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http://www.mat.uniroma1.it/seminari/geo-alg/

Andrea Bruno	bruno@mat.uniroma3.it
Alberto Canonaco	canonaco@mat.uniroma1.it
Domenico Fiorenza	fiorenza@mat.uniroma2.it
Michele Grassi	grassi@dm.unipi.it
Marco Manetti	manetti@mat.uniroma1.it
Riccardo Murri	riccardo.murri@gmx.it
Edoardo Sernesi	sernesi@mat.uniroma3.it

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An Overview of Classical Deformation Theory

Edoardo Sernesi

1 Generalities

Deformation theory is closely related to the problem of classification in algebraic geometry. If we have a class $\mathcal{M}$ of algebro-geometric objects, e.g.

$$
\begin{aligned}
\mathcal{M} &= \{\text{projective nonsingular curves of genus } g\}/(\text{isomorphism}) \\
\mathcal{M} &= \{\text{closed subschemes of } \mathbb{P}^r \text{ with given Hilbert polynomial}\} \\
\mathcal{M} &= \{\text{vector bundles of given rank and Chern} \\
&\qquad \text{classes on a smooth projective variety } X\}
\end{aligned}
$$

the problem is: to describe $\mathcal{M}$.

The interest and the difficulty of this problem come from the existence of families. Roughly speaking, the existence of families of objects in $\mathcal{M}$ implies that $\mathcal{M}$ is not just a set but has some kind of "structure", hopefully will be a scheme, which will be the *moduli space* of the classification problem. In most cases $\mathcal{M}$ is not a scheme but has a weaker structure.

In order to make this statement more precise we have to specify the notion of family. This notion is different for every different class $\mathcal{M}$ but in each case it is related to the natural fact that all objects of algebraic geometry can be "deformed" by varying the coefficients of their defining equations.

If for example we want to consider a class $\mathcal{M}$ of algebraic varieties (curves, varieties of given dimension and numerical characters, etc.) a family will be a morphism:

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow{\scriptstyle \pi} \\
S
\end{array}
$$

whose fibres $\mathcal{X}(s) = \pi^{-1}(s)$, $s \in S$, are elements of $\mathcal{M}$ and with at least the extra technical condition of being *flat*; if the class $\mathcal{M}$ consists of projective and/or nonsingular varieties, then π will be also required to be proper and/or smooth. Here $\mathcal{X}$ and S are called the *total space* and the *parameter space* of the family. If S is connected then π is called a *family of deformations* of $\mathcal{X}(s_0)$ for any $s_0 \in S$.

If $\mathcal{X}$ and S are complex manifolds with S connected, and π is proper and smooth then all fibres $\mathcal{X}(s)$ are diffeomorphic and we are just considering a family of compact complex structures on a fixed differentiable manifold.

If instead we want to consider a class $\mathcal{M}$ of closed subschemes of a given scheme Y a family will be a commutative diagram:

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[d, "\pi"'] & S \times Y \\
S
\end{tikzcd}
$$

where π is the restriction of the first projection, the inclusion is closed, and all fibres of π are in $\mathcal{M}$.

Typically, a family of hypersurfaces of degree d in $\mathbb{P}^r$ parametrized by an affine space $\mathbb{A}^n = \mathrm{Spec}(k[t_1, \dots, t_n])$, k a field, will be a hypersurface $H \subset \mathbb{A}^n \times \mathbb{P}^r$ defined by a polynomial $P(\underline{t}, \underline{X}) \in k[t_1, \dots, t_n, X_0, \dots, X_r]$ homogeneous of degree d in $X_0, \dots, X_r$.

A less ambitious goal is the study of local deformations of a given object $m \in \mathcal{M}$. This means to consider deformations of m parametrized by spectra of local rings so that m is the fibre over the closed point. This will lead to the understanding of the local structure of $\mathcal{M}$ at m. This was the point of view of Kodaira-Spencer who initiated modern deformation theory in a series of papers published in 1958 on Annals of Mathematics, where they studied local deformations of compact complex manifolds, i.e. local deformations of complex structures on a fixed compact differentiable manifold.

In each different case the notion of family has the fundamental property of being funtorial. Let's consider, to fix ideas, the case of a class $\mathcal{M}$ of isomorphism classes of projective varieties defined over a fixed algebraically closed field k, and families of objects in $\mathcal{M}$. For each scheme S we call two different such families over S

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] & \text{and} & \mathcal{X}' \arrow[d, "\pi'"] \\ S & & S \end{tikzcd}$$

isomorphic if there is an isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ such that $\pi = \pi' \circ \varphi$. We can define a contravariant functor

$$F : (\text{Schemes}/k) \longrightarrow (\text{Sets})$$

by

$$F(S) = \{\text{isomorphism classes of families of objects of } \mathcal{M} \text{ over } S\}$$

For each morphism $f : T \to S$ we have an induced

$$F(f) : F(S) \longrightarrow F(T)$$

by pulling back families with f:

$$F(f)([\mathcal{X} \to S]) = [T \times_S \mathcal{X} \to T]$$

where $[-]$ denotes the isomorphism class of $-$ and

$$\begin{tikzcd} T \times_S \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

is the induced pullback diagram.

This observation was the starting point of the development of deformation theory under the influence of Grothendieck. According to his point of view we may ask whether this functor is represented by a scheme M, namely if there is an isomorphism of functors:

$$\mu : \mathrm{Hom}(\cdot, M) \longrightarrow F$$

Such an isomorphism will be induced by pulling back a uniquely determined family $\xi : \mathcal{Y} \to M$, called the *universal family* (Infact $[\xi] = \mu(M)(id_M) \in F(M)$). If this is the case M will be a moduli space for $\mathcal{M}$ in the strongest sense. In particular its closed points will be in one-to-one correspondence with the element of $\mathcal{M}$ by the chain of bijections:

$$\begin{aligned}\mathcal{M} \longleftrightarrow \{\text{families parametrized by}\mathrm{Spec}(k)\} \longleftrightarrow \\ \longleftrightarrow \mathrm{Hom}(\mathrm{Spec}(k), M) \longleftrightarrow \{\text{closed points of } M\}\end{aligned}$$

Such a moduli space very seldom exists. Most of the time $\mathcal{M}$ will have a weaker structure corresponding to a property of the functor F weaker than representability. But let's suppose for a moment that M exists in our case. Then in principle all informations concerning its structure and all its properties are encoded in the functor F. In particular we can investigate its infinitesimal, local and formal properties around a point $m \in M$ by looking at various special families of deformations of the fibre $\mathcal{Y}(m)$ of the universal family. For example the tangent space $T_{M,m}$ can be recovered considering "first order deformations".

A *first order deformation* of a scheme X is a commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(k[\epsilon]) \end{tikzcd}$$

where π is a flat morphism, $\mathrm{Spec}(k[\epsilon]) = \mathrm{Spec}(k[t]/(t^2))$, and such that the induced morphism

$$X \longrightarrow \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[\epsilon])} \mathcal{X}$$

is an isomorphism. First order deformations can be viewed as derivatives of $\mathcal{Y}(m)$ along a tangent vector of M at m. Infact we have the following chain of bijections:

$$\begin{aligned}T_{M,m} \longleftrightarrow \mathrm{Hom}_m(\mathrm{Spec}(k[\epsilon]), M) \longleftrightarrow \\ \longleftrightarrow \{\text{first order deformations of } \mathcal{Y}(m)\}/(\text{isomorphism})\end{aligned}$$

where we have denoted by $\mathrm{Hom}_m(\mathrm{Spec}(k[\epsilon]), M)$ the set of morphisms

$$\mathrm{Spec}(k[\epsilon]) \longrightarrow M$$

mapping the unique closed point of $\mathrm{Spec}(k[\epsilon])$ to m, and where the last bijection is $\mu(\mathrm{Spec}(k[\epsilon]))$.

More generally an *infinitesimal deformation* of a scheme X is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd} \tag{1}$$

where π is a flat morphism, A is a local artinian k-algebra and the morphism $X \to \mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} \mathcal{X}$ induced by the diagram is an isomorphism. Then, in the same vein as above, infinitesimal deformations of X give informations on the infinitesimal structure of M at the point $m = \mu(\mathrm{Spec}(k))^{-1}([X])$ because we have bijections

$$\begin{aligned}\mathrm{Hom}_m(\mathrm{Spec}(A), M) \longleftrightarrow \\ \longleftrightarrow \{\text{infinitesimal deformations of } X \text{ parametrized by } \mathrm{Spec}(A)\}.\end{aligned}$$

An infinitesimal deformation (1) is called *trivial* if $\mathcal{X} = X \times \mathrm{Spec}(A)$.

Deformation theory is the study of infinitesimal deformations as a tool to understand the local structure of the moduli space. The goal is to be able to describe the restriction of the universal family to a small neighborhood of $m \in \mathcal{M}$, or, more precisely, its restriction to the germ of M at m.

What is interesting here is that we can study first order and infinitesimal deformations even though the functor F is not representable or simply we don't yet know it is. This is the most frequent case. Such an investigation will reveal the infinitesimal properties at $[X]$ of a yet unknown global structure on $\mathcal{M}$ which will be hopefully understood at a subsequent stage of the investigation. In order words it turns out to be possibile and convenient to separate the *global moduli problem* from the *local moduli problem*, and deformation theory studies the latter, with the purpose of constructing a family of deformations of a given object parametrized by the spectrum of a local ring, and having properties as close as possible to a universal property.

2 First order deformations

The first consequence of the local point of view is that, whenever we want to study infinitesimal deformations of some object, we don't need to specify the global class $\mathcal{M}$, i.e. the global moduli problem, inside which we are going to move it: all we have to do is to define what we mean by an infinitesimal deformation of it. Of course our definition will often be tailored to some specific global problem, but not always.

Let's apply these ideas to the study of first order deformations. We will only consider algebraic k-schemes where k is an algebraically closed field. We will see that isomorphism classes of first order deformations are elements of a cohomology vector space. It is a technical easy fact to check that this vector space structure coincides with the structure of tangent space in the corresponding moduli problem (whatever this means).

2.1 Nonsingular affine varieties

Let $X = \mathrm{Spec}(R)$ be a nonsingular affine variety. Then every first order deformation of X is trivial. Infact let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(k[\epsilon]) \end{tikzcd}$$

be such a deformation. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \mathcal{X} \arrow[d, "\pi"] \\ X \times \mathrm{Spec}(k[\epsilon]) \arrow[r] & \mathrm{Spec}(k[\epsilon]) \end{tikzcd}$$

and the nonsingularity of X implies the existence of a morphism $\phi : X \times \mathrm{Spec}(k[\epsilon]) \to \mathcal{X}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \mathcal{X} \arrow[d, "\pi"] \\ X \times \mathrm{Spec}(k[\epsilon]) \arrow[r] \arrow[ur, "\phi"] & \mathrm{Spec}(k[\epsilon]) \end{tikzcd}$$

is still commutative. One easily checks that ϕ is an isomorphism, and this proves that the given deformation is trivial.

2.2 Nonsingular varieties.

Lemma 2.1. *Let $\pi : \mathcal{X} \to S$ be a morphism of schemes, $\phi : X \subset \mathcal{X}$ a closed embedding defined by a sheaf of ideals $J \subset \mathcal{O}_{\mathcal{X}}$ such that $J^2 = 0$. Then there is a canonical 1-1 correspondence:*

$$\{S\text{-automorphisms of } \mathcal{X} \text{ inducing the identity on } X\} \longleftrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\phi^*\Omega^1_{\mathcal{X}/S}, J)$$

Proof. The question is local. Therefore we may assume that everything is affine and we have a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B/J \\ A \arrow[u, "\tilde{\pi}"] \arrow[r, "\tilde{\pi}"'] & B \arrow[u] \end{tikzcd}$$

Every A-automorphism ψ of B inducing the identity on B/J is of the form $\psi = 1_B + D$, where $D : B \to J$ is A-linear and satisfies

$$\begin{aligned} D(b_1b_2) = (\psi - 1_B)(b_1b_2) = \psi(b_1b_2) - \psi(b_1)b_2 + \psi(b_1)b_2 - b_1b_2 = \\ = \psi(b_1)(\psi(b_2) - b_2) + (\psi(b_1) - b_1)b_2 = \psi(b_1)D(b_2) + D(b_1)b_2 = \\ = b_1D(b_2) + D(b_1)b_2. \end{aligned}$$

In other words D is an A-derivation of B in J. Therefore the set of A-automorphisms of B inducing the identity on B/J is in $1-1$ correspondence with

$$\mathrm{Der}_A(B, J) = \mathrm{Hom}_B(\Omega_{B/A}, J) = \mathrm{Hom}_{B/J}(\Omega_{B/A} \otimes_B B/J, J)$$

□

Consider now a nonsingular variety X and a first order deformation of X:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(k[\epsilon]) \end{tikzcd} \tag{2}$$

Let $\{U_\alpha\}$ be an affine open cover of X. Then by the previous case there are $\mathrm{Spec}(k[\epsilon])$-isomorphisms:

$$\theta_\alpha : \mathcal{X}_{|U_\alpha} \cong U_\alpha \times \mathrm{Spec}(k[\epsilon])$$

inducing the identity on the central fibre $U_\alpha = U_\alpha \times \mathrm{Spec}(k)$. Therefore by the lemma:

$$\theta_\beta\theta_\alpha^{-1} \in \Gamma(U_{\alpha\beta}, Hom(\Omega^1_{\mathcal{X}/\mathrm{Spec}(k[\epsilon])} \otimes \mathcal{O}_X, \mathcal{O}_X)) = \Gamma(U_{\alpha\beta}, \Theta_X)$$

where we denote, as usual, $U_{\alpha\beta} = U_\alpha \cap U_\beta$. It follows that the system

$$\{\theta_{\alpha\beta} = \theta_\beta\theta_\alpha^{-1}\}$$

defines a Čech 1-cocycle in Θ_X and this defines an element of $H^1(X, \Theta_X)$. One easily checks that this element is independent of the chosen affine cover. Therefore we have defined a map

$$T_{\mathcal{M},[X]} \longrightarrow H^1(X, \Theta_X)$$

which is easily seen to be a bijection.

