Numerical study of singular behavior in compressible flows

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Main question:

Can vacuum appear in a compressible Navier-Stokes fluid?

or

Does there exist a weak solution to Navier-Stokes where the density ρ reaches zero in finite time, assuming $\rho(\cdot, 0)$ bounded away from zero?

Why should you care?

- open problem
- need to clarify notion of solution
- inviscid case is understood (Euler)
- cool numerical problem

Present work: numerical study of this theoretical question.

Disclaimer: No attempt is made at modeling interstellar matter and/or low density fluids.

Simplifications and notation

Symmetric flows, no swirl:

- $\rho(x, t) = \rho(r, t)$: density
- $\vec{u}(x, t) = \frac{x}{r} u(r, t)$: velocity

x point in space, $r = |x|$, t is time

This talk: barotropic flows: pressure depends solely on ρ

Navier-Stokes

$$\begin{aligned} \rho_t + (\rho u)_\xi &= 0 && \text{mass} \\ \rho(u_t + u u_r) + \frac{1}{\gamma \mathrm{M}^2}(\rho^\gamma)_r &= \frac{1}{\mathrm{Re}} u_{\xi r} && \text{momentum} \end{aligned}$$

where

- *M* Mach number
- *Re* Reynolds number
- γ adiabatic coefficient
- $\partial_\xi = \partial_r + \frac{n-1}{r}$
- n spatial dimension ($n = 1,2,3$)

Euler

Inviscid fluid: $Re \to \infty$

$$\begin{aligned} \rho_t + (\rho u)_\xi &= 0 && \text{mass} \\ \rho(u_t + uu_r) + \frac{1}{\gamma \mathrm{M}^2}(\rho^\gamma)_r &= 0 && \text{momentum} \end{aligned}$$

Riemann data ($r > 0$)

$$\begin{cases} \rho(r,0) &= 1, \\ u(r,0) &= 1, \end{cases} \qquad 1D: \qquad u(r,0) = \begin{cases} -1 & \text{if } r < 0, \\ 1 & \text{if } r > 0. \end{cases}$$

"strength of the pull" is measured by M

Riemman solution $\mathrm{M} > \frac{2}{\gamma-1}$

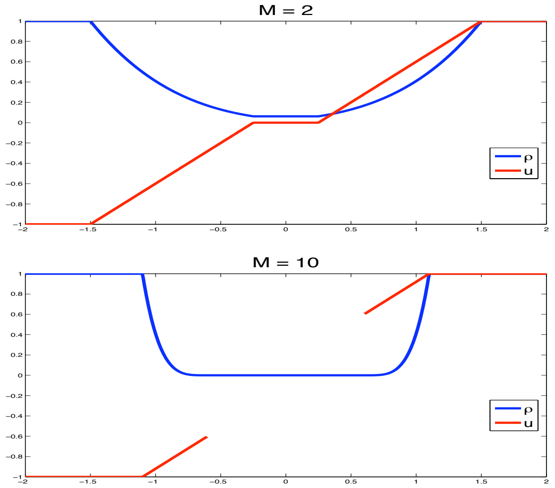
$$\begin{bmatrix}\rho\\u\end{bmatrix}(r,t) = \begin{cases} \begin{bmatrix}1\\-1\end{bmatrix} & \text{if } \frac{r}{t} < -1 - \frac{1}{\mathrm{M}}, \\ \begin{bmatrix}\left(\frac{2}{\gamma+1} - \mathrm{M}\,\frac{\gamma-1}{\gamma+1}(1+\frac{r}{t})\right)^{2/(\gamma-1)} \\ \frac{1}{\mathrm{M}(\gamma+1)}\left(2+(1-\gamma)\mathrm{M}+2\mathrm{M}\frac{r}{t}\right)\end{bmatrix} & \text{if } -1-\frac{1}{\mathrm{M}} < \frac{r}{t} < -1 + \frac{2}{\gamma-1}\frac{1}{\mathrm{M}}, \\ \begin{bmatrix}0\\\emptyset\end{bmatrix} & \text{if } -1+\frac{2}{\gamma-1}\frac{1}{\mathrm{M}} < \frac{r}{t} < 1 - \frac{2}{\gamma-1}\frac{1}{\mathrm{M}}, \\ \begin{bmatrix}\left(\frac{2}{\gamma+1} + \mathrm{M}\,\frac{\gamma-1}{\gamma+1}(-1+\frac{r}{t})\right)^{2/(\gamma-1)} \\ \frac{1}{\mathrm{M}(\gamma+1)}\left(-2+(-1+\gamma)\mathrm{M}+2\mathrm{M}\frac{r}{t}\right)\end{bmatrix} & \text{if } 1-\frac{2}{\gamma-1}\frac{1}{\mathrm{M}} < \frac{r}{t} < 1+\frac{1}{\mathrm{M}}, \\ \begin{bmatrix}1\\1\end{bmatrix} & \text{if } 1+\frac{1}{\mathrm{M}} < \frac{r}{t}. \end{cases}$$

