formally satisfies the equation

$$-\Delta u = |\nabla u|^2 + f + \frac{\mu_s}{1+v}\,.$$

The proof of Theorem 2.15 shows that the fraction $\dfrac{\mu_s}{1+v}$ is zero, which corresponds to saying that "$v(x) = +\infty$ on the set on which the singular measure μ_s is different from zero", a result which is obvious in the case where μ_s is a Dirac delta concentrated on some point of Ω. For results on the behavior of solutions of elliptic equations with measure data, one should check the papers [23] and [40].

3 The case of increasing β: variational setting and regular solutions

Consider problem

$$\begin{cases} -\Delta u &= \beta(u)|\nabla u|^2 + \lambda f(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{31}$$

where $\Omega \subset I\!R^N$ is a bounded domain, $f \in L^r(\Omega)$, with $r > \frac{N}{2}$, and

$$\beta : [0,+\infty) \longrightarrow [0,+\infty)$$

is a continuous nondecreasing function such that

$$\lim_{t\to+\infty} \beta(t) = +\infty. \tag{32}$$

We will perform a change in the dependent variable in such a way that the problem becomes semi-linear. We set

$$\gamma(t) = \int_0^t \beta(s)ds, \qquad \Psi(t) = \int_0^t e^{\gamma(s)}ds, \tag{33}$$

then we define

$$v(x) = \Psi(u(x)).$$

Then problem (31) becomes

$$\begin{cases} -\Delta v &= \lambda f(x)(1+g(v)) \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{34}$$

where

$$g(t) = e^{\gamma(\Psi^{-1}(t))} - 1 = \int_0^t \beta(\Psi^{-1}(s))ds\ . \tag{35}$$

The main properties of the differentiable function $g : [0,+\infty) \longrightarrow [0,+\infty)$ are:

1. $g(0) = 0$, and g is increasing and convex

2. $\lim_{s\to 0} \frac{g(s)}{s} = g'(0) = \beta(0)$

3. $\lim_{s\to +\infty} \frac{g(s)}{s} = +\infty$

4. $\int_0^{+\infty} \frac{ds}{1+g(s)} = +\infty$; indeed

$$\int_0^{+\infty} \frac{ds}{1+g(s)} = \int_0^{+\infty} \frac{ds}{e^{\gamma(\psi^{-1}(s))}} = \int_0^{+\infty} \frac{e^{\gamma(t)}}{e^{\gamma(t)}} dt = +\infty \;..$$

Proposition 3.1 *Assume that g verifies the assumptions above. There exists λ_0 such that for $\lambda \leq \lambda_0$, problem* (34) *has at least a positive solution $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and then $u = \Psi^{-1}(v) \in W_0^{1,2}(\Omega)$ is a positive solution of* (31)*.*

Proof. We look for a super-solution in the form $\overline{v} = tw$, where w is the solution to problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega \\ \quad w \in W_0^{1,2}(\Omega) \; . \end{cases}$$

Notice that the function

$$h(t) = \frac{t}{1 + g(t||w||_\infty)}$$

admits a positive maximum in $I\!R^+$. If $0 < \lambda \leq \lambda_0 = \max_{I\!R^+} h(t)$, fixed t such that $t \geq \lambda(1 + g(t||w||_\infty))$, then since g is increasing

$$-\Delta \overline{v} = tf \geq \lambda f\,(1 + g(t||w||_\infty) \geq \lambda f\,(1 + g(\overline{v})).$$

To have a sub-solution we consider $\underline{v} = t_1\phi_1$ where ϕ_1 is the normalized positive eigenfunctions corresponding to the first eigenvalue to problem

$$\lambda_1(f) = \inf_{\phi \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \phi|^2\, dx}{\int_\Omega f\, \phi^2\, dx} \tag{36}$$

Since $g > 0$, it suffices to have $\lambda_1(f) t_1 ||\phi_1||_\infty \leq \lambda$ then

$$-\Delta \underline{v} = t_1 \lambda_1(f) f\, \phi_1 \leq \lambda f\,(1 + g(\underline{v}))$$

Moreover $\underline{v} \leq \overline{v}$ for t_1 small enough, by Hopf's Lemma. The result is a consequence of the usual iteration argument. ■

Theorem 3.2 *There exists $\Lambda > 0$ such that, if $\lambda > \Lambda$, then problem* (34) *has no positive solution $v \in W_0^{1,2}(\Omega)$.*

Proof. Using the properties of g, we get the existence of a positive constant $c > 0$ such that $g(s) \geq cs - 1$. Consider now ϕ_1 defined as in (36), then multiplying equation (34) by ϕ_1 and using the hypothesis on g we obtain that

$$\lambda_1(f) \int_\Omega f v \phi_1 \, dx = \lambda \int_\Omega f \,(g(v) + 1)\, \phi_1 \, dx \geq \lambda c \int_\Omega f v \phi_1 \, dx \ .$$

Hence we conclude that

$$\lambda_1(f) \int_\Omega f v \phi_1 \, dx \geq \lambda c \int_\Omega f v \phi_1 \, dx \ .$$

Choosing λ such that $c\lambda > \lambda_1(f)$ we obtain that $\int_\Omega f v \phi_1 \, dx = 0$; therefore the strong maximum principle implies $v \equiv 0$. Hence problem (34) has no positive solution for $\lambda > \Lambda = \dfrac{\lambda_1(f)}{c}$. ■

Corollary 3.3 *Let Λ be as in Theorem 3.2, then for $\lambda > \Lambda$ problem* (31) *has no solution u such that $\Psi(u) \in W_0^{1,2}(\Omega)$.*

We will see in section 4, Proposition 4.8 that the nonexistence result for λ large remain true even in the distributional framework.

Remark 3.4 In the case where β is a decreasing function, it is easy to conclude that $\dfrac{g(s)}{s}$ is also decreasing. In this case problem (34) has a unique solution for λ small enough. The existence can be proved as in Proposition 3.1, while for uniqueness we refer to [5]. If, moreover, $\beta(s) \downarrow 0$, then $\dfrac{g(s)}{s} \downarrow 0$ as $s \to \infty$ then there exist a unique solution for all $\lambda \in \mathbb{R}^+$. These observations motivates the hypotheses of β nondecreasing to have two solutions to problem (34).

Next we will prove the existence of a second positive solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ under the following extra hypotheses on β and f. We assume that β satisfies

$$(\mathbf{H}) \qquad \lim_{t \to +\infty} \frac{\beta(t)}{e^{a \int_0^t \beta(s) ds}} = 0, \text{ for some } a < \frac{4}{N+2}$$

or its equivalent form

$$\lim_{t \to +\infty} \frac{g'(t)}{(1 + g(t))^a} = 0, \text{ for some } a < \frac{4}{N+2}$$

then, using the expression of g and De L'Hôpital's rule, it is easy to check that

$$\lim_{t \to +\infty} \frac{g(t)}{t^q} = 0, \ 1 - \frac{1}{q} = a. \tag{37}$$

By direct calculation we check that condition (**H**) is satisfied for the elementary functions such as $\beta(s) = (\log(1+s))^\alpha$, $\beta(s) = s^\alpha$, $\beta(s) = e^s$, $\beta(s) = e^{e^s}$, etc.