Another equivalent way to define this map is the following. To a first order deformation (2) we can associate the exact sequence:

$$0 \longrightarrow \pi^*\Omega^1_{\mathrm{Spec}(k[\epsilon])} \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/S} \longrightarrow 0$$

which tensored by $\mathcal{O}_X$ (i.e. restricted to X) gives the exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_{\mathcal{X}} \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0$$

This is an element of $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) = H^1(X, \Theta_X)$ which can be checked to be the same as the one defined above.

If $\pi : \mathcal{X} \to S$ is an infinitesimal deformation of $X = \pi^{-1}(0)$ then the differential at 0 of the functorial morphism $S \to \mathcal{M}$ is a linear map

$$KS : T_{S,0} \longrightarrow H^1(X, \Theta_X)$$

called the *Kodaira-Spencer map* of π, and $KS(v) \in H^1(X, \Theta_X)$ is the *Kodaira-Spencer class* of $v \in T_{S,0}$.

It follows that if $H^1(X, \Theta_X) = (0)$ then every first order deformation of X is trivial. It turns out that every infinitesimal deformation of X is trivial as well, i.e. X is *rigid*. For example $\mathbb{P}^r$ is rigid because $H^1(\mathbb{P}^r, \Theta_{\mathbb{P}^r}) = (0)$.

2.3 Line bundles on a fixed nonsingular projective variety

Let L be a line bundle on a nonsingular projective variety X. A first order deformation of L is a line bundle L_ϵ on $X \times \mathrm{Spec}(k[\epsilon])$ which restricts to L on the closed fibre $X = (X \times \mathrm{Spec}(k[\epsilon])) \times_{\mathrm{Spec}(k[\epsilon])} \mathrm{Spec}(k)$. Assume that L is given by a system of transition functions $\{f_{\alpha\beta}\}$ with respect to an open covering $\{U_\alpha\}$ of X, $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*_X)$. Then L_ϵ can be represented, in the same covering $\{U_\alpha\}$ of $X \times \mathrm{Spec}(k[\epsilon])$ by transition functions:

$$\tilde{f}_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, O^*_{X \times \mathrm{Spec}(k[\epsilon])})$$

such that

$$\tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma} = \tilde{f}_{\alpha\gamma} \tag{3}$$

and wich restrict to the $f_{\alpha\beta}$'s modulo ϵ.

Since $\mathcal{O}^*_{X \times \mathrm{Spec}(k[\epsilon])} = \mathcal{O}^*_X + \epsilon\mathcal{O}_X$ we can write

$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta}(1 + \epsilon\Phi_{\alpha\beta})$$

for suitable $\Phi_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Identity (3) gives

$$\Phi_{\alpha\beta} + \Phi_{\beta\gamma} = \Phi_{\alpha\gamma}$$

and therefore the system $\{\Phi_{\alpha\beta}\}$ defines an element of $H^1(X, \mathcal{O}_X)$. It is easy to check that this element does not depend on the choices made and that conversely each element of $H^1(X, \mathcal{O}_X)$ defines a first order deformation of L.

The class of all line bundles on X has the structure of a locally finite type scheme, denoted $\mathrm{Pic}(X)$, and we have computed its Zariski tangent space at L:

$$T_{\mathrm{Pic}(X),L} \cong H^1(X, \mathcal{O}_X)$$

3 Higher order deformations – Obstructions

So far we have discovered that we can compute various tangent spaces to deformation problems as cohomology vector spaces. This is of course only a first step towards the description of the local structure of our moduli problems. For the next step we need to push the local point of view a little further.

Suppose that we need to study infinitesimal deformations of a geometrical object X inside a class $\mathcal{M}$. Let's assume that a moduli space M for $\mathcal{M}$ exists and let $\xi \in F(M)$ be the universal family. Then letting $[X] = m \in M$ be the point corresponding to X, to every infinitesimal deformation of X there corresponds a morphism

$$\begin{aligned} \varphi : \mathrm{Spec}(A) &\longrightarrow M \\ \text{closed pt} &\longmapsto m \end{aligned}$$

which induces the given deformation by pullback. In turn φ corresponds to a homomorphism of local k-algebras

$$\tilde{\varphi} : \mathcal{O} = \mathcal{O}_{M,m} \longrightarrow A$$

Since A is artinian, $\tilde{\varphi}$ factors through the completion $\mathcal{O} \to \hat{\mathcal{O}}$ with respect to the maximal ideal and therefore the properties of $\mathcal{O}$ detected by the study of infinitesimal deformations will be *analytic properties*, i.e. properties preserved under completion.

For example, if $A = k[\epsilon]$ then $F(\mathrm{Spec}(k[\epsilon]))$ is the Zariski tangent space of $\mathcal{O}$, which coincides with that of $\hat{\mathcal{O}}$.

We can rephrase all the above by considering the category

$$\mathcal{A} = (\text{local artinian } k\text{-algebras with residue field } k)$$

and saying that our deformation problem defines a covariant functor

$$F_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{S}ets$$

i.e. a *functor of Artin rings*, defined by

$$F_{\mathcal{A}}(A) = \{\text{infinitesimal deformations of } X \text{ over } \mathrm{Spec}(A)\} = \mathrm{Hom}(\mathcal{O}, A)$$

The most important analytic property is nonsingularity. We can investigate the nonsingularity of M at m by means of the functor $F_{\mathcal{A}}$ and applying the following

Lemma 3.1. *Let $\mathcal{O}$ be a local noetherian k-algebra with residue field k. The following conditions are equivalent:*

1. *$\mathcal{O}$ is a regular local ring.*
2. *$\hat{\mathcal{O}}$ is a regular local ring.*
3. *There is an isomorphism*
$$\hat{\mathcal{O}} \cong k[[X_1, \ldots, X_d]]$$
where d is the Krull dimension of $\mathcal{O}$, and $X_1, \ldots, X_d$ are indeterminates.
4. *For every commutative diagram:*

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A' \arrow[d] \\ \mathcal{O} \arrow[r] & A \end{tikzcd}$$

where the right vertical arrow is a surjection of local artinian k-algebras, there is a k-algebra homomorphism $\mathcal{O} \to A'$ keeping the diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A' \arrow[d] \\ \mathcal{O} \arrow[r] \arrow[ur] & A \end{tikzcd}$$

commutative.

Condition 4 of the Lemma states that $\mathrm{Hom}(\mathcal{O}, A') \to \mathrm{Hom}(\mathcal{O}, A)$ is surjective for all surjections $A' \to A$ in $\mathcal{A}$. This condition has an immediate translation into a property of the functor $F_{\mathcal{A}}$:

Proposition 3.2. *M is nonsingular at m if and only if for every surjection $A' \to A$ in $\mathcal{A}$ the corresponding map*

$$F_{\mathcal{A}}(A') \longrightarrow F_{\mathcal{A}}(A)$$

is surjective. If this condition is satisfied the functor $F_{\mathcal{A}}$ is said to be smooth*.*

The condition of the Proposition has the following deformation-theoretic interpretation. Given a surjection $A' \to A$ in $\mathcal{A}$ and any deformation (1) there is a deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(A') \end{tikzcd} \tag{4}$$

extending (1), i.e. such that (1) is induced by (4) by pulling it back via $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$. If the extension (4) exists for each surjection $A' \to A$ the deformation (1) is called *unobstructed*; otherwise it is *obstructed*.

If all infinitesimal deformations of X are unobstructed then X is called *unobstructed*; otherwise X is *obstructed*.

It turns out that in order to check (un)obstructedness it suffices to consider surjections $q : A' \to A$ in $\mathcal{A}$ such that $\ker(q) \cong k$ (called *small extensions*).

Let's denote by t_R the Zariski tangent space $(m_R/m_R^2)^\vee$ of a local ring (R, m_R). We have the following

Definition 3.3. *Let (R, m_R) be a complete local k-algebra with residue field k. Write $R = k[[X_1, \dots, X_n]]/J$ where $J \subset (\underline{X})^2$. Then the k-vector space*

$$o(R) := (J/(\underline{X})J)^\vee$$

is called the obstruction space *of R.*

Clearly $o(R) = 0$ if and only if $R \cong k[[X_1, \dots, X_n]]$. We have the following inequalities:

$$\dim(t_R) \geq \dim(R) \geq \dim(t_R) - \dim(o(R))$$

Moreover for each A in $\mathcal{A}$ and for each $\varphi : R \to A$ there is a map which associates to each small extension $q : A' \to A$ an element $v(q) \in o(R)$ which is 0 if and only if φ can be lifted to $\varphi' : R \to A'$.

If we have a "sufficiently well behaved" deformation functor $F_{\mathcal{A}}$ then it is possible to define the *obstruction* to find an extension in $F_{\mathcal{A}}(A')$ of a given $\eta \in F_{\mathcal{A}}(A)$; this obstruction is usually an element of a cohomology vector space H. The deformation η will then be unobstructed precisely if the obstruction vanishes for each small extension q. If the deformation functor is $F_{\mathcal{A}} = \mathrm{Hom}(\mathcal{O}, -)$ where $\mathcal{O} = \mathcal{O}_{M,m}$ as above, then it follows by general nonsense that $o(\hat{\mathcal{O}}) \subset H$. This implies that M is nonsingular at m if the vector space H vanishes and, more generally, that

$$\dim(\mathcal{O}) \geq \dim(t_{\mathcal{O}}) - \dim(H) = \dim(F_{\mathcal{A}}(k[\epsilon]) - \dim(H)$$

Let's illustrate this principle with an example.

Nonsingular varieties

Assume that we have a class $\mathcal{M}$ of nonsingular varieties for which the moduli space M exists. Let X be in $\mathcal{M}$. Assume that we have a small extension $A' \to A$ and an infinitesimal deformation (1) of X. We want to find conditions for the extendability to a deformation of X over $\mathrm{Spec}(A')$.

Let $\{U_\alpha\}$ be an affine open cover of X, $\theta_\alpha : \mathcal{X}_{|U_\alpha} \cong U_\alpha \times \mathrm{Spec}(A)$ be $\mathrm{Spec}(A)$-isomorphisms inducing the identity on U_α, and let

$$\theta_{\alpha\beta} = \theta_\alpha \theta_\beta^{-1} : U_{\alpha\beta} \times \mathrm{Spec}(A) \longrightarrow U_{\alpha\beta} \times \mathrm{Spec}(A)$$

be the induced $\mathrm{Spec}(A)$-automorphisms. Then the existence of a deformation $\pi' : \mathcal{X}' \to \mathrm{Spec}(A')$ extending (1) is equivalent to the existence of a system of automorphisms

$$\theta'_{\alpha\beta} : U_{\alpha\beta} \times \mathrm{Spec}(A') \longrightarrow U_{\alpha\beta} \times \mathrm{Spec}(A')$$

which restrict to the automorphisms $\theta_{\alpha\beta}$ on $U_{\alpha\beta} \times \mathrm{Spec}(A)$, and such that

$$\theta'_{\alpha\beta}\theta'_{\beta\gamma} = \theta'_{\alpha\gamma} \tag{5}$$

on $U_{\alpha\beta\gamma}$. Let's choose arbitrarily automorphisms $\theta'_{\alpha\beta}$ which extend the $\theta_{\alpha\beta}$'s (they exist by the nonsingularity of the affine varieties $U_{\alpha\beta}$), and let's consider the $\mathrm{Spec}(A')$-automorphisms of $U_{\alpha\beta\gamma} \times \mathrm{Spec}(A')$:

$$\theta'_{\alpha\beta\gamma} := \theta'_{\alpha\beta}\theta'_{\beta\gamma}(\theta'_{\alpha\gamma})^{-1}$$

Each of these restricts to the identity on $U_{\alpha\beta\gamma} \times \mathrm{Spec}(A)$ and therefore, by the Lemma, is an element of $\Gamma(U_{\alpha\beta\gamma}, \Theta_X)$. The system $\{\theta'_{\alpha\beta\gamma}\}$ is therefore a 2-cocycle with coefficients in Θ_X and defines an element $\theta \in H^2(X, \Theta_X)$.

