Problemi numerici per il flusso di traffico su reti

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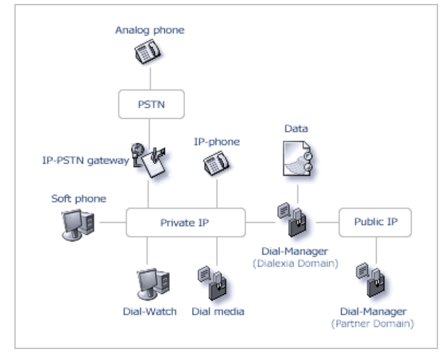
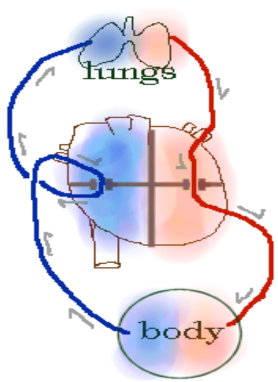
Flows on networks

Car Traffic

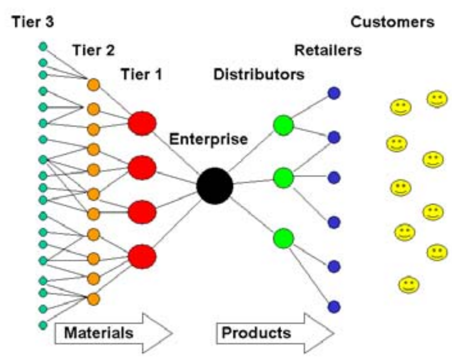
Irrigation Channels



Tlc and data networks



Blood circulation



Supply chains

Gas pipelines

Fluidodynamical models for traffic flow

- LWR model M.J. Lighthill, G.B. Whitham, Richards 1955

$$\rho_t + f(\rho)_x = 0$$

$\rho = \rho(t,x) \in [0, \rho_{max}]$ is the *density* of cars

$f(\rho) = v\,\rho$ is the flux

- Example

$$f(\rho) = v_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right)$$

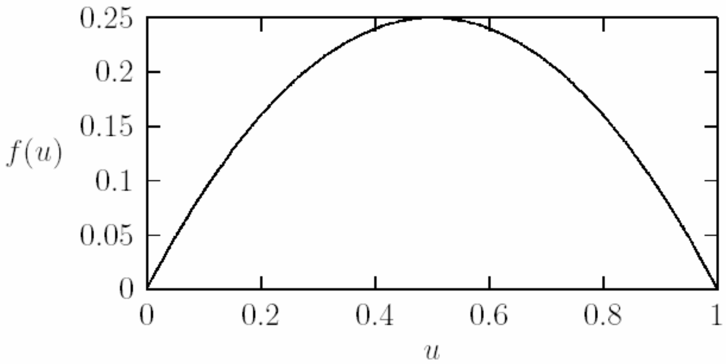


FIGURA 1. Funzione di flusso.

Traffic lights and Viale del Muro Torto

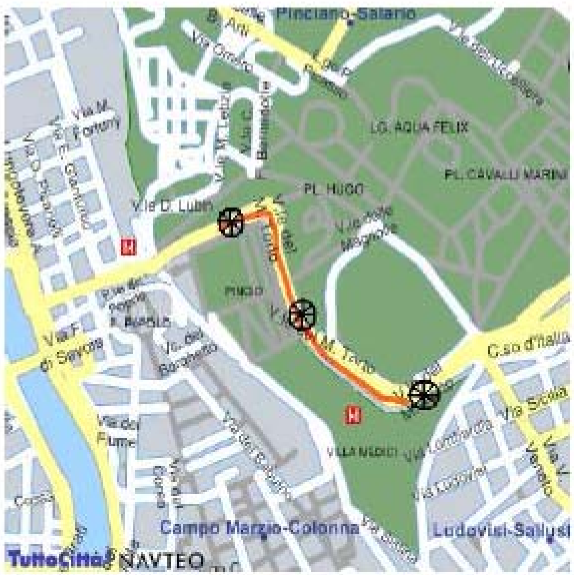
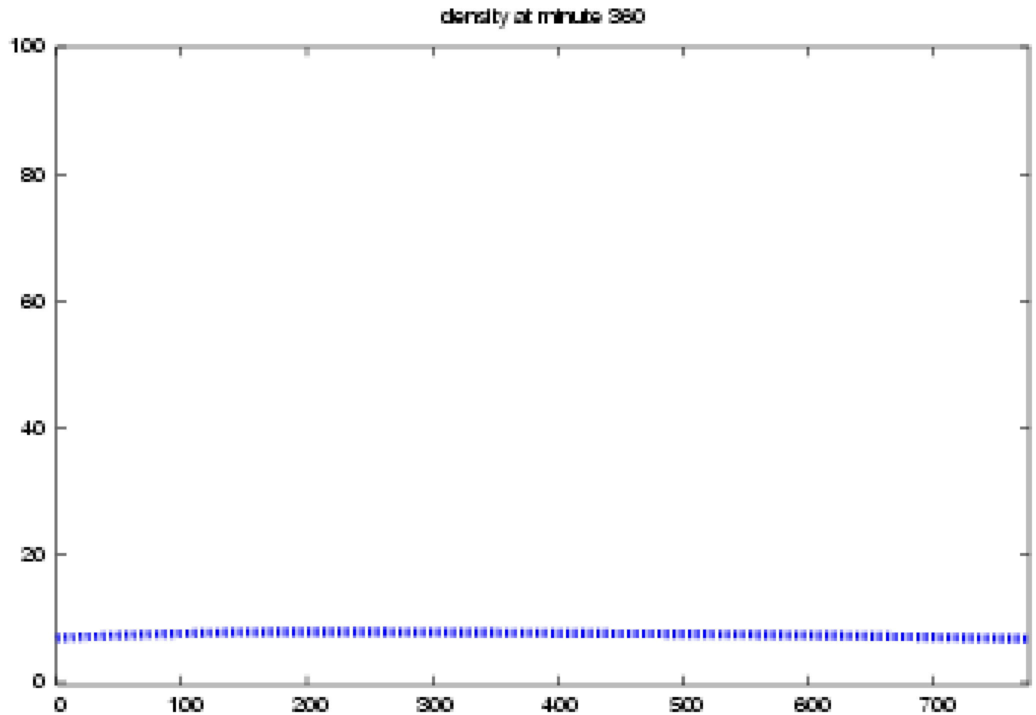


