Numerical treatment of stochastic delay differential equations

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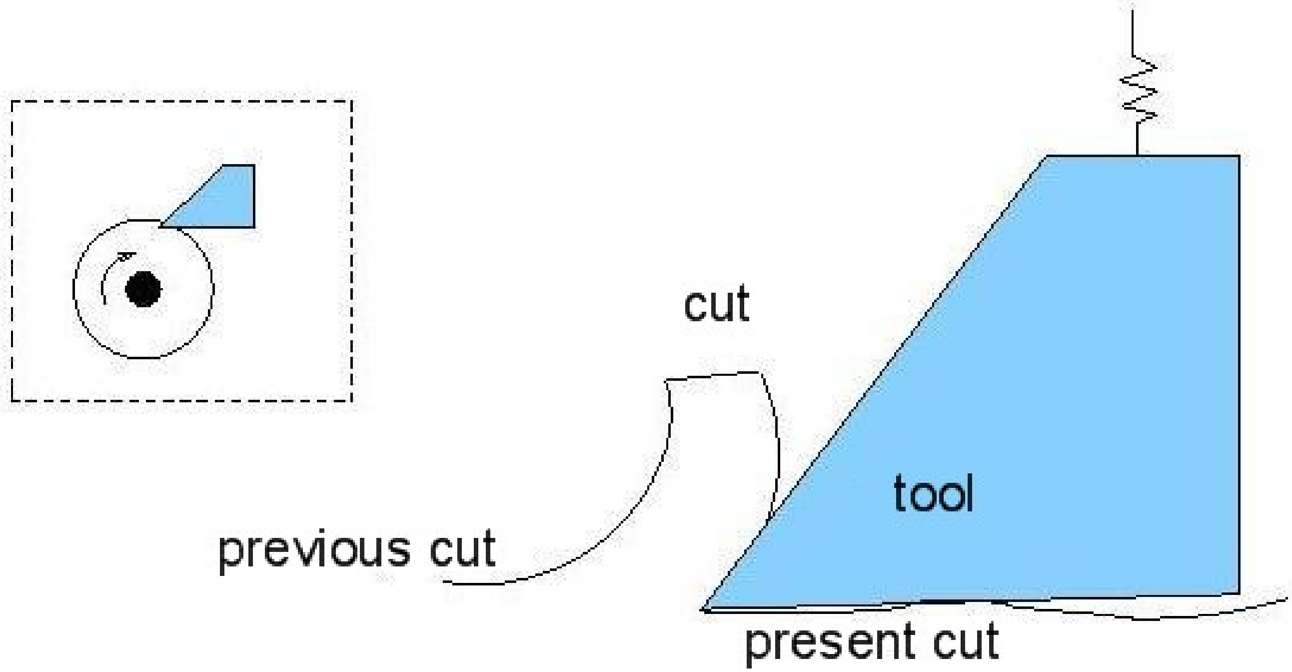
Overview

- ▷ An example: machine chatter
- ▷ Stochastic delay differential equations
- ▷ Numerics for stochastic delay differential equations
- ▷ Weak convergence analysis

An example: machine chatter



The model

'Original': 1-DOF model for machine tool dynamics (Stépán & Kalmar-Nagy, 1997), a delay differential equation containing a constant K_0 that depends on material properties.

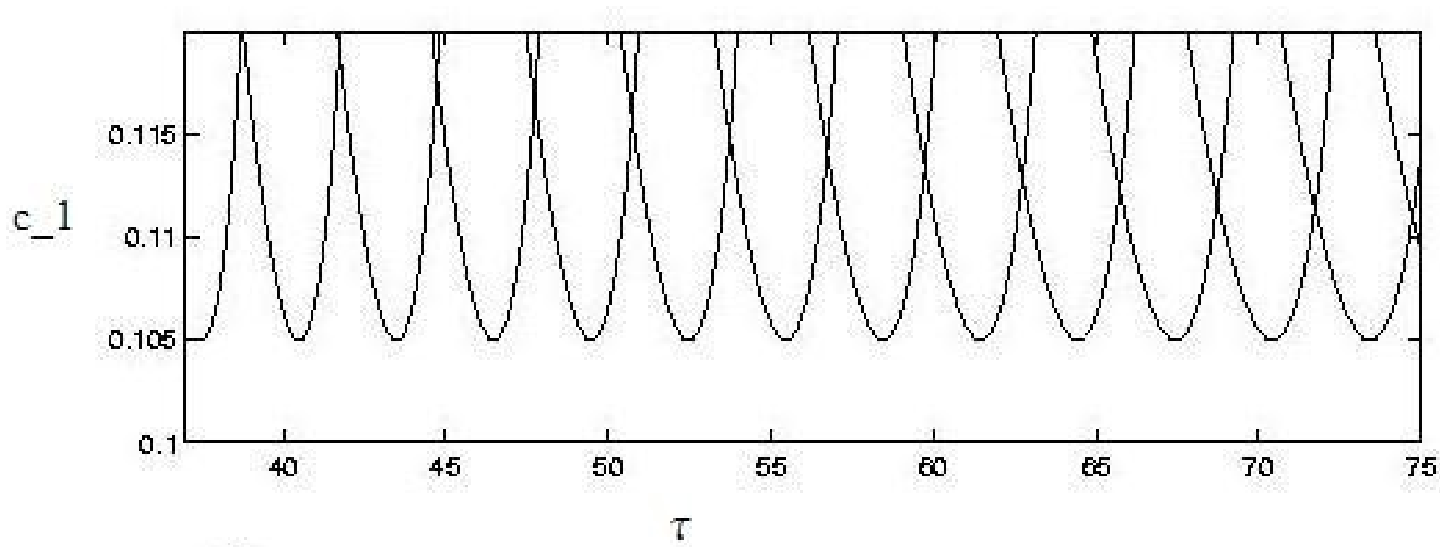
Now assume that the material is NOT homogenous
$\Rightarrow$ random variations in the material properties

$\Rightarrow K = K_0(1+\eta)$, 'noise' η viewed as a percentage of K_0,
model η by $\delta\times$ white noise , δ in the range $.01 < \delta < .15$.