Another choice of the automorphisms $\theta'_{\alpha\beta}$ is of the form

$$\bar{\theta}'_{\alpha\beta} = \theta'_{\alpha\beta}\delta_{\alpha\beta}$$

for some $\delta_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \Theta_X)$. Therefore:

$$\bar{\theta}'_{\alpha\beta\gamma} = \theta'_{\alpha\beta\gamma}\delta_{\alpha\beta}\delta_{\beta\gamma}(\delta_{\alpha\gamma})^{-1}$$

and therefore $\{\theta'_{\alpha\beta\gamma}\}$ and $\{\bar{\theta}'_{\alpha\beta\gamma}\}$ define the same cohomology class in $H^2(X,\Theta_X)$.

The class $\theta \in H^2(X, \Theta_X)$ is the *obstruction* to extend the deformation (3) to $\mathrm{Spec}(A')$. In particular we see that *if* $H^2(X, \Theta_X) = 0$ *then* M *is nonsingular at* $[X]$. For example, nonsingular projective curves are unobstructed.

4 Versal and universal formal families

We have seen how one can study the infinitesimal properties of a moduli space M at a point m using functorial methods and cohomological techniques. We now want to consider a local moduli problem and see whether it is possible to study its infinitesimal properties and to give it a local structure of some kind. From an infinitesimal point of view a local moduli problem corresponds to a (covariant) functor of Artin rings

$$F : \mathcal{A} \longrightarrow \mathcal{S}ets$$

such that $F(k)$ consists of one element. In the best possible case there will be a local k-algebra $\mathcal{O}$ with residue field k and an isomorphism of functors

$$\mathrm{Hom}(\mathcal{O}, \cdot) = \mathrm{Hom}(\hat{\mathcal{O}}, \cdot) \longrightarrow F$$

(the equality on the left is because, as we observed already, $A = \hat{A}$ for every A in $\mathcal{A}$). Since $\hat{\mathcal{O}}$ is not in $\mathcal{A}$, such a functor is not quite representable: it is called *prorepresentable*. Representable functors of Artin rings are not so interesting in this context, but prorepresentable ones are, and prorepresentability is the reachest structure such a functor can have.

Weaker structures can be introduced by requiring that there exists a morphism of functors (a "natural transformation")

$$f : \mathrm{Hom}(R, \cdot) \longrightarrow F$$

for some complete local k-algebra R with residue field k, which is not quite an isomorphism, but has some weaker property. Before discussing these properties let's see for a moment how a morphism f as above can be interpreted.

Let's denote by $\hat{\mathcal{A}}$ the category of complete local k-algebras with residue field k. Every functor of Artin rings $F : \mathcal{A} \to \mathcal{S}ets$ can be extended to a functor

$$\hat{F} : \hat{\mathcal{A}} \longrightarrow \mathcal{S}ets$$

by letting, for every (R, m) in $\hat{\mathcal{A}}$:

$$\hat{F}(R) = \varprojlim F(R/m^{n+1})$$

and for every $\varphi : (R, m) \to (S, p)$:

$$\hat{F}(\varphi) : \hat{F}(R) \longrightarrow \hat{F}(S)$$

to be the map induced by the maps $F(R/m^n) \to F(S/p^n)$, $n \geq 1$.

An element $\hat{u} \in \hat{F}(R)$ is called a *formal element* of F. By definition $\hat{u}$ can be represented as a system of elements $\{u_n \in F(R/m^{n+1})\}_{n \geq 0}$ such that for every $n \geq 0$ the map

$$F(R/m^{n+1}) \longrightarrow F(R/m^n)$$

induced by the projection $R/m^{n+1} \to R/m^n$ sends

$$u_n \longmapsto u_{n-1} \tag{6}$$

If for example F is the functor of infinitesimal deformations of a nonsingular variety X, each u_n is an infinitesimal deformation of X parametrized by $\mathrm{Spec}(R/m^{n+1})$. The compatibility condition (6) is that u_n pulls back to u_{n-1} under the closed embedding

$$\mathrm{Spec}(R/m^n) \subset \mathrm{Spec}(R/m^{n+1})$$

In this case the formal element $\hat{u}$ is also called a *formal family of deformations* of X.

If $f : F \to G$ is a morphism of functors of Artin rings then it can be extended in an obvious way to a morphism of functors $\hat{f} : \hat{F} \to \hat{G}$.

Lemma 4.1. *Let R be in $\hat{\mathcal{A}}$. There is a 1-1 correspondence between $\hat{F}(R)$ and the set of morphisms of functors*

$$\mathrm{Hom}(R, \cdot) \longrightarrow F \tag{7}$$

Proof. To a formal element $\hat{u} \in \hat{F}(R)$ there is associated a morphism of functors (6) in the following way. Each $u_n \in F(R/m^{n+1})$ defines a morphism of functors $\mathrm{Hom}(R/m^{n+1}, \cdot) \to F$. The compatibility conditions (6) imply that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(R/m^n, \cdot) \arrow[r] \arrow[dr] & \mathrm{Hom}(R/m^{n+1}, \cdot) \arrow[d] \\ & F \end{tikzcd}$$

for every n. Since for each A in $\mathcal{A}$

$$\mathrm{Hom}(R/m^n, A) \longrightarrow \mathrm{Hom}(R/m^{n+1}, A)$$

is a bijection for all $n \gg 0$ we may define

$$\mathrm{Hom}(R, A) \longrightarrow F(A)$$

as

$$\lim_{n\to\infty} [\mathrm{Hom}(R/m^{n+1}, A) \longrightarrow F(A)]$$

Conversely each morphism (7) defines a formal element $\hat{u} \in \hat{F}(R)$, where $u_n \in F(R/m^{n+1})$ is the image of the canonical projection $R \to R/m^{n+1}$ via the map

$$\mathrm{Hom}(R, R/m^{n+1}) \longrightarrow F(R/m^{n+1})$$

□

If $\hat{u} \in \hat{F}(R)$ is such that the induced morphism (7) is an isomorphism, then F is prorepresentable, and we say that F is *prorepresented by the pair* $(R, \hat{u})$. In this case $\hat{u}$ is called a *universal formal element* for F, and $(R, \hat{u})$ is a *universal pair*.

If for example F is the functor of infinitesimal deformations of a nonsingular variety X belonging to a class $\mathcal{M}$ which has a moduli space M, then the universal family $\mathcal{Y} \to M$ induces by restriction to the schemes $\mathrm{Spec}(\hat{O}/m^{n+1})$ a universal formal element for F (or a *universal formal family*).

Note that all prorepresentable functors have the following property:

N_0) $F(k)$ contains exactly one element.

All functors we will consider will have property N_0 and from now on this will be implicitly assumed unless otherwise specified.

Definition 4.2. *Let $f : F \to G$ be a morphism of functors of Artin rings. f is called* smooth *if for every surjection $\mu : B \to A$ in $\mathcal{A}$ the natural map:*

$$F(B) \longrightarrow F(A) \times_{G(A)} G(B)$$

induced by the diagram:

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & G(B) \arrow[d] \\ F(A) \arrow[r] & G(A) \end{tikzcd}$$

is surjective.

Note that the smoothness condition applied to the surjection $k[\epsilon] \to k$ states that the map

$$F(k[\epsilon]) \longrightarrow G(k[\epsilon])$$

is surjective. This map is denoted *df* and called the *differential* of f.

Let F be a functor of Artin rings. A formal element $\hat{u} \in \hat{F}(R)$, for some R in $\hat{\mathcal{A}}$, is called *versal* if the morphism $\mathrm{Hom}(R, \cdot) \to F$ defined by $\hat{u}$ is smooth; $\hat{u}$ is called *semiuniversal* if it is versal and moreover the differential $\mathrm{Hom}(R, \mathbf{k}[\epsilon]) \to F(\mathbf{k}[\epsilon])$ is an isomorphism.

We will call the pair $(R, \hat{u})$ a *versal pair* (respectively a *semiuniversal pair*, a *universal pair*) if $\hat{u}$ is versal (respectively semiuniversal, universal).

It is clear from the definitions that:

$$\hat{u} \text{ universal} \Longrightarrow \hat{u} \text{ semiuniversal} \Longrightarrow \hat{u} \text{ versal}$$

but none of the inverse implications is true.

What does it mean that a functor F has a versal pair $(R, \hat{u})$? From the definition of smoothness it follows easily that the map

$$\mathrm{Hom}(R, S) \longrightarrow \hat{F}(S) \tag{8}$$

induced by $\hat{u}$ is surjective for every S in $\hat{A}$. This means that every formal element $\hat{v} \in F(S)$ is induced by $\hat{u} \in F(R)$ by pullback. So we see that this is a property, weaker than universality, which is a sort of "completeness" of the formal element $\hat{u}$, in the sense that it induces every other by pullback.

Semiuniversality is stronger than versality: the bijectivity of the differential implies a sort of minimality among all possible versal pairs.

A theorem of Schlessinger gives conditions, easy to verify in practice, for the existence of a formal semiuniversal element for a functor F. It turns out that most functors of Artin rings arising in deformation theory satisfy Schlessinger's conditions, even though they seldom have a universal formal element; therefore all such functors have a structure weaker than prorepresentability, but very close to it.

Examples of functors satisfying Schlessinger conditions are:

- $F = \mathrm{Pic}(X)_L$ = deformations of a line bundle L on a fixed scheme X (the *local Picard functor* of X at L)
- F = deformations of a projective scheme X
- F = deformations of an affine variety with isolated singularities
- $F = \mathrm{Hilb}^Y_X$ = the local Hilbert functor of a closed embedding $X \subset Y$ of proj. schemes
- $F = \mathrm{Quot}^F_G$ = the local Quot scheme of a quotient $F \to G$ of sheaves on a projective scheme X

5 Algebrization

Suppose we know that a functor of Artin rings F has a (semi)universal pair $(R, \hat{u})$, and that F extends to the category $\mathcal{A}^*$ of local noetherian k-algebras. Then we should ask if there is a pair (S, u), where $u \in F(S)$, having the following properties:

1. S is in $\mathcal{A}^*$, and has some finiteness properties (e.g. it is essentially of finite type, it is henselian, etc.).
2. $\hat{S} = R$.

3. u induces $\hat{u}$.

This question is an abstract version of a natural problem in local deformation theory. Consider for example a projective nonsingular variety X. We can consider local deformations of X, i.e. families of the form

$$\xi : \begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \operatorname{Spec}(S) \end{array}$$

where (S, m_S) is in $\mathcal{A}^*$, and with an isomorphism $X \cong \mathcal{X}(m_S)$. Then we want to know if there is a (semi)universal such family ξ, i.e. a family which induces every other by pullback, and has a (semi)universal property. Applying the theory outlined before to the functor of Artin rings defined by X we obtain a formal (semi)universal pair $(R, \hat{u})$, and we now want to see if we can lift this pair to a pair (S, u) as above.

This is an algebraic version of the original problem studied and solved by Kodaira, Niremberg, Spencer and Kuranishi in the analytic case. Their final result is the following.

Theorem 5.1. *Let X be a compact complex manifold. Then there is a germ of complex space $(B, 0)$, with $\dim(B) \geq h^1(X, \Theta_X) - h^2(X, \Theta_X)$ and a smooth and proper family*

$$\xi : \begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ B \end{array}$$

such that $X \cong \mathcal{X}(0)$, which is a semiuniversal family of deformations of X. If $H^2(X,\Theta_X) = 0$ then B is nonsingular of dimension $h^1(X, \Theta_X)$. If $H^0(X, \Theta_X) = 0$ then ξ is universal.

In the algebraic case there is no such general result. The most general algebrizability result is due to M. Artin. It gives sufficient conditions for the existence of a pair (S, u) as above with S an henselian ring, i.e. the local ring of an algebraic space (for an exposition see [1]).