Notice that in this way $q < \frac{N+2}{N-2} = 2^* - 1$, and problem (34) becomes variational in nature. Moreover this variational problem has a subcritical concave-convex structure. We will look for positive solutions to problem (34) as critical points of the associated energy functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f\, u_+ dx - \lambda \int_\Omega f\, G(u_+) dx, \tag{38}$$

where

$$G(s) = \int_0^s g(t)dt\,,$$

which is well defined in in $W_0^{1,2}(\Omega)$. As far as $f(x)$ is concerned, for simplicity we will prove the result in the case where it is a non-negative, bounded function. However all the results can be easily proved under the assumption that

$$\textbf{(F)} \qquad f(x) \in L^r(\Omega), \quad \text{for } r > \frac{2^*}{2^* - (q+1)},$$

where q is defined by (37).

As a first step, we will prove the existence of at least two positive solutions for λ small enough. Precisely we have the following result.

Theorem 3.5 *Assume that* (32) *and* (**H**) *hold, that $f(x)$ is bounded and non-negative, and that the functional J_λ has the geometry of the mountain pass, that is, there exist two points $v_1, v_2 \in W_0^{1,2}(\Omega)$ such that, setting*

$$\Gamma = \{\gamma \in C([0,1]; W_0^{1,2}),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there holds

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}. \tag{39}$$

Then problem (34) *has a mountain-pass type positive solution u.*

Corollary 3.6 *There exists λ_0 such that if $0 < \lambda \le \lambda_0$, then the functional J_λ has the geometry of the mountain pass and then problem* (34) *has at least two positive solutions.*

PROOF. Since $J_\lambda(0) = 0$, using (**H**) one can easily prove that for λ small enough there exists a number $R = R(\lambda) > 0$ such that $J_\lambda(v) \ge \rho_0 > 0$ for every v satisfying $\|v\| = R$. On the other hand, using the superlinearity at ∞ of $g(s)$, it is easy to prove, for every $\lambda > 0$, the existence of a function $w \in W_0^{1,2}(\Omega)$, with norm arbitrarily large, such that $J_\lambda(w) < 0$. Therefore

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0 = \max\{J_\lambda(0), J_\lambda(w)\}.$$

Therefore, applying Theorem 3.5 to the points 0 and w, we obtain the existence of a positive solution v_1 to problem (34) such that $J_\lambda(v_1) = c(\lambda) > 0$. We have to prove that $v_1 \ne v$, where v is the minimal solution obtained by Proposition 3.1. It is sufficient to prove that $J_\lambda(v) \le 0$.

We follow closely the argument used in [5]. Setting $a(x) = \lambda f(x) g'(v(x)) \in L^r(\Omega)$, $r > \frac{N}{2}$, we are able to define the first eigenvalue m_1 to problem

$$\begin{cases} -\Delta\phi_1 - a(x)\phi_1 &= m_1\phi_1 \quad \text{in } \Omega \\ \phi_1 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{40}$$

Here ϕ_1 is the associated eigenfunction and then we can take $\phi_1 > 0$. We will prove that $m_1 \geq 0$. We argue by contradiction. Assume that $m_1 < 0$. We claim that there exists $\alpha > 0$ such that $v - \alpha\phi_1$ is a supersolution to problem (34). Indeed, we have

$$-\Delta(v - \alpha\phi_1) - \lambda f\,(1 + g(v - \alpha\phi_1)) = \lambda f\,[g(v) - g(v - \alpha\phi_1) - \alpha g'(v)\phi_1] - \alpha m_1\phi_1.$$

Since $\lambda f\,[g(v) - g(v - \alpha\phi_1) - \alpha g'(v)\phi_1] = o(\alpha\phi_1)$ and using the fact that $m_1 < 0$, we get

$$\lambda f\,[g(v) - g(v - \alpha\phi_1) - \alpha g'(v)\phi_1] - \alpha m_1\phi_1 \geq 0.$$

Hence by comparison the claim follows. Moreover, by Hopf's lemma, one has $v - \alpha\phi_1 > 0$ for α small enough. Then by an iteration argument we obtain that problem (34) has a positive solution $w \leq v - \alpha\phi_1$, a contradiction with the definition of v as the minimal solution. Hence $m_1 \geq 0$.

In particular we have

$$\int\limits_\Omega |\nabla v|^2 dx - \lambda \int\limits_\Omega f\, g'(v) v^2 dx \geq 0. \tag{41}$$

We prove now that $J_\lambda(v) \leq 0$. Since v is a solution to (34) we obtain that

$$\begin{aligned} J_\lambda(v) &= J_\lambda(v) - \frac{1}{2}\langle J'_\lambda(v), v\rangle = \frac{\lambda}{2}\int\limits_\Omega f\,\Big(g(v)v - 2G(v) - v\Big)dx \\ &= -\frac{\lambda}{2}\int\limits_\Omega f\,\Big((1+g(v))v - v^2 g'(v)\Big)dx - \frac{\lambda}{2}\int\limits_\Omega f\,\Big(2G(v) - 2g(v)v + v^2 g'(v)\Big)dx \end{aligned} \tag{42}$$

Using (41) we obtain that

$$\frac{\lambda}{2}\int\limits_\Omega f\,\Big((1+g(v))v - v^2 g'(v)\Big)dx = \frac{1}{2}\int\limits_\Omega \Big(|\nabla v|^2 - \lambda f\, v^2 g'(v)\Big)dx \geq 0.$$

We deal with the second term in (42). We set $h(s) = 2G(s) - 2sg(s) + s^2 g'(s)$, then $h(0) = 0$ and $h'(s) \geq 0$. Hence we conclude that $h(s) \geq 0$ for all s. In particular

$$\frac{\lambda}{2}\int\limits_\Omega f(x)\Big(2G(v) - 2g(v)v + v^2 g'(u)\Big)dx = \frac{\lambda}{2}\int\limits_\Omega f\, h(v)dx \geq 0.$$

Hence we conclude that $J_\lambda(v) \leq 0$. ∎

For the proof of Theorem 3.5 we use the following general result proved in [30].

Theorem 3.7 *Let X be a Banach space endowed with the norm $||.||$ and let $\mathcal{J} \subset I\!R^+$ be an interval. Let $\{J_\alpha\}_{\alpha\in\mathcal{J}}$ be a family of functionals on X of the form*

$$J_\alpha(u) = A(u) - \alpha B(u)$$

where $B(u) \geq 0$ and such that $A(u)$ or $B(u) \to +\infty$ as $||u|| \to \infty$. We assume that there exist two points $v_1, v_2 \in X$ such that, setting

$$\Gamma = \{\gamma \in C([0,1]; X),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there hold, for all $\alpha \in \mathcal{J}$,

$$c(\alpha) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\alpha(\gamma(t)) > \max\{J_\alpha(v_1), J_\alpha(v_2)\}.$$

Then for almost every $\alpha \in \mathcal{J}$, there exists a sequence $\{v_k\} \subset X$ such that : i) $\{v_k\}$ is bounded; ii) $J_\alpha(v_k) \to c(\alpha)$ and iii) $J'_\alpha(v_k) \to 0$ in X', the dual of X.