FIGURE 4. Viale del Muro Torto.



Data reconstruction error: 9% free phase, 19% congested phase

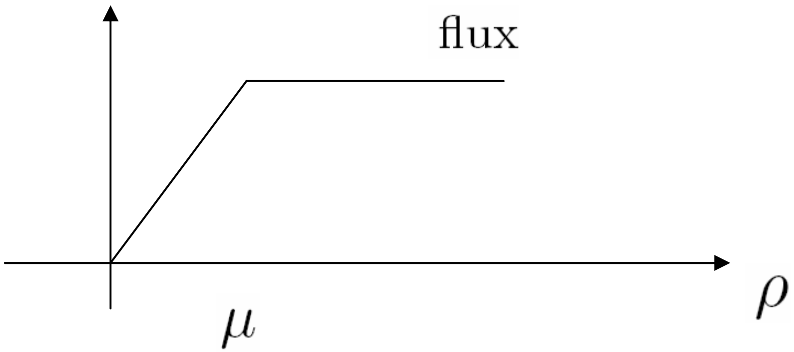
Fluid dynamic models for SC (Armbruster-Degond-Ringhofer et alii)

Conservation law for part density:

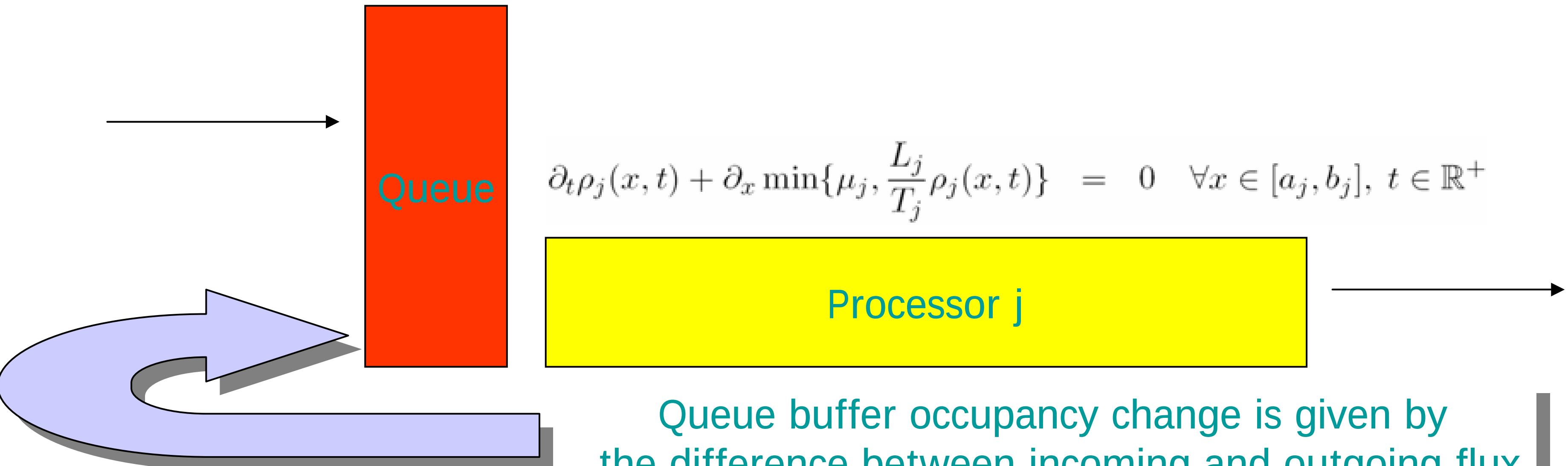
$$\rho_t + (\min\{\mu(t,x), \rho\})_x = 0,$$

$\rho \;=\; \rho(t,x) \;\in\; [0, \rho_{max}]$ part density

$\mu(t,x)$ processing rate



Processor with queue model (Goettlich-Herty-Klar)



$$\partial_t \rho_j(x,t) + \partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} \quad = \quad 0 \quad \forall x \in [a_j, b_j],\ t \in \mathbb{R}^+$$