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Deformations of singularities via differential graded Lie algebras

Marco Manetti

1 Introduction

Let $\mathbb{K}$ be a fixed algebraically closed field of characteristic 0, $X \subset \mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ a closed subscheme. Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$.

Definition 1.1. *An* infinitesimal deformation *of X over $A \in$ **Art** is a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "f_A"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

such that f_A is flat and the induced morphism $X \to X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{K})$ is an isomorphism.

It is not difficult to see (cf. [1]) that X_A is affine and more precisely it is isomorphic to a closed subscheme of $\mathbb{A}^n \times \mathrm{Spec}(A)$. Two deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \mathrm{Spec}(A)$, $X \xrightarrow{j} \tilde{X}_A \xrightarrow{g_A} \mathrm{Spec}(A)$ are isomorphic if there exists a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & X_A \arrow[d, "f_A"] \arrow[dl, "\theta"] \\ \tilde{X}_A \arrow[r, "g_A"'] & \mathrm{Spec}(A) \end{tikzcd}$$

It is easy to prove that necessarily θ is an isomorphism (cf. [5]). Since flatness commutes with base change, for every deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \mathrm{Spec}(A)$ and every morphism $A \to B$ in the category **Art**, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(B) \end{tikzcd}$$

is a deformation of X over $\mathrm{Spec}(B)$; it is then defined a covariant functor $\mathrm{Def}_X \colon$ **Art** $\to$ **Set**,

$$\mathrm{Def}_X(A) = \{ \text{ isomorphism classes of deformations of } X \text{ over } A \}.$$

The set $\mathrm{Def}_X(\mathbb{K})$ contains only one point.

In a similar way we can define the functor $\mathrm{Hilb}_X \colon$ **Art** $\to$ **Set** of embedded deformations of X into $\mathbb{A}^n$: $\mathrm{Hilb}_X(A)$ is the set of closed subschemes $X_A \subset \mathbb{A}^n \times \mathrm{Spec}(A)$ such that the restriction to X_A of the projection on the second factor is a flat map $X_A \to \operatorname{Spec}(A)$ and $X_A \cap (\mathbb{A}^n \times \operatorname{Spec}(\mathbb{K})) = X \times \operatorname{Spec}(\mathbb{K})$.

In these notes we give a recipe for the construction of two differential graded Lie algebras $\mathcal{H}$, $\mathcal{L}$ together two isomorphism of functors

$$\operatorname{Def}_{\mathcal{L}} = \frac{MC_{\mathcal{L}}}{\text{gauge}} \to \operatorname{Def}_X, \qquad \operatorname{Def}_{\mathcal{H}} = \frac{MC_{\mathcal{H}}}{\text{gauge}} \to \operatorname{Hilb}_X.$$

The DGLAs $\mathcal{L}$, $\mathcal{H}$ are unique up to quasiisomorphism and their cohomology can be computed in terms of the cotangent complex of X. For the notion of differential graded Lie algebra, Maurer-Cartan functors and gauge equivalence we refer to [3], [5], [2].

Moreover we can choose $\mathcal{H}$ as a differential graded Lie subalgebra of $\mathcal{L}$ such that $\mathcal{H}^i = \mathcal{L}^i$ for every $i > 0$.

2 Flatness and relations

In this section $A \in \mathbf{Art}$ is a fixed local artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$.

Lemma 2.1. *Let M be an A-module, if $M \otimes_A \mathbb{K} = 0$ then $M = 0$.*

Proof. If M is finitely generated this is Nakayama's lemma. In the general case consider a filtration of ideals $0 = I_0 \subset I_1 \subset \ldots \subset I_n = A$ such that $I_{i+1}/I_i = \mathbb{K}$ for every i. Applying the right exact functor $\otimes_A M$ to the exact sequences of A-modules

$$0 \longrightarrow \mathbb{K} = \frac{I_{i+1}}{I_i} \longrightarrow \frac{A}{I_i} \longrightarrow \frac{A}{I_{i+1}} \longrightarrow 0$$

we get by induction that $M \otimes_A (A/I_i) = 0$ for every i. □

The following is a special case of the *local flatness criterion* [6, Thm. 22.3]

Theorem 2.2. *For an A-module M the following conditions are equivalent:*

1. *M is free.*
2. *M is flat.*
3. *$\operatorname{Tor}_1^A(M, \mathbb{K}) = 0$.*

Proof. The only nontrivial assertion is 3) $\Rightarrow$ 1). Assume $\operatorname{Tor}_1^A(M, \mathbb{K}) = 0$ and let F be a free module such that $F \otimes_A \mathbb{K} = M \otimes_A \mathbb{K}$. Since $M \to M \otimes_A \mathbb{K}$ is surjective there exists a morphism $\alpha\colon F \to M$ such that its reduction $\overline{\alpha}\colon F \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is an isomorphism. Denoting by K the kernel of α and by C its cokernel we have $C \otimes_A \mathbb{K} = 0$ and then $C = 0$; $K \otimes_A \mathbb{K} = \operatorname{Tor}_1^A(M, \mathbb{K}) = 0$ and then $K = 0$. □

Corollary 2.3. *Let $h\colon P \to L$ be a morphism of flat A-modules, $A \in \mathbf{Art}$. If $\overline{h}\colon P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is injective (resp.: surjective) then also h is injective (resp.: surjective).*

Proof. Same proof of Theorem 2.2. □

Corollary 2.4. *Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules with N flat. Then:*

1. *$M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective $\Rightarrow$ P flat.*
2. *P flat $\Rightarrow$ M flat and $M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective.*

Proof. Take the associated long $\mathrm{Tor}_*^A(-, \mathbb{K})$ exact sequence and apply 2.2 and 2.3. $\square$

Corollary 2.5. *Let*

$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0 \tag{1}$$

be a complex of A-modules such that:

1. *P, Q, R are flat.*
2. *$Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0$ is exact.*
3. *$P \otimes_A \mathbb{K} \xrightarrow{\overline{f}} Q \otimes_A \mathbb{K} \xrightarrow{\overline{g}} R \otimes_A \mathbb{K} \xrightarrow{\overline{h}} M \otimes_A \mathbb{K} \longrightarrow 0$ is exact.*

Then M is flat and the sequence (1) is exact.

Proof. Denote by $H = \ker h = \mathrm{Im}\, g$ and $g = \phi\eta$, where $\phi \colon H \to R$ is the inclusion and $\eta \colon Q \to H$; by assumption we have an exact diagram

$$\begin{tikzcd}
P \otimes_A \mathbb{K} \arrow[r, "\overline{f}"] & Q \otimes_A \mathbb{K} \arrow[rr, "\overline{g}"] \arrow[dr, bend left, "\overline{\eta}"'] & & R \otimes_A \mathbb{K} \arrow[r, "\overline{h}"] & M \otimes_A \mathbb{K} \arrow[r] & 0 \\
& & H \otimes_A \mathbb{K} \arrow[ur, bend left, "\overline{\phi}"] \arrow[dr, bend right] & & & \\
& & & 0 & &
\end{tikzcd}$$

which allows to prove, after an easy diagram chase, that $\overline{\phi}$ is injective. According to Corollary 2.4 H and M are flat modules. Denoting $L = \ker g$ we have, since H is flat, that also L is flat and $L \otimes_A K \to Q \otimes_A \mathbb{K}$ injective. This implies that $P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is surjective. By Corollary 2.3 $P \to L$ is surjective. $\square$

Corollary 2.6. *Let $n > 0$ and*

$$0 \longrightarrow I \longrightarrow P_0 \xrightarrow{d_1} P_1 \longrightarrow \dots \xrightarrow{d_n} P_n,$$

a complex of A-modules with $P_0, \dots, P_n$ flat. Assume that

$$0 \longrightarrow I \otimes_A \mathbb{K} \longrightarrow P_0 \otimes_A \mathbb{K} \xrightarrow{\overline{d_1}} P_1 \otimes_A \mathbb{K} \longrightarrow \dots \xrightarrow{\overline{d_n}} P_n \otimes_A \mathbb{K}$$

is exact; then $I, \mathrm{coker}(d_n)$ are flat modules and the natural morphism $I \to \ker(P_0 \otimes_A \mathbb{K} \to P_1 \otimes_A \mathbb{K})$ is surjective.

Proof. Induction on n and Corollary 2.5. $\square$

3 Differential graded algebras, I

Unless otherwise specified by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$. We denote by:

- **G** the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector space; given an object $V = \oplus V_i$, $i \in \mathbb{Z}$, of **G** and a homogeneous element $v \in V_i$ we denote by $\overline{v} = i$ its degree.
- **DG** the category of $\mathbb{Z}$-graded differential $\mathbb{K}$-vector space (also called complexes of vector spaces).

Given (V, d) in $\mathbf{DG}$ we denote as usual by $Z(V) = \ker d$, $B(V) = d(V)$, $H(V) = Z(V)/B(V)$.

Given an integer n, the shift functor $[n]\colon \mathbf{DG} \to \mathbf{DG}$ is defined by setting $V[n] = \mathbb{K}[n] \otimes V$, $V \in \mathbf{DG}$, $f[n] = Id_{\mathbb{K}[n]} \otimes f$, $f \in \mathrm{Mor}_{\mathbf{DG}}$, where

$$\mathbb{K}[n]_i = \begin{cases} \mathbb{K} & \text{if } i + n = 0 \\ 0 & \text{otherwise} \end{cases}$$

More informally, the complex $V[n]$ is the complex V with degrees shifted by n, i.e. $V[n]_i = V_{i+n}$, and differential multiplied by $(-1)^n$.

Given two graded vector spaces V, W, the "graded Hom" is the graded vector space

$$\mathrm{Hom}^*_{\mathbb{K}}(V, W) = \oplus_n \mathrm{Hom}^n_{\mathbb{K}}(V, W) \in \mathbf{G},$$

where by definition $\mathrm{Hom}^n_{\mathbb{K}}(V, W)$ is the set of $\mathbb{K}$-linear map $f\colon V \to W$ such that $f(V_i) \subset W_{i+n}$ fore every $i \in \mathbb{Z}$. Note that $\mathrm{Hom}^0_{\mathbb{K}}(V, W) = \mathrm{Hom}_{\mathbf{G}}(V, W)$ is the space of morphisms in the category $\mathbf{G}$ and there exist natural isomorphisms

$$\mathrm{Hom}^n_{\mathbb{K}}(V, W) = \mathrm{Hom}_{\mathbf{G}}(V[-n], W) = \mathrm{Hom}_{\mathbf{G}}(V, W[n]).$$

A morphism in $\mathbf{DG}$ is called a quasiisomorphism if it induces an isomorphism in homology. A commutative diagram in $\mathbf{DG}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

is called cartesian if the morphism $A \to C \times_D B$ is an isomorphism; it is an easy exercise in homological algebra to prove that if f is a surjective (resp.: injective) quasiisomorphism, then g is a surjective (resp.: injective) quasiisomorphism.

Definition 3.1. *A graded (associative, $\mathbb{Z}$-commutative) algebra is a graded vector space $A = \oplus A_i \in \mathbf{G}$ endowed with a product $A_i \times A_j \to A_{i+j}$ satisfying the properties:*

1. *$a(bc) = (ab)c$.*
2. *$a(b + c) = ab + ac$, $(a + b)c = ac + bc$.*
3. *(Koszul sign convention) $ab = (-1)^{\bar{a}\,\bar{b}} ba$ for a, b homogeneous.*

The algebra A is unitary if there exists $1 \in A_0$ such that $1a = a1 = a$ for every $a \in A$.

Let A be a graded algebra, then A_0 is a commutative $\mathbb{K}$-algebra in the usual sense; conversely every commutative $\mathbb{K}$-algebra can be considered as a graded algebra concentrated in degree 0. If $I \subset A$ is a homogeneous left (resp.: right) ideal then I is also a right (resp.: left) ideal and the quotient A/I has a natural structure of graded algebra.

Example 3.2. *Polynomial algebras.* Given a set $\{x_i\}$, $i \in I$, of homogeneous indeterminates of integral degree $\overline{x_i} \in \mathbb{Z}$ we can consider the graded algebra $\mathbb{K}[\{x_i\}]$. As a $\mathbb{K}$-vector space $\mathbb{K}[\{x_i\}]$ is generated by monomials in the indeterminates x_i. Equivalently $\mathbb{K}[\{x_i\}]$ can be defined as the symmetric algebra $\bigoplus_{n \geq 0} \bigodot^n V$, where $V = \oplus_{i \in I} \mathbb{K}\, x_i \in \mathbf{G}$. In some cases, in order to avoid confusion about terminology, for a monomial $x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n}$ it is defined:

- The *internal degree* $\sum_h \overline{x_{i_h}} \alpha_h$.
- The *external degree* $\sum_h \alpha_h$.