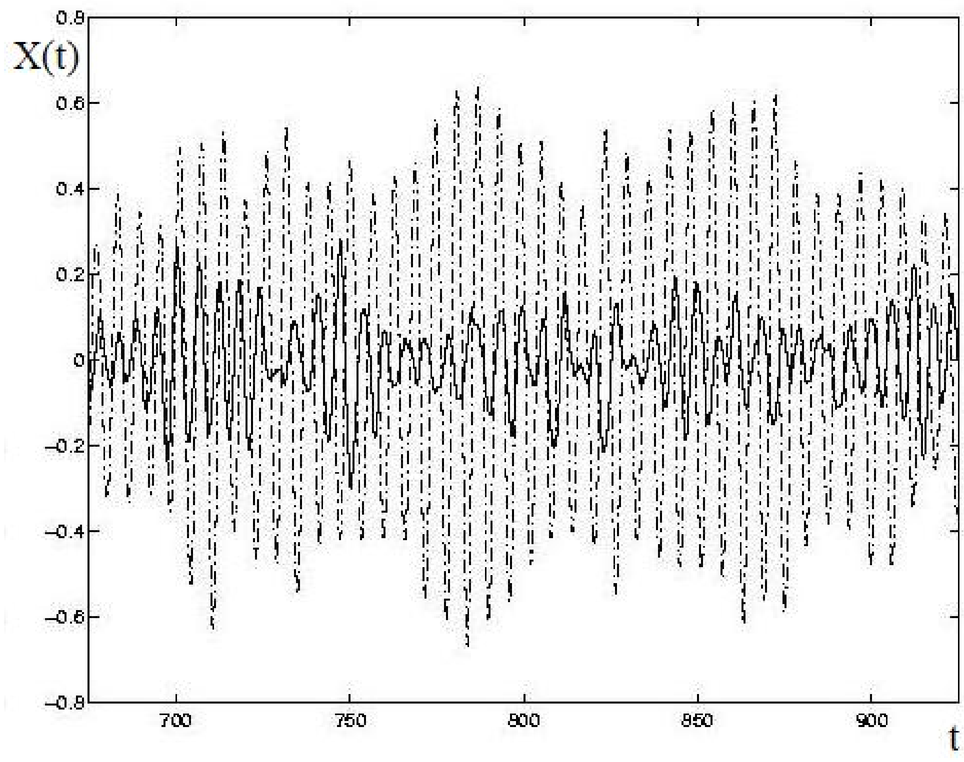
Proper derivation yields:

$$
\begin{aligned}
\mathrm{d}X &= Y\,\mathrm{d}s \\
\mathrm{d}Y &= (-2\kappa Y - X + \textstyle\sum_{j=1}^{3} c_j\,[X(s-\tau) - X(s)]^j)\mathrm{d}s + \delta \mathrm{d}W \\
&\qquad\qquad + \textstyle\sum_{j=1}^{3} c_j\,[X(s-\tau) - X(s)]^j\,\delta \mathrm{d}W
\end{aligned}
$$

Linear stability of the deterministic equations (Stépán)



Paths of additive noise linear equation, with $\kappa = 0.05, \delta = 0.5, \tau = 66$, solid line corresponds to $c_1 = 0.07$, dash-dotted to $c_1 = 0.11$



Further Applications

- Population dynamics (Carletti, Beretta, Tapaswi & Mukhopadhyay);
- Laser dynamics (Masoller, Pikovsky, Tsimring);
- Biophysics of the nervous system (Beuter, Longtin, Tass)
- Finance (Hobson & Rogers, Chang & Youree)
- Economy (Boucekkine, Benhabib)

Stochastic delay differential equations

$$\mathrm{d}X(t) = F(t, X(t), X(t-\tau))\mathrm{d}t + G(t, X(t), X(t-\tau))\mathrm{d}W(t)$$

- delay: $\tau \in (0, \infty)$;
- initial data: $X(t) = \psi(t)$ for $t \in [-\tau, 0]$;
- coefficients: (globally Lipschitz) $F : [0, T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$,
  $G = (G_1, \ldots, G_m) : [0, T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^{n \times m}$;
- Wiener process: $W = \{W(t, \omega), t \in [0, T], \omega \in \Omega\}$ is an m-dim. Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$.

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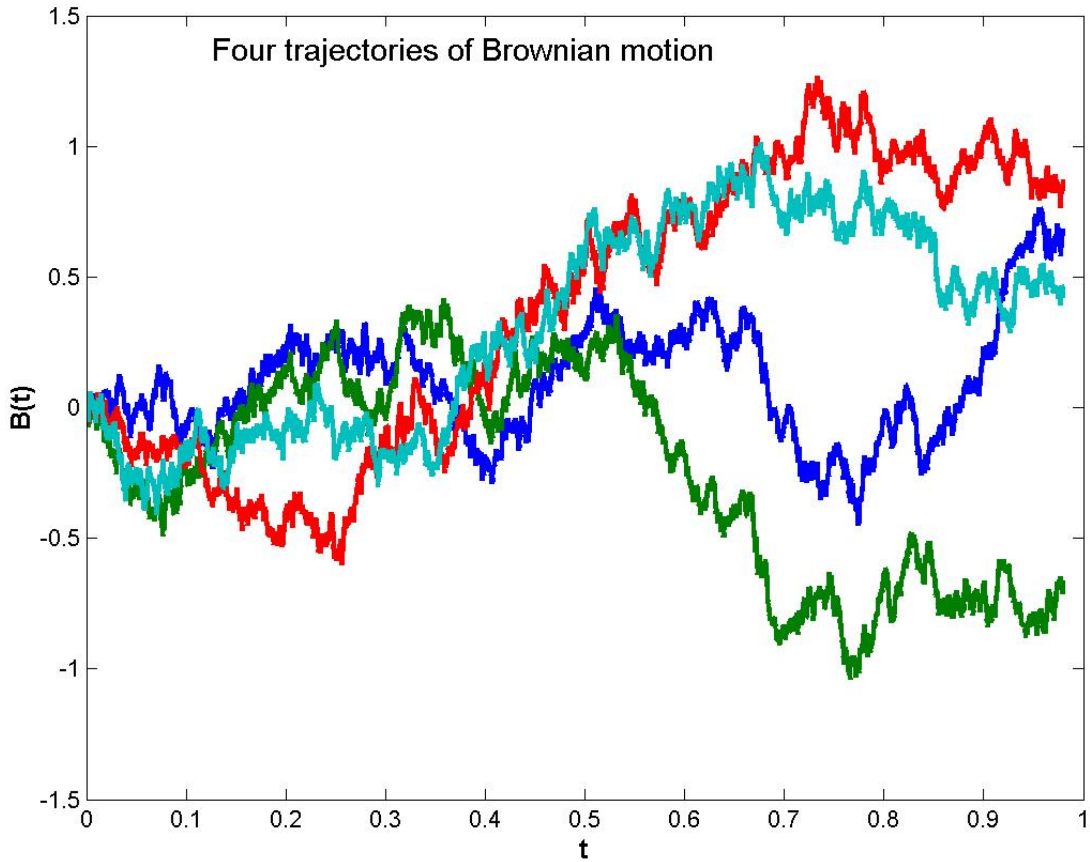
Some background