Proof of Theorem 3.5 Assume that (39) holds. By a continuity argument we get the existence of $\varepsilon > 0$ such that for all $\alpha \in \mathcal{J} = [1-\varepsilon, 1+\varepsilon]$, the family of functional $\{J_{\lambda,\alpha}\}_{\alpha\in\mathcal{J}}$ defined by

$$J_{\lambda,\alpha}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda\alpha\Big(\int_\Omega f\, u_+ dx + \int_\Omega f\, G(u_+)dx\Big)$$

have the same geometry, namely

$$c(\lambda,\alpha) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\alpha}(\gamma(t)) > \max\{J_{\lambda,\alpha}(v_1), J_{\lambda,\alpha}(v_2)\}.$$

Notice that (v_1, v_2) are independent of $\alpha \in \mathcal{J}$. By Theorem 3.7 we obtain that for almost every $\alpha \in \mathcal{J}$ there exists a sequence $\{v_k^{(\alpha)}\}$ such that: *i)* $\{v_k^{(\alpha)}\}$ is bounded; *ii)* $J_{\lambda,\alpha}(v_k^{(\alpha)}) \to c(\lambda,\alpha)$ and *iii)* $J'_{\lambda,\alpha}(v_k^{(\alpha)}) \to 0$ in $W^{-1,2}(\Omega)$. Since g verifies (**H**), then using a compactness argument we obtain that the Palais-Smale compactness condition holds, namely, up to a subsequence, $v_k^{(\alpha)} \to v^{(\alpha)}$ strongly in $W_0^{1,2}(\Omega)$, where $v^{(\alpha)}$ is a positive solution to problem

$$\left\{\begin{array}{rcll} -\Delta v^{(\alpha)} &=& \lambda\alpha f\,(1+g(v^{(\alpha)})) & \text{in } \Omega \\ v^{(\alpha)} &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{43}$$

such that $J_{\lambda,\alpha}(v^{(\alpha)}) = c(\lambda,\alpha)$. We have to prove that the conclusion in Theorem 3.7 holds for $\alpha = 1$. Let $\{\alpha_n\}$ be a decreasing sequence in $\mathcal{J}$ such that $\alpha_n \downarrow 1$ as $n \to \infty$ and consider $v^{(\alpha_n)}$ the corresponding solution to problem (43). We will prove that $\{v^{(\alpha_n)}\}$ is bounded in $W_0^{1,2}(\Omega)$. For the simplicity of notation we set $v_n = v^{(\alpha_n)}$. If $||v_n||_\infty \leq C$ for all n, then using (43) and by the condition on f and g we conclude that $||v_n||_{W_0^{1,2}} \leq C_1$. Assume now that $||v_n||_\infty \to +\infty$ as $n \to \infty$. Notice that

$$\int_\Omega f\, v_n dx \leq C. \tag{44}$$

Indeed, consider ϕ_1 the positive eigenfunction associated to the first eigenvalue

$$\begin{cases} -\Delta\phi_1 = \lambda_1(f) f\,\phi_1 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{45}$$

By taking ϕ_1 as a test function in (43) we obtain that

$$\lambda_1(f)\int_\Omega f\,\phi_1 v_n dx = \lambda\alpha_n \int_\Omega f\,\phi_1 + \lambda\alpha_n \int_\Omega f\,g(v_n)\phi_1 dx.$$

Since the hypothesis 3. on g it is easy to check that there exists a constant C_1 such that

$$\int_\Omega f\,\phi_1 v_n dx \le C_1 \qquad \text{and} \qquad \int_\Omega f\,\phi_1 g(v_n) dx \le C_1.$$

Let now ϕ_2 be the solution to problem

$$\begin{cases} -\Delta\phi_2 = f & \text{in } \Omega \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{46}$$

Notice that, by Hopf Lemma, there exist $c_1, c_2 > 0$ such that $c_1\phi_1 \le \phi_2 \le c_2\phi_1$. Taking ϕ_2 as a test function in (43) we obtain that

$$\int_\Omega f\,v_n dx = \lambda\alpha_n \int_\Omega f\,\phi_2 + \lambda\alpha_n \int_\Omega f\,g(v_n)\phi_2 dx. \tag{47}$$

Since $\phi_2 \le c_2\phi_1$ we conclude that

$$\int_\Omega f\,v_n dx \le \lambda\alpha_n \int_\Omega f\,\phi_2 + c_2\lambda\alpha_n \int_\Omega f\,g(v_n)\phi_1 dx.$$

Hence $\int_\Omega f\,v_n dx \le C$. As $J_{\lambda,\alpha_n}(v_n) = c(\lambda,\alpha_n) \le c(\lambda)+1$, by using (44) we obtain that

$$\int_\Omega f\,(g(v_n)v_n - 2G(v_n))dx \le C. \tag{48}$$

We now prove the energy estimate. Assume by contradiction that $||v_n||_{W_0^{1,2}} \to \infty$ as $n \to \infty$. We set $w_n = \dfrac{v_n}{||v_n||_{W_0^{1,2}}}$, then $||w_n||_{W_0^{1,2}} = 1$, hence we get the existence of $w_0 \in W_0^{1,2}(\Omega)$ such that, up to subsequences, $w_n \rightharpoonup w_0$ weakly in $W_0^{1,2}(\Omega)$ and $w_n \to w_0$ strongly in $L^a(\Omega)$ for all $a < \frac{2N}{N-2}$. Moreover w_n verifies

$$-\Delta w_n = \frac{\alpha_n\lambda f}{||v_n||_{W_0^{1,2}}} + \alpha_n\lambda\frac{f\,g(v_n)}{||v_n||_{W_0^{1,2}}}.$$

Since $w_n \rightharpoonup w_0$ weakly in $W_0^{1,2}$ we obtain that

$$\int_\Omega -\Delta w_0 \phi = \lim_{n\to\infty} \lambda \int_\Omega \frac{f\, g(v_n)}{||v_n||_{W_0^{1,2}}} \phi \text{ for all } \phi \in \mathcal{C}_0^\infty(\Omega). \tag{49}$$

From (47) and (44) we obtain that $fg(v_n)$ is bounded in $L^1_{\text{loc}}(\Omega)$. Therefore (49) implies $w_0 = 0$. Let $z_n = t_n v_n$ where t_n is defined as

$$t_n = \inf\left\{ t \in [0,1] \mid J_{\lambda,\alpha_n}(tv_n) = \max_{t\in[0,1]} J_{\lambda,\alpha_n}(tv_n) \right\}.$$

We prove that $t_n \in (0,1)$ for n large enough. That $t_n \neq 0$ is obvious because $J_{\lambda,\alpha_n}(0) = 0$ for all values of α_n. To show that $t \neq 1$ we claim that

$$\lim_{n\to\infty} J_{\lambda,\alpha_n}(z_n) = +\infty\,. \tag{50}$$

We argue by contradiction; if $\liminf\limits_{n\to\infty} J_{\lambda,\alpha_n}(z_n) \le M$, we set $u_n = \sqrt{4M} w_n$, then $u_n \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega)$, hence $\int\limits_\Omega f\, G(u_n) dx$, $\int\limits_\Omega f\, u_n dx \to 0$ as $n \to \infty$. Therefore we obtain that

$$J_{\lambda,\alpha_n}(u_n) = 2M - \frac{\alpha_n \lambda}{2} \int_\Omega f u_n dx - \alpha_n \lambda \int_\Omega f\, G(u_n) dx \ge \frac{3}{2} M \text{ as } n \to \infty. \tag{51}$$

On the other hand, using the definition of z_n and observing that $u_n = \frac{\sqrt{4M}}{||v_n||_{W_0^{1,2}}} v_n$, we obtain that

$$J_{\lambda,\alpha_n}(u_n) \le J_{\lambda,\alpha_n}(z_n) \le M$$

a contradiction with (51). Hence (50) is proved.
Therefore, taking into account that $J_{\lambda,\alpha_n}(v_n) = c_{\lambda,\alpha_n} \le c(\lambda) + 1$ and by the claim, we conclude $t_n \neq 1$ for n large enough. As a consequence by the definition of z_n we have $\langle J'_{\lambda,\alpha_n}(z_n), z_n \rangle = 0$, hence we conclude that

$$J_{\lambda,\alpha_n}(z_n) = \frac{\alpha_n \lambda}{2} \int_\Omega f\, (g(z_n) z_n - 2G(z_n))\, dx - \frac{\alpha_n \lambda}{2} \int_\Omega f\, z_n\, dx\,.$$

