Some vanishing and finiteness results on complete manifolds: a generalization of the Bochner technique

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La Sapienza, January 24, 2007

Abstract

In this talk we present some results, recently obtained in collaboration with M. Rigoli and A.G. Setti, that extend the original Bochner technique to the case of L^p harmonic forms on geodesically complete manifolds and in the presence of a negative amount of curvature.

Contents

1 The case of a closed manifold

Geometric Analysis. Roughly speaking: you are given a geometric problem. Summarize it into a family of functions (of geometric content) which, in turn, are governed by a system of differential (in)equalities. Obtain information on the qualitative and quantitative properties of solutions of these differential systems. Geometry, in general, will impose some further constrains and guide the analysis of solutions. Apply this information to the given geometric functions and get a conclusion about the original problem.

1.1 Bochner original argument

A prototypical example is represented by the celebrated Bochner technique, originally introduced by S. Bochner in the '50s to investigate the relation between the topology and the curvature of a closed (i.e. compact and without boundary) Riemannian manifold. In fact, it is a simple matter to realize that, given a smooth, compact manifold M, it is always possible to find a contractible, dense, open set $\mathcal{E} \subseteq M$ supporting a metric with constant curvature of a prescribed sign. Simply fix any metric $(,)$ on M, a reference origin $p \in M$ and delete from M the corresponding cut-locus $cut\,(p)$, which is a closed (hence compact) set of zero-measure. Thus $\mathcal{E} = M - cut\,(p)$ is diffeomorphic to the star-shaped, relatively compact, open set $0 \in E \subseteq T_pM \approx \mathbb{R}^m$ via the exponential map $\exp_p$. To conclude, fix a constant curvature metric on E and pull it back on $\mathcal{E}$. Note that, in a (quite strong) sense, the topology of M

is contained in the (apparently evanescent) removed set $cut\,(p)$. For instance, it is not difficult to show that the inclusion $i : cut\,(p) \hookrightarrow M$ induces isomorphisms between homology (and cohomology) groups

$$H_k\left(cut\,(p)\,;\mathbb{Z}\right) \simeq H_k\left(M;\mathbb{Z}\right)$$

at least for $k \neq m, m-1$. Now, closing $M - cut\,(p)$ by addition of $cut\,(p)$ produces a non-trivial topology that, in general, may represent an obstruction for M to support a Riemannian metric with some curvature bound, e.g., given sign. Bochner result goes precisely in this direction. Let us recall the argument.

Theorem 1 (Bochner) *Let $(M, \langle , \rangle)$ be an oriented, closed, connected Riemannian manifold of dimension $m = \dim M$. Suppose that $Ric \geq 0$ on M. Then the first (real) Betti number of M satisfies*

$$b^1\left(M;\mathbb{R}\right) \leq m$$

the equality holding if and only if M is a flat torus. Furthermore, if $Ric > 0$ at some $p \in M$, then

$$b^1\left(M;\mathbb{R}\right) = 0.$$

Proof. By the Hodge-de Rham theory we know that the simplicial/singular and the de Rham cohomology theories with real coefficients are isomorphic. Furthermore, every de Rham cohomology class is represented by a unique harmonic form. Namely, given $[\omega] \in H^1_{dR}\left(M;\mathbb{R}\right)$ there exists one and only one $\bar{\omega} \in [\omega]$ such that $\Delta_H \bar{\omega} = (\delta d + \delta d)\,\bar{\omega} = 0$. Here, d is the exterior differential while δ stands for the (formal) adjoint of d with respect to the L^2 inner product of k-forms. Thus, denoting by $\mathcal{H}^k\left(M\right)$ the vector space of harmonic k-forms on M, we have

$$H^1\left(M;\mathbb{R}\right) \simeq H^1_{dR}\left(M;\mathbb{R}\right) \simeq \mathcal{H}^1\left(M\right).$$

It suffices to deal with the space $\mathcal{H}^1\left(M\right)$. A formula of Weitzenbock, independently rediscovered by Bochner, states that

$$\frac{1}{2}\Delta\left|\bar{\omega}\right|^2 = \left|D\bar{\omega}\right|^2 + Ric\left(\bar{\omega}^{\#}, \bar{\omega}^{\#}\right), \tag{1}$$

where Δ is the Laplace-Beltrami operator (with the sign convention $+d^2/dx^2$) and D denotes the extension to 1-forms of the Levi-Civita connection of M. Suppose $Ric \geq 0$. With this assumption we get the differential inequality

$$\Delta\left|\bar{\omega}\right|^2 \geq 0,$$

i.e., $|\bar{\omega}|$ is sub-harmonic. Since M is closed, we easily conclude that $|\bar{\omega}|$ =const. This can be done using two different viewpoints, (i) the L^∞ and (ii) the $L^{p<+\infty}$ one. As for (i) note that the smooth function $|\bar{\omega}|$ attains its maximum at some point and, therefore, by the Hopf maximum principle we conclude that $|\bar{\omega}|$ =const. In case (ii) we use the divergence theorem to deduce

$$0 = \int_M \operatorname{div}\left(\left|\bar{\omega}\right|^2 \nabla\left|\bar{\omega}\right|^2\right) = \int_M \left|\nabla\left|\bar{\omega}\right|^2\right|^2 + \int_M \left|\bar{\omega}\right|^2 \Delta\left|\bar{\omega}\right|^2 \geq \int_M \left|\nabla\left|\bar{\omega}\right|^2\right|^2 \geq 0.$$

This again implies $|\bar{\omega}|$ =const.

