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March 18,2008

A posteriori error estimation and adaptivity for Hamilton-Jacobi equations

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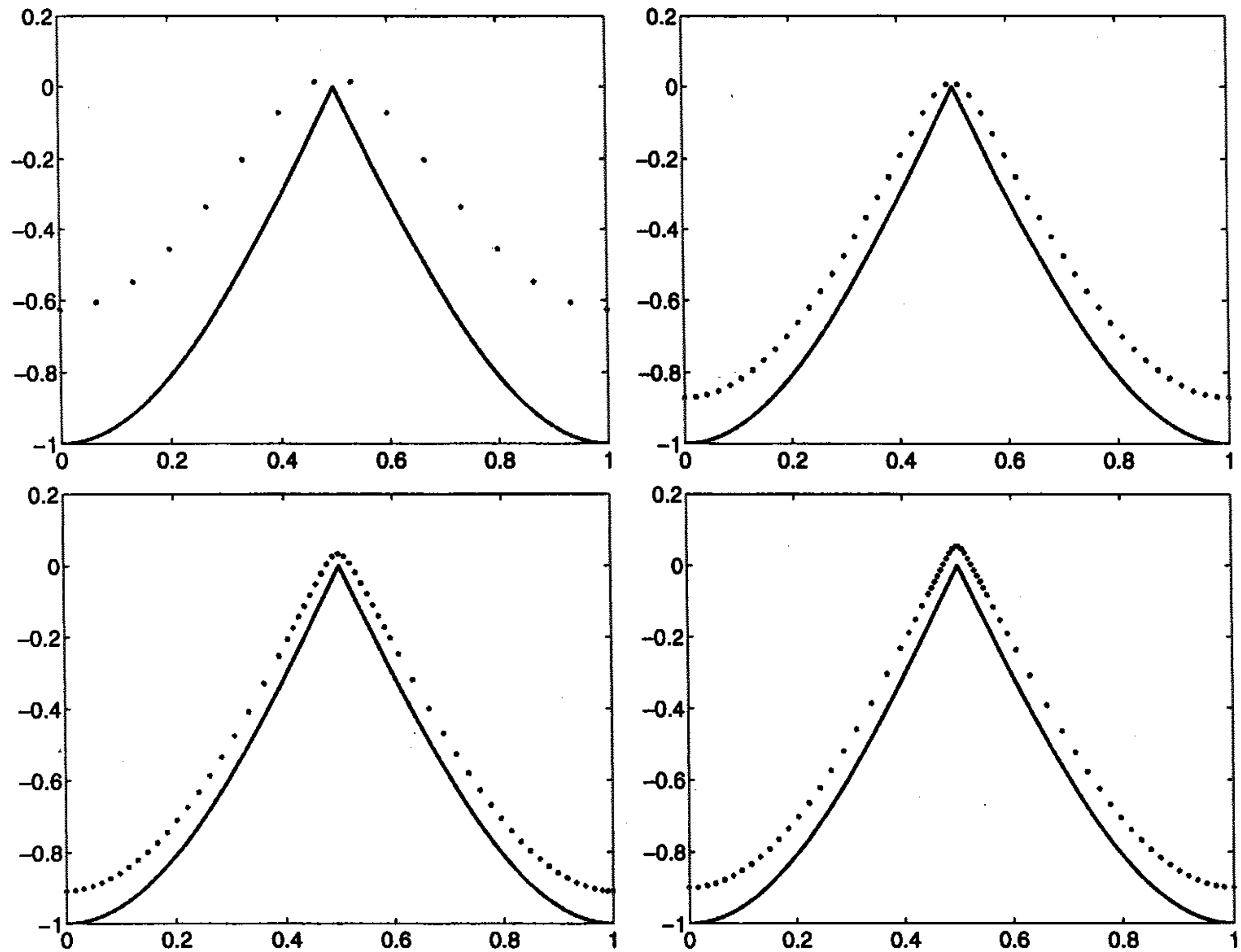
Content

- <u>The main idea</u>

- Viscosity solution of $u + H(\nabla u) = f$ in Ω
- $\| u - v \| \leq \Phi(v) \qquad \forall\, v$
- $u_\nu + H(\nabla u_\nu) - \nu \Delta u = f$
- $\| u_\nu - v \| \leq \Phi_\nu(v) \qquad \forall\, v$
- Characterization of $u = \lim_{\nu \to 0} u_\nu$:

  $\| u - v \| \leq \Phi(v) = \lim_{\nu \to 0} \Phi_\nu(v)$
- Devise adaptive methods.



- <u>A first continuous dependence result</u>

Set

$$|\, u - v \,|_- = \sup_{x \in \Omega} (\, u(x) - v(x) \,)^+,$$

$$|\, u - v \,|_+ = \sup_{x \in \Omega} (\, v(x) - u(x) \,)^+,$$

where $w^+ \equiv \max\{0, w\}$. For any v in $\mathcal{C}^2(\mathbf{R}^d)$ define its residual by

$$R(v; x) = v(x) + H(\nabla v(x)) - \nu \Delta v(x) - f(x).$$

Then, for $\sigma \in \{-, +\}$, we have

$$|\, u - v \,|_\sigma \leq \Phi_\sigma(v),$$

where

$$\Phi_\sigma(v) = \sup_{x \in \mathbf{R}^d} \big(\sigma\, R(v; x) \big)^+.$$

Proof

Set $\psi(x) = u(x) - v(x)$ and choose $\widehat{x} \in \Omega$ so that $\psi(\widehat{x}) \geq \psi(y)$, $\forall y \in \Omega$. Set $\widehat{p} = \nabla u(\widehat{x}) = \nabla v(\widehat{x})$.

Assume that $|\,u - v\,|_{-} > 0$. Then

$$
\begin{aligned}
|\,u - v\,|_{-} &= \sup_{x\in\Omega} \left\{u(x) - v(x)\right\} \\
&= \sup_{x\in\Omega} \psi(x) \\
&= u(\widehat{x}) - v(\widehat{x}) \\
&= \quad \left[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\right] \\
&\quad\; -\left[v(\widehat{x}) + H(\widehat{p}) - \nu\Delta v(\widehat{x}) - f(\widehat{x})\right] \\
&\quad\; +\left[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\right] \\
&= R(u;\widehat{x}) - R(v;\widehat{x}) + \left[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\right] \\
&\leq (-R(v;\widehat{x}))^{+}.
\end{aligned}
$$

- <u>Breakdown of the estimate</u>

Consider

$$v_\nu(x) = -\nu \ln\big(\exp(x/\nu) + 2 + \exp(-x/\nu)\big),$$

which is the smooth solution of the parabolic equation

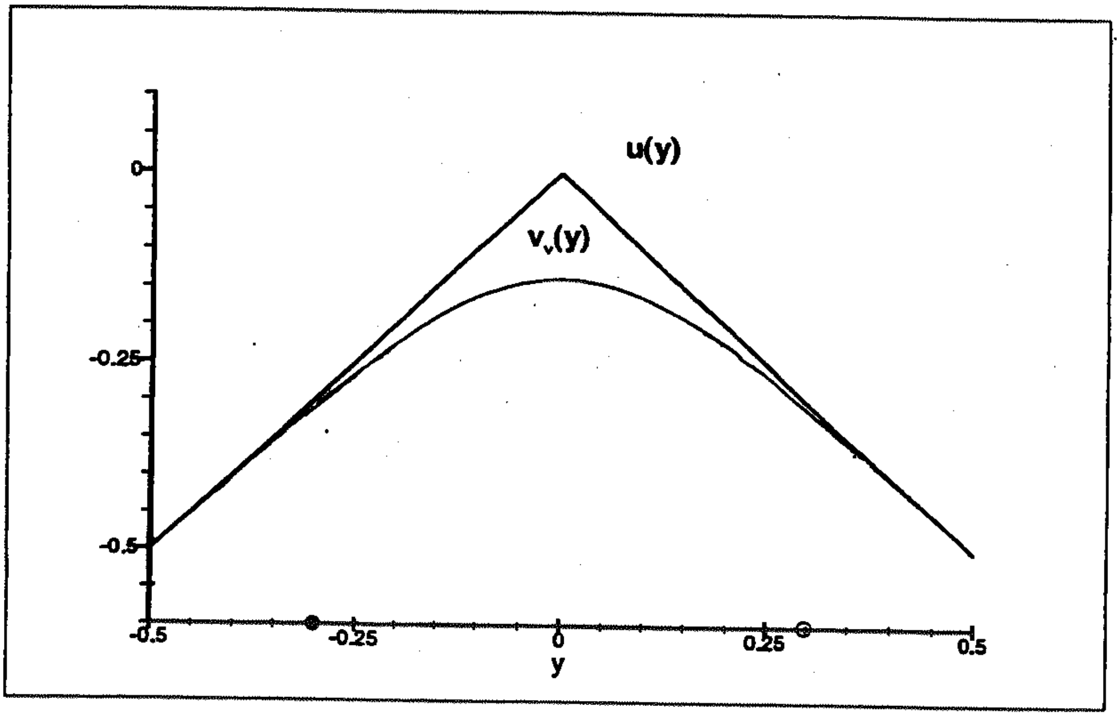
$$u + \frac{1}{2}(u')^2 - \nu\, u'' = v_\nu(x) + 1/2.$$

Then

$$\lim_{\nu \downarrow 0} |\, v_\nu - v_{\bar\nu} \,|_- = |\, u - v_{\bar\nu} \,|_- = \bar\nu \ln(2),$$

and

$$\lim_{\nu \downarrow 0} \Phi_-(v_{\bar\nu}) = \bar\nu \ln(2) + \frac{1}{2}.$$



u(y)
$v_v(y)$
0
-0.25
-0.5
-0.5
-0.25
0
0.25
0.5
y

- <u>A second continuous dependence result</u>

Define the *generalized residual* $R_\epsilon^{\bar\nu}(u; x, p)$ by

$$u(x) + H(p) - \bar\nu \Delta u(x) - f(x - \epsilon\, p) - \frac{\epsilon}{2}\,|\,p\,|^2 .$$

Define the *paraboloid* P_v by

$$P_v(x, p, \kappa; y) = v(x) + (y - x) \cdot p + \frac{\kappa}{2}|\,y - x\,|^2 .$$