Since $\int\limits_\Omega f\, z_n\, dx \le \int\limits_\Omega f\, v_n\, dx \le C$, by (50) we conclude that

$$\int_\Omega f\, (g(z_n) z_n - 2G(z_n))\, dx \to +\infty, \quad \text{as } n \to \infty.$$

By the fact that the function $l(s) = g(s)s - 2G(s)$ is increasing we obtain that

$$g(z_n) z_n - 2G(z_n) \le g(v_n) v_n - G(v_n)$$

and then

$$\int\limits_{\Omega} f\,(g(v_n)v_n - 2G(v_n))\,dx \to \infty \qquad \text{as } n \to \infty\,,$$

a contradiction with (48). As a consequence we conclude that

$$||v_n||_{W_0^{1,2}} \le C_1.$$

Therefore $v_n \to v$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^\theta(\Omega)$ for all $\theta < \frac{2N}{N-2}$. Using again the hypotheses on g and by a simple compactness argument we obtain that v is a weak solution to problem (34) and then we get easily that $v_n \to v$ strongly in $W_0^{1,2}(\Omega)$. Therefore we conclude that v is a non-negative solution to problem (34) such that

$$c(\lambda, \alpha_n) = J_{\lambda,\alpha_n}(v_n) \to J_\lambda(v) \quad \text{as } n \to \infty.$$

Hence we get the existence of a positive solution v to problem (34) with $J_\lambda(v) = c(\lambda)$ and the proof is complete. ■

We now prove the general existence result.

Theorem 3.8 *Under the same assumptions of Theorem 3.5, let λ^* be defined by*

$$\lambda^* = \sup\{\lambda \ge 0 \text{ such that problem (34) has a positive solution }\}. \tag{52}$$

Then for all $\lambda \in (0, \lambda^*)$, problem (34) *has at least two positive solutions. If $\lambda = \lambda^*$, then problem* (34) *has at least one positive solution.*

Proof. Consider the case $\lambda < \lambda^*$; then problem (34) has a minimal solution v_λ, and as in the proof of Corollary 3.5 one can show that $J_\lambda(v_\lambda) \le 0$. Using the hypothesis on g and integrating by parts we conclude that

$$\frac{g(s)}{s} \le g'(s) \le \frac{g(2s)}{s},$$

therefore

$$\lim_{t\to\infty} \frac{g'(t)}{t^{q_1}} = 0, \text{ for some } q_1 < \frac{4}{N-2}. \tag{53}$$

Fixed $\lambda_1 < \lambda^*$, let $\lambda_1 < \lambda_2 < \lambda^*$ and consider $\overline{v}_1$ and $\overline{v}_2$ the minimal solutions to problem (34) with $\lambda = \lambda_1,\ \lambda_2$, respectively; since $\lambda_1 < \lambda_2$, we get that $\overline{v}_2$ is a strict super-solution to problem with λ_1 and $\overline{v}_2 > \overline{v}_1$ by the strong maximum principle. We set

$$M = \{u \in W_0^{1,2}(\Omega) \,:\, 0 \le u \le \overline{v}_2 \text{ a.e. in } \Omega\} \qquad \text{and} \qquad I = \inf_{u\in M} J_{\lambda_1}(u).$$

Since M is a convex closed subset of $W_0^{1,2}(\Omega)$, using the fact that J_{λ_1} is bounded and weakly lower-semicontinuous in M, we get the existence of $\vartheta \in M$ such that $J_{\lambda_1}(\vartheta) = I$. Notice that $I < 0$ and then $\vartheta \ne 0$. using a similar argument as in Theorem 2.4 of [49] we can prove that ϑ is a weak solution to problem (34) with $\lambda = \lambda_1$. If $\vartheta \ne \overline{v}_1$ we obtain the existence of at least two positive solutions. If $\vartheta = \overline{v}_{\lambda_1}$, then will prove that ϑ is a local minimum for J_{λ_1}. We follow closely the argument used in [5] (see also [4]). By contradiction, suppose that ϑ is not a local minimum for J_{λ_1}, then there exists $\{v_n\} \subset W_0^{1,2}(\Omega)$ such that $||v_n - \vartheta||_{W_0^{1,2}} \to 0$ and $J_{\lambda_1}(v_n) < J_{\lambda_1}(\vartheta)$. We set $w_n = (v_n - \overline{v}_2)^+$ and $u_n = \max\{0, \min\{v_n, \overline{v}_2\}\}$. Then $u_n \in M$ and

$$u_n(x) = \begin{cases} 0 & \text{if } v_n(x) \le 0 \\ v_n(x) & \text{if } 0 \le v_n(x) \le \overline{v}_2(x) \\ \overline{v}_2(x) & \text{if } \overline{v}_2(x) \le v_n(x). \end{cases}$$

We define

$$T_n = \{x \in \Omega : \, 0 < v_n(x) \le \overline{v}_2(x)\}\,, \qquad S_n = \{x \in \Omega : \, v_n(x) > \overline{v}_2(x)\}\,.$$

Notice that $u_n > 0$ only on $T_n \cup S_n$. We will prove that $|S_n| \to 0$ as $n \to \infty$, where $|\cdot|$ is the Lebesgue measure. For $\varepsilon > 0$, we define

$E_n = \{x \in \Omega : \, v_n(x) \ge \overline{v}_2(x) > \vartheta(x)+\delta\}$ and $F_n = \{x \in \Omega : \, v_n(x) \ge \overline{v}_2(x) \text{ and } \overline{v}_2(x) \le \vartheta(x)+\delta\}\,,$

where δ is a positive constant that we will choose later. Using the fact that $\overline{v}_2(x) > \vartheta(x)$ for all $x \in \Omega$, then

$$0 = \left| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega : \, \overline{v}_2(x) \le \vartheta(x) + \frac{1}{j} \right\} \right| = \lim_{j \to \infty} \left| \left\{ x \in \Omega : \, \overline{v}_2(x) \le \vartheta(x) + \frac{1}{j} \right\} \right| , \tag{54}$$

and we obtain the existence of $\delta_0 = \dfrac{1}{j_0}$ such that if $\delta < \delta_0$, we have

$$|F_n| \le |\{x \in \Omega : \, \overline{v}_2(x) \le \vartheta(x) + \delta\}| \le \frac{\varepsilon}{2}.$$

Since $||v_n - \vartheta||_{L^2(\Omega)} \to 0$ as $n \to \infty$, setting $\eta = \frac{\delta^2 \varepsilon}{2}$, we obtain that for n large enough

$$\frac{\delta^2 \varepsilon}{2} \ge \int_\Omega |v_n - \vartheta|^2 dx \ge \int_{E_n} |v_n - \vartheta|^2 dx \ge \delta^2 |E_n|.$$