Now, using this information into formula (1) shows that $\bar{\omega}$ is parallel, i.e., $D\bar{\omega} = 0$. As a consequence, $\bar{\omega}$ is completely determined by its value at given point, say $p \in M$. The evaluation map $\varepsilon_p : \mathcal{H}^1\left(M\right) \to \Lambda^1\left(T^*_p M\right)$ defined by

$$\varepsilon_p\left(\bar{\omega}\right) = \bar{\omega}_p$$

is an injective homomorphism, proving that, in general,

$$\dim \mathcal{H}^1\left(M\right) \leq m.$$

Note that (1) yields

$$0 = Ric\left(\bar{\omega}^{\#}_p, \bar{\omega}^{\#}_p\right) \text{ at } p.$$

Therefore, if $Ric\,(p) > 0$, we get $\bar{\omega}_p = 0$ which, in turn, implies $\bar{\omega} = 0$. This shows that, when Ricci is positive somewhere,

$$\dim \mathcal{H}^1\left(M\right) = 0.$$

∎

Remark 2 The assumption $Ric \geq 0$ is vital in order to deduce that $|\bar{\omega}|$ is subharmonic, i.e., $\Delta |\bar{\omega}| \geq 0$. A negative amount of Ric could destroy this happy picture. For instance, if S is an orientable, closed Riemann surface of genus $g \geq 2$, by uniformization (and recalling the Gauss-Bonnet theorem) we can endow S with a Riemannian metric of Gauss curvature -1. Therefore, in general, there is no control on the topology of a negatively curved manifold. The analytical counterpart of this fact is that, setting

$$-a(x) = \min_{v \in \mathbb{S}^{m-1} \subset T_x M} Ric_x(v, v), \tag{2}$$

from (1) we have

$$\frac{1}{2}\Delta |\bar{\omega}|^2 + a(x) |\bar{\omega}|^2 \geq |D\bar{\omega}|^2 \geq 0 \tag{3}$$

and, in general, the maximum principle fails to hold for inequalities of this type.

Remark 3 The previuous simple example shows that, in the negatively curved case, a uniform curvature bound is in general not sufficient to give a uniform limitation on the Betti numbers of a close manifold. According to a celebrated result by M. Gromov, later extended in various ways, such a universal topological limitation can be obtained up to requiring a control on a further Riemannian invariant, namely, the diameter; see [13]. From a different (more analytic) perspective, we shall see momentarily how one could think of extending Bochner estimating theorem in the presence of (a little amount of) negative curvature.

1.2 Generalized maximum principle

As a matter of fact, a form of the maximum principle still holds for a function $\psi \geq 0$ satisfying

$$\Delta\psi + q(x)\psi \geq 0, \tag{4}$$

provided M supports a solution $\varphi > 0$ of

$$\Delta\varphi + q(x)\varphi \leq 0. \tag{5}$$

This is named the "generalized maximum principle". We shall see later that the existence of φ is related to spectral properties of the operator $-\Delta - q(x)$. The generalized maximum principle relies on the idea of absorbing the linear term of (4) using a combination of the solutions ψ and φ. Indeed, setting

$$u = \frac{\psi}{\varphi}$$

one can verify by direct computations that

$$\Delta u + \langle \nabla u, \nabla \log \varphi \rangle \geq 0.$$

Now, in the last inequality there is no linear (i.e. zero-order) term in u and therefore the Hopf maximum principle applies. In case M is closed, u must attain its maximum so that, by Hopf, u is constant. Equivalently,

$$\psi = C\varphi,$$

for some constant $C \geq 0$. Note that, in particular, using this latter into (4) and (5), we deduce

$$\Delta\psi + q(x)\psi = 0. \tag{6}$$

In case $\psi = |\bar{\omega}|^2$ and $q(x) = 2a(x)$ as in (2), comparing (6) with (3) yields, once again, that $\bar{\omega}$ is parallel, thus extending the original Bochner vanishing result.

Remark 4 As a matter of fact the same conclusion can be reached by taking $\psi = |\bar{\omega}|$ and $q(x) = a(x)$. This latter, according to (5), implies a less stringent condition on the manifold. To recover the result, compute $\Delta |\bar{\omega}|$ and use the Kato inequalities as explained below.

Remark 5 On a closed manifold M, if $q(x) \geq 0$, the validity of (5) for some $\varphi > 0$ is equivalent to saying that $q(x) \equiv 0$. Indeed, according to (5), φ is superharmonic, hence constant. Using this information into (5) gives the vanishing of $q(x)$. As a consequence (in the compact setting) the request (5) represents a genuine extension of Bochner condition $Ric \geq 0$ only in case the Ricci tensor changes its sign.

2 The setting of open manifolds

What does of the previous survive picture in the case of a non-compact manifold $(M, \langle , \rangle)$?