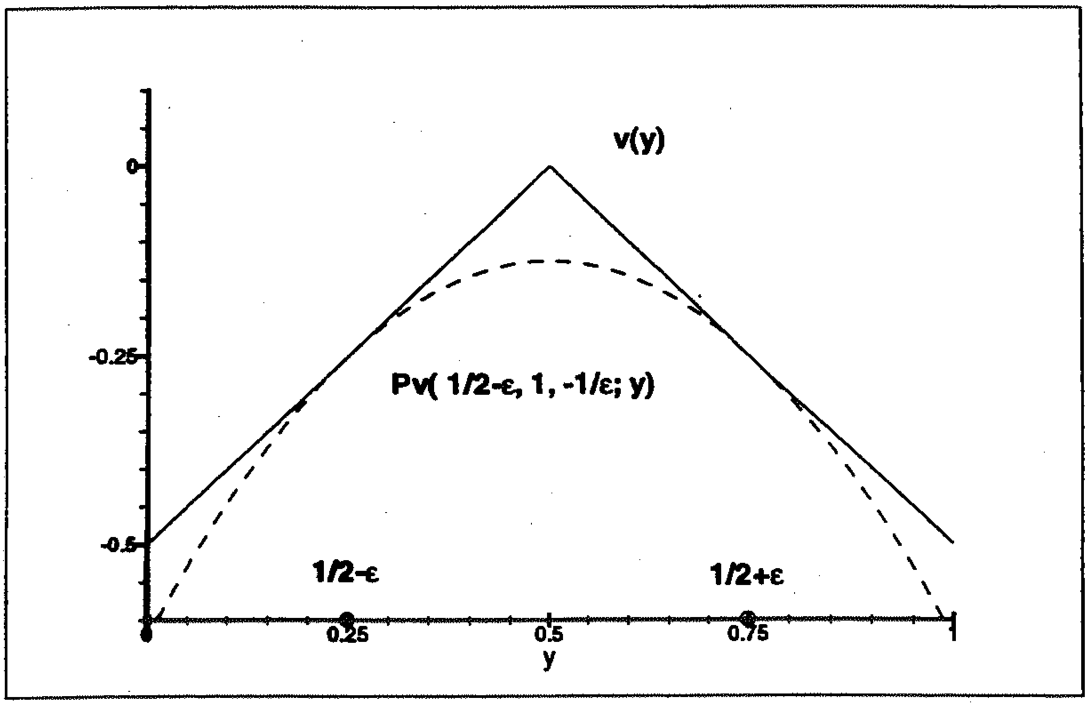


FIGURE 1. The parabola $y \mapsto Pv(1/2 - \epsilon, 1, -1/\epsilon; y)$ for $\epsilon = 1/4$.

Let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function that is periodic in each coordinate with period 1. Then for $\sigma \in \{-,+\}$, we have

$$|\, u - v\,|_\sigma \leq \inf_{\bar{\nu}\geq 0, \epsilon > 0} \Phi_\sigma^{\bar{\nu}}(v;\epsilon),$$

where $\Phi_\sigma^{\bar{\nu}}(v;\epsilon)$ is given by

$$\sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}^{\bar{\nu}}(v;x,p) + \frac{(\sqrt{\nu}-\sqrt{\bar{\nu}})^2}{\epsilon}\, d\right)^+,$$

and the set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of points $(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$ satisfying

$$\sigma\,\{v(y) - P_v(x,p,\sigma/\epsilon\,;\,y)\} \leq 0, \qquad \forall\, y \in \mathbf{R}^d.$$

Consider

$$v_\nu(x) = -\nu \ln\big(\exp(x/\nu) + 2 + \exp(-x/\nu)\big),$$

which is the smooth solution of the parabolic equation

$$u + \frac{1}{2}(u')^2 - \nu\, u'' = v_\nu(x) + 1/2.$$

Then

$$\lim_{\nu\downarrow 0} |\, v_\nu - v_{\bar\nu} \,|_- = |\, u - v_{\bar\nu} \,|_- = \bar\nu \ln(2),$$

and

$$\lim_{\nu\downarrow 0} \inf_{\epsilon>0} \Phi^\nu_\sigma(v_{\bar\nu};\epsilon) = \frac{\bar\nu}{2}\,(\,1 + \ln(4/\bar\nu)\,)(1 + \mathcal{O}(\bar\nu)).$$

- <u>The viscosity solution</u>

Since for any $p \in \mathbf{R}^d$,

$$\begin{aligned} R^{\bar\nu}_{\sigma\epsilon}(u_{\bar\nu}; x, p) &= f(x) - f(x - \sigma\epsilon\, p) - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \epsilon\,|\,p\,|\,|\,f\,|_{W^{1,\infty}(\Omega)} - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)}, \end{aligned}$$

the above result implies that

$$\begin{aligned} |\,u_\nu - u_{\bar\nu}\,|_\sigma &\le \inf_{\epsilon\ge 0} \; \Phi^{\bar\nu}_\sigma(u_{\bar\nu}; \epsilon) \\ &\le \inf_{\epsilon\ge 0} \; \left(\frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)} + \frac{(\sqrt{\nu} - \sqrt{\bar\nu})^2}{\epsilon}\, d \right) \\ &= \sqrt{2\,d}\,|\,f\,|_{W^{1,\infty}(\Omega)} \left|\, \sqrt{\nu} - \sqrt{\bar\nu}\, \right|. \end{aligned}$$

A continuous dependence result for the viscosity solution

Let u be the viscosity solution and let v be any $\mathcal{C}^0(\mathbf{R}^d)$ function that is periodic in each coordinate with period 1. Then for $\sigma \in \{-,+\}$, we have

$$| u - v |_\sigma \leq \inf_{\epsilon > 0} \Phi_\sigma(v; \epsilon),$$

where

$$\Phi_\sigma(v;\epsilon) = \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}(v;x,p)\right)^+ ,$$

and the set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of points $(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$ satisfying

$$\sigma\,\{v(y) - P_v(x,p,\sigma/\epsilon\,;\,y)\} \leq 0, \qquad \forall y \in \mathbf{R}^d.$$

Note that the above <u>a posteriori error estimate</u>

1. Holds for any Hamiltonian H.
2. Automatically takes into account the form of the non-linearity in $H(|\ln(h)|$ versus $h^{-1/2})$.
3. Is independent of how v has been computed.
4. Does not need to solve an adjoint equation.
5. Its evaluation is parallelizable.

Characterization of the viscosity solution

The viscosity solution is the only function u in $C^0(\mathbf{R}^d)$ that is periodic in each coordinate with period 1, such that for all x in $\mathbf{R}^d$,

$$u(x) + H(p) - f(x) \leq 0, \quad \forall\, p \in D^+ u(x),$$

and

$$u(x) + H(p) - f(x) \geq 0, \quad \forall\, p \in D^- u(x).$$

Indeed, we have

$$\begin{aligned}
|\, u - v \,|_- &\leq \inf_{\epsilon>0} \Phi_-(v;\epsilon) \\
&= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(-R_{-\epsilon}(v;x,p)\right)^+ \\
&= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(f(x+\epsilon\, p) - f(x)\right)^+ \\
&\leq \inf_{\epsilon>0} |\, f \,|_{W^{1,\infty}(\Omega)} \,|\, v \,|_{W^{1,\infty}(\Omega)}\, \epsilon \\
&= 0.
\end{aligned}$$

• An adaptive method

Given an arbitrary tolerance, $\tau > 0$, we want to find a mesh $\{x_i\}$ such that

$$\| u - v \|_{L^\infty(0,1)} \leq \tau,$$

where the approximate solution v is obtained by using the Lax-Friedrichs scheme.

Since there is an a posteriori error estimate of the form

$$\| u - v \|_{L^\infty(0,1)} \leq \Phi(v),$$

we try to devise a adaptive method that computes the approximation v satisfying the quality constraint

$$\Phi(v) \leq \tau,$$

with optimal complexity.

The general form of the adaptive is the following:

- Pick an initial mesh G_h.
- Compute v on G_h, then, compute $\Phi_h(v)$. If the quality constraint is satisfied and the grid G_h is *reasonable*, stop.
- Otherwise, compute a new grid G_h and go to the previous step.

• Computing a new mesh

Given a mesh $\{x_i\}$, its mesh-size function is defined to be

$$h(y) = x_{i+1} - x_i \text{ for } y \in (x_i, x_{i+1}).$$

We compute a new mesh $\{y_i\}$ by using a mesh-size modification function

$$\mu : [0,1) \mapsto [0, \infty),$$

as follows:

(1) Compute the number of grid points between 0 and y, namely,

$$N(y) = \int_0^y \frac{\mu(s)}{h(s)} ds.$$

(2) Compute the grid points $y_i := N^{-1}(i)$.

Note that if $\mu = 1$ then $x_i = y_i$. If $\mu > 1$, the new mesh is more refined and if $\mu < 1$, it is coarser. We thus take

$$\mu = \Psi(\gamma)$$

where

$$\gamma(s) := \frac{1}{\tau} \min\{|R(v; s, v'(s))|, \Phi_h(v)\},$$

for $s \in \Omega_h$, and

$$\Psi(g) = \begin{cases} g\,(1 + \sqrt{g-1}), & g > 1, \\ \frac{g+3}{4}, & 0 \le g \le 1 \end{cases}$$

We take

$$\mu := \max\{1, \Psi(\gamma)\},$$

when the mesh is *reasonable*, that is, when

$$\|\,(1 - \Psi(\gamma))^+\,\|_{L^1(0,1)} \le 0.02\,|\Omega|.$$

TABLE 1. Smooth solution test problems.

Hamiltonian $H(p)$	right-hand side $f(x)$	viscosity solution $u(x)$
p (linear)	$\cos^2(\pi x) - \pi \sin(2\pi x)$	$\cos^2(\pi x)$
$-p^2/4\pi^2$ (concave)	$\cos^4(\pi x)$	$\cos^2(\pi x)$
$p^3/8\pi^3$ (non-convex)	$\sin(2\pi x) + \cos^3(2\pi x)$	$\sin(2\pi x)$

TABLE 2. Non-smooth solution test problems.