Hence $|E_n| \le \frac{\varepsilon}{2}$. Since $S_n \subset F_n \cup E_n$ we conclude that $|S_n| \to 0$ as $n \to \infty$. We define

$$H(u) = H(x,u) = \lambda_1 f(x)(u_+ + G(u_+)),$$

then we obtain

$$\begin{aligned} J_{\lambda_1}(v_n) &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \int_\Omega H(v_n)\, dx \\ &= \frac{1}{2}\int_{T_n} |\nabla u_n|^2 dx - \int_{T_n} H(u_n)\, dx + \frac{1}{2}\int_{S_n} |\nabla v_n|^2 dx - \int_{S_n} H(v_n)\, dx + \frac{1}{2}\int_\Omega |\nabla v_n^-|^2 dx \\ &\ge \frac{1}{2}\int_{T_n} |\nabla u_n|^2 dx - \int_{T_n} H(u_n)\, dx + \frac{1}{2}\int_{S_n} |\nabla (w_n + \overline{v}_2)|^2 dx - \int_{S_n} H(w_n + \overline{v}_2)\, dx\,. \end{aligned}$$

Since

$$\int_\Omega |\nabla u_n|^2 dx = \int_{T_n} |\nabla u_n|^2 dx + \int_{S_n} |\nabla \overline{v}_2|^2 dx\,,$$

$$\int_\Omega H(u_n)\,dx = \int_{T_n} H(u_n)\,dx + \int_{S_n} H(\overline{v}_2)\,dx\,,$$

using the fact that $\overline{v}_2$ is a supersolution to (34) with $\lambda = \lambda_1$, we conclude that

$$\begin{aligned} J_{\lambda_1}(v_n) &\geq J_{\lambda_1}(u_n) + \frac{1}{2}\int_{S_n} (|\nabla(w_n + \overline{v}_2)|^2 - |\nabla \overline{v}_2|^2)\,dx - \int_{S_n} (H(w_n + \overline{v}_2) - H(\overline{v}_2))\,dx \\ &\geq J_{\lambda_1}(u_n) + \frac{1}{2}||w_n||^2_{W_0^{1,2}} - \int_\Omega \Big\{ H(w_n + \overline{v}_2) - H(\overline{v}_2) - H_u(\overline{v}_2) w_n \Big\}\,dx \\ &\geq J_{\lambda_1}(\vartheta) + \frac{1}{2}||w_n||^2_{W_0^{1,2}} - \int_\Omega \Big\{ H(w_n + \overline{v}_2) - H(\overline{v}_2) - H_u(\overline{v}_2)\,w_n \Big\}\,dx. \end{aligned}$$

Notice that

$$H(w_n + \overline{v}_2) - H(\overline{v}_2) - H_u(\overline{v}_2) w_n = \lambda_1 f(x) \Big(G(w_n + \overline{v}_2) - G(\overline{v}_2) - w_n g(\overline{v}_2) \Big)\,.$$

Hence we conclude that

$$H(w_n + \overline{v}_2) - H(\overline{v}_2) - H_u(\overline{v}_2) w_n \leq \lambda_1 f(x) w_n^2 g'(w_n + \overline{v}_2)\,.$$

From (53) one obtains

$$g'(s) \leq C(1 + s^{q_1}), \qquad q_1 < \frac{4}{N-2}.$$

Hence we conclude that

$$w_n^2 g'(w_n + \overline{v}_2) \leq C(w_n^2 + w_n^{q_1+2} + \overline{v}_2^{q_1} w_n^2)\,.$$

On the other hand

$$\int_\Omega w_n^{2+q_1}\,dx \leq C\Big(\int_\Omega |\nabla w_n|^2 dx \Big)^{\frac{2+q_1}{2}} = o(1)||w_n||^2_{W_0^{1,2}}$$

and

$$\int_\Omega w_n^2 \overline{v}_2^{q_1}\,dx \leq C\Big(\int_{\mathrm{Supp}\, w_n} w_n^{2^*}\,dx \Big)^{\frac{2}{2^*}} \Big(\int_{\mathrm{Supp}\, w_n} \overline{v}_2^{q_1 N/2} dx \Big)^{\frac{2}{N}}.$$

Since $\frac{q_1 N}{2} < 2^*$ we obtain that

$$\int_\Omega w_n^2 \overline{v}_2^{q_1}\,dx \leq C\Big(\int_\Omega |\nabla w_n|^2 dx \Big) \Big(\int_\Omega \overline{v}_2^{2^*}\,dx \Big)^{\frac{q_1}{2^*}} |\mathrm{supp}\ (w_n)|^{\frac{4-q_1(N-2)}{2N}}.$$

Since $|\text{supp }(w_n)| = |S_n| \to 0$ as $n \to \infty$ and $\frac{4-q_1(N-2)}{2N} > 0$ we conclude that

$$\int\limits_\Omega w_n^2 \overline{v}_2^{q_1} dx \le o(1)||w_n||^2_{W_0^{1,2}}.$$

Moreover

$$\int\limits_\Omega w_n^2 dx \le |\text{supp}(w_n)|^{\frac{2}{N}} \Big(\int\limits_\Omega w_n^{2^*} dx\Big)^{\frac{2}{2^*}} \le C|\text{supp}(w_n)|^{\frac{2}{N}} \int\limits_\Omega |\nabla w_n|^2 dx = o(1)||w_n||^2_{W_0^{1,2}}.$$

Hence we conclude that

$$\int\limits_\Omega \Big\{ H(w_n + \overline{v}_2) - H(\overline{v}_2) - H_u(\overline{v}_2)w_n \Big\} dx \le o(1)||w_n||^2_{W_0^{1,2}}.$$

Hence

$$J_{\lambda_1}(v_n) \ge J_{\lambda_1}(\vartheta) + \frac{1}{2}||w_n||^2_{W_0^{1,2}}(1 - o(1)).$$

Therefore we obtain that $J_{\lambda_1}(\vartheta) > J_{\lambda_1}(v_n) \ge J_{\lambda_1}(\vartheta)$ for n large enough, which is a contradiction. Hence we conclude that ϑ is a local minimum for J_{λ_1}. Since now J_λ has a local minimum, then we get easily that J_{λ_1} has the geometry of the *Mountain Pass Theorem*, i.e., the existence of $(\overline{v}_1, \overline{v}_2)$ such that

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(\overline{v}_1), J_\lambda(\overline{v}_2)\}.$$

Then using Theorem 3.5 we get the multiplicity result. Let now $\lambda = \lambda^*$ and consider a sequence of increasing numbers λ_n such that $\lambda_n \in (0, \lambda^*)$ and $\lambda_n \uparrow \lambda^*$ as $n \to \infty$. Let $\{v_{\lambda_n}\}$ be the family of minimal solution to problem (34) with $\lambda = \lambda_n$. Then we obtain that $\{v_{\lambda_n}\}$ is an increasing sequence in n and $J_{\lambda_n}(v_{\lambda_n}) \le 0$. Using the same argument as in the proof of Theorem 3.5 we get the existence of a constant $C > 0$ such that $\int\limits_\Omega f\, v_n dx \le C$. Since

$$J_{\lambda_n}(v_n) = \frac{\lambda_n}{2} \int\limits_\Omega f\Big(g(v_n)v_n - 2G(v_n)\Big)\, dx - \frac{\lambda_n}{2} \int\limits_\Omega f\, v_n dx$$

we conclude that $|J_{\lambda_n}(v_n)| \le C_1$. Hence following again the idea of the proof of Theorem 3.5 we obtain that $||v_{\lambda_n}||_{W_0^{1,2}} \le C_1$ and then $v_{\lambda_n} \rightharpoonup v_0$ weakly in $W_0^{1,2}(\Omega)$. Since $\{v_{\lambda_n}\}$ is an increasing sequence we conclude that v_0 verifies

$$\left\{ \begin{array}{rcll} -\Delta v_0 &=& \lambda^* f(x)(1 + g(v_0)) & \text{in } \Omega \\ v_0 &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{55}$$

Hence we conclude. ∎

Remark 3.9

1. Notice that the nonlinear term $g(u)$ has slightly super-linear growth and, in general, doesn't verify Ambrosetti-Rabinowitz assumption ensuring that all Palais-Smale sequences for the associated energy functional are bounded. Namely we must prove this boundedness by alternative estimates.