Queue buffer occupancy change is given by the difference between incoming and outgoing flux

$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_j(\rho_j(a_j, t))$$

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)), \mu_j\} & q_j(t) = 0 \\ \mu_j & q_j(t) > 0 \end{cases}$$

Other fluidodynamical models for car traffic flow

- Aw Rascle model (modified gas dynamics)

$$\begin{cases} \partial_t \rho + \partial_x (y - \rho^{\gamma+1}) = 0 \\ \partial_t y + \partial_x (\frac{y^2}{\rho} - y\rho^{\gamma}) = 0 \end{cases}$$

$y = \rho v + \rho^{\gamma+1}$ is the momentum

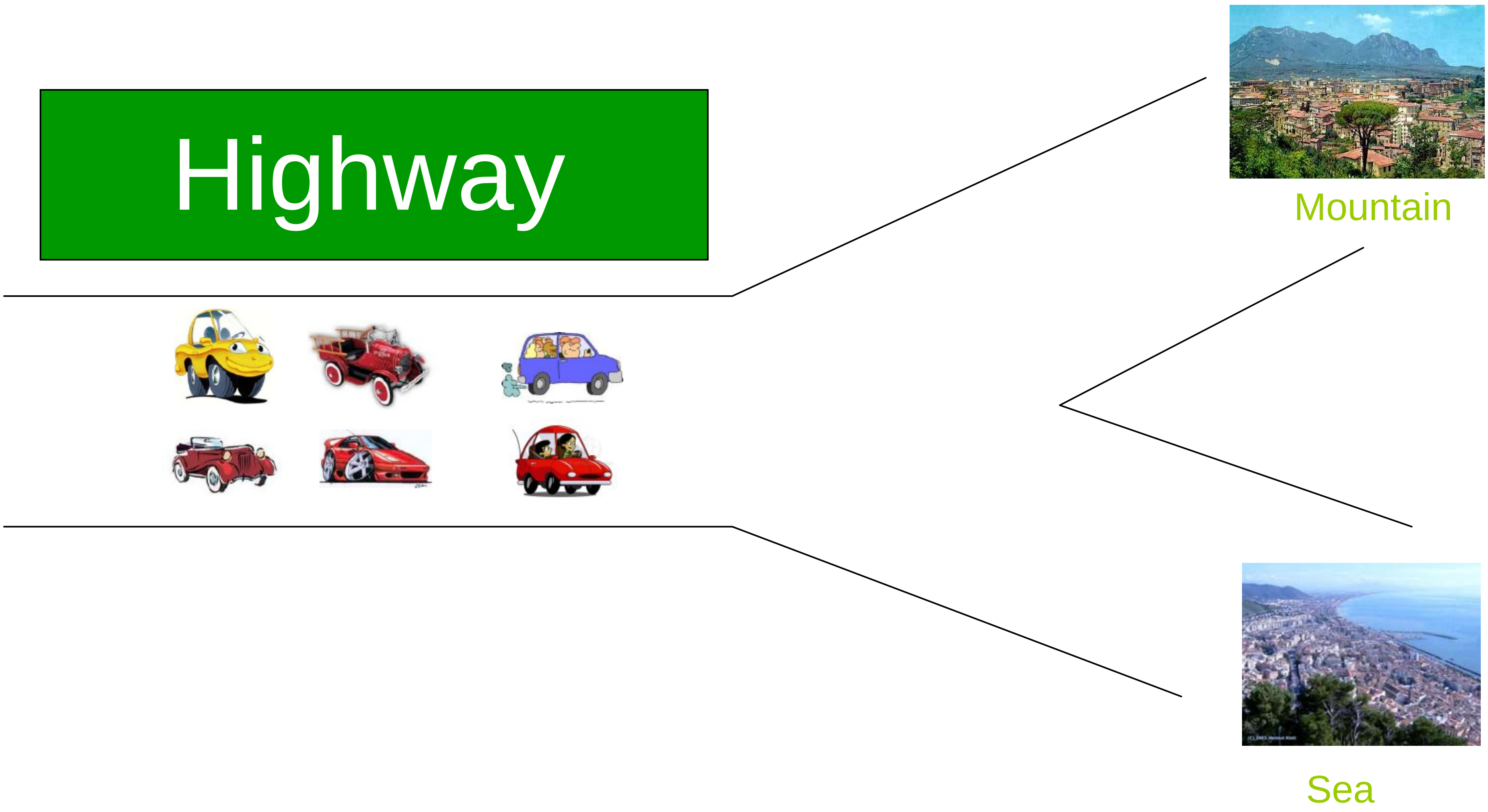
Aw Rascle model solves typical problems of second order models:

Cars going backwards!

Other models: Greenberg, Zhang, Helbing, Klar, Rascle, Benzoni - Colombo, etc.