Hamiltonian $H(p)$	right-hand side $f(x)$	viscosity solution $u(x)$
p^2/π^2 (convex)	$-\lvert \cos(\pi x) \rvert + \sin^2(\pi x)$	$-\lvert \cos(\pi x) \rvert$
$-p^4 + 2p^2 - 1$ (non $-$ convex)	$u(x) + H(u'(x))$ if $x \neq 1/2$	$\begin{cases} x^2, & \text{if } 0 \le x \le \frac{1}{2}, \\ (x-1)^2, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$

TABLE 3. Monotone scheme on smooth solution test problems.

Hamiltonian	$1/\Delta x$	error	order	$ei_h(u,v)$
linear	40	3.8e-2	-	6.4
	80	1.9e-2	0.99	6.4
	160	9.7e-3	1.00	6.4
	320	4.8e-3	1.00	6.4
	640	2.4e-3	1.00	6.4
	1280	1.2e-3	1.00	6.4
concave	40	1.4e-1	-	1.0
	80	7.5e-2	0.91	1.0
	160	3.8e-2	0.96	1.0
	320	1.9e-2	0.98	1.0
	640	9.8e-3	0.99	1.0
	1280	4.9e-3	1.00	1.0
non-convex	40	5.0e-1	-	1.3
	80	3.1e-1	0.69	1.4
	160	1.8e-1	0.81	1.4
	320	9.7e-2	0.86	1.3
	640	5.2e-2	0.90	1.2
	1280	2.7e-2	0.93	1.1

TABLE 4. Monotone scheme on non-smooth solution test problems.

Hamiltonian	$1/\Delta x$	error	order	$ei_h(u,v)$
convex	40	1.5e-1	-	1.8
	80	7.7e-2	0.98	1.7
	160	3.9e-2	0.99	2.0
	320	2.0e-2	1.00	2.3
	640	9.8e-3	1.00	2.4
	1280	4.9e-3	1.00	2.9
non-convex	40	3.9e-2	-	1.5
	80	1.9e-2	1.00	1.8
	160	9.6e-3	1.00	2.6
	320	4.8e-3	1.00	3.5
	640	2.4e-3	1.00	4.6
	1280	1.2e-3	1.00	6.3

TABLE 5. Monotone scheme on non-smooth solution test problems: The set $[a, b]$ for which the parabola test failed, the number of grid points in the interval, and the optimal value ϵ_{opt}.

Hamiltonian	$1/\Delta x$	a	b	$\lvert b-a \rvert/\Delta x$	ϵ_{opt}	ϵ_{opt}/E	$N \cdot \epsilon_{opt}/E$
convex	40	.425	.550	5	3.1e-2	0.24	4
	80	.450	.538	7	1.6e-2	0.21	4
	160	.475	.519	7	8.0e-3	0.18	4
	320	.484	.513	9	5.0e-3	0.20	5
	640	.492	.506	9	2.5e-3	0.18	5
	1280	.495	.504	11	1.5e-3	0.20	6
non-convex	40	.475	.500	1	2.9e-2	0.20	3
	80	.475	.513	3	2.4e-2	0.27	5
	160	.488	.506	3	1.2e-2	0.23	5
	320	.491	.506	5	9.6e-3	0.33	8
	640	.494	.505	7	6.0e-3	0.37	10
	1280	.496	.503	9	3.9e-3	0.44	13

Table 3: Results of the adaptive method.

Problem	τ	n	$\| u - u_h \|_{L^\infty(\Omega_h)}$	order	ei_τ	ei_{adap}	steps	$cmplxr$
(1)	10.0E-2	392	8.2E-2	–	1.20	1.01	3	1.96
	5.0E-2	1475	4.1E-2	1.06	1.21	1.00	3	1.82
	2.5E-2	7032	1.9E-2	1.01	1.34	1.06	4	2.43
(2)	10.0E-2	17499	4.6E-2	–	2.16	1.10	6	3.78
	5.0E-2	64830	2.3E-2	1.06	2.16	1.06	5	2.99
	2.5E-2	292324	1.0E-2	1.02	2.33	1.20	5	2.84
(3)	10.0E-2	14483	6.9E-2	–	1.44	1.08	4	2.23
	5.0E-2	65241	3.0E-2	1.09	1.64	1.16	5	2.60
	2.5E-2	186660	1.6E-2	1.16	1.51	1.03	4	2.30
(4)	10.0E-2	3690	6.5E-2	–	1.53	1.22	4	2.03
	5.0E-2	11863	3.4E-2	1.07	1.44	1.07	3	1.42
	2.5E-2	41569	1.7E-2	1.14	1.48	1.08	3	1.66

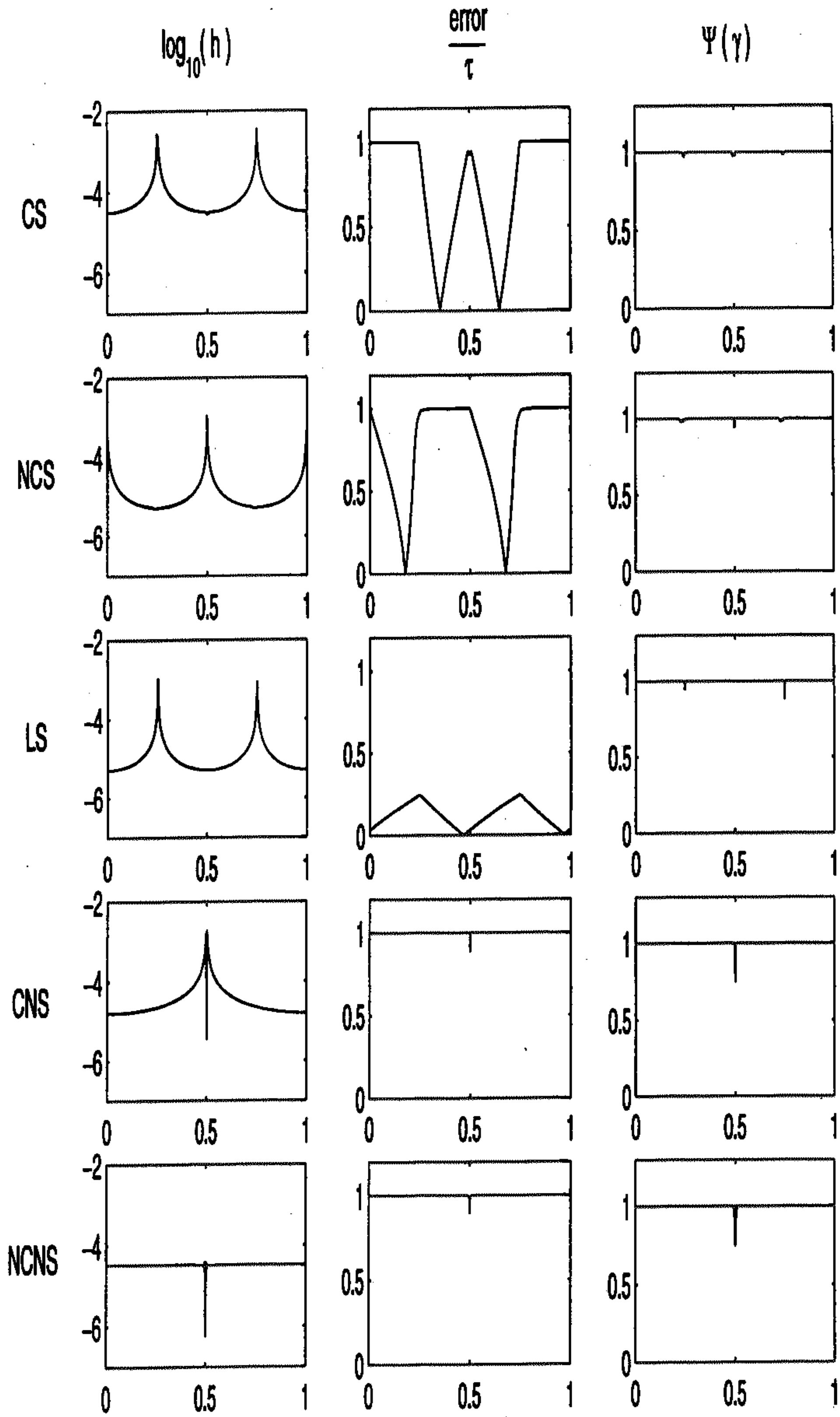


Fig. 3. Convergence step, $\tau = 10^{-4}$.