In a similar way it is defined $A[\{x_i\}]$ for every graded algebra A.

Definition 3.3. *A* dg-algebra *(differential graded algebra) is the data of a graded algebra A and a $\mathbb{K}$-linear map $s\colon A \to A$, called* differential*, with the properties:*

1. *$s(A_n) \subset A_{n+1}$, $s^2 = 0$.*
2. *(graded Leibnitz rule) $s(ab) = s(a)b + (-1)^{\overline{a}} a s(b)$.*

A morphism of dg-algebras is a morphism of graded algebras commuting with differentials; the category of dg-algebras is denoted by **DGA**.

In the sequel, for every dg-algebra A we denote by $A_\sharp$ the underlying graded algebra.

Exercise 3.4. Let (A, s) be a unitary dg-algebra; prove:

1. $1 \in Z(A)$.
2. $1 \in B(A)$ if and only if $H(A) = 0$.
3. $Z(A)$ is a graded subalgebra of A and $B(A)$ is a homogeneous ideal of $Z(A)$.
4. If A is local with maximal ideal M then $s(M) \subset M$ if and only if $H(A) \neq 0$.

△

A differential ideal of a dg-algebra (A, s) is a homogeneous ideal I of A such that $s(I) \subset I$; there exists an obvious bijection between differential ideals and kernels of morphisms of dg-algebras.

On a polynomial algebra $\mathbb{K}[\{x_i\}]$ a differential s is uniquely determined by the values $s(x_i)$.

Example 3.5. Let t, dt be inderminates of degrees $\overline{t} = 0$, $\overline{dt} = 1$; on the polynomial algebra $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ there exists an obvious differential d such that $d(t) = dt$, $d(dt) = 0$. Since $\mathbb{K}$ has characteristic 0, we have $H(\mathbb{K}[t, dt]) = \mathbb{K}$. More generally if (A, s) is a dg-algebra then $A[t, dt]$ is a dg-algebra with differential $s(a \otimes p(t)) = s(a) \otimes p(t) + (-1)^{\overline{a}} a \otimes p'(t)dt$, $s(a \otimes q(t)dt) = s(a) \otimes q(t)dt$.

Definition 3.6. *A morphism of dg-algebras $B \to A$ is a* quasiisomorphism *if the induced morphism $H(B) \to H(A)$ is an isomorphism.*

Given a morphism of dg-algebras $B \to A$ the space $\mathrm{Der}^n_B(A, A)$ of B-derivations of degree n is by definition

$$\mathrm{Der}^n_B(A, A) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}.$$

We also consider the graded vector space

$$\mathrm{Der}^*_B(A, A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n_B(A, A) \in \mathbf{G}.$$

There exists a structure of differential graded Lie algebra on $\mathrm{Der}^*_B(A, A)$ with differential

$$d\colon \mathrm{Der}^n_B(A, A) \to \mathrm{Der}^{n+1}_B(A, A), \qquad d\phi = d_A\phi - (-1)^n \phi d_A$$

and bracket

$$[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf.$$

Exercise 3.7. Verify that $d[f,g] = [df,g] + (-1)^{\overline{f}}[f,dg]$. $\triangle$

Exercise 3.8. Let A be graded algebra: if every $a \neq 0$ is invertible then $A = A_0$ is a field. $\triangle$

Exercise 3.9. Let A be a graded algebra and let $I \subset A$ be a left ideal. Then the following conditions are equivalent:

1. I is the unique left maximal ideal.
2. A_0 is a local ring with maximal ideal M and $I = M \oplus_{i\neq 0} A_i$.

$\triangle$

4 The DG-resolvent

Let $X \subset \mathbb{A}^n$ be a closed subscheme, $R_0 = \mathbb{K}[x_1, \ldots, x_n]$ the ring of regular functions on $\mathbb{A}^n$, $I_0 \subset R_0$ the ideal of X and $\mathcal{O}_X = R_0/I$ the function ring of X.

Our aim is to construct a dg-algebra (R, d) and a quasiisomorphism $R \to \mathcal{O}_X$ such that $R = R_0[y_1, y_2, \ldots]$ is a countably generated graded polynomial R_0-algebra, every indeterminate y_i has negative degree and, if $R = \oplus_{i\leq 0} R_i$, then R_i is a finitely generated free R_0 module.

Choosing a set of generators $f_1, \ldots, f_{s_1}$ of the ideal I_0 we first consider the graded-commutative polynomial dg-algebra

$$R(1) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_{s_1}] = R_0[y_1, \ldots, y_{s_1}], \qquad \overline{x_i} = 0, \quad \overline{y_i} = -1$$

with differential d defined by $dx_i = 0$, $dy_j = f_j$. Note that $(R(1), d)$, considered as a complex of R_0 modules, is the Koszul complex of the sequence $f_1, \ldots, f_{s_1}$. By construction the complex of R_0-modules

$$\ldots \longrightarrow R(1)_{-2} \xrightarrow{d} R(1)_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is exact in R_0 and $\mathcal{O}_X$. If $(R(-1), d) \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras (e.g. if X is a complete intersection) the construction is done. Otherwise let $f_{s_1+1}, \ldots, f_{s_2} \in \ker d \cap R(1)_{-1}$ be a set of generators of the R_0 module $(\ker d \cap R(1)_{-1})/dR(1)_{-2}$ and define

$$R(2) = R(1)[y_{s_1+1}, \ldots, y_{s_2}], \qquad \overline{y_j} = -2, \quad dy_j = f_j, \quad j = s_1+1, \ldots, s_2.$$

Repeating in a recursive way the above argument (step by step killing cycles) we get a chain of polynomial dg-algebras

$$R_0 = R(0) \subset R(1) \subset \ldots \subset R(i) \subset \ldots$$

such that $(R(i), d) \to \mathcal{O}_X$ is a quasiisomorphism in degree $> -i$. Setting

$$R = \cup R(i) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots] = \bigoplus_{i\leq 0} R_i,$$

the projection $\pi\colon R \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras; in particular

$$\ldots \xrightarrow{d} R_{-i} \xrightarrow{d} \ldots \xrightarrow{d} R_{-2} \xrightarrow{d} R_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is a free resolution of the R_0 module $\mathcal{O}_X$.

We denote by:

1. $Z_i = \ker d \cap R_i$.

2. $\mathcal{L} = \mathrm{Der}^*_{\mathbb{K}}(R, R)$.

3. $\mathcal{H} = \mathrm{Der}^*_{R_0}(R, R) = \{g \in \mathcal{L} \mid g(R_0) = 0\}$.

It is clear that, since $gR_i \subset R_{i+j}$ for every $g \in \mathcal{L}^j$, $\mathcal{L}^i = \mathcal{H}^i$ for every $i > 0$ and then the DGLAs $\mathcal{L}$, $\mathcal{H}$ have the same Maurer-Cartan functor $MC_{\mathcal{H}} = MC_{\mathcal{L}}$. Moreover R is a free graded algebra and then $\mathcal{L}^j$ is in bijection with the maps of "degree j" $\{x_i, y_h\} \to R$.

Consider a fixed $\eta \in MC_{\mathcal{H}}(A)$. Recalling the definition of $MC_{\mathcal{H}}$ we have that $\eta = \sum \eta_i \otimes a_i \in \mathrm{Der}^1_{R_0}(R, R) \otimes m_A$ and the A-derivation

$$d + \eta \colon R \otimes A \to R \otimes A, \qquad (d + \eta)(x \otimes a) = dx \otimes a + \sum \eta_i(x) \otimes a_i a$$

is a differential. Denoting by $\mathcal{O}_A$ the cokernel of $d + \eta \colon R_{-1} \otimes A \to R_0 \otimes A$ we have by Corollary 2.5 that $(R \otimes A, d + \eta) \to \mathcal{O}_A$ is a quasiisomorphism, $\mathcal{O}_A$ is flat and $\mathcal{O}_A \otimes \mathbb{K} = \mathcal{O}_X$. Therefore we have natural transformations of functors

$$MC_{\mathcal{H}} = MC_{\mathcal{L}} \to \mathrm{Hilb}_X \to \mathrm{Def}_X .$$

Lemma 4.1. *The above morphisms of functors are surjective.*

Proof. Let $\mathcal{O}_A$ be a flat A-algebra such that $\mathcal{O}_A \otimes_A \mathbb{K} = \mathcal{O}_X$; since R_0 is a free $\mathbb{K}$-algebra, the projection $R_0 \xrightarrow{\pi} \mathcal{O}_X$ can be extended to a morphism of flat A-algebras $R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A$. According to Corollary 2.3 π_A is surjective; this proves that $\mathrm{Hilb}_X(A) \to \mathrm{Def}_X(A)$ is surjective (in effect it is possible to prove directly that $\mathrm{Hilb}_X \to \mathrm{Def}_X$ is smooth, cf. [1]). An element of $\mathrm{Hilb}_X(A)$ gives an exact sequence of flat A-modules

$$R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A \longrightarrow 0.$$

Denoting by $I_{0,A} \subset R_0 \otimes A$ the kernel of π_A we have that $I_{0,A}$ is A-flat and the projection $I_{0,A} \to I_0$ is surjective. We can therefore extend the restriction to $R(1)$ of the differential d to a differential d_A on $R(1) \otimes A$ by setting $d_A(y_j) \in I_{0,A}$ a lifting of $d(y_j)$, $j = 1, \dots, s_1$. Again by local flatness criterion the kernel $Z_{-1,A}$ of $R_{-1} \otimes A = R(1)_{-1} \otimes A \xrightarrow{d_A} R_0 \otimes A$ is flat and surjects onto Z_{-1}. The same argument as above, with $I_{0,A}$ replaced by $Z_{-1,A}$ shows that d can be extended to a differential d_A on $R(2)$ and then by induction to a differential d_A on $R \otimes A$ such that $(R \otimes A, d_A) \to \mathcal{O}_A$ is a quasiisomorphism. If $a_1, \dots, a_r$ is a $\mathbb{K}$-basis of the maximal ideal of A we can write $d_A(x \otimes 1) = dx \otimes 1 + \sum \eta_i(x) \otimes a_i$ and then $\eta = \sum \eta_i \otimes a_i \in MC_{\mathcal{H}}(A)$. $\square$

If $\xi \in \mathrm{Der}^0_{R_0}(R, R) \otimes m_A$, $A \in \mathbf{Art}$, then $e^{\xi} \colon R \otimes A \to R \otimes A$ is an automorphism inducing the identity on R and $R_0 \otimes A$. Therefore the morphism $MC_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$ factors through $\mathrm{Def}_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$. Similarly the morphism $MC_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$ factors through $\mathrm{Def}_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$.

Theorem 4.2. *The natural transformations*

$$\mathrm{Def}_{\mathcal{H}} \to \mathrm{Hilb}_X, \qquad \mathrm{Def}_{\mathcal{L}} \to \mathrm{Def}_X$$

are isomorphisms of functors.

Proof. We have already proved the surjectivity. The injectivity follows from the following lifting argument. Given $d_A, d'_A \colon R \otimes A \to R \otimes A$ two liftings of the differential d and $f_0 \colon R_0 \otimes A \to R_0 \otimes A$ a lifting of the identity on R_0 such that $f_0 d_A(R_{-1} \otimes A) \subset d'_A(R_{-1} \otimes A)$ there exists an isomorphism $f \colon (R \otimes A, d_A) \to (R \otimes A, d'_A)$ extending f_0 and the identity on R. This is essentially trivial because $R \otimes A$ is a free $R_0 \otimes A$ graded algebra and $(R \otimes A, d'_A)$ is exact in degree < 0. Thinking f as an automorphism of the graded algebra $R \otimes A$ we have, since $\mathbb{K}$ has characteristic 0, that $f = e^{\xi}$ for some $\xi \in \mathcal{L}^0$ and $\xi \in \mathcal{H}^0$ if and only if $f_0 = Id$. By the definition of gauge action $d'_A - d = exp(\xi)(d_A - d)$; the injectivity follows. $\square$

Proposition 4.3. *If $I \subset R_0$ is the ideal of $X \subset \mathbb{A}^n$ then:*

1. $H^i(\mathcal{H}) = H^i(\mathcal{L}) = 0$ *for every* $i < 0$.
2. $H^0(\mathcal{H}) = 0$, $H^0(\mathcal{L}) = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$.
3. $H^1(\mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$ *and* $H^1(\mathcal{L})$ *is the cokernel of the natural morphism*

$$\mathrm{Der}_{\mathbb{K}}(R_0, \mathcal{O}_X) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X).$$

Proof. There exists a short exact sequence of complexes

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{L} \longrightarrow \mathrm{Der}^*_{\mathbb{K}}(R_0, R) \longrightarrow 0.$$

Since R_0 is free and R is exact in degree < 0 we have:

$$H^i(\mathrm{Der}^*_{\mathbb{K}}(R_0, R)) = \begin{cases} 0 & i \neq 0, \\ \mathrm{Der}_{\mathbb{K}}(R_0, \mathcal{O}_X) & i = 0. \end{cases}$$

Moreover $\mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$ is the kernel of α and then it is sufficient to compute $H^i(\mathcal{H})$ for $i \leq 1$.