2.1 Bochner-type inequalities

On a (generic) manifold of dimension $m = \dim M$, any differential k-form $\omega \in \Lambda^k(M)$ satisfies the Bochner-Weitzenbock identity

$$\Delta_H \omega = \Delta_B \omega + \mathcal{R}(\omega) \tag{7}$$

where $\Delta_B = -trD^2$, the Bochner Laplacian, and $\mathcal{R}$ is a suitable (symmetric) endomorphism of the vector bundle $\Lambda^k(M)$. In particular, if ω is harmonic, i.e., $\omega \in \mathcal{H}^k(M)$, we deduce that

$$\frac{1}{2}\Delta |\omega|^2 = |D\omega|^2 + \langle \mathcal{R}(\omega), \omega \rangle . \tag{BW}$$

For instance, in case $k = 1$, $\langle \mathcal{R}(\omega), \omega \rangle = Ric(\omega^{\#}, \omega^{\#})$. For a general k, denoting by $\rho_x : \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ the curvature operator of $(M, \langle , \rangle)$, condition $\rho_x \geq -a(x)$ implies

$$\langle \mathcal{R}(\omega), \omega \rangle \geq -C \cdot a(x) |\omega|^2 ,$$

with $C = C(m, k) > 0$ a suitable constant; see [12]. Therefore, if we set $\psi = |\omega|$, from (BW) we deduce

$$\psi \{\Delta\psi + q(x)\psi\} \geq |D\omega|^2 - |\nabla |\omega||^2 ,$$

with $q(x) = Ca(x)$. Now, in general, one has the Kato inequality

$$|D\omega|^2 - |\nabla |\omega||^2 \geq 0,$$

the equality holding if and only if

$$\omega = |\omega| \omega_0,$$

for some parallel form ω_0. However, in case ω is both closed and co-closed, i.e.,

$$d\omega = 0, \quad \delta\omega = 0,$$

the Kato inequality refines to

$$|D\bar{\omega}|^2 - |\nabla |\bar{\omega}||^2 \geq A |\nabla |\bar{\omega}||^2 \tag{8}$$

for a suitable constant $A = A(m, k) > 0$; see e.g. [5]. For instance, in case of a 1-form it is easy to see that $A = 1/(m-1)$. Note that, in general, ($L^{p\neq 2}$) harmonic forms on non-compact manifolds are neither closed nor co-closed; see [24]. In contrast, using a Gaffney cut-off trick, one can show that L^2-harmonic forms on complete manifolds (hence any harmonic form in the compact setting) enjoy this property. We thus conclude that $\psi = |\omega|$ satisfies an inequality of the form

$$\psi \{\Delta\psi + q(x)\psi\} \geq A |\nabla\psi|^2 , \tag{9}$$

with $q(x)$ a continuous function and $A \in \mathbb{R}$. We shall refer to (9) as the **general Bochner-type inequality**.

2.2 General existence result

Bochner result is essentially a vanishing&estimating theorem for the space of harmonic forms. Harmonic forms on a closed manifold represent cohomology classes: therefore their vanishing or their presence is a topological question. In strong contrast, in the non-compact setting, harmonic forms may represent nothing, even for a geodesically complete manifold $(M, \langle , \rangle)$. For instance, on the flat Euclidean space $\mathbb{R}^m$ every differential k-form $h(x)\, dx^{i_1} \wedge ... \wedge dx^{i_k}$ is harmonic provided $h(x)$ is a harmonic function (in fact, every harmonic form in $\mathbb{R}^m$ is obtained from these ones). Now, the space of harmonic functions on $\mathbb{R}^m$ is not finitely generated. Simply consider the polynomials in any two independent variables $x = x^{i_1}, y = x^{i_2}$

$$\sum_{j=0}^{2k} P_{2j}(y)\, x^{2j}$$

satisfying the conditions

$$\begin{cases} P_{2k}(y) = \alpha y + \beta \\ \frac{d^2}{dy^2} P_{2j}(y) + (2j+2)(2j+1) P_{2(j+1)}(y) = 0. \end{cases}$$

Whence, we conclude that, for the vector space $\mathcal{H}^k(\mathbb{R}^m)$ of harmonic k-forms on $\mathbb{R}^m$, one has

$$\dim \mathcal{H}^k(\mathbb{R}^m) = +\infty.$$

In fact, a generic open manifold $(M, \langle,\rangle)$ always supports non-zero harmonic forms of every degree. To see this, fix a smooth, compact domain Ω. According to a classical result by Duff and Spencer, [7], for any differential k-form ω_0 on $\overline{\Omega}$ with non-zero tangential (or normal) part on $\partial\Omega$, we can uniquely solve the Dirichlet problem

$$\begin{cases} \Delta_H \omega = 0, & \text{on } \Omega \\ \omega = \omega_0, & \text{on } \partial\Omega, \end{cases}$$

thus obtaining a non-zero harmonic form on the region Ω. Now, choose a domain $D \subset\subset \Omega$ so that $M - D$ has no compact components (e.g. choose D to be a small geodesic ball). Note that, by (7), $\Delta_H \omega = 0$ represents an elliptic differential system of second order. By unique continuation, $\omega \not\equiv 0$ on D. Using the Runge-type theorem by Malgrange, [18], we can uniformly approximate ω on D by a harmonic k-form on M, say $\eta \in \mathcal{H}^k(M)$. If η is sufficiently close to ω on D, we have $\eta \not\equiv 0$, proving that