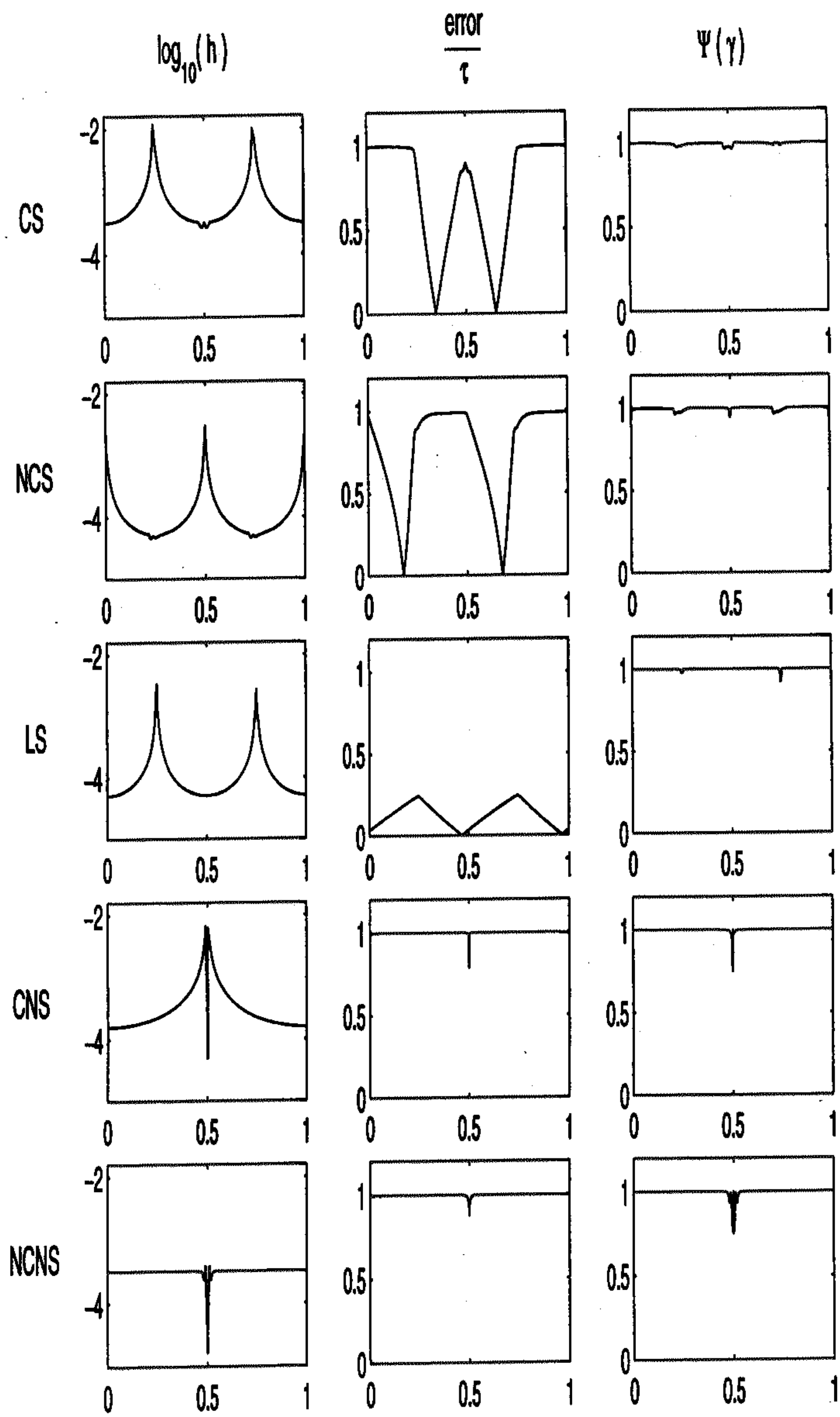


Fig. 2. Convergence step, $\tau = 10^{-3}$.

History of convergence of the self-adaptive method.

Problem	τ	n	$\|u-u_h\|_{L^\infty(G_h)}$	order	$ei_\tau(\tau;u,v)$	$ei_{adap}(\tau;v)$	steps	*cmplxr*
CS	1.0E-2	212	9.93E-3	–	1.01	1.001	6	4.88
	1.0E-3	2040	9.98E-4	1.01	1.00	1.001	4	2.77
	1.0E-4	20071	1.00E-4	1.01	1.00	1.000	6	4.78
NCS	1.0E-2	1301	9.36E-3	–	1.07	1.01	5	3.54
	1.0E-3	12320	9.87E-4	1.00	1.01	1.00	5	3.58
	1.0E-4	120749	9.98E-5	1.00	1.00	1.00	4	2.61
LS	1.0E-2	1271	2.35E-3	–	4.26	1.00	4	2.89
	1.0E-3	12587	2.42E-4	0.99	4.13	1.00	5	3.89
	1.0E-4	125690	2.44E-5	1.00	4.10	1.00	5	3.89
CNS	1.0E-2	454	9.95E-3	–	1.01	1.00	3	2.05
	1.0E-3	4085	9.98E-4	1.05	1.00	1.00	3	2.13
	1.0E-4	40148	9.99E-5	1.01	1.00	1.00	5	4.15
NCNS	1.0E-2	357	9.97E-3	–	1.00	1.00	4	2.73
	1.0E-3	3378	9.97E-4	1.02	1.00	1.00	4	2.74
	1.0E-4	32458	9.99E-5	1.02	1.00	1.00	4	2.80

Table 5
Behavior of $n \cdot \Phi_h(u_h)$ for adaptive and uniform meshes.

Problem	$\Phi_h(u_h)$	$n_{adap} \cdot \Phi_h(u_h)/(\omega \cdot \|u''\|_{L^1(0,1)})$	$\Phi_h(u_h)$	$n_{unif} \cdot \Phi_h(u_h)/(\omega \cdot \|u''\|_{L^\infty(0,1)})$
CS	1.0E-2	1.06	9.8E-3	1.00
	1.0E-3	1.02	1.2E-3	0.98
	1.0E-4	1.00	1.5E-4	0.98
NCS	9.9E-3	1.07	1.5E-2	1.01
	1.0E-3	1.02	1.8E-3	0.98
	1.0E-4	1.01	4.6E-4	1.00
LS	1.0E-2	1.01	3.1E-2	0.99
	1.0E-3	1.00	1.9E-3	0.99
	1.0E-4	1.00	4.8E-4	1.00

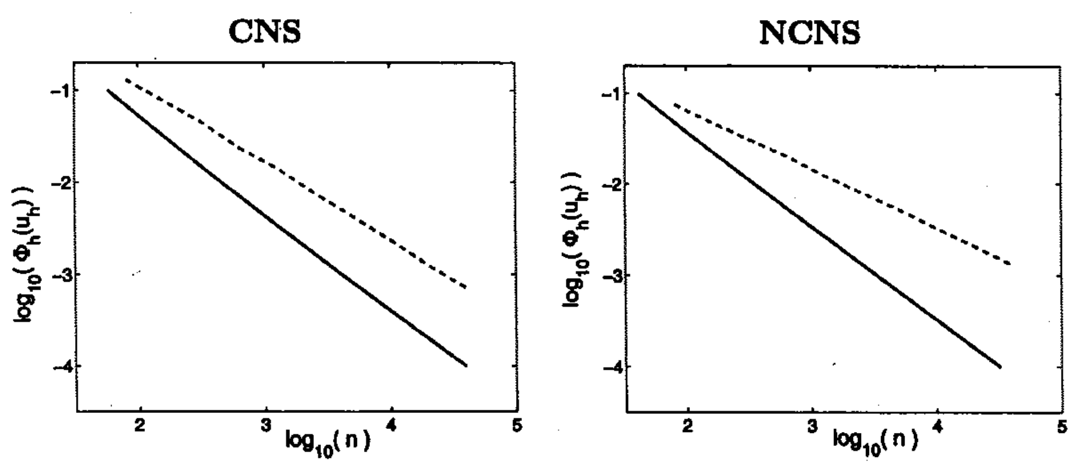


Fig. 7. Comparison of convergence rates: Uniform (dashed line) and adaptive (solid line) refinement.

- <u>Adaptivity versus uniform refinement</u>

We have that

$$TE \approx \omega\, h\, u''.$$

This suggests to take $\tau = \omega\, h\, u''$. The corresponding mesh is called *optimal* and is given by $\{\mathcal{N}^{-1}(i)\}$ where

$$\mathcal{N}(y) = \int_0^1 \frac{\omega\, u''(s)}{\tau}\, ds.$$

Then

$$N_o\, \tau = \omega\, \|\, u''\|_{L^1}$$

If the mesh is uniform, then

$$N_u\, \tau = \omega\, \|\, u''\|_{L^\infty}\, |\Omega|$$

Hence, $N_u > N_o$.

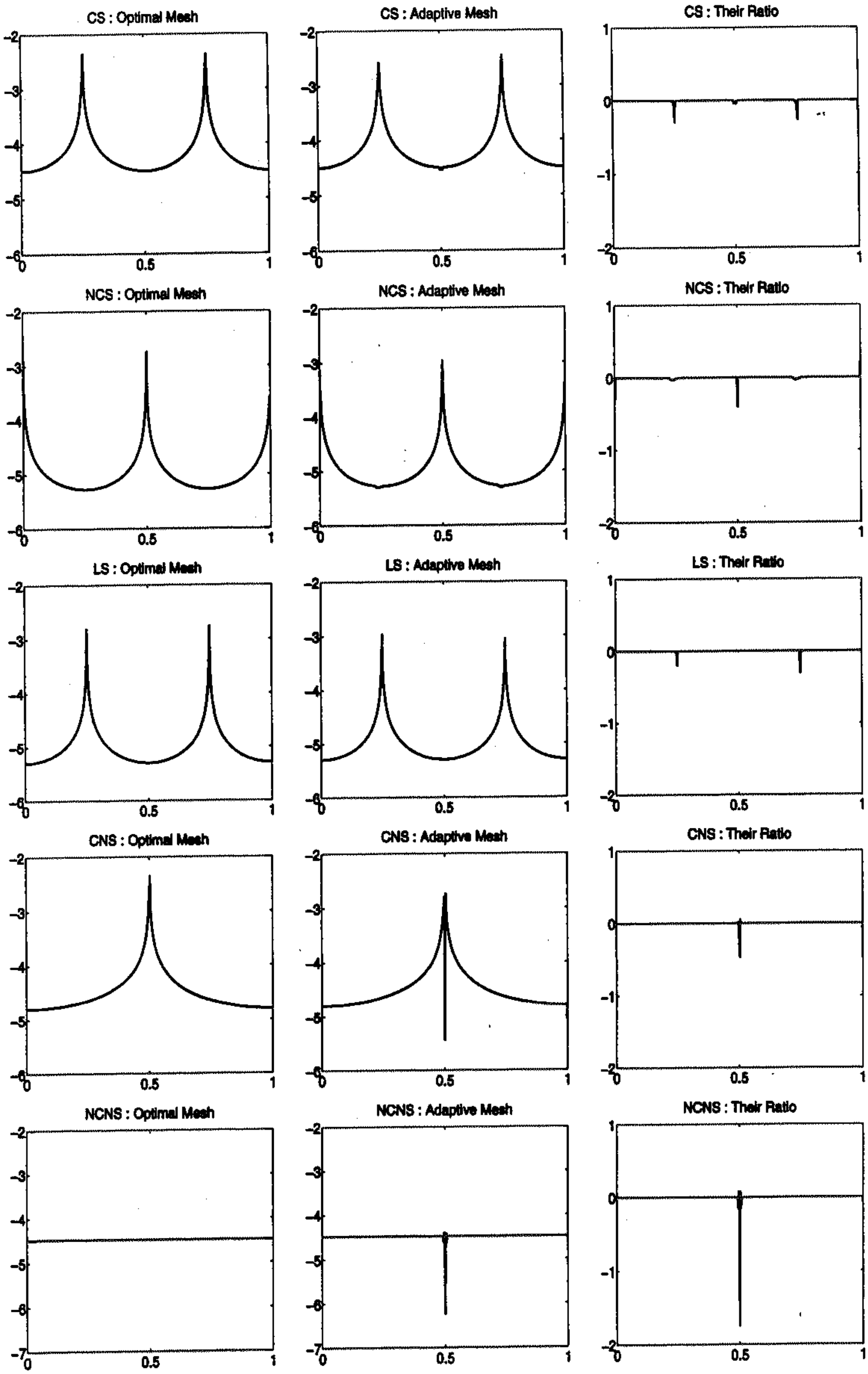


Fig. 5. Optimal and adaptive mesh-size functions for $\tau = 10^{-4}$.