A Wiener process is a stochastic process $W(t,\omega),\ \omega \in \Omega$ with

- $W(0) = 0$
- $W(t) - W(s)$ is a $\sqrt{t-s}\mathcal{N}(0,1)$ distr. random variable, $0 \leq s < t \leq T$,
- $W(t) - W(s)$, $W(v) - W(u)$ independent for $0 \leq s \leq t \leq u \leq v \leq T$
- $W(t)$ has continuous paths
- $W(t)$ is nowhere differentiable
- total variation $= \infty$

Construction: $W(t_n) = W(0) + \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))$


Four trajectories of Brownian motion
B(t)
t
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Stochastic delay differential equations

$$\mathrm{d}X(t) = F(t, X(t), X(t-\tau))\mathrm{d}t + G(t, X(t), X(t-\tau))\mathrm{d}W(t)$$

- delay: $\tau \in (0, \infty)$;
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- coefficients: (globally Lipschitz) $F : [0,T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$,
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Itô Integral for $f(t)$ a stochastic process: (h grid-width)

$$\int_0^t f(s)\mathrm{d}W(s) = \quad \text{(conv. in } L^2\text{)} \quad \lim_{h\to 0} \sum_{i=0}^{N-1} f(t_i)[W(t_{i+1}) - W(t_i)]$$

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$\Uparrow$

evaluation at left endpoint !

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$\Uparrow$

evaluation at left endpoint !

Properties:

- usual linearity and additivity
- Martingale
- $\mathbb{E}(\int_0^t f(s)\mathrm{d}W(s)) = 0$
- $\mathbb{E}(|\int_0^t f(s)\mathrm{d}W(s)|)^2 = \mathbb{E}\int_0^t |f(s)|^2\mathrm{d}s$ "Itô isometry"

Stochastic delay differential equations

$$\mathrm{d}X(t) = F(t, X(t) \qquad\qquad)\mathrm{d}t + G(t, X(t) \qquad\qquad)\mathrm{d}W(t)$$

- delay: $\tau \in (0, \infty)$;
- initial data: $X(t) = \psi(t)$ for $t \in [-\tau, 0]$;
- coefficients: (globally Lipschitz) $F : [0, T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$, $G = (G_1, \ldots, G_m) : [0, T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^{n \times m}$;
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Stochastic differential equations

defined as $X(t) = X(0) + \int_0^t f(s, X(s))\ \mathrm{d}s + \int_0^t g(s, X(s))\ \mathrm{d}W(s)$

abbreviated as $\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t))\ \mathrm{d}W(t)$

"Itô formula" (stochastic chain rule), for $\phi(x)$ function, suff. differentiable, scalar:

$$\begin{aligned}\phi(X(t)) &= \phi(X(0)) + \int_0^t \phi_x(X(s)) f(s, X(s)) \mathrm{d}s + \phi_x(X(s)) g(s, X(s)) \mathrm{d}W(s) \\ &\quad + \frac{1}{2} \int_0^t \phi_{xx}(X(s))\ g^2(s, X(s))\ \mathrm{d}s\end{aligned}$$

Stochastic delay differential equations

$$\mathrm{d}X(t)=F(t,X(t),X(t-\tau))\mathrm{d}t+G(t,X(t),X(t-\tau))\mathrm{d}W(t)$$