$$\dim \mathcal{H}^k(M) \neq 0.$$

2.3 The need of some integrability condition

Let us take a look at the analytic counterpart of the above non-vanishing. Suppose we are in the case of a manifold whose curvature operator ρ satisfies $\rho \geq -a(x)$. According to Bochner-Weitzenbock, a harmonic k-form $\omega \in \mathcal{H}^k(M)$ satisfies

$$\frac{1}{2}\Delta |\omega|^2 + Ca(x)|\omega|^2 \geq |D\omega|^2 \geq 0,$$

with $C = C(m,k) > 0$. Restrict now to $\rho \geq 0$, so that $a(x) \equiv 0$ and the above reduces to

$$\Delta |\omega|^2 \geq 0.$$

Following Bochner original argument, one tries to reach the Liouville-type conclusion $|\omega| = \text{const}$ and, hence, that ω is parallel. However, in general, $|\omega|$ is neither (i) bounded nor (ii) in some $L^{p<+\infty}$ integrability class and we have no Liouville property at all.

It happens that the geometry of the underlying manifold $(M, \langle,\rangle)$ enters the game when we consider harmonic functions and forms with these special integrability properties. The L^∞ and $L^{p<+\infty}$ situations are substantially different. They are dealt with completely different methods. The L^∞ case is often considered in the perspective of the weak maximum principle at infinity. We shall not consider this situation here. For an extensive study of the subject, we refer to the paper [20]. In this talk, following [21], [22] and [23], we will focus our attention on the L^p case.

Example. The case of $\mathbb{R}^m$ is easy to handle and, therefore, can be used to exemplify the situation. Let $L^p\mathcal{H}^k(\mathbb{R}^m)$ be the vector space of the harmonic k-forms ω on $\mathbb{R}^m$ satisfying $|\omega| \in L^p(\mathbb{R}^m)$. We show that

$$\dim L^p\mathcal{H}^k(\mathbb{R}^m) = 0.$$

To see this, take any $\omega \in L^p\mathcal{H}^k(\mathbb{R}^m)$. Then, by Bochner-Weitzenbock, $|\omega| \in L^p(\mathbb{R}^m)$ is a subharmonic function. Since non-negative subharmonic functions in $\mathbb{R}^m$ enjoy the L^p mean-value property we get, for any fixed $x_0 \in \mathbb{R}^m$ and for every $R > 0$,

$$|\omega|^p(x_0) \leq \frac{\int_{B_R(x_0)} |\omega|^p}{vol\,(B_R(x_0))}.$$

Letting $R \to +\infty$ we deduce $|\omega|(x_0) = 0$. Since x_0 was an arbitrary point, we conclude $\omega \equiv 0$, as claimed.

2.4 Controlling the negative part of the curvature

We now want to take under consideration a possible contribution of the negative part of the curvature. In the compact setting, we remarked that hyperbolic surfaces are critical. The situation is similar in the non-compact, complete setting.

Example. Let $\mathbb{H}^m$ be the standard hyperbolic space of dimension m and constant curvature -1. We realize it as the Poincarè disc

$$\left(\mathbb{B}_1^m(0), \frac{4\sum dx^i \otimes dx^i}{\left(1-|x|^2\right)^2}\right),$$

where $\mathbb{B}_1^m(0)$ is the Euclidean unit ball of $\mathbb{R}^m$. Note that the hyperbolic metric is a pointwise conformal deformation of the Euclidean one. More generally, say that $\widetilde{\langle,\rangle}$ is a pointwise conformal deformation of the Riemannian metric $\langle,\rangle$ of M, if

$$\widetilde{\langle,\rangle} = \lambda^2(x)\langle,\rangle$$

for some function $0 < \lambda \in C^\infty$. It can be shown that

$$L^2\mathcal{H}^k(M, \langle,\rangle) \simeq L^2\mathcal{H}^k\left(M, \widetilde{\langle,\rangle}\right),$$

provided

$$2k = m = \dim M.$$

Specifying the situation to the Hyperbolic space we deduce

$$L^2\mathcal{H}^k(\mathbb{H}^m) \simeq L^2\mathcal{H}^k(\mathbb{B}_1^m(0)). \tag{10}$$

Now, we have already observed that $\dim \mathcal{H}^k(\mathbb{R}^m) = +\infty$. Since $\mathbb{B}_1^m$ has finite Euclidean volume every $\omega \in \mathcal{H}^k(\mathbb{R}^m)$ restricts to a form $\omega' \in L^2\mathcal{H}^k(\mathbb{B}_1^m(0))$. By unique continuation, the restriction map is injective. It follows that $\dim L^2\mathcal{H}^k(\mathbb{B}_1^m(0)) = +\infty$. According to (10), we conclude

$$\dim L^2\mathcal{H}^k\left(\mathbb{H}^{2k}\right) = +\infty.$$

2.4.1 Bottom of the spectrum and Morse index

We introduce a "measure" of the negative part of the curvature in such a way that, "small" negative curvature implies finite dimensionality and, furthermore, small enough gives vanishing. This will be done using spectral properties of a suitable Schrodinger operator. From now on, $(M, \langle,\rangle)$ will denote a geodesically complete, non-compact, connected Riemannian manifold of dimension $m = \dim M$.