Riemman solution $0 < \mathrm{M} < \frac{2}{\gamma-1}$

$$\begin{bmatrix} \rho \\ u \end{bmatrix}(r,t) = \begin{cases} \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \text{if } \frac{r}{t} < -1 - \frac{1}{\mathrm{M}}, \\ \begin{bmatrix} \left(\frac{2}{\gamma+1} - \mathrm{M}\,\frac{\gamma-1}{\gamma+1}(1+\frac{r}{t})\right)^{2/(\gamma-1)} \\ \frac{1}{\mathrm{M}(\gamma+1)}\left(2 + (1-\gamma)\mathrm{M} + 2\mathrm{M}\frac{r}{t}\right) \end{bmatrix} & \text{if } -1 - \frac{1}{\mathrm{M}} < \frac{r}{t} < -\frac{1}{\mathrm{M}} + \frac{\gamma-1}{2}, \\ \begin{bmatrix} \left(1 - \frac{\mathrm{M}}{2}(\gamma-1)\right)^{\frac{2}{\gamma-1}} \\ 0 \end{bmatrix} & \text{if } -\frac{1}{\mathrm{M}} + \frac{\gamma-1}{2} < \frac{r}{t} < \frac{1}{\mathrm{M}} - \frac{\gamma-1}{2}, \\ \begin{bmatrix} \left(\frac{2}{\gamma+1} + \mathrm{M}\,\frac{\gamma-1}{\gamma+1}(-1+\frac{r}{t})\right)^{2/(\gamma-1)} \\ \frac{1}{\mathrm{M}(\gamma+1)}\left(-2 + (-1+\gamma)\mathrm{M} + 2\mathrm{M}\frac{r}{t}\right) \end{bmatrix} & \text{if } \frac{1}{\mathrm{M}} - \frac{\gamma-1}{2} < r/t < 1 + \frac{1}{\mathrm{M}}, \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } 1 + \frac{1}{\mathrm{M}} < \frac{r}{t}. \end{cases}$$

Known Euler results: 1D

Explicit Riemann solution: vacuum $\Leftrightarrow M > \frac{2}{\gamma-1}$



2, 3 D "Riemann problem"

$$\rho = \rho(s), \qquad u = u(s), \qquad s = \frac{t}{r}$$

$\Rightarrow$

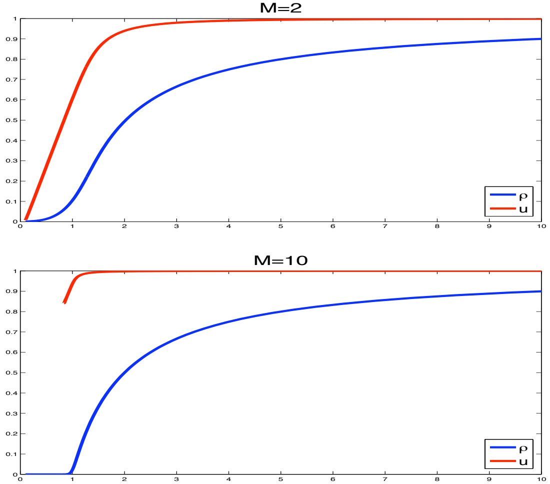
$$\begin{aligned} \rho_s &= (n-1)\frac{\rho u(1-su)}{s^2c^2-(1-su)^2} \\ u_s &= (n-1)\frac{sc^2u}{s^2c^2-(1-su)^2} \end{aligned}$$