2. The result of Theorem 3.8 is true if we assume that f satisfies hypothesis (**F**).

The following result shows that the assumption $f \in L^{\frac{N}{2}}(\Omega)$ is optimal as far as existence of a solution to problem (34) is concerned.

Proposition 3.10 *Let f be a positive function such that $|x|^2 f(x) \geq c > 0$ for $x \in B_r(0)$, then for all $\lambda > 0$, problem* (34) *has no positive weak solution.*

Proof. We argue by contradiction. Assume that v is a non negative weak solution to problem (34), namely $v,\ f(x)g(v) \in L^1(\Omega)$ and for all $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int\limits_\Omega v(-\Delta\phi)dx = \lambda \int\limits_\Omega f(x)(1+g(v))\phi dx.$$

Since $\lambda > 0$, using the hypothesis on f we will show that $\lim\limits_{x\to 0} v(x) = +\infty$. Indeed,

$$\lambda f(x)(1+g(v)) \geq \lambda \frac{c}{|x|^2} \text{ in } B_r(0),$$

and an appropriate choice of c_1 provides

$$-c_1\Delta(\log r - \log|x|) = \lambda\frac{c}{|x|^2}.$$

Therefore the weak comparison principle implies

$$v(x) \geq c_1(\log r - \log|x|), \quad x \in B_r(0).$$

Since $\lim\limits_{s\to+\infty} \dfrac{g(s)}{s} = +\infty$, we get the existence of $\eta(\lambda, g, v, f) > 0$ such that if $x \in B_\eta(0)$, then

$$\lambda f(x)(1+g(v)) \geq C\frac{v}{|x|^2}$$

where C can be chosen large enough, in particular we can choose $\eta > 0$ such that $C > \Lambda_N \equiv (\frac{N-2}{2})^2$, the critical constant in the Hardy inequality(see [27]). Then we conclude that

$$-\Delta v \geq C\frac{v}{|x|^2} \text{ with } C > \Lambda_N$$

a contradiction with the result of [27] (see also [2]). ■

As a direct application of Theorem 3.8 we get the following result.

Corollary 3.11 *Assume that β is an increasing function such that* (32) *and* (**H**) *hold. Let λ^* be defined as in* (52)*. Then for every $\lambda < \lambda^*$, problem* (31) *admits a least two solutions u_1, u_2 such that $\Psi(u) \in W_0^{1,2}(\Omega)$ where Ψ is defined as in* (33)*. For $\lambda = \lambda^*$ problem* (31) *admits at least one solution u with $\Psi(u) \in W_0^{1,2}(\Omega)$ and for $\lambda > \lambda^*$ problem* (31) *has no solution such that $\Psi(u) \in W_0^{1,2}(\Omega)$.*

Remark 3.12 We will show in the next section that no distributional solution exists for problem (31) for $\lambda > \lambda^*$. We will also prove that for λ small an infinite number of solutions appears. As in the previous section, we will show that each of these solutions is related, via a change of variable, to a semilinear problem with measure datum.

4 The case of continuous β: Regularity and existence of weaker solutions.

In this section we deal with problem (31) where β satisfies more general hypotheses than in the previous section. Precisely we will only assume that,

- $b1)$ β is a continuous non-negative function on $[0, +\infty)$.
- $b2)$ $\liminf_{t\to\infty} \beta(t) \in (0, +\infty]$.

In the existence result, Theorem 4.4, we will use an extra hypothesis, that is,

$$\lim_{t\to\infty} \frac{\beta(t)}{e^{a\gamma(t)}} = 0, \quad \begin{cases} a < \dfrac{2}{N} \text{ if } N \geq 3 \\ \\ a < 1 \text{ if } N = 1, 2 \end{cases} \tag{56}$$

or its equivalent form $\lim_{t\to\infty} \dfrac{g'(t)}{(1+g(t))^a} = 0$. Then it is easy to check that

$$\lim_{t\to\infty} \frac{g(t)}{t^q} = 0, \; q = \frac{1}{1-a} < \frac{N}{N-2} \text{ if } N \geq 3$$

and

$$\lim_{t\to\infty} \frac{g(t)}{t^q} = 0, \; q < \infty \text{ if } N = 1, 2$$

(we recall that the functions γ and g have been defined at the beginning of the previous section). This condition is verified if β is any elementary function. We will also suppose that $f \in L^1(\Omega)$ is a positive function.
By a solution to problem (31) we mean a function $u \in W_0^{1,2}(\Omega)$ such that $\beta(u)|\nabla u|^2 \in L^1(\Omega)$ and u is a solution in distribution sense to problem

$$\begin{cases} -\Delta u = \beta(u)|\nabla u|^2 + f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{57}$$

Once u is fixed one can consider (57) as a problem with L^1 right-hand side. Notice that, in this case the renormalized solution coincide with the distributional solution (see [23] for details). As a consequence we obtain that $T_k(u) \in W_0^{1,2}(\Omega)$ for all $k > 0$. Since $\beta(t) > A > 0$ as $\tau \to \infty$ and by the fact that $\beta(u)|\nabla u|^2 \in L^1(\Omega)$ we conclude that $u \in W_0^{1,2}(\Omega)$. We start with the following regularity result.

Theorem 4.1 *Assume that $u \in W_0^{1,2}(\Omega)$ is a solution of problem* (31)*, where $f(x) \in L^1(\Omega)$ satisfies $f(x) \geq 0$ a.e. in Ω. Then*

$$\Psi_\delta(u) \in W_0^{1,2}(\Omega) \qquad \text{for every } \delta < \frac{1}{2}, \quad \text{where} \quad \Psi_\delta(s) = \int_0^s \sqrt{\beta(t)} e^{\delta\gamma(t)} dt. \tag{58}$$

Proof. By using $w_\varepsilon = e^{\frac{\delta\gamma(u)}{1+\varepsilon\gamma(u)}} - 1$ as a test function in (31) and by passing to the limit as $\varepsilon \to 0$, we can conclude with a similar argument to the one used in the proof of Theorem 2.6. ■

The main result of this section is the following.