Suppose the curvature operator of M satisfies $\rho \geq -a(x)$, $a(x) \geq 0$, so that, for every $\omega \in \mathcal{H}^k(M)$, its length $\psi = |\omega|$ satisfies

$$\psi\{\Delta\psi + Ca(x)\psi\} \geq A|\nabla\psi|^2,$$

where $C = C(k,m) > 0$ and $A = A(k,m) \geq 0$, with $A > 0$ if ω is both closed and co-closed. For instance, in the Hyperbolic case,

$$C = k(m-k),\ a(x) = -1.$$

Let us consider the Schrodinger operator

$$\mathcal{L} = -\Delta - q(x)$$

where we have set $q(x) = Ca(x)$. For any smooth domain $\Omega \subset\subset M$, the operator $\mathcal{L}$ acting on $C_c^\infty(\Omega)$ has discrete spectrum

$$-\infty < \lambda_1(\mathcal{L},\Omega) < \lambda_2(\mathcal{L},\Omega) \leq \lambda_3(\mathcal{L},\Omega) \leq ... \leq \lambda_s(\mathcal{L},\Omega) < 0 \leq \lambda_{s+1}(\mathcal{L},\Omega) \leq ...$$

From the variational characterization,

$$\lambda_k(\mathcal{L},\Omega) = \inf_{\substack{V \subset C_c^\infty(\Omega) \\ \dim V = k}} \frac{\int |\nabla\varphi|^2 - \int q(x)\varphi^2}{\int \varphi^2}.$$

Define the **Morse index** of $\mathcal{L}$ on Ω as

$$Ind(\mathcal{L},\Omega) = s = \#\text{negative eigenvalues}.$$

In particular, $Ind(\mathcal{L},\Omega) = 0$ if and only if $\lambda_1(\mathcal{L},\Omega) \geq 0$. Now, by (strict) domain monotonicity we have that

$$\Omega_1 \subset \Omega_2 \Longrightarrow \lambda_k(\mathcal{L},\Omega_2) < \lambda_k(\mathcal{L},\Omega_1),$$

and therefore

$$\Omega_1 \subset \Omega_2 \Longrightarrow Ind(\mathcal{L},\Omega_1) \leq Ind(\mathcal{L},\Omega_2).$$

Define the **generalized Morse index of $\mathcal{L}$ on** M to be

$$Ind(\mathcal{L},M) = \sup_{\Omega \nearrow M} Ind(\mathcal{L},\Omega) \leq +\infty.$$

Also, define the **bottom of the spectrum of $\mathcal{L}$ on** M to be

$$\lambda_1(\mathcal{L},M) = \inf_{\Omega \nearrow M} \lambda_1(\mathcal{L},\Omega) = \inf_{\varphi \in Lip_c(M)} \frac{\int |\nabla\varphi|^2 - \int q(x)\varphi^2}{\int \varphi^2} \geq -\infty.$$

In case $\lambda_1(\mathcal{L},M) > -\infty$ we say that $\mathcal{L}$ is **semi-bounded**.

Remark 6 The generalized Morse index and the bottom of the spectrum are obviously related by the following

$$Ind(\mathcal{L},M) = 0 \Longleftrightarrow \lambda_1(\mathcal{L},M) \geq 0.$$

Finiteness and vanishing of the Morse index essentially reflect the fact that the (positive part of the) potential is small in some integral sense. This is made precise in the following classical theorem due to G.V. Rosenbljum, W. Cwikel and E. Lieb. The present statement, is a version by P. Li and S.T. Yau.

Theorem 7 *Let $(M^m, \langle,\rangle)$ be a complete Riemannian manifold supporting the following global L^2-Sobolev inequality*

$$S_\mu \left(\int_M v^{2\mu}\right)^{\frac{1}{\mu}} \leq \int_M |\nabla v|^2,\ \forall v \in C_0^\infty(M)$$

for some $\mu > 1$ and some constant $S_\mu > 0$. Given a smooth function $q(x)$ satisfying

$$q_+(x) = \max(q(x),0) \in L^{\frac{\mu}{\mu-1}}(M),$$

consider the Schrodinger operator $\mathcal{L} = -\Delta - q(x)$ acting on $C_0^\infty(M)$. Then, $\mathcal{L}$ is a semi-bounded, essentially self-adjoint operator on $L^2(M)$ with non-negative essential spectrum. Let N_0 be the number (counting multiplicity) of strictly negative eigenvalues of $\mathcal{L}$. Then, $N_0 = Ind(\mathcal{L},M)$ and there exists a constant $C = C(m,S_\mu) > 0$ such that

$$Ind(\mathcal{L},M) \leq C \|q_+\|_{L^{\frac{\mu}{\mu-1}}(M)}.$$

2.4.2 PDE counterparts of the spectral properties

The following theorem is the result of contributions of many authors: D. Fisher-Colbrie [9], and R. Gulliver [14] (finite index part) and Fisher-Colbrie and R. Schoen [10], and W. Moss and J. Piepenbrink [19] (zero index part). See also [21] for regularity questions concerning potential and solutions.