Similar models for : irrigation channels, blood circulation, gas pipelines

Dynamics at junctions



Dynamics at junctions

Highway

Mountain

Sea

Rule 1 : Traffic distribution coefficients

Rule 2 : Through flux

Solutions at junctions : Riemann problems

(A) There are prescribed preference of drivers, i.e. traffic from incoming roads distribute on outgoing roads according to fixed (probabilistic) coefficients

$$A \doteq \{\alpha_{ji}\}_{j=n+1,\dots n+m,\ i=1,\dots,n} \in \mathbb{R}^{m\times n} \qquad 0 < \alpha_{ji} < 1, \qquad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1$$

(B) Respecting rule (A) drivers behave so as to maximize flow (entropy).

REMARKS:

- The only conservation of cars does not give uniqueness
- Rule (A) implies conservation of cars
-The only rule (A) does not give uniqueness
- Rules (A) and (B) determine a LP problem

Previous work: Holden-Risebro, Lebacque, Daganzo

Another Riemann solver at junctions (tlc)

Maximize the fluxes over incoming and outgoing lines:

then apply rule (A) and priority rule

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}), & \text{if } \rho_{i,0} \in [0,\sigma], \\ f(\sigma), & \text{if } \rho_{i,0} \in]\sigma,1], \end{cases}$$

$$\gamma_j^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{3,0} \in [0,\sigma], \\ f(\rho_{3,0}), & \text{if } \rho_{3,0} \in]\sigma,1], \end{cases}$$

$$\Gamma = \min\left\{\Gamma_{in}^{\max}, \Gamma_{out}^{\max}\right\}$$

Numerical schemes

Approximation schemes (explicit schemes):
Godunov's scheme (first order)
Kinetic scheme (kinetic scheme with 2 or 3 velocities) of first order (Aregba-Driollet – Natalini)
Kinetic scheme with 3 velocities of second order (Aregba-Driollet – Natalini)

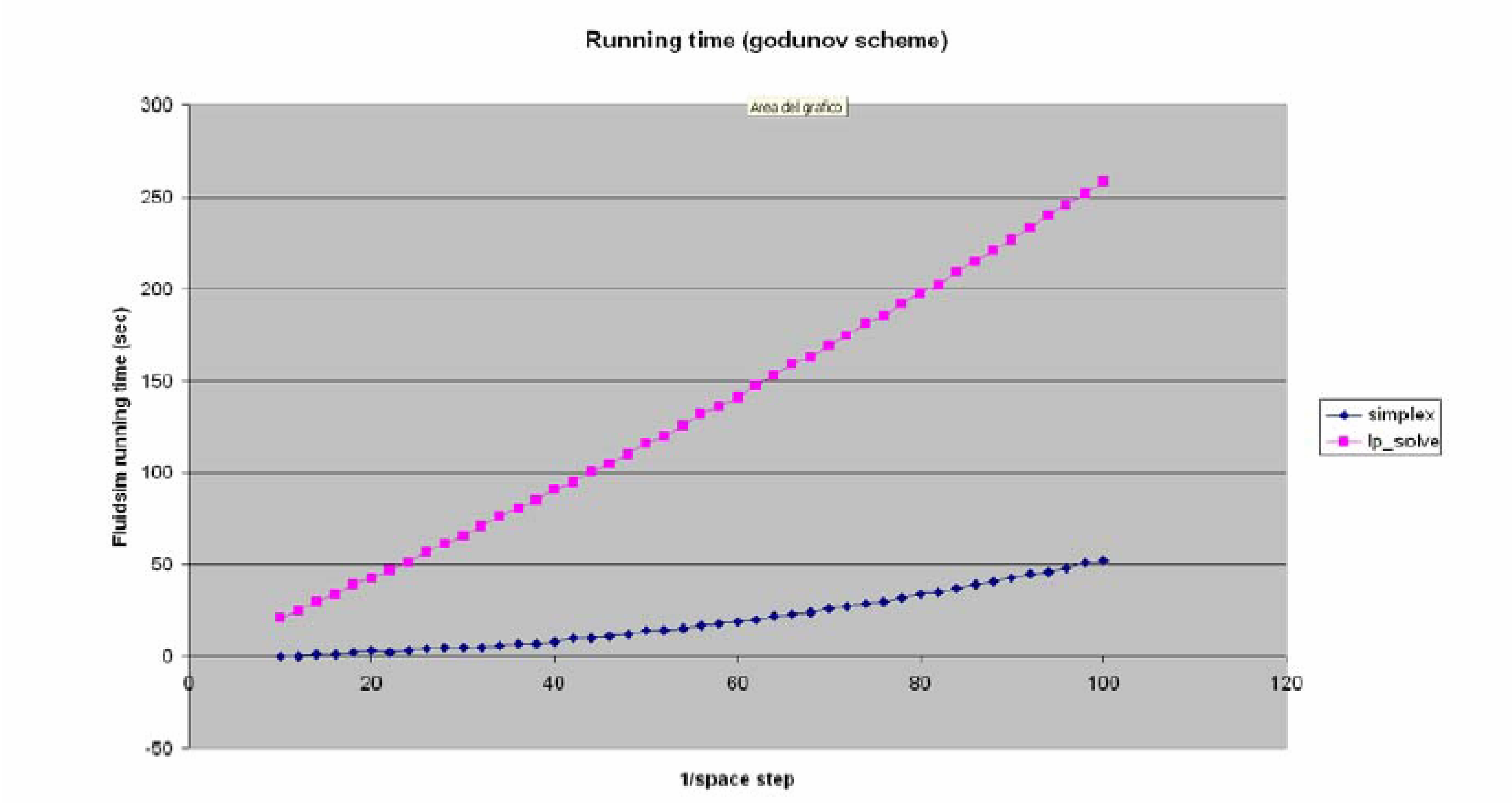
Godunov scheme reads:

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\left(g^G(v_j^n, v_{j+1}^n) - g^G(v_{j-1}^n, v_j^n)\right)$$

$f(\rho) = \rho(1-\rho)$, then the numerical flux :

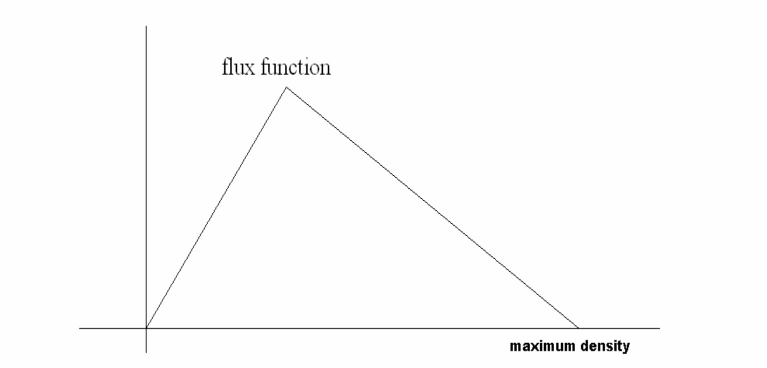
$$g^G(u,v) = \begin{cases} \min(f(u), f(v)) & \text{if } u \le v, \\ f(u) & \text{if } v < u < \sigma \\ f_M & \text{if } v < \sigma < u \\ f(v) & \text{if } \sigma < v < u \end{cases}$$

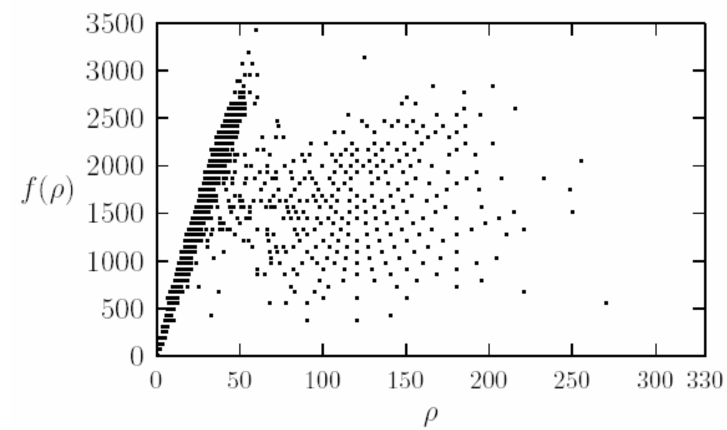
LP solvers



Running time (godunov scheme)
Area del grafico
Fluidsim running time (sec)
300
250
200
150
100
50
0
-50
0
20
40
60
80
100
120
1/space step
simplex
lp_solve

Bilinear model





- Simple model with reasonable properties

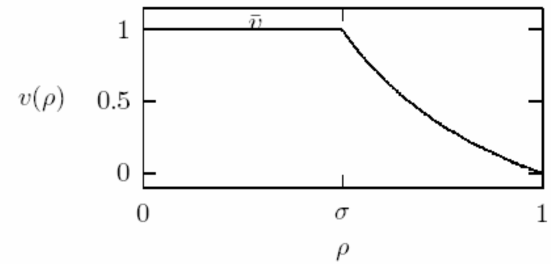
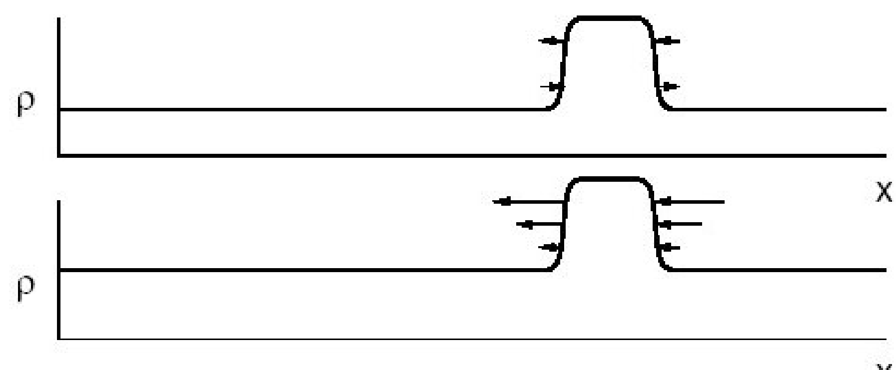


Figure 1: Velocity as a function of the density.