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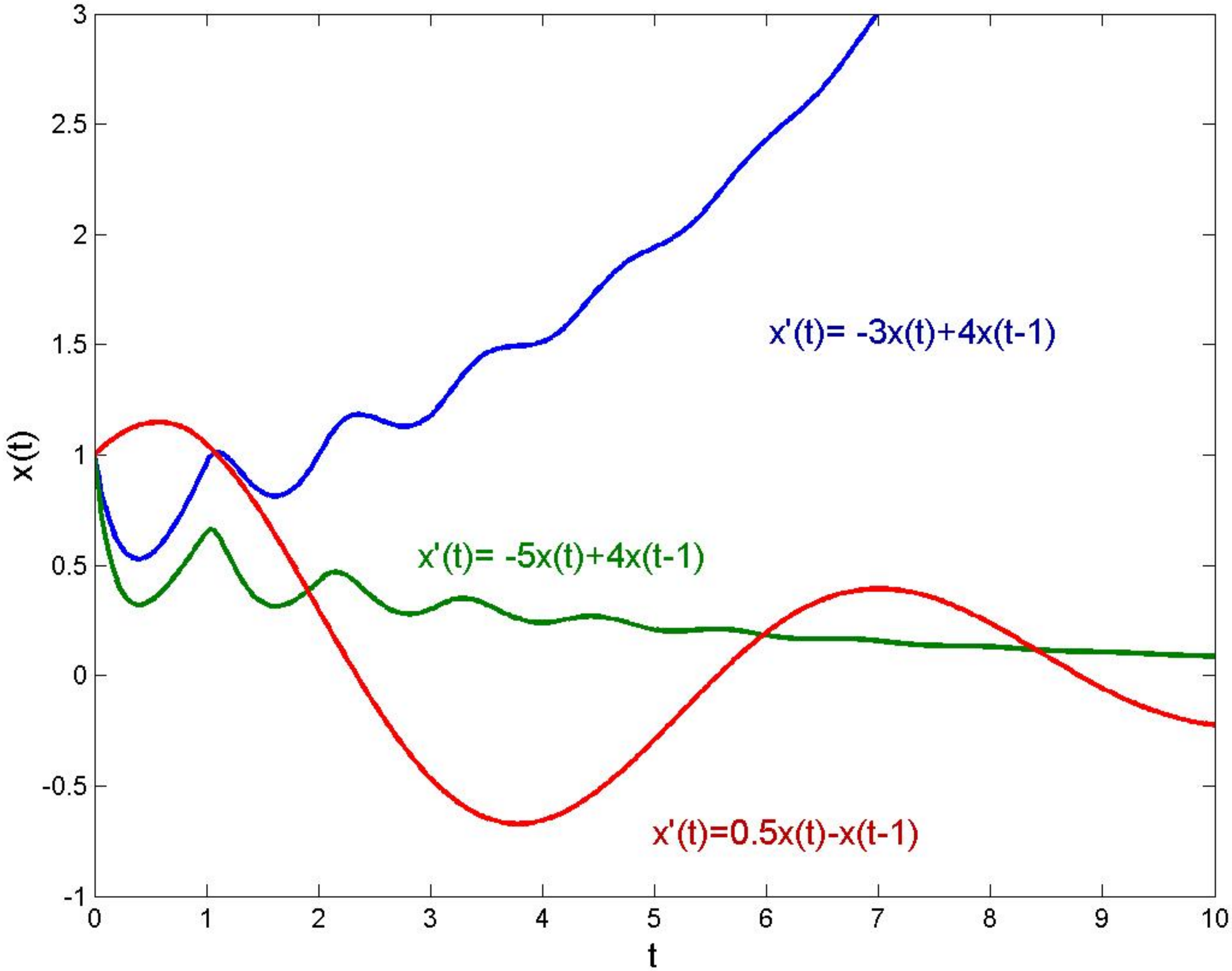
What is the effect of the delays?

Example:

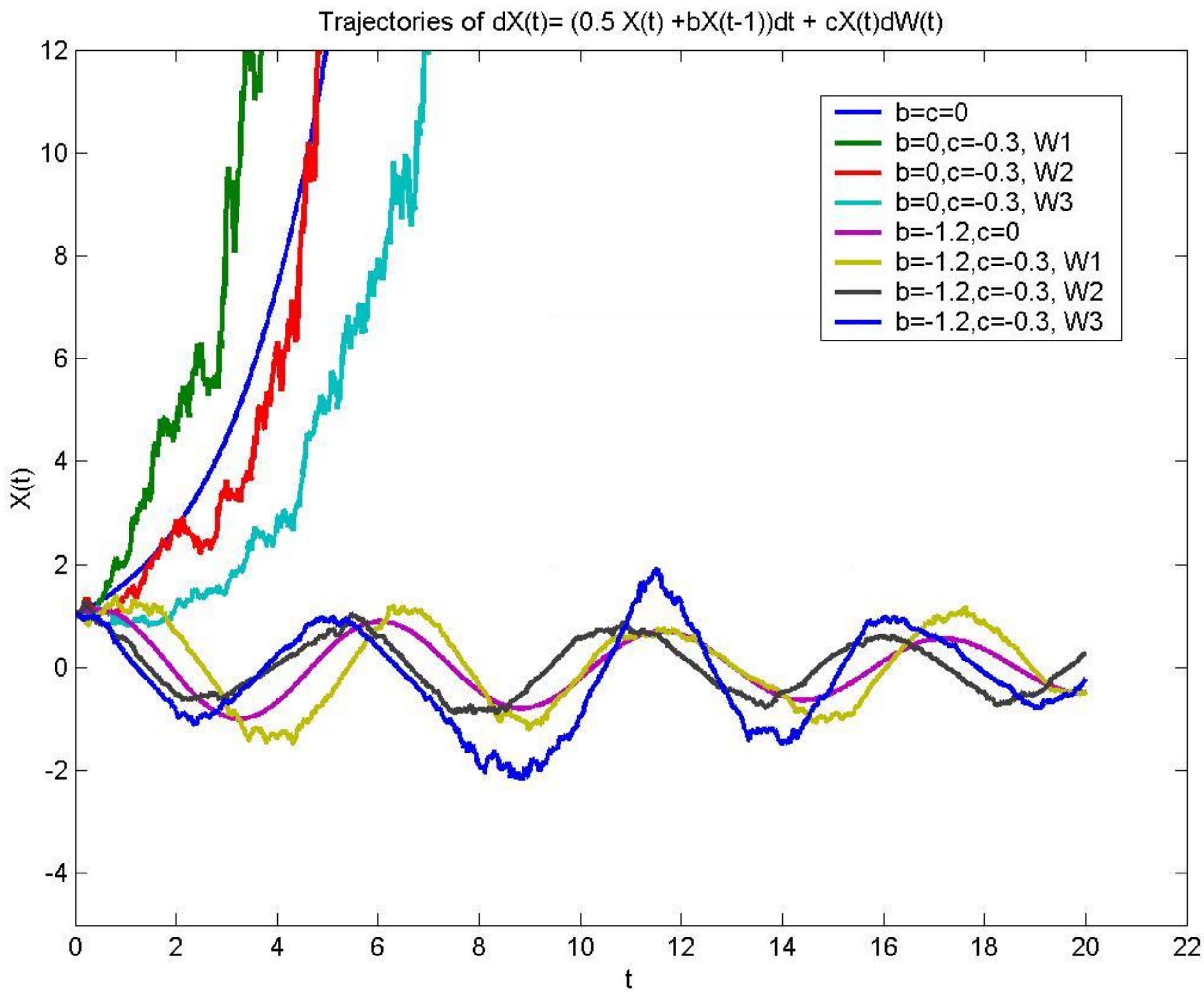
$$y'(t) = \lambda y(t), \quad t \geq 0, \lambda \in \mathbb{R}, \quad y(0) = 1$$