where $\rho(0) = 1$, $u(0) = 1$, $c = \frac{1}{M}\rho^{\frac{\gamma-1}{2}}$

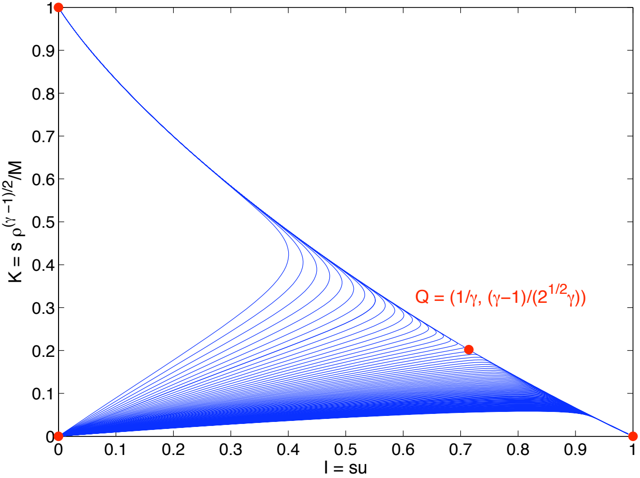
Phase space analysis (Zheng, 2001) shows existence of critical Mach number $M^\star$

Known Euler results: 2, 3 D

Zheng (2001): vacuum $\Leftrightarrow M > M^\star$

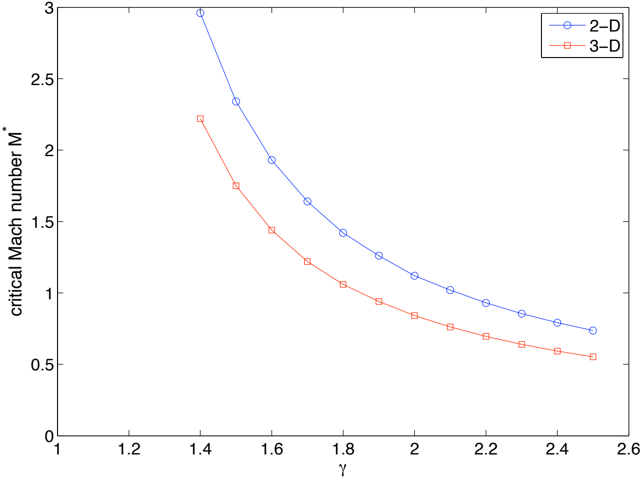


Euler: phase diagram



$(s = t/r)$

Euler: critical Mach number



Known Navier-Stokes results

- theory far from complet, Danchin (2005), Feireisl (2004), Hoff (1997), P.L. Lions (1998)
- unique uniqueness result for discont. sol., Hoff (2006)
- Hoff & Smoller (2001): no vacuum formation for 1D NS
- Xin & Yuan (2006): 2, 3D sufficient conditions to rule out vacuum
- results below are consistent with the above

Numerics

- equations are split

$$(\rho_n, u_n) \xrightarrow{Euler} (\rho^\star, u^\star) \xrightarrow{\rho^* u_t = \frac{1}{Re} u_{\xi r}} (\rho_{n+1}, u_{n+1})$$

- diffusive step solved by Chebyshev-Gauss-Radau collocation (avoid coord. singularity at 0)
- diffusive step advanced in time by BDF (can manage “infinite stiffness" when $\rho = 0$, i.e., index 1 DAE)
- Euler step advanced at each collocation node “à la Zheng" (ODE in $s = t/r$)

Digression on collocation

Basic collocation principles

- Work on a finite grid
- Find p such that $p(x_j) = u_j$, $\forall x_j \in$ grid
- approximate derivative is $p'(x_j)$.

Non periodic problems

- algebraic polynomials on non-uniform grids
- Chebyshev T_N optimality
- $T_N(x) = \cos(N\theta)$ with $\theta = \arccos x$ inherits fast convergence from periodic case