- Two characteristic velocities

- Respect phenomenon of backward moving clusters



Modified Godunov

IDEA: Use bilinear model to have simplified choices of numerical fluxes

$$v_m^{n+1} = v_m^n - \frac{\Delta t}{\Delta x}\big(g^G(v_m^n, v_{m+1}^n) - g^G(v_{m-1}^n, v_m^n)\big).$$

(1,1) $(v_{j-1}^n, v_j^n), (v_j^n, v_{j+1}^n) \in A_1$: $v_j^{n+1} = v_{j-1}^n$;

(1,3) $(v_{j-1}^n, v_j^n) \in A_1, (v_j^n, v_{j+1}^n) \in A_3$: $v_j^{n+1} = v_{j-1}^n$;

(1,4) $(v_{j-1}^n, v_j^n) \in A_1, (v_j^n, v_{j+1}^n) \in A_4$: $v_j^{n+1} = v_{j-1}^n + v_j^n + v_{j+1}^n - 1$;

(2,2) $(v_{j-1}^n, v_j^n), (v_j^n, v_{j+1}^n) \in A_2$: $v_j^{n+1} = v_{j+1}^n$;

(2,5) $(v_{j-1}^n, v_j^n) \in A_2, (v_j^n, v_{j+1}^n) \in A_5$: $v_j^{n+1} = \frac{1}{2}$;

(3,2) $(v_{j-1}^n, v_j^n) \in A_3, (v_j^n, v_{j+1}^n) \in A_2$: $v_j^{n+1} = v_{j-1}^n + v_j^n + v_{j+1}^n - 1$;

(3,5) $(v_{j-1}^n, v_j^n) \in A_3, (v_j^n, v_{j+1}^n) \in A_5$: $v_j^{n+1} = v_{j-1}^n + v_j^n - \frac{1}{2}$;

(4,2) $(v_{j-1}^n, v_j^n) \in A_4, (v_j^n, v_{j+1}^n) \in A_2$: $v_j^{n+1} = v_{j+1}^n$;

(4,5) $(v_{j-1}^n, v_j^n) \in A_4, (v_j^n, v_{j+1}^n) \in A_5$: $v_j^{n+1} = \frac{1}{2}$;

(5,1) $(v_{j-1}^n, v_j^n) \in A_5, (v_j^n, v_{j+1}^n) \in A_1$: $v_j^{n+1} = \frac{1}{2}$;

(5,3) $(v_{j-1}^n, v_j^n) \in A_5, (v_j^n, v_{j+1}^n) \in A_3$: $v_j^{n+1} = \frac{1}{2}$;

(5,4) $(v_{j-1}^n, v_j^n) \in A_5, (v_j^n, v_{j+1}^n) \in A_4$: $v_j^{n+1} = v_{j+1}^n + v_j^n - \frac{1}{2}$.

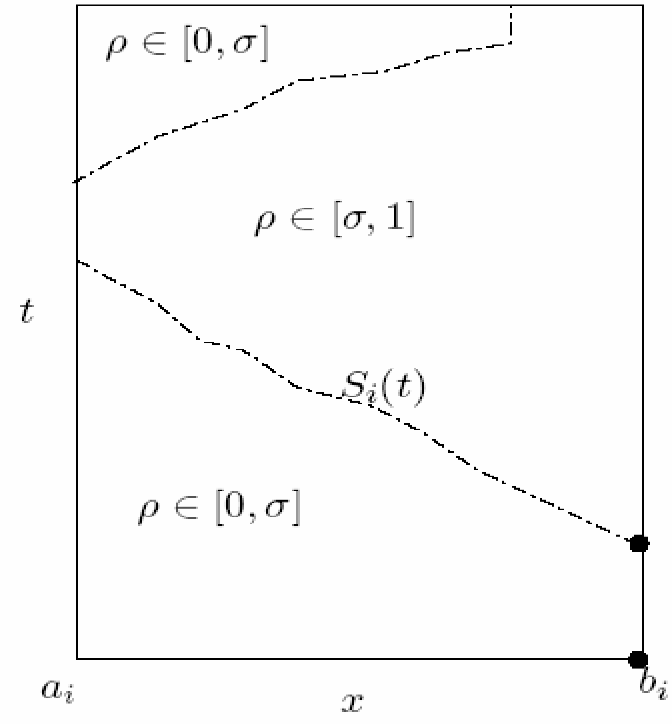
FSF scheme

1. Use simplified flux function with two characteristic speeds

Lemma 2.2 *If road I_i of a junction J has a good datum, then it remains good after interactions with J of waves coming from other roads. Then, no big shock can be produced in this way. If road I_i has a bad datum, then after interactions with J of waves coming from other roads, either the datum of I_i is unchanged or a big shock is produced (and the new datum is good).*

2. Make use of theoretical results to bound the number of regimes changes

3. Track exactly regimes changes or separating shocks and use simple dynamics for one-sided zones



Comparison of schemes

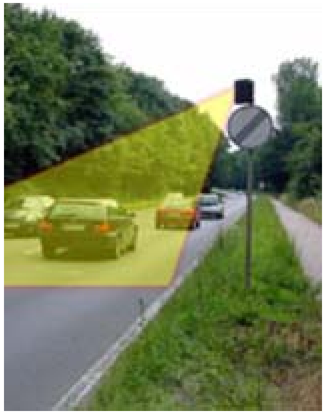
CPU time						
	$T = 10$			$T = 30$		
h	MG	K3V	FVST	MG	K3V	FVST
0.2	1.12 sec	29.37 sec	0.60 sec	3.32 sec	85.71 sec	1.83 sec
0.1	3.05 sec	104.74 sec	1.35 sec	9.12 sec	309.55 sec	4.05 sec
0.05	9.30 sec	394.03 sec	3.20 sec	27.93 sec	1171.10 sec	9.58 sec
0.025	31.40 sec	1515.32 sec	8.39 sec	95.38 sec	4527.80 sec	25.07 sec

TABLE T10: CPU time for modified Godunov scheme (MG), 3-velocities kinetic method (K3V), shock-tracking method (FVST) for Brownian-motion boundary data (4.3), $T = 10, T = 30$, 5000 roads.