versus

$$x'(t) = \lambda x(t) + \mu x(t-1),\ t \geq 0, \lambda, \mu \in \mathbb{R},\ x(t) = 1 + t,\ t \leq 0$$



x'(t)= -3x(t)+4x(t-1)
x'(t)= -5x(t)+4x(t-1)
x'(t)=0.5x(t)-x(t-1)
x(t)
t
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10


Trajectories of dX(t)= (0.5 X(t) +bX(t-1))dt + cX(t)dW(t)
X(t)
t
b=c=0
b=0,c=-0.3, W1
b=0,c=-0.3, W2
b=0,c=-0.3, W3
b=-1.2,c=0
b=-1.2,c=-0.3, W1
b=-1.2,c=-0.3, W2
b=-1.2,c=-0.3, W3
12
10
8
6
4
2
0
-2
-4
0
2
4
6
8
10
12
14
16
18
20
22

Solutions of SODEs ($X(0) = 1$):

$\triangleright$ $\mathrm{d}X(t) = aX(t)\mathrm{d}t + b\mathrm{d}W(t)$, $\triangleright X(t) = e^{at}(1 + b\int_0^t e^{-as}\mathrm{d}W(s))$

$\triangleright$ $\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}W(t)$, $\triangleright X(t) = \exp((a - \frac{1}{2}b^2)t + bW(t))$

Solutions of SDDEs via the "method of steps"

$\triangleright$ $\mathrm{d}X(t) = X(t-1)\mathrm{d}t + \beta\mathrm{d}W(t)$, $t \geq 0$ und $X(t) = \Phi_1(t) = 1 + t$, $t \in [-1, 0]$

$$\begin{array}{lrcll}
\to \; t \in [0,1] & \mathrm{d}X(t) & = & \Phi_1(t-1)\mathrm{d}t + \beta\mathrm{d}W(t) & = t\;\mathrm{d}t + \beta\mathrm{d}W(t) \\
\Rightarrow & X(t) & = & 1 + \dfrac{t^2}{2} + \beta W(t) & =: \Phi_2(t) \\
\to \; t \in [1,2] & \mathrm{d}X(t) & = & \Phi_2(t-1)\mathrm{d}t + \beta\mathrm{d}W(t) & \\
\Rightarrow & X(t) & = & \dfrac{1}{6}t^3 - \dfrac{1}{2}t^2 + \dfrac{3}{2}t + \dfrac{1}{3} + \beta(\displaystyle\int_1^t W(s-1)\mathrm{d}s + W(t)) &
\end{array}$$

Numerics for Stochastic Delay Differential Equations

Integral equation on $t \in [0,T]$:

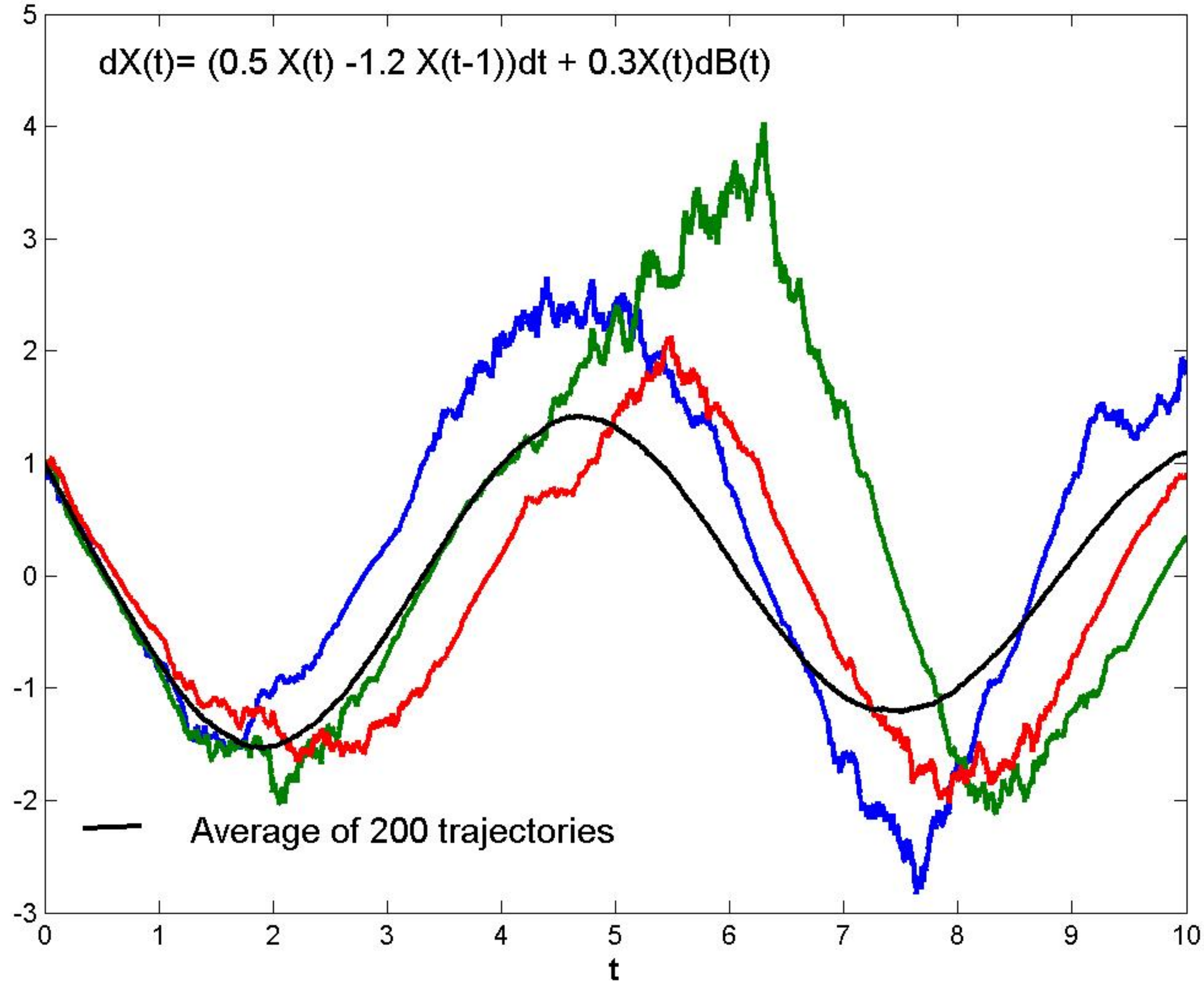
$$X(t) = \psi(0) + \int_0^t F(s, X(s), X(s-\tau))\mathrm{d}s + \int_0^t G(s, X(s), X(s-\tau))\mathrm{d}W(s)$$