Theorem 8 *Let $\mathcal{L} = -\Delta - q(x)$. Then*

(a) For any domain $W \subseteq M$,

$$\lambda_1(\mathcal{L}, W) \geq 0 \Longleftrightarrow \exists\varphi > 0 \text{ solution of } -\mathcal{L}\varphi = 0 \text{ on } W.$$

(b) It holds the implication

$$Ind(\mathcal{L}, M) < +\infty \Longrightarrow \exists K \subset\subset M \text{ such that } \lambda_1(\mathcal{L}, M - K) \geq 0.$$

In particular

Corollary 9 *Let $\mathcal{L} = -\Delta - q(x)$. Then*

$$Ind(\mathcal{L}, M) = 0 \Longleftrightarrow \exists\varphi > 0 : -\mathcal{L}\varphi = 0 \text{ on } M$$

and

$$Ind(\mathcal{L}, M) < +\infty \Longrightarrow \exists\varphi > 0 : -\mathcal{L}\varphi = 0 \text{ on } M - K, \text{ for some } K \subset\subset M.$$

Some observations are in order.

Remark 10 The bottom-of-the-spectrum condition on $M - K$ is slightly weaker than the finiteness of the Morse index. Moreover, it is easier to handle. For instance, suppose $(M, \langle,\rangle)$ enjoys a global L^2-Sobolev inequality

$$\left(\int |u|^{2\mu}\right)^{\frac{1}{\mu}} \leq S_\mu^{-1} \int |\nabla u|^2, \ \forall u \in C_c^\infty(M)$$

with $\mu > 1$ and $S_\mu > 0$ a suitable constant. Then, it is readily seen that

$$\|q\|_{L^{\frac{\mu}{\mu-1}}(M)} < +\infty \Longrightarrow \lambda_1(\mathcal{L}, M - K) \geq 0,$$

for a sufficiently large $K \subset\subset M$. Furthermore

$$\|q\|_{L^{\frac{\mu}{\mu-1}}(M)} \leq S_\mu \Longrightarrow \lambda_1(\mathcal{L}, M) \geq 0.$$

Indeed, fix $K \subset\subset M$, possibly $K = \emptyset$. Then, for any $\varphi \in C_c^\infty(M - K)$, using Holder inequality,

$$\begin{aligned} \int |\nabla\varphi|^2 - \int q(x)\varphi^2 &\geq \int |\nabla\varphi|^2 - \|q\|_{L^{\frac{\mu}{\mu-1}}(M-K)} \left(\int \varphi^{2\mu}\right)^{\frac{1}{\mu}} \\ &\geq \left(1 - S_\mu^{-1}\|q\|_{L^{\frac{\mu}{\mu-1}}(M-K)}\right) \int |\nabla\varphi|^2. \end{aligned}$$

Now, you can choose K large enough so that

$$\left(1 - S_\mu^{-1}\|q\|_{L^{\frac{\mu}{\mu-1}}(M-K)}\right) \geq 0.$$

Remark 11 As remarked above, the spectral properties of $-\Delta - q(x)$, in a sense, give a measure of the positivity of $q(x)$, hence a measure of the amount of negativity of the curvature. On the other hand, the PDE reformulation of the spectral properties is very suitable for an analytic approach to vanishing and finiteness results in the spirit of the generalized maximum principle.

2.5 Main theorems: Bochner generalized

In Theorems 12, 13 below, we shall generalize Bochner vanishing and finiteness results. The range of application of our theorems goes beyond the domain of harmonic forms but, when specified to this situation, can be essentially stated as follows.

Suppose the curvature operator satisfies $\rho_x \geq -a(x)$, so that, for any harmonic k-form ω, we have

$$|\omega| \{\Delta |\omega| + Ca(x) |\omega|\} \geq |D\omega|^2 - |\nabla |\omega||^2 \geq A |\nabla |\omega||^2,$$

where $C = C(m,k) > 0$ and $A = A(m,k) \geq 0$. Moreover $A > 0$ e.g. if $|\omega| \in L^2(M)$. For any fixed $H \geq 1$ define

$$\mathcal{L}_H = -\Delta - HCa(x).$$

Then

$$Ind(\mathcal{L}_H, M) = 0 \Longrightarrow \dim L^{2p}\mathcal{H}^k(M) = 0, \ \forall p \in [1, H],$$

and

$$Ind(\mathcal{L}_H, M) < +\infty \Longrightarrow \dim L^{2p}\mathcal{H}^k(M) < +\infty, \ \forall p \in [1, H].$$

It is worth noting that the range of the allowed integrability exponents depends in a rather explicit way on the "spectral smallness" of the potential. In case of closed and co-closed forms, due to the presence of a refined Kato inequality, such a range can be further extended below $2p = 2$.