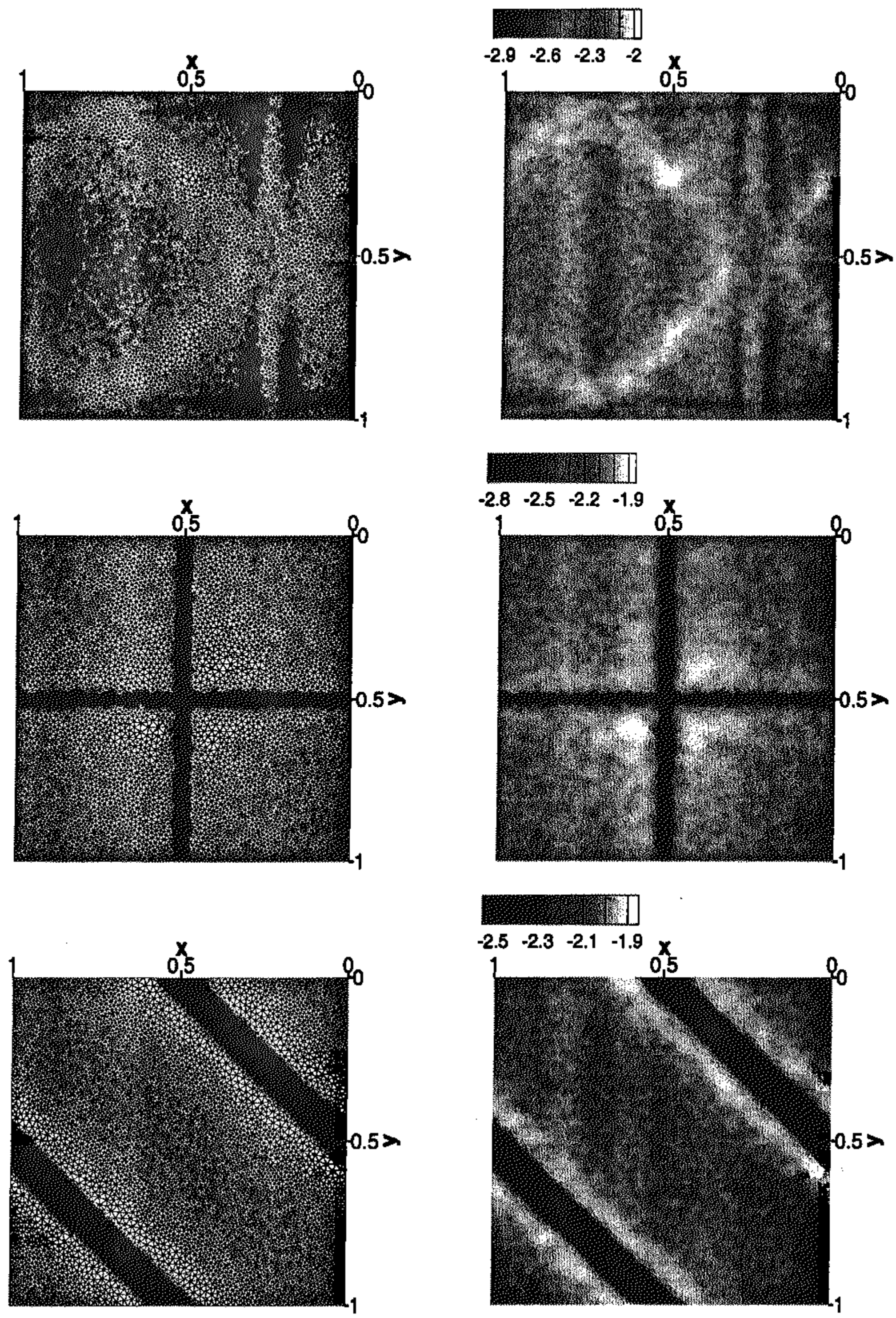


Figure 3: Actual adaptive meshes (left column) and their mesh-size functions $\log_{10}(h)$ (right column) for Problems 2 (top), 3 (middle), and 4 (bottom).

$\log_{10}(h)$	$\lvert u - u_h \rvert / \tau$	$\Psi(\gamma^k)$

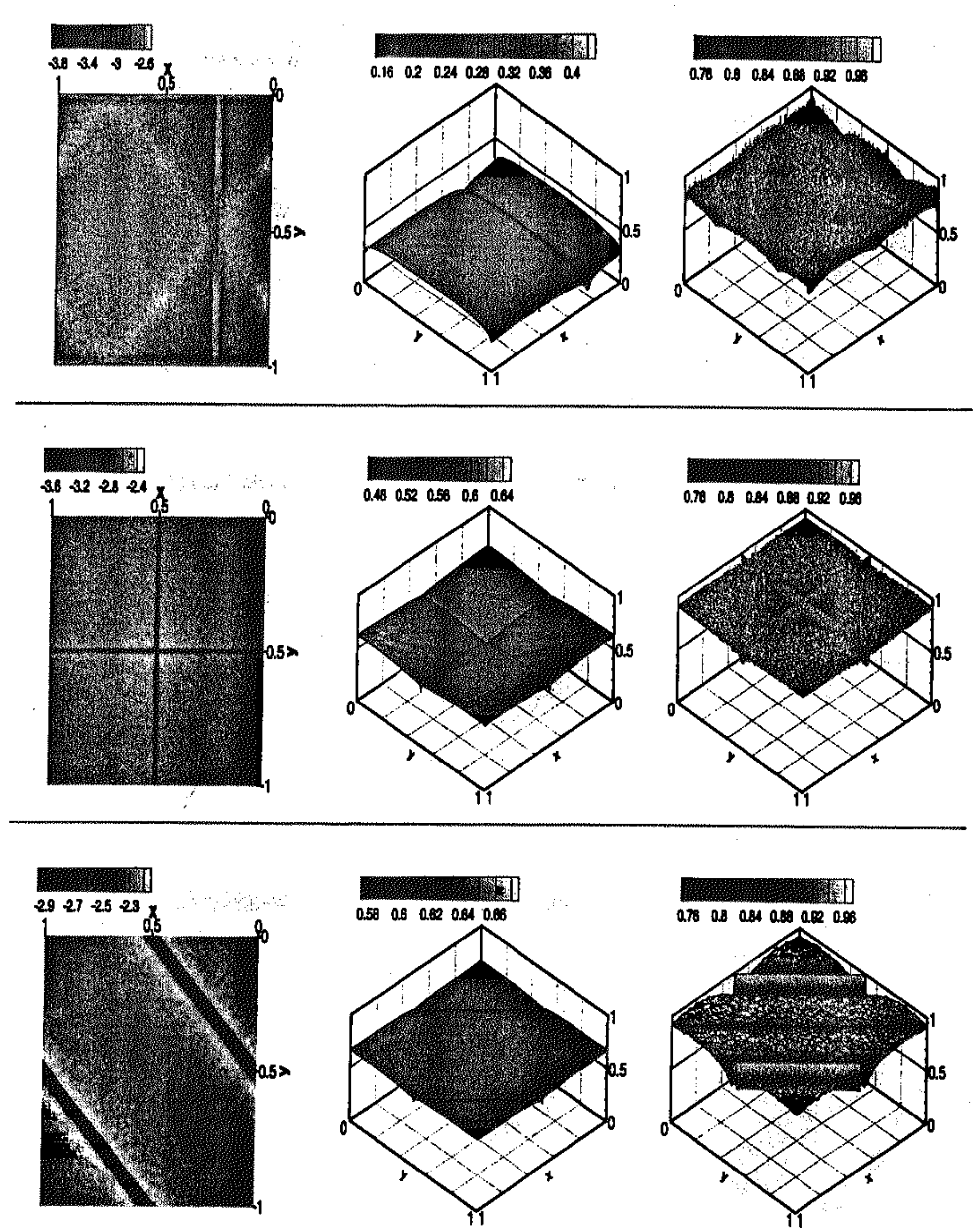


Figure 5: The final step for τ = 2.5E-2, for Problem 2 (top), Problem 3 (center), and Problem 4 (bottom).