Define:

Grid on $[0,T]$, step-size $h := T/N$, $N_\tau h = \tau$ and $t_n = n \cdot h,\ n = 0, \ldots, N$,
$I^{t_n,t_{n+1}} = \int_{t_n}^{t_{n+1}} \mathrm{d}W(s) = W(t_{n+1}) - W(t_n) \sim \sqrt{h}\mathcal{N}(0,1)$.

Simplest numerical method: Euler-Maruyama-method

$$X_{n+1} \;=\; X_n \;+ h\, F(t_n, X_n, X_{n-N_\tau}) + G(t_n, X_n, X_{n-N_\tau})\, I^{t_n,t_{n+1}}$$



dX(t)= (0.5 X(t) -1.2 X(t-1))dt + 0.3X(t)dB(t)
Average of 200 trajectories
t
5
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
7
8
9
10

Two Objectives - Two Modes of Convergence

Strong Approximations: Compute (several to many) single paths, strong convergence criterion (mean-square convergence), p order of method:

$$\max_{1\le n\le N} \left(\mathbb{E}|X(t_n) - X_n|^2\right)^{\frac{1}{2}} \quad \le \quad C\ h^p, \quad \text{for } h \to 0\,.$$

Weak Approximations: Compute (using many paths) the expectation of a function Ψ of the solution, weak convergence criterion, p order of method:

$$\max_{1\le n\le N} |\mathbb{E}\Psi(X(t_n)) - \mathbb{E}\Psi(X_n)| \quad \le \quad C\ h^p, \quad \text{for } h \to 0\,.$$

approx. $\mathbb{E}(\Psi(X_n))$ by M realisations $\frac{1}{M}\sum_{i=1}^{M}\Psi(X_n^{(i)})$

Euler-Maruyama method: strong order $\frac{1}{2}$, weak order 1.

Weak convergence analysis

Assume you need to calculate $\mathbb{E}\,\Psi(y(t_n))$ for
$\mathrm{d}y(t) = f(y(t))\mathrm{d}t + g(y(t))\mathrm{d}W(t),\ t \in [0,T],\ y(0) = y_0.$

Major ingredients for classic weak convergence analysis (Milstein, Talay):

i) Itô's formula.

ii) Set $u(t_n, y(t_n)) := \mathbb{E}\,\Psi(y(t_{n+1}))$,

then $u(t,y)$ satisfies a deterministic parabolic partial differential equation, the Fokker-Planck equation.

For stochastic delay differential equations:

Problem 1.: Itô-formula for $F(X(t), X(t-\tau))$ needs Malliavin calculus

Problem 2.: a PDE is not available, at least not in a practicable version

Solution to Problem 1.

Itô-formula from Hu, Mohammed & Yan (AoP **32**(1A), 2004)

for SDDE with $m = 1$, $n = 1$, $r = 2$, $s = 2$, $\tau_1 = \sigma_1 = 0$

$\mathrm{d}X(t) = F(X(t), X(t - \tau_2))\mathrm{d}t + G(X(t), X(t - \sigma_2))\mathrm{d}W(t)$, $t > 0$

$X(t) = \psi(t), \quad -\tau < t < 0, \quad \tau := \tau_2 \vee \sigma_2,$

and function $\phi(X(t), X(t - \delta))$ $\delta > 0$

$$
\begin{aligned}
&\mathrm{d}\phi\big(X(t), X(t-\delta)\big)\\
&\quad= \frac{\partial \phi}{\partial x_2}(X(t), X(t-\delta))\, 1_{[0,\delta)}(t)\mathrm{d}\psi(t-\delta)\\
&\quad+ \frac{\partial \phi}{\partial x_2}(X(t), X(t-\delta))1_{[\delta,\infty)}(t)\, \Big[F\big(X(t-\delta), X(t-\tau_2-\delta)\big)\mathrm{d}t\\
&\qquad\qquad\qquad\qquad + G\big(X(t-\delta), X(t-\sigma_2-\delta)\big)\mathrm{d}W(t-\delta)\Big]\\
&\quad+ \frac{\partial^2 \phi}{\partial x_1 \partial x_2}\big(X(t), X(t-\delta)\big)G\big(X(t-\delta), X(t-\sigma_2-\delta)\big)1_{[\delta,\infty)}(t)\mathcal{D}_{t-\delta}X(t)\mathrm{d}t\\
&\quad+ \frac{1}{2}\frac{\partial^2 \phi}{\partial x_2^2}\big(X(t), X(t-\delta)\big)G\big(X(t-\delta), X(t-\sigma_2-\delta)\big)^2 1_{[\delta,\infty)}(t)\mathrm{d}t\\
&\quad+ \frac{\partial \phi}{\partial x_1}\big(X(t), X(t-\delta)\big)\, \Big[F\big(X(t), X(t-\tau_2)\big)\mathrm{d}t + G\big(X(t), X(t-\sigma_2)\big)\mathrm{d}W(t)\Big]\\
&\qquad\qquad + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_1^2}\big(X(t), X(t-\delta)\big)G\big(X(t), X(t-\sigma_2)\big)^2\mathrm{d}t,
\end{aligned}
$$