Every $g \in Z^i(\mathcal{H})$, $i \leq 0$, is a R_0-derivation $g\colon R \to R$ such that $g(R) \subset \oplus_{i<0} R_i$ and $gd = \pm dg$. As above R is free and exact in degree < 0, a standard argument shows that g is a coboundary. If $g \in Z^1(\mathcal{H})$ then $g(R_{-1}) \subset R_0$ and, since $gd + dg = 0$, g induces a morphism

$$\overline{g}\colon \frac{R_{-1}}{dR_{-2}} = I \longrightarrow \frac{R_0}{dR_{-1}} = \mathcal{O}_X.$$

The easy verification that $Z^1(\mathcal{H}) \to \mathrm{Hom}_{R_0}(I, \mathcal{O}_X)$ induces an isomorphism $H^1(\mathcal{H}) \to \mathrm{Hom}_{R_0}(I, \mathcal{O}_X)$ is left to the reader. $\square$

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Obstruction Calculus in Deformation Theory

Alberto Canonaco

Introduction and notation

The study of functors of Artin rings is a classical subject of deformation theory (see [6]). On the other hand, although it was well known that for functors coming from deformation problems a (vector) space of obstructions nearly always exists, even a precise definition of obstruction space has been unclear for a long time. This gap has been recently filled by B. Fantechi and M. Manetti in [1]. In this paper, after reviewing the basic facts about functors of Artin rings, we present the main results of their work. In particular, for every morphism of functors of Artin rings, an obstruction theory is defined, which consists of an obstruction space (which is in general only a pointed set) together with some obstruction maps. It can be proved that, under mild hypotheses, there exists an obstruction theory which is complete (every obstruction map vanishes on an element if and only if that element can be lifted) and linear (the obstruction space is a vector space). In the last part of the paper we determine a complete linear obstruction theory in some examples of deformation problems.

Set and $\mathbf{Set}_*$ will be the categories of sets and of pointed sets, respectively, and **Group** the category of groups. By abuse of notation we will always denote by $*$ the chosen point in a set $S \in \mathbf{Set}_*$, unless S is a group, in which case the chosen point will be the identity. The kernel of a morphism in $\mathbf{Set}_*$ is the inverse image of $*$. A sequence $S \xrightarrow{f} T \xrightarrow{g} U$ in $\mathbf{Set}_*$ is exact if $\ker(g) = \operatorname{im}(f)$.

If $\mathbf{C}$ is a category, then $\mathbf{C}^\circ$ is the opposed category. If $\nu : F \to G$ is a natural transformation of functors (say from $\mathbf{C}$ to $\mathbf{D}$) and A is an object of $\mathbf{C}$, we will usually write ν instead of $\nu(A)$.

Unless otherwise stated $\mathbb{K}$ is an arbitrary fixed field. **Fvs** will be the category of finite dimensional $\mathbb{K}$–vector spaces. If V, W are vector spaces, we will write $V \otimes W$ instead of $V \otimes_{\mathbb{K}} W$; $V^\vee$ will be the $\mathbb{K}$–dual of V.

If A is a local ring, $\mathbf{m}_A$ will denote its maximal ideal, $\pi_A : A \to A/\mathbf{m}_A$ the natural projection and $\widehat{A}$ the completion of A (with respect to the $\mathbf{m}_A$–adic topology). $\mathbb{K}[\epsilon]$ will be the artinian ring $\mathbb{K}[x]/(x^2)$.

If $\{U^i \mid i \in I\}$ is a covering of a topological space, $U^{i_1,\dots,i_n} := U^{i_1} \cap \cdots \cap U^{i_n}$.

If $f : X \to Y$ is a morphism of schemes, $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ will be the associated morphism of sheaves of rings on X.

1 Functors of Artin rings

$\mathbf{A}$ is the category of noetherian local $\mathbb{K}$–algebras with residue field $\mathbb{K}$ (with local homomorphisms of $\mathbb{K}$–algebras as morphisms). **Art** (respectively, $\widehat{\mathbf{Art}}$) is the full subcategory of $\mathbf{A}$ consisting of those rings which are artinian (respectively, complete). Notice that $\mathbf{Art} \subset \widehat{\mathbf{Art}}$ and that, given $A \in \mathbf{A}$, $A \in \mathbf{Art}$ if and only if $\mathbf{m}_A$ is a nilpotent ideal, if and only if $\dim_{\mathbb{K}}(A) < \infty$. Observe moreover that if $\pi_i : A_i \to A$ are morphisms in $\mathbf{A}$ for $i = 1, 2$, then

$$A_1 \times_A A_2 := \{(a_1, a_2) \in A_1 \times A_2 \mid \pi_1(a_1) = \pi_2(a_2)\}$$

represents the fibre product of π_1 and π_2 in $\mathbf{A}$, and that $A_1 \times_A A_2$ is an object of $\mathbf{Art}$ (respectively, $\widehat{\mathbf{Art}}$) if π_1 and π_2 are morphisms of $\mathbf{Art}$ (respectively, $\widehat{\mathbf{Art}}$).

Definition 1.1. *A* functor of Artin rings *(or simply a functor, if no confusion is possible) is a functor $F : \mathbf{Art} \to \mathbf{Set}$ such that $F(\mathbb{K}) = \{*\}$. The collection of all such functors with natural transformations as morphisms forms a category, which will be called* $\mathbf{Fun}$.

Remark 1.2. *A functor of Artin rings F can be considered as a functor $F : \mathbf{Art} \to \mathbf{Set}_*$ (denoting, for $A \in \mathbf{Art}$, by $i_A : \mathbb{K} \to A$ the structure morphism, the chosen point in $F(A)$ is $F(i_A)(*)$).*

Remark 1.3. *A functor $F : \mathbf{Art} \to \mathbf{Set}$ extends naturally to a functor which will be denoted by $\widehat{F} : \widehat{\mathbf{Art}} \to \mathbf{Set}$ (see [5]). Similarly, a natural transformation $\nu : F \to G$ extends to a natural transformation $\widehat{\nu} : \widehat{F} \to \widehat{G}$.*

Example 1.4. *If $R \in \mathbf{A}$, $h_R := \mathrm{Hom}_{\mathbf{A}}(R, -)$ is a functor of Artin rings. Clearly $h_R \cong h_{\widehat{R}}$. In particular, $h_{\mathbb{K}}$ is the* trivial *functor (i.e., $h_{\mathbb{K}}(A) = \{*\}\ \forall\, A \in \mathbf{Art}$) and will be denoted by $\overline{*}$.*

Definition 1.5. *A functor of Artin rings is* prorepresentable *if it is isomorphic to h_R for some $R \in \widehat{\mathbf{Art}}$.*

Obviously the map $R \mapsto h_R$ extends to a functor $h : \mathbf{A}^\circ \to \mathbf{Fun}$.

Lemma 1.6. *(See [5]) $\forall\, R \in \widehat{\mathbf{Art}}$ and $\forall\, F \in \mathbf{Fun}$ the natural map*

$$\mathrm{Hom}_{\mathbf{Fun}}(h_R, F) \longrightarrow \widehat{F}(R), \qquad \nu \mapsto \widehat{\nu}(\mathrm{id}_R)$$

is a bijection. In particular, if $F = h_S$ for some $S \in \widehat{\mathbf{Art}}$, then $\mathrm{Hom}_{\mathbf{Fun}}(h_R, h_S) \cong \widehat{h_S}(R) = \mathrm{Hom}_{\widehat{\mathbf{Art}}}(S, R)$, whence h embeds $\widehat{\mathbf{Art}}^\circ$ as a full subcategory of $\mathbf{Fun}$.

Definition 1.7. *A* small extension *e in $\mathbf{Art}$ (respectively, $\widehat{\mathbf{Art}}$) is a short exact sequence*

$$e:\ 0 \longrightarrow J \longrightarrow B \xrightarrow{f} A \longrightarrow 0$$

where f is a morphism in $\mathbf{Art}$ (respectively, $\widehat{\mathbf{Art}}$) and J is an ideal of B such that $\mathbf{m}_B J = (0)$ (hence J is a finite dimensional $\mathbb{K}$–vector space). Given such a small extension, we will often denote J by $K(e)$, B by $S(e)$ and A by $T(e)$.

A small extension e is called principal *if $K(e) \cong \mathbb{K}$.*

A morphism of small extensions $\alpha : e_1 \to e_2$ is a commutative diagram

$$\begin{tikzcd}
e_1 :\ 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "K(\alpha)"'] & B_1 \arrow[r] \arrow[d, "S(\alpha)"'] & A_1 \arrow[r] \arrow[d, "T(\alpha)"'] & 0 \\
e_2 :\ 0 \arrow[r] & J_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

where $S(\alpha), T(\alpha)$ are morphisms in $\widehat{\mathbf{Art}}$.

We will denote by $\mathbf{Smex}$ (respectively, $\widehat{\mathbf{Smex}}$) the category of small extensions in $\mathbf{Art}$ (respectively, $\widehat{\mathbf{Art}}$).

In the following we will often say that a morphism $f : B \to A$ in $\widehat{\mathbf{Art}}$ is a small extension, meaning that $0 \to \ker(f) \to B \xrightarrow{f} A \to 0$ is a small extension.

Remark 1.8. *Since a morphism f in $\widehat{\mathbf{Art}}$ is a principal small extension if and only if it is surjective and $\dim_{\mathbb{K}}(\ker(f)) = 1$, it is easy to prove that every surjective morphism in* **Art** *can be expressed as the composition of a finite number of principal small extensions.*

Given $A \in \widehat{\mathbf{Art}}$ and $J \in \mathbf{Fvs}$, we define the set of small extensions of A by J in the following way:

$$\mathrm{Ex}(A, J) := \{e \in \widehat{\mathbf{Smex}} \,|\, K(e) = J, T(e) = A\}/\sim$$

where $e_1 \sim e_2$ if and only if there exists a morphism (necessarily an isomorphism) $\alpha : e_1 \to e_2$ in $\widehat{\mathbf{Smex}}$ such that $K(\alpha) = \mathrm{id}_J$, $T(\alpha) = \mathrm{id}_A$.

Let $A \oplus J$ be the element of $\widehat{\mathbf{Art}}$ with multiplication defined by

$$(a, v) \cdot (a', v') := (aa', \pi_A(a)v' + \pi_A(a')v)$$

and let $p : A \oplus J \to A$ be the natural projection. Then a small extension of A by J is called *trivial* if it represents the same element as p in $\mathrm{Ex}(A, J)$. It is straightforward to prove that a small extension $f : B \to A$ is trivial if and only if there exists $s : A \to B$ in $\widehat{\mathbf{Art}}$ such that $f \circ s = \mathrm{id}_A$.

Given e in $\widehat{\mathbf{Smex}}$ by abuse of notation we will denote by e (and call small extension) also its equivalence class in $\mathrm{Ex}(T(e), K(e))$. $\overline{\epsilon}$ will be the (trivial) principal small extension $0 \to \mathbb{K} \xrightarrow{\cdot\epsilon} \mathbb{K}[\epsilon] \to \mathbb{K} \to 0$.

Lemma 1.9. $\forall\, A \in \widehat{\mathbf{Art}}$ *and* $\forall\, J \in \mathbf{Fvs}$ *the set* $\mathrm{Ex}(A, J)$ *has a natural structure of* $\mathbb{K}$*–vector space (of finite dimension: see 2.15) such that* 0 *is the trivial extension.*

Morphisms $f : A' \to A$ *in* $\widehat{\mathbf{Art}}$ *and* $\varphi : J \to J'$ *in* **Fvs** *induce linear maps*

$$f^* : \mathrm{Ex}(A, J) \longrightarrow \mathrm{Ex}(A', J) \qquad \text{and} \qquad \varphi_* : \mathrm{Ex}(A, J) \longrightarrow \mathrm{Ex}(A, J').$$

This determines a functor $\mathrm{Ex}(-,-) : \widehat{\mathbf{Art}}^{\circ} \times \mathbf{Fvs} \to \mathbf{Fvs}$,[1] *which is additive in the second argument. Moreover, given a morphism* $\alpha : e_1 \to e_2$ *in* $\widehat{\mathbf{Smex}}$, $K(\alpha)_*(e_1) = T(\alpha)^*(e_2) \in \mathrm{Ex}(T(e_1), K(e_2))$.