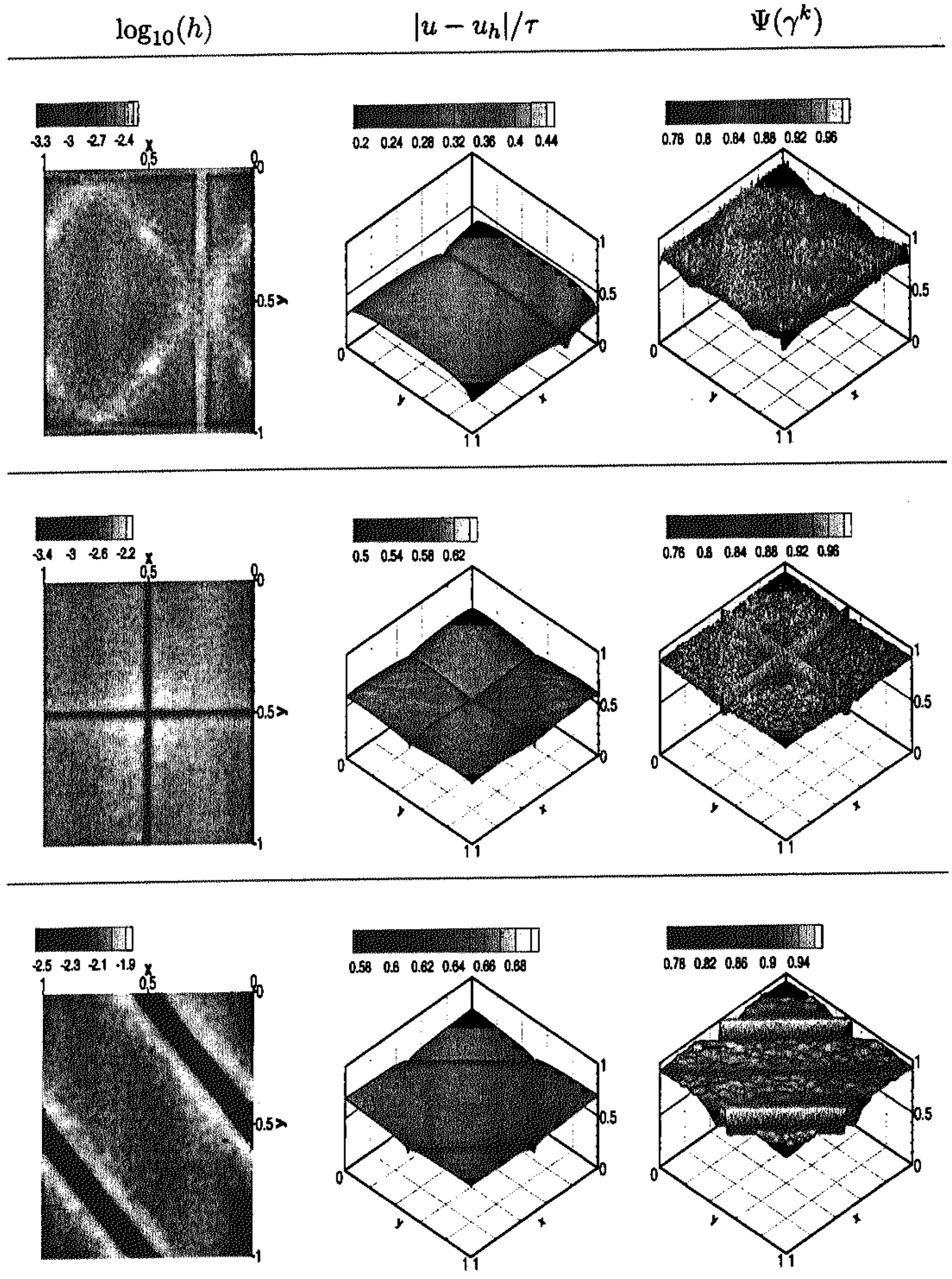
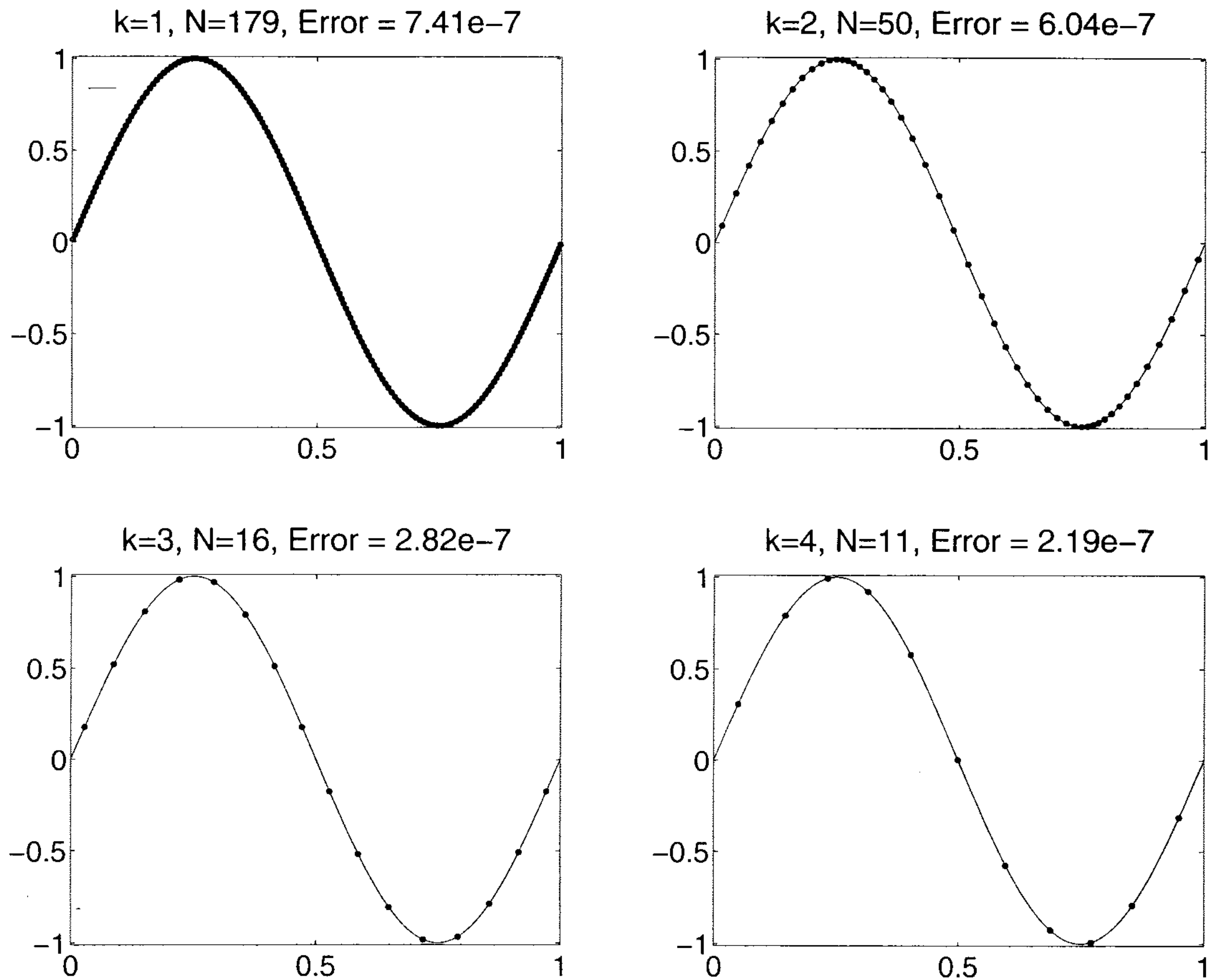


Figure 4: The final step for τ = 5.0E-2, for Problem 2 (top), Problem 3 (center), and Problem 4 (bottom).

Two Test Problems

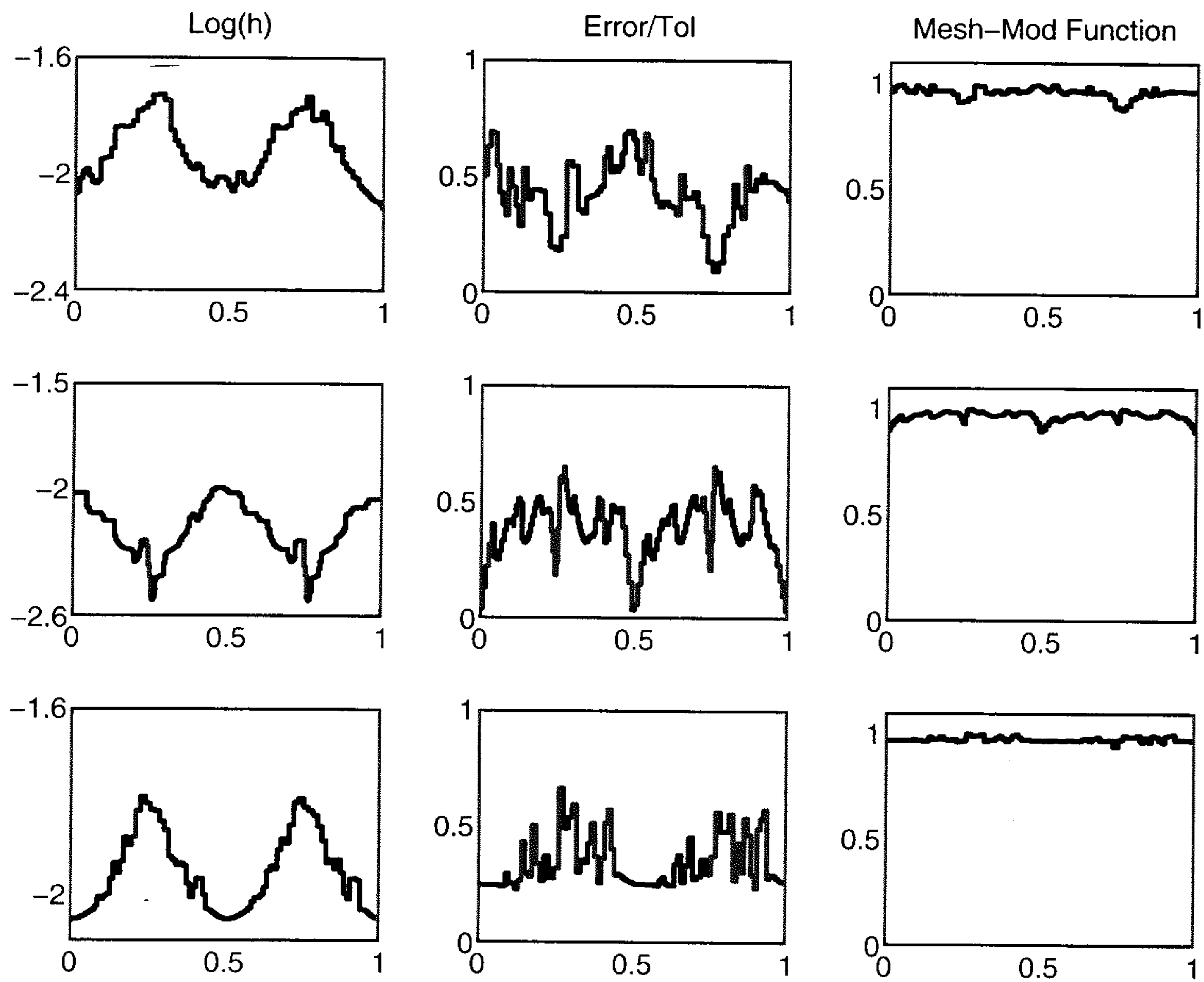
Problem	Hamiltonian $H(p)$	right-hand side $f(x)$	viscosity solution $u(x)$
NCS	$\frac{p^3}{8\pi^3}$ (Non-Convex)	$\sin(2\pi x) + \cos^3(2\pi x)$	$\sin(2\pi x)$
CNS	$\frac{p^2}{\pi^2}$ (Convex)	$-\lvert \sin\left(\pi\left(x - \frac{\pi}{4}\right)\right) \rvert + \cos^2\left(\pi\left(x - \frac{\pi}{4}\right)\right)$	$-\lvert \sin\left(\pi\left(x - \frac{\pi}{4}\right)\right) \rvert$



$k =$ polynomial degree $\quad$ tolerance $= 10^{-6}$ $\quad$ $N =$ number of elements.