Theorem 4.2 *Let $u \in W_0^{1,2}(\Omega)$ be a solution to problem* (31)*, where $f \in L^1(\Omega)$ is a positive function. Consider $v = \Psi(u)$ where Ψ is defined in* (33)*, then there exists a measure μ_s, which is concentrated on a set of zero capacity, such that*

$$\begin{cases} -\Delta v = f(x)(1 + g(v)) + \mu_s \text{ in } \mathcal{D}'(\Omega) \\ \\ v \in W_0^{1,q}(\Omega) \text{ for all } q < \dfrac{N}{N-1}. \end{cases} \tag{59}$$

Moreover μ_s can be characterized as a weak limit in the space of bounded Radon measures, as follows

$$\mu_s = \lim_{\varepsilon \to 0} |\nabla u|^2 \beta(u) e^{\frac{\gamma(u)}{(1+\varepsilon\gamma(u))}} \left(1 - \frac{1}{(1+\varepsilon\gamma(u))^2}\right). \tag{60}$$

Proof. Since $u \in W_0^{1,2}(\Omega)$ and satisfies (58) we conclude that the truncation of v, $T_k(v) \in W_0^{1,2}(\Omega)$. We take $e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}} - 1 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ as test function in (31). Then, by similar calculation as in the proof of Theorem 2.10, we obtain

$$\int_\Omega f\,(e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}} - 1)\,dx \leq \int_\Omega \beta(u)|\nabla u|^2 dx$$

which implies by monotone convergence that

$$\int_\Omega f\,(e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}} - 1)\,dx \to \int_\Omega f g(v) dx \leq \int_\Omega \beta(u)|\nabla u|^2 dx < +\infty. \tag{61}$$

Moreover,

$$\int_\Omega |\nabla u|^2 \beta(u) e^{\frac{\gamma(u)}{(1+\varepsilon\gamma(u))}} \left(1 - \frac{1}{(1+\varepsilon\gamma(u))^2}\right) dx \leq \int_\Omega \beta(u)|\nabla u|^2 dx \tag{62}$$

therefore, up to a subsequence,

$$\beta(u)|\nabla u|^2 e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right) \rightharpoonup \mu_s$$

a positive Radon measure. Notice that μ_s is concentrated on the set $A = \{x \in \Omega : u(x) = +\infty\}$, which has capacity zero. This follows from the fact that

$$\int_{u\leq k} \beta(u)|\nabla u|^2 e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\left(1-\frac{1}{(1+\varepsilon\gamma(u))^2}\right) dx \to 0 \text{ as } \varepsilon \to 0.$$

We now define

$$v_\varepsilon(x) = \int_0^{u(x)} e^{\frac{\gamma(s)}{1+\varepsilon\gamma(s)}} ds \in W_0^{1,2}(\Omega),$$

in this way $v_\varepsilon(x) \nearrow v(x)$ a.e. as $\varepsilon \to 0$. It is easy to check that v_ε solves

$$\begin{aligned} -\Delta v_\varepsilon \quad &= e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}(-\Delta u) - e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\beta(u)\frac{|\nabla u|^2}{(1+\varepsilon u)^2} \\ &= e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}}\beta(u)|\nabla u|^2(1-\frac{1}{(1+\varepsilon\gamma(u))^2}) + f(x)e^{\frac{\gamma(u)}{1+\varepsilon\gamma(u)}} \end{aligned} \tag{63}$$

in the sense of distributions. By taking $T_k(v_\varepsilon)$ as a test function in (63), and using the arguments in [12], we conclude that $v_\varepsilon \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ with $||v_\varepsilon||_{W_0^{1,q}(\Omega)} \leq C(q, \Omega, M, N)$ where M is the uniform bound in $L^1(\Omega)$ of the right hand side of (63),

$$M = 2\int_\Omega \beta(u)|\nabla u|^2 dx + \int_\Omega f dx.$$

Since $v_\varepsilon(x) \nearrow v(x)$ as $\varepsilon \to 0$ and up to a subsequence v_ε converges weakly in $W_0^{1,q}(\Omega)$, we conclude that $v \in W_0^{1,q}(\Omega)$ for $q < \frac{N}{N-1}$ and, in particular, $v_\varepsilon \to v$ in $L^1(\Omega)$. The end of the proof follows closely the arguments in Theorem 2.10.

■

We now consider the reverse problem, namely we have the following result.

Theorem 4.3 *Let μ_s be a bounded positive measure which is concentrated on a set with zero capacity. Let v be a solution to problem*

$$\begin{cases} -\Delta v = f(x)(1+g(v)) + \mu_s \ \textit{in } \mathcal{D}'(\Omega) \\ v \in W_0^{1,q}(\Omega) \ \textit{for all } q < \dfrac{N}{N-1}, \\ f(x)(g(v)+1) \in L^1(\Omega) \end{cases} \tag{64}$$

If we define $u = \Psi^{-1}(v)$, where Ψ is given by (33), then u solves

$$\begin{cases} -\Delta u = \beta(u)|\nabla u|^2 + f(x) \ \textit{in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,2}(\Omega) \\ \beta(u)|\nabla u|^2 \in L^1(\Omega). \end{cases} \tag{65}$$

Proof. We begin by proving that $\beta(u)|\nabla u|^2 \in L^1(\Omega)$. Let $\{h_n\}$ be a sequence of a bounded positive function such that $h_n \to \mu_s$ in $\mathcal{M}_0(\Omega)$. Let v_n be the unique solution to problem

$$\begin{cases} -\Delta v_n &= T_n(f\,(1+g(v))) + h_n \text{ in } \Omega \\ \quad v_n &\in W_0^{1,2}(\Omega). \end{cases} \tag{66}$$

Notice that $v_n \to v$ in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$ and $||T_k(v_n)||_{W_0^{1,2}} \leq A_k$ for all $k > 0$. By taking $\dfrac{g(v_n)}{1+g(v_n)}$ as a test function in (66), we obtain that

$$\int_\Omega \frac{g'(v_n)|\nabla v_n|^2}{(1+g(v_n))^2}\,dx \leq C.$$

Hence we conclude that

$$\int_\Omega \frac{g'(v)|\nabla v|^2}{(1+g(v))^2}\,dx \leq C.$$

Since $\beta(u)|\nabla u|^2 = \dfrac{g'(v)|\nabla v|^2}{(1+g(v))^2}$ we conclude that $\beta(u)|\nabla u|^2 \in L^1(\Omega)$. Notice that by taking $w_n = 1 - \dfrac{1}{(1+g(v_n))^\delta}$, where $\delta > 0$, as a test function in (66), we obtain that

$$\int_\Omega \frac{g'(v_n)|\nabla v_n|^2}{(1+g(v_n))^{1+\delta}}\,dx \leq C \quad \text{and then} \quad \int_\Omega \frac{g'(v)|\nabla v|^2}{(1+g(v))^{1+\delta}}\,dx \leq C \quad \text{for all } \delta > 0.$$

Since $g'(s) = \beta(\Psi^{-1}(s))$, the hypothesis on β implies $g'(s) \geq C_1 > 0$ for s large enough; recalling that $T_k(v_n)$ is bounded in $W_0^{1,2}(\Omega)$ for every fixed k, we conclude that

$$\int_\Omega \frac{|\nabla v_n|^2}{(1+g(v_n))^{1+\delta}}\,dx \leq C \quad \text{and} \quad \int_\Omega \frac{|\nabla v|^2}{(1+g(v))^{1+\delta}}\,dx \leq C \quad \text{for all } \delta > 0.$$

We set $u_n = \Psi^{-1}(v_n)$, then by a direct computation one can obtain that

$$-\Delta u_n = \beta(u_n)|\nabla u_n|^2 + \frac{T_n(f\,(1+g(v)))}{1+g(v_n)} + \frac{h_n}{1+g(v_n)} \quad \text{in } \mathcal{D}'(\Omega).$$

Notice that $\dfrac{T_n(f\,(1+g(v)))}{1+g(v_n)} \to f$ in $L^1(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. (The last estimate follows by the fact that $|\nabla u_n| = \dfrac{|\nabla v_n|}{1+g(v_n)}$ is bounded in $L^2(\Omega)$, hence $u_n \to u$ in $L^a(\Omega)$ for all $a < \frac{2N}{N-2}$). We claim that

$$\frac{h_n}{1+g(v_n)} \to 0 \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{67}$$