Proof. Given $e_1, e_2 \in \mathrm{Ex}(A, J)$ represented by

$$e_i : \ 0 \longrightarrow J \xrightarrow{j_i} B_i \xrightarrow{p_i} A \longrightarrow 0\,,$$

the sum $e_1 + e_2$ is represented by

$$e_1 + e_2 : \ 0 \longrightarrow J \xrightarrow{j} B \xrightarrow{p} A \longrightarrow 0\,,$$

where $B := (B_1 \times_A B_2)/\{(j_1(x), -j_2(x)) \,|\, x \in J\}$, $j(x) := [(j_1(x), 0)] = [(0, j_2(x))]$ and $p([(b_1, b_2)]) := p_1(b_1) = p_2(b_2)$. If $e \in \mathrm{Ex}(A, J)$ is represented by

$$e : \ 0 \longrightarrow J \xrightarrow{j} B \xrightarrow{p} A \longrightarrow 0$$

and $\lambda \in \mathbb{K} \setminus \{0\}$, then $\lambda \cdot e$ is represented by

$$\lambda \cdot e : \ 0 \longrightarrow J \xrightarrow{\lambda^{-1}j} B \xrightarrow{p} A \longrightarrow 0$$

(of course, $0 \cdot e$ is defined to be the trivial extension). It is easy to prove that these operations are well defined and endow $\mathrm{Ex}(A, J)$ with a structure of $\mathbb{K}$–vector space.

[1]This implies, in particular, that $f^* \circ \varphi_* = \varphi_* \circ f^* : \mathrm{Ex}(A, J) \to \mathrm{Ex}(A', J')$.

Given $e: \ 0 \to J \xrightarrow{j} B \xrightarrow{p} A \to 0$ in $\mathrm{Ex}(A, J)$, f^*e is represented by

$$f^*e: \ 0 \longrightarrow J \xrightarrow{j'} B \times_A A' \xrightarrow{p'} A' \longrightarrow 0 \,,$$

where $j'(x) := (j(x), 0)$ and $p'((b, a')) := a'$. $\varphi_* e$ is represented by

$$\varphi_* e: \ 0 \longrightarrow J' \xrightarrow{j''} B \coprod_J J' \xrightarrow{p''} A \longrightarrow 0 \,,$$

where $B \coprod_J J' := (B \oplus J')/\{(j(x), -\varphi(x)) \,|\, x \in J\}$, $j''(x') := [(0, x')]$ and $p''([(b, x')]) := p(b)$. Again, we leave as exercise to check that f^* and φ_* are well defined and have the required properties. □

Given $F \in \mathbf{Fun}$ and $\pi_i : A_i \to A$ in $\mathbf{Art}$ for $i = 1, 2$, let

$$\eta : F(A_1 \times_A A_2) \longrightarrow F(A_1) \times_{F(A)} F(A_2) \tag{1}$$

be the natural map.

Definition 1.10. *F is called* left–exact *if η is always bijective.*
F is called homogeneous *if η is bijective whenever π_2 is surjective.*

Remark 1.11. *It follows from the universal property of fibre product that a prorepresentable functor is left–exact.*

Now we introduce some conditions, weaker than prorepresentability, that a functor may satisfy. Conditions $(H1)$ to $(H4)$ are called *Schlessinger conditions* (see [6]), whereas conditions $(H2')$ and (L) (introduced in [1]) will be useful when dealing with obstruction theories.

$(H1)$ η is surjective if π_2 is a principal small extension.

$(H2)$ η is bijective if $A = \mathbb{K}$ and $A_2 = \mathbb{K}[\epsilon]$.

$(H2')$ η is bijective if $A = \mathbb{K}$.

$(H3)$ F satisfies $(H1)$, $(H2)$ and $\dim_{\mathbb{K}}(t_F)$ is finite (see 1.15).

$(H4)$ η is bijective if $\pi_1 = \pi_2$ is a principal small extension.

(L) For every principal small extension $0 \to J \to B \to A \to 0$, let

$$C := B \times_{\mathbb{K}} B/\{(b, b) \,|\, b \in J\}$$

and let p, q be the natural maps

$$F(C) \xrightarrow{p} F(A \times_{\mathbb{K}} A) \xrightarrow{q} F(A) \times F(A).$$

Then $q^{-1}(\{(x, x) \,|\, x \in F(A)\}) \subset p(F(C))$.

Lemma 1.12. *If $F \in \mathbf{Fun}$ satisfies $(H2')$, then it satisfies (L), too.*

Proof. If $y \in F(A \times_{\mathbb{K}} A)$ is such that $q(y) = (x, x)$ for some $x \in F(A)$, then by $(H2')$ $y = F(\delta)(x)$, where $\delta : A \to A \times_{\mathbb{K}} A$ is the diagonal morphism. Then the statement follows from the fact that δ factors through C. □

Remark 1.13. *It follows easily from 1.8 that if F satisfies $(H1)$ then η is surjective whenever π_2 is surjective.*

If F satisfies $(H2)$ then η is bijective whenever $A = \mathbb{K}$ and π_2 is a small extension (this is a consequence of the fact that every small extension e with $T(e) = \mathbb{K}$ is trivial).

It can be proved that if F satisfies $(H2)$ and $(H4)$, then η is bijective when $\pi_1 = \pi_2$ is a small extension.

We will denote by **Gdt** ("good deformation theory") the full subcategory of **Fun** whose objects satisfy $(H1)$ and $(H2)$, and by **Gdot** ("good deformation and obstruction theory") the full subcategory of **Gdt** whose objects satisfy (L). As we will see, in many concrete cases one can verify that a functor is in **Gdot** by proving that it satisfies $(H1)$ and $(H2')$.

Definition 1.14. *The* tangent space *to* $F \in$ **Fun** *is the pointed set* $t_F := F(\mathbb{K}[\epsilon])$. *Given* $\nu : F \to G$ *in* **Fun**, *the* relative tangent space *to* ν *is* $t_\nu := \ker(\nu : t_F \to t_G)$. *Notice that if* $G = \bar{*}$, *then* $t_\nu = t_F$.

Proposition 1.15. *Assume that* $F \in$ **Fun** *satisfies* $(H2)$.

1. t_F *has a natural structure of vector space (with* $0 = *$*) such that if* $\nu : F \to G$ *is a morphism in* **Fun** *with* G *also satisfying* $(H2)$, *then* $\nu : t_F \to t_G$ *is linear. Moreover,* $\forall\, V \in$ **Fvs** *the pointed set* $F(\mathbb{K} \oplus V)$ *is canonically in bijection with* $t_F \otimes V$ *(and so it also has a natural structure of vector space).*

2. *For every* $0 \to J \to B \xrightarrow{f} A \to 0$ *in* **Smex** *there is a natural action of* $t_F \otimes J$ *on each fibre of* $F(f) : F(B) \to F(A)$. *These actions are compatible with morphisms of small extensions and with morphisms of functors.*

3. *If* F *satisfies* $(H1)$ *then the actions are transitive.*

4. *The actions are transitive and free if and only if* F *satisfies* $(H4)$.

Proof. 1. Given $v_1, v_2 \in t_F$, consider v_i as an element of $F(\mathbb{K}[\epsilon_i])$ and let $v := \eta^{-1}((v_1, v_2))$, where $\eta : F(\mathbb{K}[\epsilon_1, \epsilon_2]) \to F(\mathbb{K}[\epsilon_1]) \times F(\mathbb{K}[\epsilon_2])$ is an isomorphism by $(H2)$. Given $\lambda_1, \lambda_2 \in \mathbb{K}$, let $\phi : \mathbb{K}[\epsilon_1, \epsilon_2] \to \mathbb{K}[\epsilon]$ be the morphism of **Art** defined by $\phi(\epsilon_i) := \lambda_i \epsilon$. Then it is easy to see that $\lambda_1 v_1 + \lambda_2 v_2 := F(\phi)(v)$ gives t_F a structure of vector space with the required properties. The last statement follows by induction on $\dim_{\mathbb{K}}(V)$, since (always by $(H2)$) $F(\mathbb{K} \oplus V_1 \oplus V_2) \xrightarrow{\sim} F(\mathbb{K} \oplus V_1) \times F(\mathbb{K} \oplus V_2)$.

2. Consider the morphism of small extensions

$$\begin{tikzcd}
0: & 0 \arrow[r] & J \arrow[r] \arrow[d, "\mathrm{id}"'] & B \oplus J \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[r] \arrow[d, "f"'] & 0 \\
e: & 0 \arrow[r] & J \arrow[r] & B \arrow[r, "f"] & A \arrow[r] & 0
\end{tikzcd}$$

where p is the projection and $q((b, c)) := b + c$. Since $B \oplus J \cong B \times_{\mathbb{K}} (\mathbb{K} \oplus J)$, by $(H2)$ and by 1 there is a natural isomorphism $F(B \oplus J) \xrightarrow{\sim} F(B) \times (t_F \otimes J)$, whence we obtain a commutative diagram

$$\begin{tikzcd}
F(B) \times (t_F \otimes J) \arrow[r, "\pi"] \arrow[d, "\tau"'] & F(B) \arrow[d, "F(f)"] \\
F(B) \arrow[r, "F(f)"'] & F(A)
\end{tikzcd}$$

(where π is the projection). It is not difficult to prove that τ defines an action of $t_F \otimes J$ on $F(B)$, which clearly restricts to an action on each fibre of $F(f)$. It is straightforward to check that these actions are compatible with morphisms of small extensions and with morphisms of functors.

3. By the last statement of 1.9, $0 = f^* e \in \mathrm{Ex}(B, J)$, so that $B \oplus J \cong B \times_A B$. Since $F(B \times_A B) \to F(B) \times_{F(A)} F(B)$ is surjective by $(H1)$ (taking into account 1.13), the claim follows.

4. The proof is completely analogous to that of 3.

□

Example 1.16. *It is easy to see that if $F = h_R$ for some $R \in \widehat{\mathbf{Art}}$, then $t_F \cong t_R := (\frac{\mathbf{m}_R}{\mathbf{m}_R^2})^\vee$. More generally, if $\nu : h_R \to h_S$ is induced by $f : S \to R$ in $\widehat{\mathbf{Art}}$, then $t_\nu \cong t_f := (\frac{\mathbf{m}_R}{\mathbf{m}_R^2 + \mathbf{m}_S R})^\vee$.*

Corollary 1.17. *If $F \in \mathbf{Gdt}$, then $F = \overline{*}$ if and only if $t_F = (0)$.*

Proof. Let $F \in \mathbf{Gdt}$ be such that $t_F = (0)$: $\forall\, A \in \mathbf{Art}$ we prove that $F(A) = \{*\}$ by induction on $\dim_{\mathbb{K}}(A)$. The case $\dim_{\mathbb{K}}(A) = 1$ being obvious, we can assume that there is a principal small extension $\pi : A \to B$ and that $F(B) = \{*\}$. Then $F(A) = \{*\}$ because $t_F = (0)$ acts transitively on the unique fibre $F(A)$ of $F(\pi)$. □

Corollary 1.18. *$F \in \mathbf{Fun}$ is homogeneous if and only if it satisfies $(H1)$, $(H2)$, $(H4)$.*

Proof. The other implication being trivial, we assume that F satisfies $(H1)$, $(H2)$, $(H4)$, and we have to prove that η as in (1) is bijective if π_2 is surjective. As usual, using 1.8 we can assume that π_2 is a principal small extension, and, since $(H1)$ holds, it remains to show that η is injective. Setting $A' := A_1 \times_A A_2$, consider the morphism in **Smex**

$$\begin{tikzcd}
\pi_1^* e : & 0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[d, "\mathrm{id}"] & A' \arrow[r, "p_1"] \arrow[d, "p_2"] & A_1 \arrow[r] \arrow[d, "\pi_1"] & 0 \\
e : & 0 \arrow[r] & \mathbb{K} \arrow[r] & A_2 \arrow[r, "\pi_2"] & A \arrow[r] & 0
\end{tikzcd}$$

and let $y, y' \in F(A')$ be such that $\eta(y) = \eta(y') := (x_1, x_2)$. Since t_F acts transitively and freely on the fibres of $F(p_1)$, $\exists!\, v \in t_F$ such that $y' = v \cdot y$ (here $\cdot$ denotes the action). As the actions are compatible with morphisms of small extensions, it follows that $x_2 = v \cdot x_2$. On the other hand, the action on the fibres of $F(\pi_2)$ is also free, whence $v = 0$ and $y = y'$. □

2 Obstruction theories

If $\nu : F \to G$ is a morphism in **Fun** and $f : B \to A$ is a morphism in **Art**, we define $\widetilde{\nu}(f)$ to be the pointed set $F(A) \times_{G(A)} G(B)$ (notice that, if $G = \overline{*}$, then $\widetilde{\nu}(f) = F(A)$). In particular, $\widetilde{\nu}$ is defined on small extensions, and it clearly determines a functor $\widetilde{\nu} : \mathbf{Smex} \to \mathbf{Set}_*$.