Here is our L^q-vanishing theorem from [21]. We recall that vanishing results for L^2 harmonic sections on complete manifolds under spectral assumptions go back to a paper by W. Elworthy and S. Rosenberg, [8]. Their technique relies on very refined probabilistic tools and requires the additional curvature assumption $\inf_M Ric > -\infty$ in order to guarantee that Brownian paths do not explode (a.s.) in a finite time. Soon later, P. Berard, [1], generalized Elworthy-Rosenberg results by removing the curvature condition. Moreover, his proof is completely elementary and makes a direct use of the spectral assumption. His direct approach, however, forces him to consider only L^2-energies.

Theorem 12 (P.-Rigoli-Setti, [21]) *Let $(M, \langle\, , \rangle)$ be a complete manifold, $a(x) \in L^\infty_{loc}(M)$ and let $0 \leq \psi \in Lip_{loc}(M)$ satisfy the differential inequality*

$$\psi\Delta\psi + a(x)\psi^2 \geq A|\nabla\psi|^2 \qquad \textit{weakly on } M. \tag{11}$$

for some $A \in \mathbb{R}$. Suppose that there exists $\varphi \in Lip_{loc}(M)$ satisfying

$$\Delta\varphi + Ha(x)\varphi \leq 0 \ \textit{weakly on } M \tag{12}$$

for some H such that

$$H \geq -A + 1, \ H > 0 \tag{13}$$

If

$$\psi \in L^{2p}(M) \tag{14}$$

for some

$$-A + 1 \leq p \leq H, \ p > 0,$$

then there exist a constant $C \geq 0$ such that

$$C\varphi = \psi^H. \tag{15}$$

Further,

(i) If $H - 1 > -A$, then ψ is constant on M, and if in addition, $a(x)$ does not vanish identically, then ψ is identically zero;

(ii) If $H-1=-A$, and ψ does not vanish identically, then φ and therefore ψ^H satisfy (12) with equality sign.

Now we state a very general L^q-finiteness result from [22]. In the special case $q=2$, finiteness results have been extensively investigated by many authors under different assumptions. We limit ourselves to quote the "minimal hypersurfaces" papers [16], [17] by P. Li and J. Wang, where Morse index assumptions are used in a way similar to the present note, and the "L^2-cohomology paper" [3] by G. Carron where quantitative dimensional estimates are obtained assuming that the underlying manifold supports a global Sobolev inequality; see also [2] and [4]. Some random remarks: the classical minimal (hyper)surface theory in Euclidean space greatly influenced the investigation of finiteness results on non-compact spaces under spectral properties of the relevant Schrodinger operator (the stability operator $\mathcal{L}=-\Delta-|\mathrm{II}|^2$). According to the (quite recent) harmonic function theory initiated by P. Li and L.F. Tam, [15], the dimension of (suitable subspaces of) $L^2\mathcal{H}^1(M)$ reflects in some sense the topology at infinity of the underlying manifold (number of non-parabolic ends). This applies in particular to minimal submanifolds of finite index (where all ends are non-parabolic). Also, on a generic complete manifold, according to the decomposition theorem by Hodge-de Rham-Kodaira, L^2 harmonic forms completely represent the (reduced) L^2 cohomology of the manifold. It has been recently observed by D. Alexandru-Rugina, [25], that, in case of manifolds with bounded geometry (e.g. co-compact coverings), the spaces of harmonic forms $L^{p<2}\mathcal{H}^k(M)$ imbeds continuously into the corresponding L^p cohomology spaces. Furthermore, it is known from works by J. Dodziuk, [6], and V.M. Gol'dshtein, V.I. Kuz'minov, I.A. Shvedov, [11], that the L^p cohomology (non reduced, in fact) of a co-compact covering is a homotopy invariant of the base (compact!) manifold.

Theorem 13 (P.-Rigoli-Setti, [22]) *Let $(M,\langle,\rangle)$ be a connected, complete, m-dimensional Riemannian manifold and E a Riemannian (Hermitian) vector bundle of rank l over M. The space of its smooth sections is denoted by $\Gamma(E)$. Having fixed*

$$a(x)\in C^0(M), \qquad A\in\mathbb{R}, \qquad H\geq p$$

satisfying the further restrictions

$$p\geq -A+1, \quad p>0, \tag{16}$$

let $V=V(a,A,p,H)\subset\Gamma(E)$ be any vector space with the following property:

***(P)** Every $\xi\in V$ has the unique continuation property, i.e., ξ is the null section whenever it vanishes on some domain; furthermore the locally-Lipschitz function $u=|\xi|$ satisfies*

$$\begin{cases} u(\Delta u+a(x)u)\geq A|\nabla u|^2 & \text{weakly on } M \\ \int_{B_r} u^{2p}=o(r^2) & \text{as } r\to+\infty. \end{cases} \tag{17}$$

If there exists a solution $0<\varphi\in Lip_{loc}$ of the differential inequality

$$\Delta\varphi+Ha(x)\varphi\leq 0 \text{ weakly on } M-K \tag{18}$$

for some compact set $K\subset M$, then

$$\dim V\leq d, \tag{19}$$

for some $d<+\infty$ depending only on the geometry of M in a neighborhood of K.

By way of example, let us point out the following consequence which extends to every $p\geq 1$ previous results obtained in case $p=1$ by G. Carron, [3], and P. Li and J. Wang, [16].