Indeed, by hypothesis, we know that μ_s is concentrated on a set A such that $\text{cap}(A) = 0$. For $\varepsilon > 0$ let U_ε be an open set such that $A \subset U_\varepsilon$ and $\text{cap}(U_\varepsilon) \le \varepsilon$. Moreover, there exists a function $\phi \in \mathcal{C}_0^\infty(\Omega)$ such that $\phi \ge 0$, $\phi \equiv 1$ in U_ε and $\int\limits_\Omega |\nabla\phi|^2 dx \le 2\varepsilon$. By using Picone type inequality (see [2]) we have

$$2\varepsilon > \int\limits_\Omega |\nabla\phi|^2 dx \ge \int\limits_\Omega \frac{-\Delta(v_n+1)}{v_n+1}\,\phi^2 dx \ge C \int\limits_{U_\varepsilon} \frac{h_n}{1+g(v_n)}\,dx.$$

Notice that the last part of the above inequality follows by the fact that $s+1 \le C_1(g(s)+1)$. Hence we conclude that

$$\int\limits_{U_\varepsilon} \frac{h_n}{1+g(v_n)}\,dx \le C\varepsilon.$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, then

$$\int\limits_\Omega \phi\,\frac{h_n}{1+g(v_n)}\,dx = \int\limits_{U_\varepsilon} \phi\,\frac{h_n}{1+g(v_n)}\,dx + \int\limits_{\Omega\backslash U_\varepsilon} \phi\,\frac{h_n}{1+g(v_n)}\,dx.$$

Hence

$$\left|\int\limits_\Omega \phi\,\frac{h_n}{1+g(v_n)}\,dx\right| \le C\varepsilon||\phi||_\infty + \int\limits_{\Omega\backslash U_\varepsilon} |\phi| h_n dx.$$

Since $h_n \to \mu_s$ in $\mathcal{M}_0(\Omega)$ and μ_s is concentrated on A, we conclude that

$$\int\limits_{\Omega\backslash U_\varepsilon} |\phi| h_n dx \to 0 \text{ as } n \to \infty.$$

Hence (67) follows.
On the other hand, since

$$\beta(u_n)|\nabla u_n|^2 = \frac{g'(v_n)|\nabla v_n|^2}{(1+g(v_n))^2} \quad \text{and} \quad \beta(u)|\nabla u|^2 = \frac{g'(v)|\nabla v|^2}{(1+g(v))^2},$$

and since $\dfrac{g(s)}{s} \ge C$ for s large enough and by the fact that $\dfrac{g'(v_n)|\nabla v_n|^2}{(1+g(v_n))^{1+\delta}}$ is bounded in $L^1(\Omega)$ for all $\delta > 0$, then using the same argument as in the proof of Theorem 2.15 we obtain the convergence result

$$\beta(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2 \text{ in } L^1(\Omega). \tag{68}$$

Hence we conclude that u solves (64). ■

We now give a fairly general example for which problem (64) has a solution.

Theorem 4.4 *Assume that $f \in L^\infty(\Omega)$ and assumption (56) holds, then problem (64) has a positive solution for λ small enough depending on μ. This implies that problem (31) admits infinitely many solutions for small λ.*

For the proof we will use the following result that can be seen in [7].

Theorem 4.5 *Consider the problem*

$$\begin{cases} -\Delta v &= v^q + \lambda\nu \ \text{in } \mathcal{D}'(\Omega) \\ v_{|\partial\Omega} &= 0. \end{cases} \tag{69}$$

Assume that $q < \frac{N}{N-2}$ if $N \geq 3$ or $q < \infty$ if $N = 1, 2$, then there exists λ^* such that problem (69) *has a solution if $\lambda < \lambda^*$.*

Proof of Theorem 4.4. Consider the problem

$$\begin{cases} -\Delta v &= \lambda(v^q + C + 1) + \mu_s \ \text{in } \mathcal{D}'(\Omega) \\ v_{|\partial\Omega} &= 0. \end{cases} \tag{70}$$

By (56), all solutions of (70) are supersolutions to problem (64), for a suitable C. Let $v_1 = \lambda^{\frac{1}{q-1}} v$, then

$$-\Delta v_1 = v_1^q + \lambda^{\frac{1}{q-1}}(\lambda c + \mu_s). \tag{71}$$

Using Theorem 4.5, we get the existence of $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ equation (71) has a solution, hence problem (70) has a solution which is a supersolution to problem (64). Using an iteration argument we get the existence result. ∎

Remark 4.6 The hypothesis on g is verified for all *elementary functions* β, such as logarithms, powers, exponential and so on.

Remark 4.7 The existence of infinitely many solutions for problem (31) (and of (3)) should be compared with a uniqueness result proved by Korkut-Pašić-Žubrinić in [33]. In that article, which extends to more general operators than the Laplacian, they prove that, in the case where $\beta(s) \in L^\infty(I\!R) \cap L^1(I\!R)$ and $f = 0$, the only solution $u \in W_0^{1,2}(\Omega)$ of (31) is zero. In the light of the change of variable used here, one can explain this uniqueness result (and also give an alternative proof in our particular framework). Indeed, assume that $u \in W_0^{1,2}(\Omega)$ is a nonzero distributional solution of (31), with $\beta(s) \in L^\infty(I\!R) \cap L^1(I\!R)$ and $f = 0$. Then we can perform the change of variable $v = \Psi(u)$, with Ψ defined as in (33), obtaining, as in Theorem 4.2 above, that v is a solution of $-\Delta v = \mu_s$, where μ_s is a bounded Radon measure which is singular with respect to the capacity. But in this case it is easy to check that Ψ is a Lipschitz function, therefore $v \in W_0^{1,2}(\Omega)$ and $\mu_s \in W^{-1,2}(\Omega)$. But this means (see [13]) that μ_s is absolutely continuous with respect to capacity: a contradiction.

Finally, we give a non-existence result which completes the statement of Theorem 3.2 and Corollary 3.3.

Proposition 4.8 *Assume that β is an nondecreasing function such that* (**H**) *and* (32) *hold. Let λ^* defined as in* (52)*, then problem* (31) *admits no distributional solution for $\lambda > \lambda^*$.*

Proof. By contradiction. Let $\lambda > \lambda^*$ be such that problem (31) has a solution. We set $v = \Psi(u)$ where Ψ is defined as in (33), then by Theorem 4.2 we obtain that v satisfies to problem

$$\left\{ \begin{array}{rcl} -\Delta v & = & \lambda f(x)(g(v)+1) + \mu_s \text{ in } \mathcal{D}'(\Omega) \\ v & \in & W_0^{1,q}(\Omega) \text{ for all } q < \frac{N}{N-1}. \end{array} \right. \tag{72}$$

where μ_s is a positive measure which only charges a set with singular measure. Consider now the problem

$$\left\{ \begin{array}{rcl} -\Delta w & = & \lambda f(x)(g(w)+1) \text{ in } \Omega \\ w_{|\partial\Omega} & = & 0. \end{array} \right. \tag{73}$$

Since $w_0 = 0$ is a strictly supsolution and $w_1 = v$ is a supersolution, then using an iteration argument we obtain that problem (73) has at least a positive solution for $\lambda > \lambda^*$, a contradiction with the definition of λ^*. Hence we conclude. ∎

Remark 4.9 The above result should be compared with the existence results by Porretta-Segura [43] in the case where $\beta(s)$ is a positive function such that $\lim_{s\to+\infty} \beta(s) = 0$. In that paper it is proved that, under this assumption, a solution of (31) exists for all $\lambda > 0$.

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