Solution to Problem 2.(joint work with R. Kuske, SEA Mohammed, T. Shardlow)

Set $X(t_n) = X(t_n; 0, \psi)$ and $Y_n = Y(t_n; 0, \psi)$,

$X_{t_i} := X_{t_i}(\,\cdot\,; t_{i-1}, X_{t_{i-1}}(\,\cdot\,; 0, \psi))$ and $Y_{t_i} := Y_{t_i}(\,\cdot\,; t_{i-1}, X_{t_{i-1}}(\,\cdot\,; 0, \psi))$

Start:

$$\mathbb{E}\Psi\big(X(t_n; 0, \psi)\big) - \mathbb{E}\Psi\big(Y(t_n; 0, \psi)\big)$$

$$= \sum_{i=1}^{n} \mathbb{E} \int_0^1 D(\Psi \circ Y)\big(t_n; t_i, \lambda X_{t_i} + (1-\lambda) Y_{t_i}\big) \cdot [X_{t_i} - Y_{t_i}] \, \mathrm{d}\lambda$$

To show: each of the terms in the sum is $O((t_i - t_{i-1})^2)$.

Next step: Apply the Itô formula.

Results in multiple Skorohod integrals:

$$J_1^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_1(v)\, dv\, dW(u),$$

$$J_2^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_2(v)\, dW(v-\tau_2)\, dW(u),$$

$$J_3^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_3(v)\, dW(v-\sigma_2)\, du.$$

where $Y(ds)$ is a random discrete measure on $[-\tau, 0]$ and the processes $\Sigma_j, j = 1, 2, 3$, are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_2), W(\cdot - \sigma_2)$.

Next step: Estimate the terms $|\mathbb{E}J_j^i|$

Use:

definition of the Skorohod integral as the adjoint of the weak differentiation operator,

estimates on higher-order moments of the Malliavin derivatives of the Σ_j's, $j = 1, 2, 3$, obtained using the corresponding higher moments of the Euler approximations Y and their linearizations.

Finally: Summing over all the steps $i = 1, \ldots, n$ and dealing with an approximation of the initial function ψ we arrive at the expected result.

Numerical example

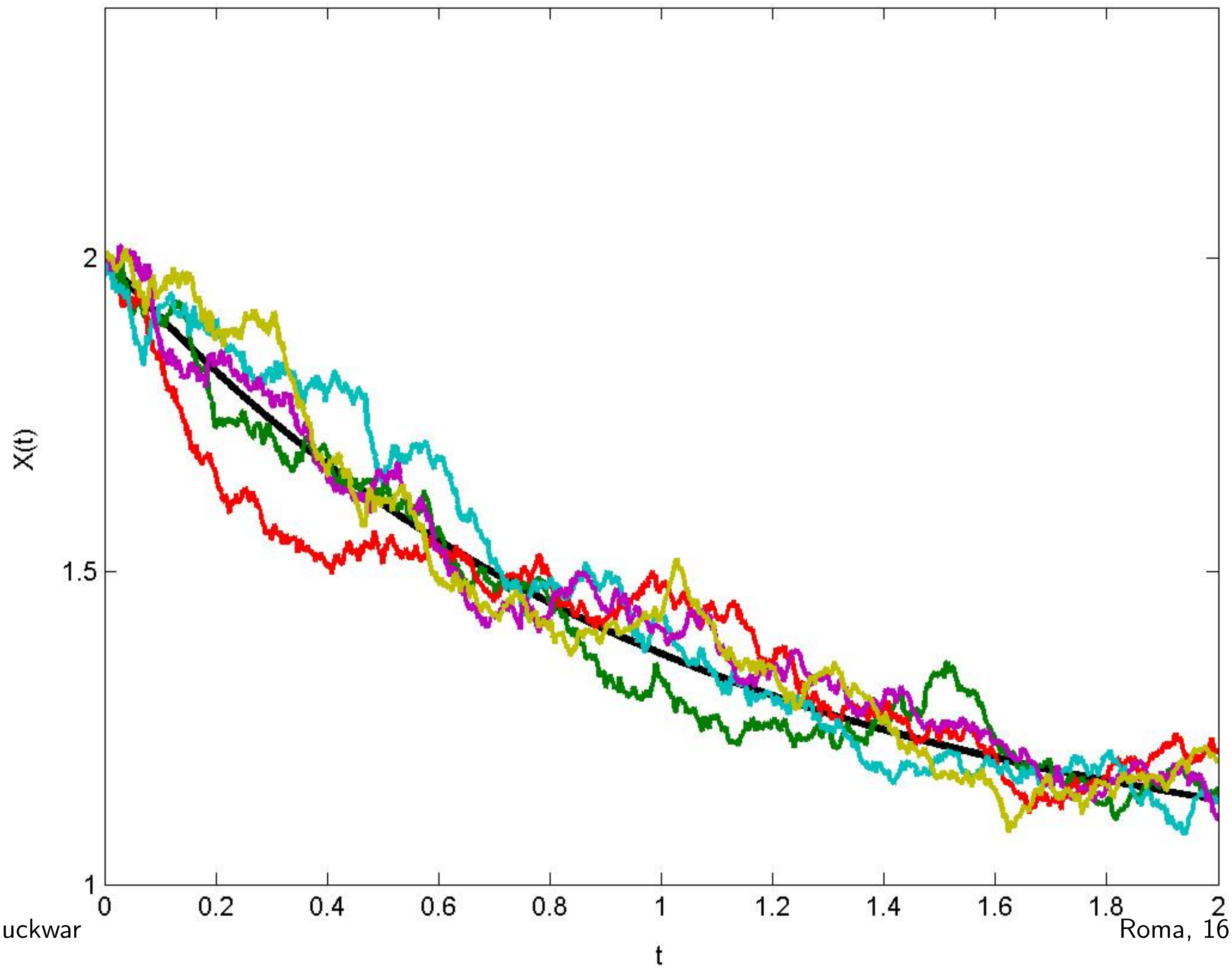
SDDE $\mathrm{d}X(t) = (3\,\mathbf{e}^{-1}\,X(t-1) - 4\,X(t) + 4 - 3\,\mathbf{e}^{-1})\mathrm{d}t$