Real data

Problems :

1. Data: measurements and elaboration
2. Dimensionality: big networks

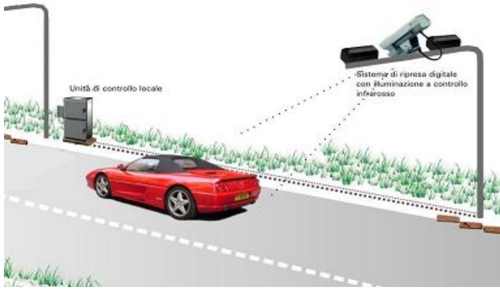


Radars

Manual counting

Satellite data

Videocameras



Plates reading

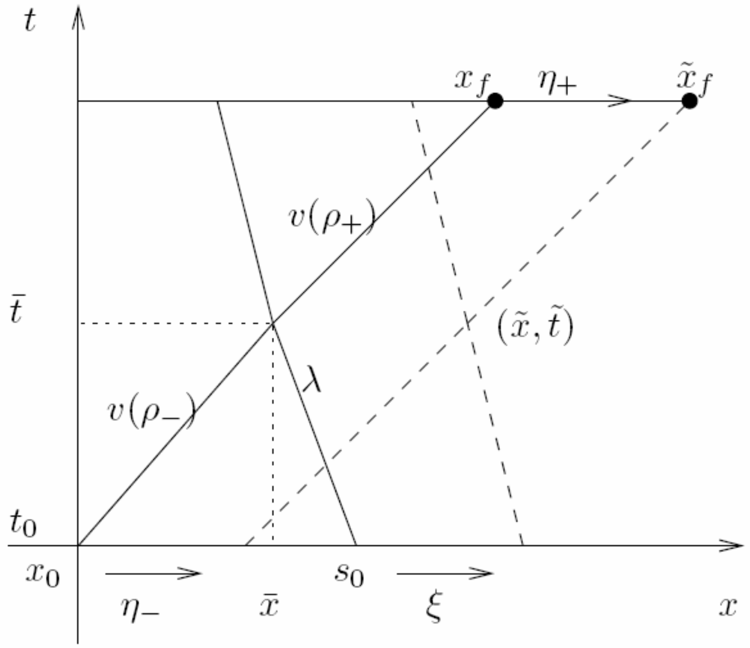
1500 arcs network

Car trajectory on network

- Determine the trajectory of a car on a loaded network

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, \\ \rho(0,x) = \rho_0(x), \end{cases}$$

$$\begin{cases} \dot{x} = v(\rho(t,x)), \\ x(\bar{t}) = \bar{x}, \end{cases}$$



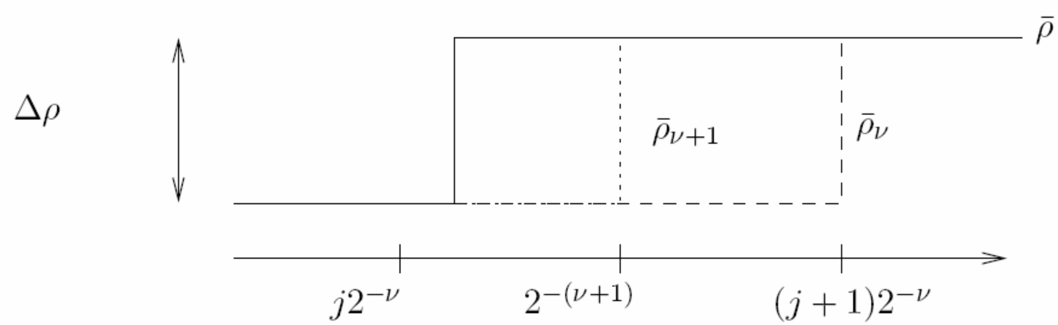
Convergence of WFT in papers by Colombo and Marson

THEOREM 4.4. *Let $\bar\rho \in BV$ and assume that WFT approximate solutions are constructed taking the initial datum $\bar\rho_\nu$ as in (4.1). Assume that $\bar\rho \geq \tilde\rho > 0$, then:*

$$|x_{\nu+1}(t) - x_\nu(t)| \leq \frac{2^{-(\nu+1)}}{\tilde\rho} TV(\bar\rho). \tag{4.15}$$

In particular, $x_\nu(t)$ converges uniformly to some $x(t)$ solution of equation (1.2) when $\nu \to +\infty$. Since the grid mesh parameter is $\Delta x = 2^{-\nu}$ as showed by (4.1), the convergence speed estimate is linear in Δx.