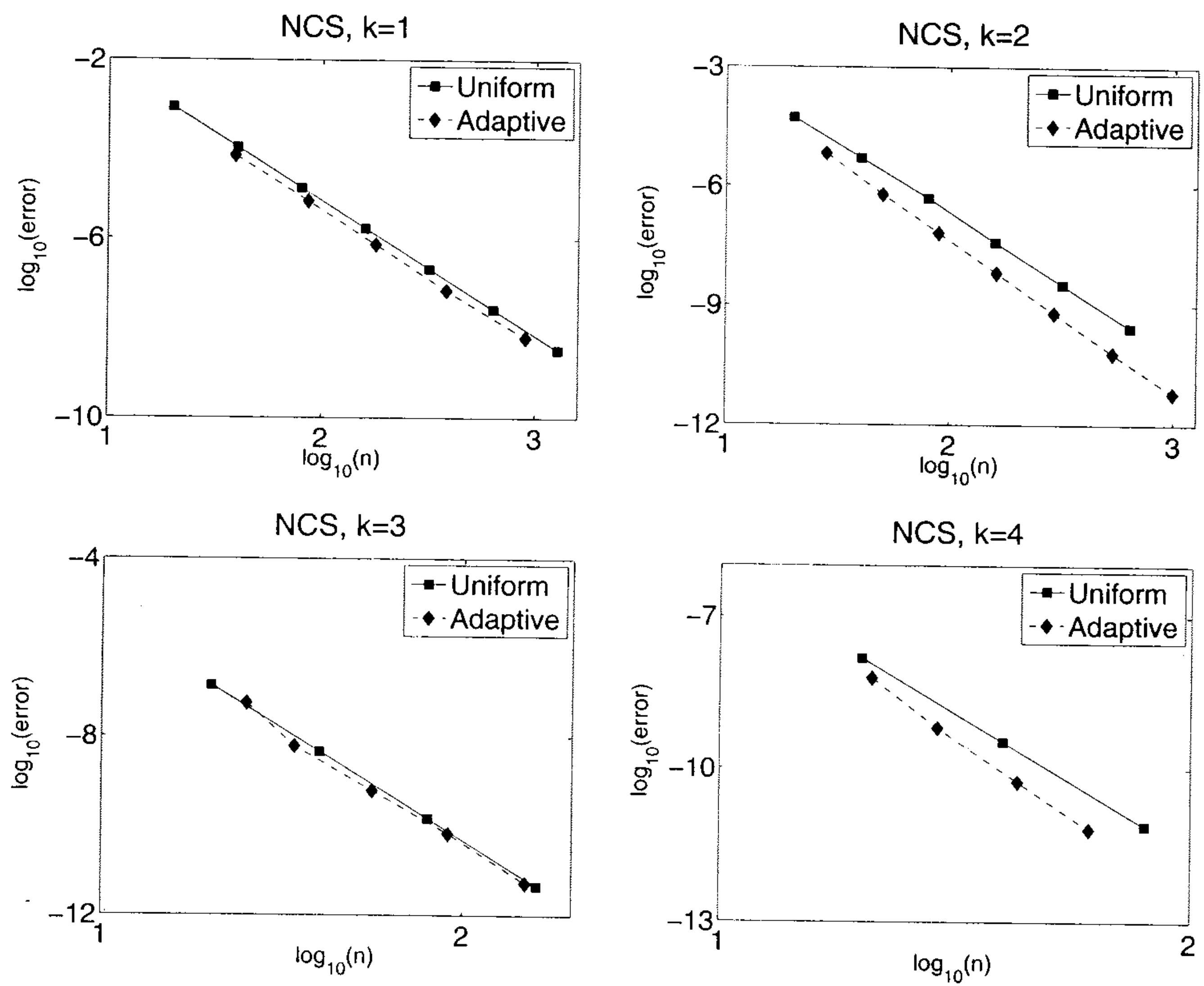
History of Convergence of the adaptive method for the problem NCS.

Deg	τ	n	$\|u - u_h\|$	order	$ei_{\mathcal{T}}(\tau; u, v)$	$ei_{ad}(\tau; v)$	Steps
1	1.0e-04	39	7.16e-05	-	1.40	1.11	4
	1.0e-05	86	6.94e-06	2.95	1.44	1.11	4
	1.0e-06	179	7.41e-07	3.05	1.35	1.05	4
	1.0e-07	385	7.06e-08	3.07	1.42	1.05	7
	1.0e-08	904	6.26e-09	2.84	1.60	1.02	7
2	1.0e-05	28	6.56e-06	-	1.52	1.02	5
	1.0e-06	50	6.04e-07	4.11	1.66	1.11	4
	1.0e-07	89	6.57e-08	3.85	1.52	1.02	6
	1.0e-08	162	6.52e-09	3.86	1.53	1.02	4
	1.0e-09	293	6.45e-10	3.90	1.55	1.02	4
	1.0e-10	536	6.27e-11	3.86	1.60	1.02	4
	1.0e-11	992	6.21e-12	3.76	1.61	1.03	4
3	1.0e-07	25	5.73e-08	-	1.74	1.17	4
	1.0e-08	34	6.07e-09	7.30	1.65	1.10	3
	1.0e-09	56	5.98e-10	4.65	1.67	1.12	4
	1.0e-10	91	6.63e-11	4.53	1.51	1.01	4
	1.0e-11	149	5.26e-12	5.14	1.90	1.16	5
4	1.0e-08	21	6.19e-09	-	1.62	1.08	3
	1.0e-09	29	6.36e-10	7.05	1.57	1.05	5
	1.0e-10	43	5.51e-11	6.21	1.82	1.21	6
	1.0e-11	61	6.49e-12	6.12	1.54	1.03	4

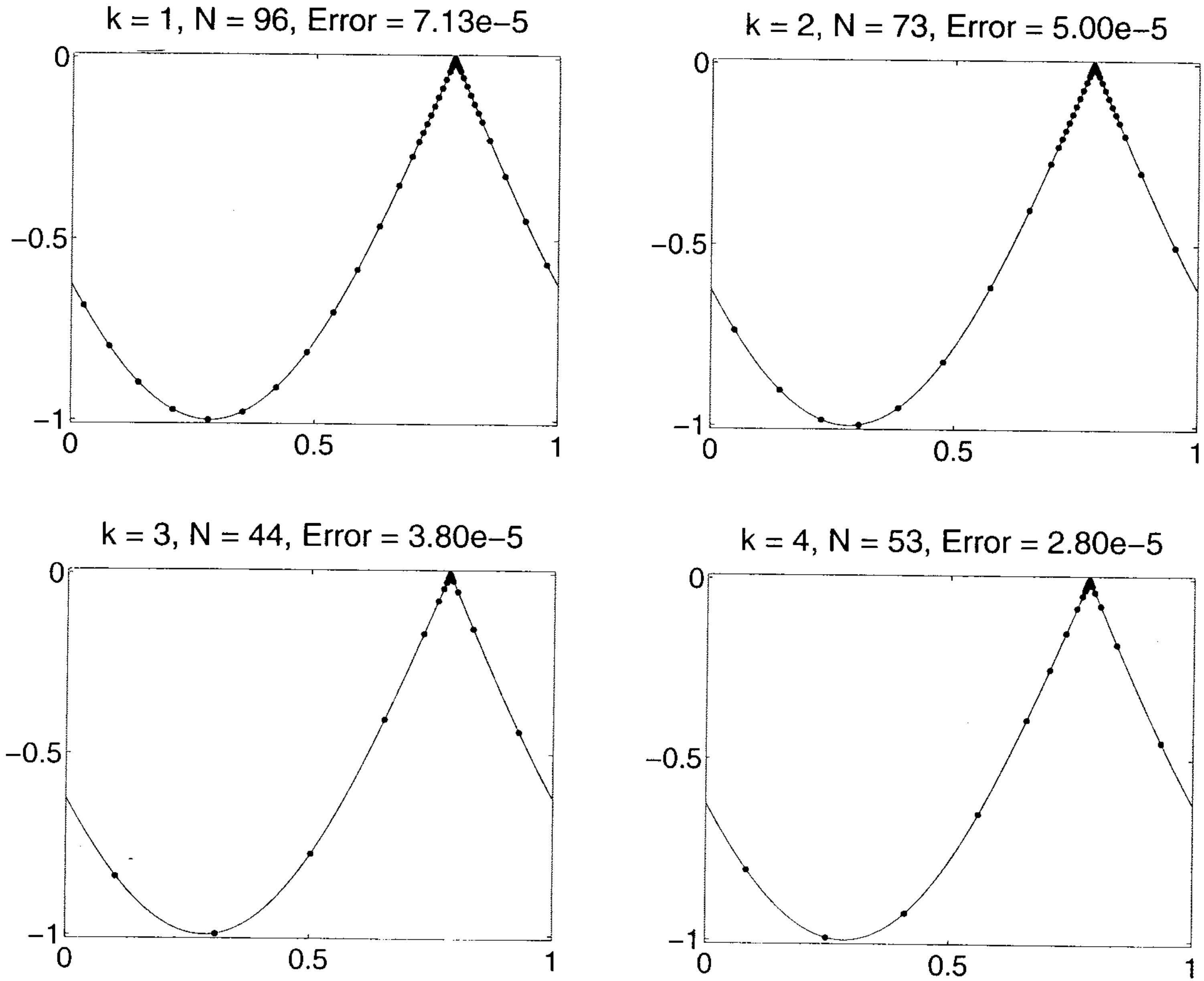


Convergence step for NCS: $\tau = 10^{-5}$, $k = 1$ for the first row, $\tau = 10^{-8}$, $k = 2$ for the second row and $\tau = 10^{-10}$, $k = 3$ for the last.