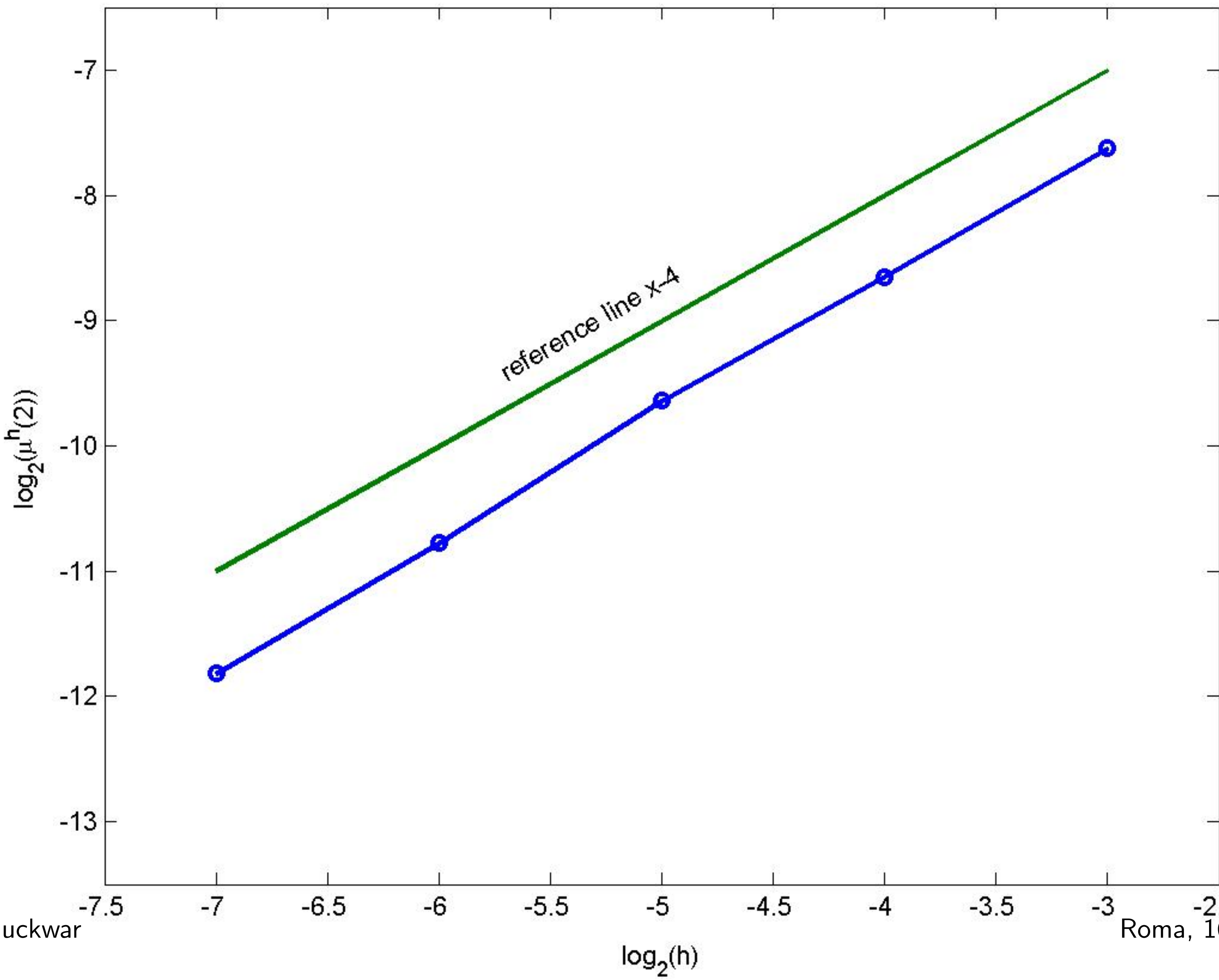
$$+\,(0.01\,X(t-\tfrac{1}{2}) + 0.1\,X(t))\mathrm{d}W(t), \quad 0 \le t \le 2,$$

$$X(t) = \psi(t) = 1 + \mathbf{e}^{-1t}, \qquad -1 \le t < 0,$$

functional $m(t) := \mathbb{E}X(t)$ satisfies $m'(t) = 3\,\mathbf{e}^{-1}\,m(t-1) - 4m(t) + 4 - 3\,\mathbf{e}^{-1}$ with $m(t) = \psi(t)$ $-1 \le t \le 0$ and has solution $m(t) = 1 + \mathbf{e}^{-1t}$

numerical tests simulation of 150000 trajectories with step-sizes $h = 2^{-3}, 2^{-4}, .., 2^{-7}$, empirical error $\mu^h(2) = |\mathbb{E}(X(2)) - \mathbb{E}(X_N(2))|$


X(t)
2
1.5
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t



reference line x-4
-7
-8
-9
-10
-11
-12
-13
-7.5
-7
-6.5
-6
-5.5
-5
-4.5
-4
-3.5
-3
-2.5
log2(μh(2))
log2(h)

Thank you for your attention