$$\bar\rho_\nu(x) = \bar\rho\left(2^\nu\left[\frac{x}{2^\nu}\right]\right), \quad \nu \in \mathbb{N}. \tag{4.1}$$

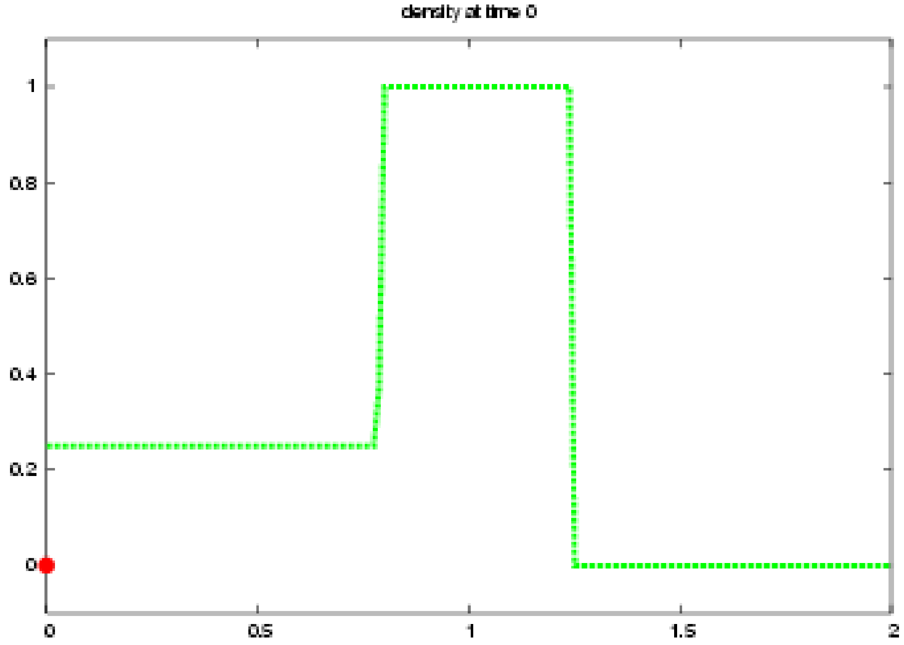


THEOREM 4.5. *Consider a single road $[a, b]$. Then, $\forall \alpha \in (0, 1)$ there exist $\rho_\nu \to \rho$ and $x_{\nu+1} \to x$ such that*

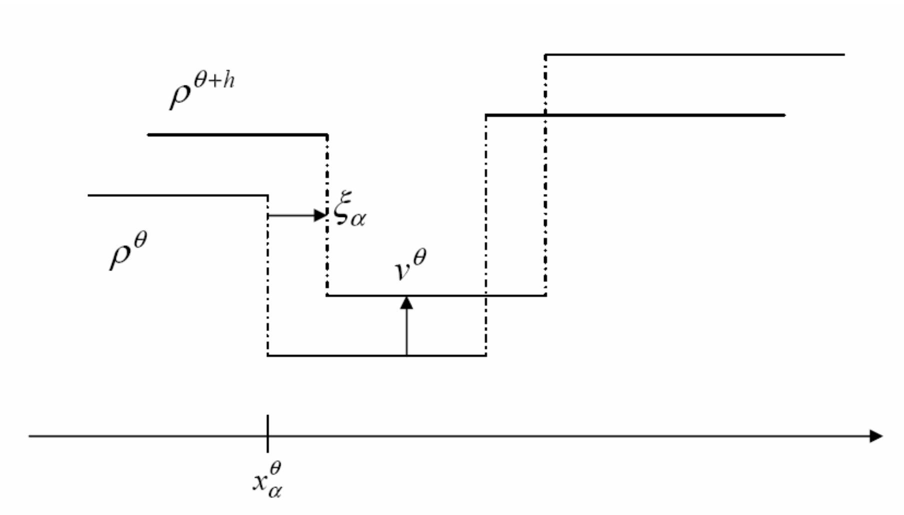
$$|\rho - \rho_\nu| < 2^{-\nu\alpha}(b-a), \tag{4.17}$$

$$|x_{\nu+1} - x_\nu| \leq 2^{-\nu(1-\alpha)} TV(\bar\rho). \tag{4.18}$$

Simulation results



Lipschitz continuous dependence (tlc and GHK supply chain model)



$$\|(v,\xi)\| \doteq \|v\|_{L^1} + \sum_{\beta=1}^{M} |\Delta\rho_\beta|\,|\xi_\beta|\,,$$

$$d(u,u') \doteq \inf\left\{\|\gamma\|_{L^1}\,,\gamma\in\Omega(u,u')\right\}\,.$$

Lemma (tlc) $\quad \left\|(v,\xi)^+\right\| \le \left\|(v,\xi)^-\right\|$

$$\|(\xi_{\beta_i^j},\eta_j)\| = \sum_{j,i} |\xi_{\beta_i^j}||\Delta\rho_{\beta_i^j}| + \sum_j |\eta_j|.$$

η_j is the shift of the queue buffer occupancy q_j

Lemma 2.7 *The norm of tangent vectors are decreasing along wave front tracking approximations.*

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Thank you for your attention!

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