Comparison of history of convergence

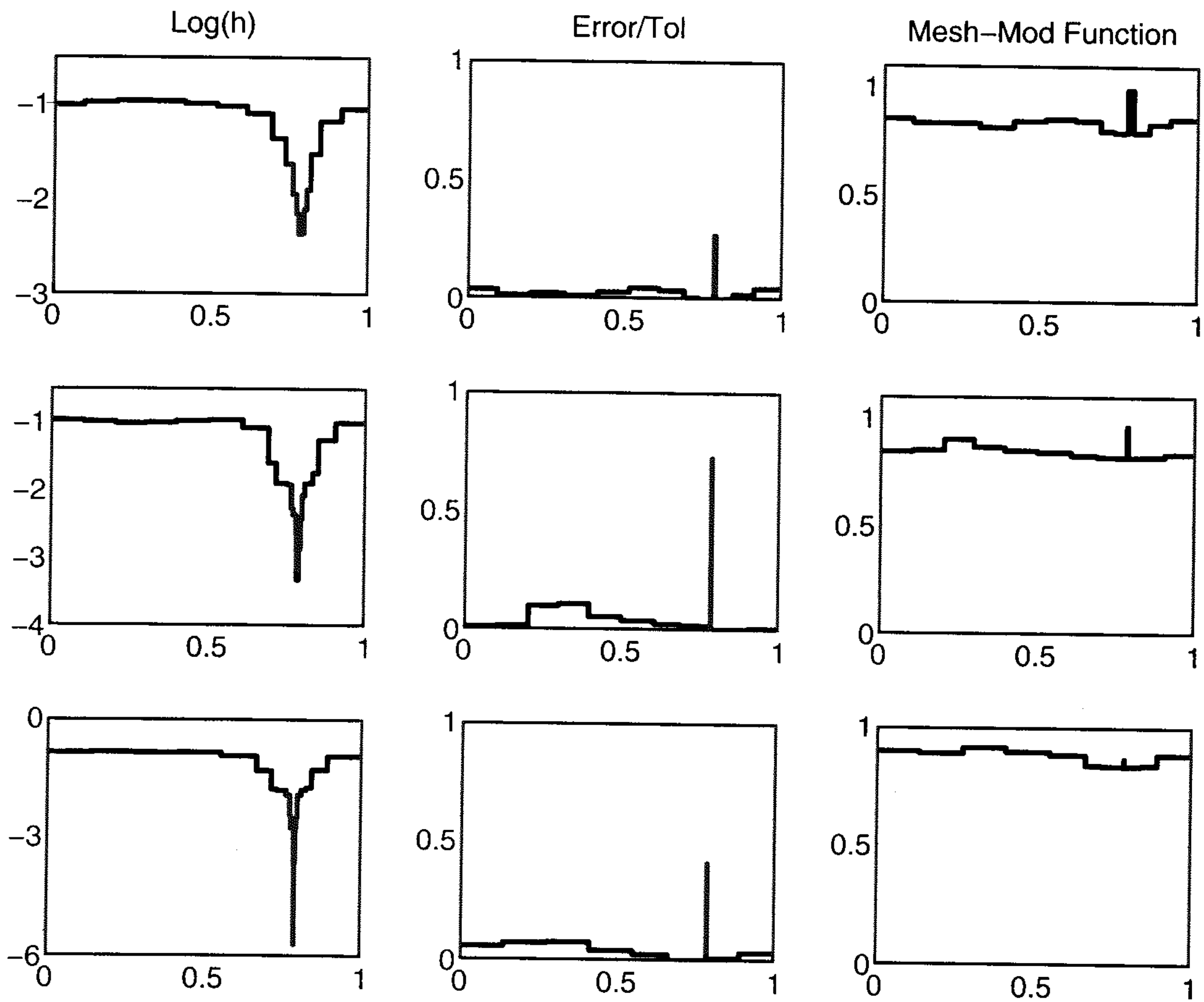


Uniform (solid line) and adaptive (dashed line) refinement for NCS.


k = 1, N = 96, Error = 7.13e−5
k = 2, N = 73, Error = 5.00e−5
k = 3, N = 44, Error = 3.80e−5
k = 4, N = 53, Error = 2.80e−5
0
−0.5
−1
0
0.5
1

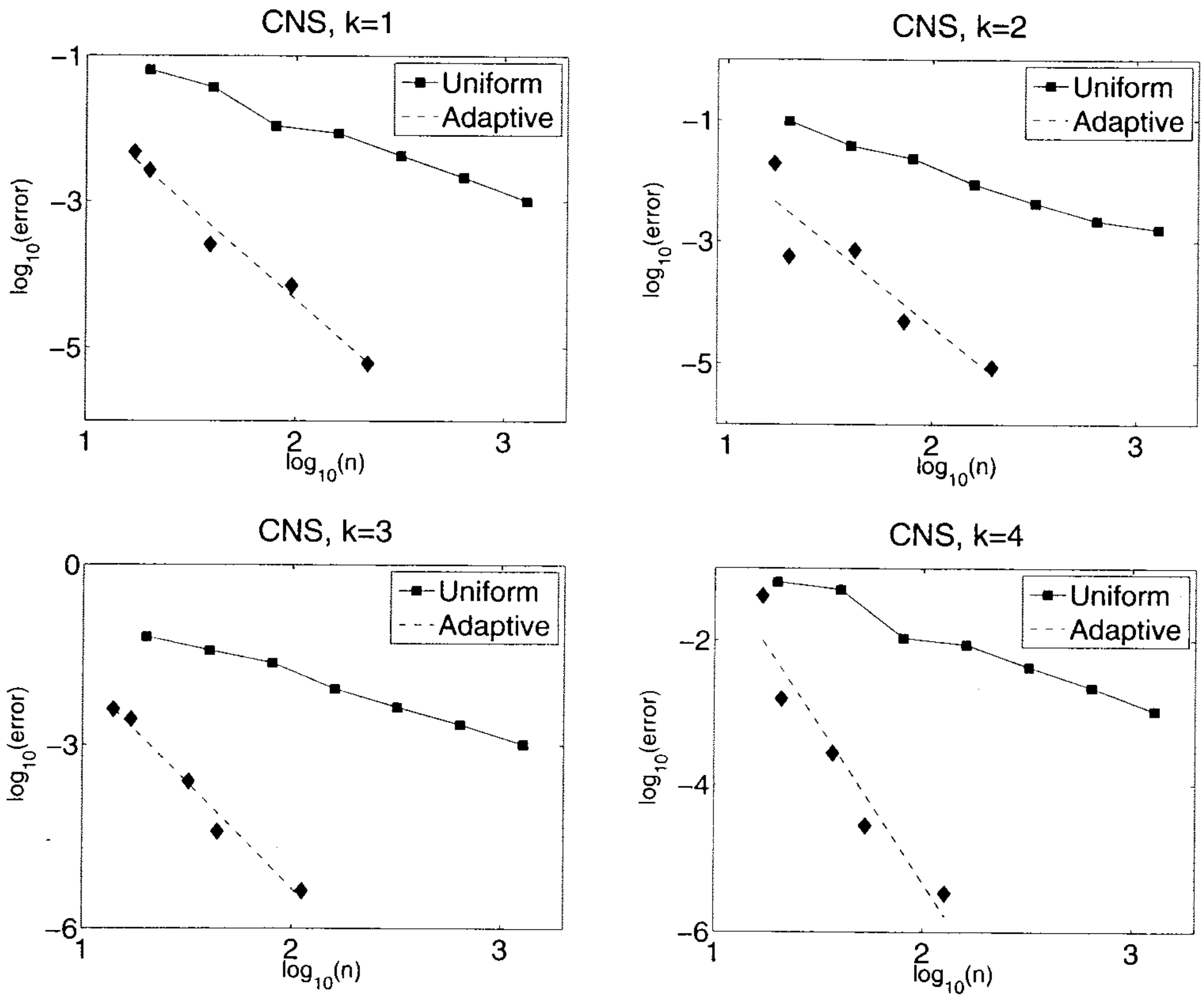
History of Convergence of the adaptive method for problems CNS.

Deg	τ	n	$\|u - u_h\|$	order	$ei_{\mathcal{T}}(\tau; u, v)$	$ei_{ad}(\tau; v)$	Steps
1	1.0e-01	17	4.78e-03	-	20.9	1.25	2
	1.0e-02	20	2.68e-03	3.55	3.72	1.07	5
	1.0e-03	39	2.60e-04	3.50	3.85	1.08	5
	1.0e-04	96	7.13e-05	1.43	1.40	1.12	6
	1.0e-05	224	6.16e-06	2.89	1.62	1.09	11
2	1.0e-01	17	1.94e-02	-	5.15	1.98	3
	1.0e-02	20	5.78e-04	21.6	17.3	1.10	5
	1.0e-03	42	7.24e-04	-0.30	1.38	1.29	5
	1.0e-04	73	5.00e-05	4.84	2.00	1.01	7
	1.0e-05	197	8.70e-06	1.76	1.15	1.00	6
3	1.0e-01	17	2.71e-03	-	36.9	1.61	2
	1.0e-02	14	3.96e-03	1.96	2.52	1.17	8
	1.0e-03	32	2.54e-04	3.32	3.93	1.73	9
	1.0e-04	44	3.80e-05	5.97	2.63	1.19	9
	1.0e-05	112	4.10e-06	2.38	2.44	2.18	7
4	1.0e-01	17	4.17e-02	-	2.40	1.39	3
	1.0e-02	21	1.57e-03	15.5	6.38	3.06	8
	1.0e-03	37	2.76e-04	3.07	3.62	1.07	7
	1.0e-04	53	2.80e-05	6.36	3.57	1.01	7
	1.0e-05	127	3.37e-06	2.43	2.97	1.10	7



Convergence step for CNS: $\tau = 10^{-2}$, $k = 1$ for the first row, $\tau = 10^{-3}$, $k = 2$ for the second row and $\tau = 10^{-5}$, $k = 3$ for the last.

Comparison of history of convergence



Uniform (solid line) and adaptive (dashed line) refinement for CNS.

- References

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