Corollary 14 *Let $(M, \langle\,,\,\rangle)$ be a complete manifold satisfying the global Sobolev inequality*

$$\left(\int |u|^{2\mu}\right)^{\frac{1}{\mu}} \leq S_\mu^{-1} \int |\nabla u|^2\,,\ \forall u \in C_c^\infty(M)\,,$$

with $\mu > 1$. Assume that $\rho_x \geq -a(x)$ for some $a(x) \geq 0$ such that

$$a(x) \in L^{\frac{\mu}{\mu-1}}(M)\,.$$

Then, for every $k = 1, ..., m$ and for every $p \geq 1$,

$$\dim L^{2p}\mathcal{H}^k(M) < +\infty.$$

Furthermore, if

$$\|a(x)\|_{L^{\frac{\mu}{\mu-1}}(M)} \leq \frac{S_\mu}{Cp}$$

for some $p \geq 1$ and with $C = C(k, m) > 0$ the constant in the Bochner-Witzenbock formula for harmonic k-forms. Then

$$L^{2p}\mathcal{H}^k(M) = 0.$$

2.5.1 About the proofs

The proofs of Theorems 12 and 13 have one basic feature in common but proceed along very different paths.

Outline of the proof of the vanishing theorem. Combine the solutions ψ of (11) and φ of (12) so to obtain a new function

$$u = \varphi^{-\frac{p}{H}}\psi^p$$

which, in turn, satisfies the easy to handle inequality

$$u\operatorname{div}\left(\varphi^{\frac{2p}{H}}\nabla u\right) \geq 0.$$

Note that

$$\varphi^{\frac{2p}{H}}u^2 \in L^1(M)\,.$$

Now, the key step is to obtain a general Liouville theorem for (possibly changing sign!) solutions of the problem

$$\begin{cases} v\operatorname{div}(w\nabla v) \geq 0 \\ w\,|v|^q \in L^1(M)\,. \end{cases}$$

In our special case, deduce $u \equiv$const. Equivalently

$$C\varphi = \psi^H.$$

Use this information into inequality (12) and get

$$H(H-1+A)\,\psi^{H-2}\,|\nabla\psi|^2 \leq 0.$$

This latter produces the dichotomy in the statement of the theorem, and combined with (11) and (12) gives the desired conclusions.

Outline of the proof of the finiteness theorem. Choose $R >> 1$ in such a way that $K \subset B_R(o)$ and, therefore, inequality (18) holds on $M - B_R(o)$. Note that, by unique continuation, the restriction map

$$\begin{array}{ccc} V & \rightarrow & \Gamma\left(E|_{B_R}\right) \\ \xi & \mapsto & \xi|_{B_R} \end{array}$$

is an injective homomorphism. Use the same symbol V to denote the image of V in $\Gamma\left(E|_{B_R}\right)$. An extension of a classical result by P. Li states that if $T \subset V$ be any finite dimensional subspace then, there exists a (non-zero) section $\bar{\xi} \in T$ such that, setting $\bar{\psi} = |\bar{\xi}|$, it holds

$$(\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p} \leq vol(B_R) \min\{l, \dim T\}^{\min(1,p)} \sup_{B_R} \bar{\psi}^{2p}. \tag{20}$$

Now, observe that, on every sufficiently small closed ball,

$$\lambda_1(\mathcal{L}_H, B_\delta(x)) > 0,$$

where $\mathcal{L}_H = -\Delta - Ha(x)$, and therefore there exists $w > 0$ solution of

$$\Delta w + Ha(x) w = 0.$$

As above deduce that

$$u = w^{-\frac{p}{H}} \psi^p$$

satisfies

$$u \mathrm{div}\left(w^{\frac{2p}{H}} \nabla u\right) \geq 0 \text{ on } B_\delta(x).$$

Obtain a local L^q-mean value inequality for solutions u of this inequality and apply it to get

$$\sup_{B_\delta} \psi^{2p} \leq C \int_{B_{2\delta}} \psi^{2p}.$$

The local inequalities patches together and, in the special case of $\bar{\psi}$, give

$$\sup_{B_R} \bar{\psi}^{2p} \leq C' \int_{B_{R+1}} \bar{\psi}^{2p}.$$

Inserting into (20) we obtain

$$(\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p} \leq Cvol(B_R) \min\{l, \dim T\}^{\min(1,p)} \left\{ \int_{B_R} \bar{\psi}^{2p} + \int_{A(R,R+1)} \bar{\psi}^{2p} \right\} \tag{21}$$

where $A(a,b)$ is the annulus $B_b - B_a$. Now, using once again the combination of ψ and φ and a careful cut-off analysis inspired by Li-Wang estimating technique, [17], obtain the estimate

$$\int_{A(R,R+1)} \bar{\psi}^{2p} \leq C'' \int_{B_R} \bar{\psi}^{2p}.$$

Inserting into (21) and simplifying gives

$$\dim T \leq C''' \min\{l, \dim T\}$$

for some C''' depending only on the geometry of B_R. This proves that any finitely generated subspace T of V have a dimension which is bounded by a universal constant, depending only on the rank l of E and on the geometry of B_R. The same bound must work for the dimension of the whole V.

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