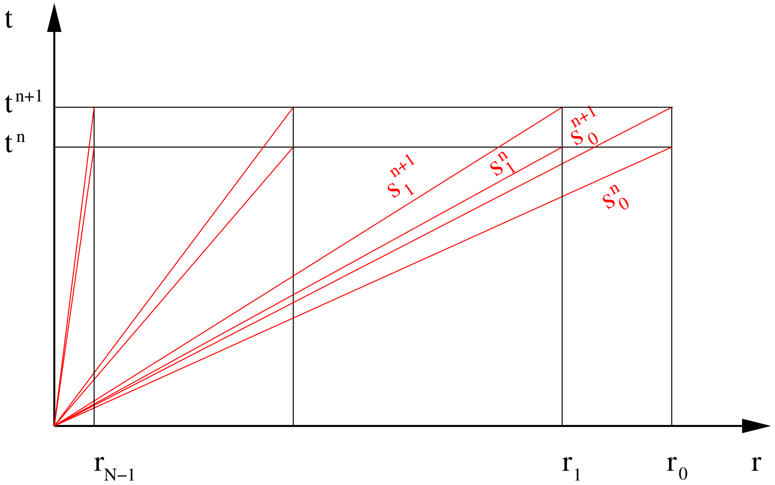
Coordinate singularity at $r = 0$

- Chebyhsev-Gauss-Radau

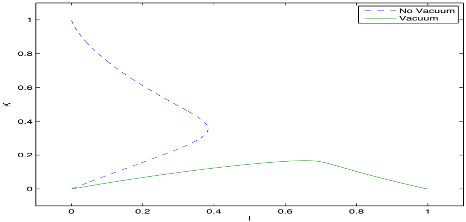
(Spatial) discretization

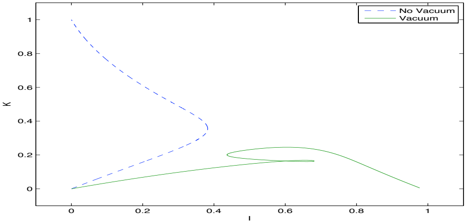
- $u_N(r,t) = \sum_{i=0}^{N-1} U_i(t)\psi_i(r)$; ψ_i Lagrange interpolation pol.

The mesh

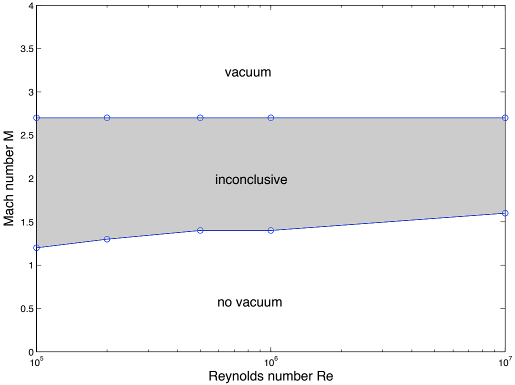


Euler vs NS, 3D, $M = 1.2/2.7$, $\mathrm{Re} = 10^6$





result: 3D



criterion: $\rho_N(r_{N-1}, t) < tol = 10^{-14}$, for some t, $0 < t < .005$

So...

numerics $\Rightarrow$ possible vacuum formation for multi-D NS flows

Another example: relativistic Euler (2D)

$$\begin{aligned}
\partial_t \hat{\rho} + \partial_x(\tilde{\rho} v_1) + \partial_y(\tilde{\rho} v_2) &= 0, \\
\partial_t(\tilde{\rho} v_1) + \partial_x(\tilde{\rho} v_1^2 + \frac{1}{\gamma M^2}\rho^\gamma) + \partial_y(\tilde{\rho} v_1 v_2) &= 0, \\
\partial_t(\tilde{\rho} v_2) + \partial_x(\tilde{\rho} v_1 v_2) + \partial_y(\tilde{\rho} v_2^2 + \frac{1}{\gamma M^2}\rho^\gamma) &= 0,
\end{aligned}$$

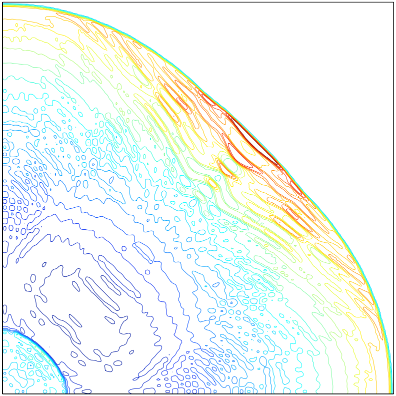
where

- $\tilde{\rho} = \frac{\rho + \frac{1}{\gamma}\frac{\beta^2}{M^2}\rho^\gamma}{1-\beta^2|v|^2}$, $\qquad \hat{\rho} = \tilde{\rho} - \frac{\beta^2}{\gamma M^2}\rho^\gamma$,
- $\beta = \frac{\bar{v}}{c}$,
- $\beta \to 0 \Rightarrow$ classical Euler

Singularity formation

- blow up of smooth compactly supported perturbations of constant states Pan & Smoller (2006)
- type of singularity is unknown
  - shock formation
  - violation of subluminal condition
  - mass concentration
- numerical difficulty: relationship between conserved and physical variables is non trivial

Preliminary results



shock formation; more to follow...

Conclusions

- analyzed two phenomena of singularity formation in compressible fluids
- discussed corresponding numerical challenges
- provided “numerical answers" to two open questions