Definition 2.1. *A morphism $\nu : F \to G$ in* **Fun** *is* smooth *if the natural map $F(B) \to \widetilde{\nu}(f)$ is surjective for every surjective morphism $f : B \to A$ in* **Art** *(it follows from 1.8 that it is enough to check this condition when f is a principal small extension).*

A functor $F \in$ **Fun** *is* smooth *if the morphism $F \to \overline{*}$ is smooth (which is true if and only if F preserves surjective morphisms).*

Remark 2.2. *If $\nu : F \to G$ is a smooth morphism, then $\nu(B) : F(B) \to G(B)$ is surjective $\forall\, B \in$* **Art** *(to see this, it is enough to take $A = \mathbb{K}$ in the above definition).*

Definition 2.3. *A* hull *for $F \in$* **Fun** *is a morphism $\nu : H \to F$ in* **Fun** *such that H is prorepresentable and ν is smooth and bijective on tangent spaces.*

It is not difficult to prove that if a functor admits a hull, then it is unique up to (non canonical) isomorphism.

Definition 2.4. *Let $\nu : F \to G$ be a morphism in* **Fun**. *A* relative obstruction theory (V, v_e) *for ν consists of an* obstruction space $V \in \mathbf{Set}_*$ *and, $\forall\, e \in$* **Smex**, *of an* obstruction map $v_e : \widetilde{\nu}(e) \times K(e)^\vee \to V$, *subject to the following conditions:*

1. $v_{\overline{\epsilon}}(*, \mathrm{id}) = *$;

2. (base change) $\forall \alpha : e_1 \to e_2$ *in* **Smex** *the diagram*

$$\begin{tikzcd}
\widetilde{\nu}(e_1) \times K(e_2)^{\vee} \arrow[r, "\widetilde{\nu}(\alpha)\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times K(\alpha)^{\vee}"'] & \widetilde{\nu}(e_2) \times K(e_2)^{\vee} \arrow[d, "v_{e_2}"] \\
\widetilde{\nu}(e_1) \times K(e_1)^{\vee} \arrow[r, "v_{e_1}"'] & V
\end{tikzcd}$$

commutes.

A morphism $(V, v_e) \to (V', v'_e)$ *of relative obstruction theories for* ν *is a map* $\alpha : V \to V'$ *such that* $v'_e = \alpha \circ v_e \ \forall\, e \in$ **Smex**.

An obstruction theory *for* F *is just a relative obstruction theory for the morphism* $F \to \overline{*}$.

Examples 2.5. 1. $V = \{*\}$ *is a relative obstruction theory (called* trivial*) for* ν.

2. *If* (V, v_e) *is a relative obstruction theory for* ν *and* $f : V \to W$ *is a morphism in* $\mathbf{Set}_*$, *then* $(W, f \circ v_e)$ *is a relative obstruction theory for* ν, *too.*

3. *If* (V, v_e) *is an obstruction theory for* G, *then* $(V, v_e \circ \nu)$ *is an obstruction theory for* F.

Definition 2.6. *Let* $\nu : F \to G$ *be a morphism in* **Fun**. *A relative obstruction theory* (V, v_e) *is* universal *if for every other relative obstruction theory* (V', v'_e) *there is a unique morphism* $(V, v_e) \to (V', v'_e)$.

Proposition 2.7. *For every morphism* $\nu : F \to G$ *in* **Fun** *there exists a unique (up to isomorphism) universal relative obstruction theory* (O_ν, ob_e). *Every element of* O_ν *is of the form* $ob_e(x, \varphi)$ *for some principal small extension* e, *some* $x \in \widetilde{\nu}(e)$ *and some* $0 \neq \varphi \in K(e)^{\vee}$.

Proof. Uniqueness follows from the universal property. Let

$$O_\nu := (\coprod_{e \in \mathbf{Smex}} \widetilde{\nu}(e) \times K(e)^{\vee})/\approx$$

where $\approx$ is the equivalence relation generated by the direct relation $\sim$ induced by the base change axiom. More precisely, if $(x_i, y_i) \in \widetilde{\nu}(e_i) \times K(e_i)^{\vee}$ for $i = 1, 2$, then $(x_1, y_1) \sim (x_2, y_2)$ if and only if there is a morphism $\alpha : e_1 \to e_2$ in **Smex** such that $x_2 = \widetilde{\nu}(\alpha)(x_1)$ and $y_1 = K(\alpha)^{\vee}(y_2)$. The maps ob_e are defined in the obvious way, and then it is easy to prove that (O_ν, ob_e) has the claimed properties. □

Of course, $O_{F\to\overline{*}}$ will be denoted by O_F.

Remark 2.8. *A morphism* $\nu : F \to G$ *in* **Fun** *induces a morphism* $\nu : O_F \to O_G$ *in* $\mathbf{Set}_*$, *defined by* $\nu(ob_e(x, \varphi)) = ob_e(\nu(x), \varphi)\ \forall\, e \in$ **Smex** *and* $\forall\, x \in F(T(e))$.

The name obstruction theory is (partially) justified by the following result ([1, prop. 3.3]).

Proposition 2.9. *Let* $\nu : F \to G$ *be a morphism in* **Fun**, *with* $G \in$ **Gdt**, *and let* (V, v_e) *be a relative obstruction theory for* ν. *If* $e \in$ **Smex** *and* $x \in \widetilde{\nu}(e)$ *is contained in the image of* $F(S(e))$, *then* $v_e(x, \varphi) = *\ \forall\, \varphi \in K(e)^{\vee}$. *In particular, if* ν *is smooth then all the maps* v_e *are trivial.*

Of course the converse does not hold in general, so we are lead to give the following definition.

Definition 2.10. *A relative obstruction theory* (V, v_e) *for* $\nu : F \to G$ *(with* $G \in \mathbf{Gdt}$*) is* complete *if* $\forall\, e \in \mathbf{Smex}$ *an element* $x \in \widetilde{\nu}(e)$ *can be lifted to* $F(S(e))$ *if and only if* $v_e(x, \varphi) = * \;\forall\, \varphi \in K(e)^\vee$.

Remark 2.11. *If* (V, v_e) *is a complete relative obstruction theory for* ν*, then* (O_ν, ob_e) *is also complete and* $\ker(O_\nu \to V) = \{*\}$*. Moreover,* $O_\nu = \{*\}$ *if and only if* ν *is smooth.*

In general the universal relative obstruction theory is not complete, but it is with some assumptions. For instance, one can prove the following result ([1, cor. 4.13]).

Proposition 2.12. *Let* $\nu : F \to G$ *be a morphism in* $\mathbf{Gdt}$*. If either* $\nu : t_F \to t_G$ *is surjective or* G *satisfies* $(H4)$*, then* O_ν *is complete. In particular,* O_F *is complete* $\forall\, F \in \mathbf{Gdt}$.

Definition 2.13. *An obstruction theory* (V, v_e) *for* $\nu : F \to G$ *(with* $G \in \mathbf{Gdt}$*) is* linear *if* V *is a vector space and* $\forall\, e \in \mathbf{Smex}$, $\forall\, x \in \widetilde{\nu}(e)$ *the map* $v_e(x, -) : K(e)^\vee \to V$ *is linear. In this case* v_e *can be regarded as a map* $v_e : \widetilde{\nu}(e) \to V \otimes K(e)$*, and* $v_e(x) \in V \otimes K(e)$ *is called the obstruction of* $x \in \widetilde{\nu}(e)$.

Remark 2.14. *It is not difficult to prove that if* O_F *and* O_G *are linear, then a morphism* $\nu : F \to G$ *induces a linear map* $\nu : O_F \to O_G$.

Example 2.15 (Universal obstruction theory of a prorepresentable functor). *Every* $R \in \widehat{\mathbf{Art}}$ *can be written as* $R = P/I$*, where* $P = \mathbb{K}[[x_1, \ldots, x_n]]$ *(*$n = \dim_{\mathbb{K}}(t_R)$*) and* $I \subset \mathbf{m}_P^2$*. It is easy to see that the small extension*

$$\xi :\; 0 \longrightarrow I/\mathbf{m}_P I \longrightarrow P/\mathbf{m}_P I \longrightarrow R = P/I \longrightarrow 0$$

is universal *in the sense that every other small extension* e *with* $T(e) = R$ *can be obtained from* ξ *by a unique push–forward. More precisely,* $\forall\, J \in \mathbf{Fvs}$ *the map*

$$\mathrm{Hom}_{\mathbf{Fvs}}(I/\mathbf{m}_P I, J) \longrightarrow \mathrm{Ex}(R, J), \qquad \varphi \longmapsto \varphi_* \xi$$

is a linear isomorphism. It is not difficult to prove that $O_{h_R} := \mathrm{Ex}(R, \mathbb{K}) \cong (I/\mathbf{m}_P I)^\vee$ *is the universal linear complete obstruction theory for* h_R*, with the obstruction maps defined* $\forall\, e \in \mathbf{Smex}$ *by*

$$ob_e : h_R(T(e)) \longrightarrow O_{h_R} \otimes K(e) \cong \mathrm{Ex}(R, K(e)), \qquad f \longmapsto f^* e.$$

Proposition 2.16 (Standard smoothness criterion). *([1, lemma 6.1]) Let* $\nu : F \to G$ *be a morphism in* $\mathbf{Gdt}$*. Then* ν *is smooth if and only if*

$$t_F \longrightarrow t_G \longrightarrow * \longrightarrow O_F \longrightarrow O_G$$

is an exact sequence in $\mathbf{Set}_*$.

Remark 2.17. *It can be proved easily that if* $\nu : F \to G$ *is a smooth morphism in* $\mathbf{Gdt}$*, then* $\nu : O_F \to O_G$ *is surjective (and then bijective if* O_F *and* O_G *are linear).*

Corollary 2.18. *Let* $\nu : F \to G$ *be a morphism in* $\mathbf{Gdt}$ *and let* (V, v_e)*,* (W, w_e) *be obstruction theories for* F *and* G *respectively, with* V *complete. If* $\phi : V \to W$ *is a morphism of obstruction theories compatible with* ν *(meaning that* $w_e \circ \nu = \phi \circ v_e \;\forall\, e \in \mathbf{Smex}$*) and if*

$$t_F \longrightarrow t_G \longrightarrow * \longrightarrow V \xrightarrow{\;\phi\;} W$$

is an exact sequence in $\mathbf{Set}_*$*, then* ν *is smooth.*

Theorem 2.19 (Factorization theorem). *([1, thm. 6.2])* [2] *Let $\nu : G \to F$ be a morphism in* **Gdt** *with G prorepresentable. Then there exists a factorization $G \xrightarrow{\mu} H \xrightarrow{\xi} F$ of ν such that:*

1. *H is prorepresentable and $\mu : t_G \to t_H$ is bijective;*
2. $\ker(\xi : O_H \to O_F) = \{*\}$.

If in addition F satisfies $(H4)$ and $\nu : t_G \to t_F$ is injective, then ξ is injective.

Corollary 2.20. *([6, thm. 2.11])*

1. *$F \in$* **Fun** *has a hull if and only if it satisfies $(H3)$.*
2. *$F \in$* **Fun** *is prorepresentable if and only if it satisfies $(H3)$ and $(H4)$.*

Proof. 1. If F has a hull it is easy to prove (without using 2.19: see [6]) that it satisfies $(H1)$ and $(H2)$, and then obviously it satisfies also $(H3)$. Conversely, if F satisfies $(H3)$, let $A := \mathbb{K} \oplus t_F^\vee \in$ **Art**. By 1.6 and 1.15 $\mathrm{Hom}_{\mathbf{Fun}}(h_A, F) \cong F(A) \cong t_F \otimes t_F^\vee$, and it is easy to see that the morphism $\nu : h_A \to F$ corresponding to $\mathrm{id} \in t_F \otimes t_F^\vee$ induces an isomorphism on tangent spaces. Then by 2.19 ν factors through a morphism $\xi : H \to F$ in **Fun** with H prorepresentable and such that $\xi : t_H \to t_F$ is bijective and $\ker(\xi : O_H \to O_F) = \{*\}$. Therefore ξ is a hull by 2.16.

2. Obviously if F is prorepresentable it satisfies $(H3)$ and $(H4)$. Conversly, if F satisfies $(H3)$ and $(H4)$, let $\xi : H \to F$ be a hull obtained as in 1: by the last statement of 2.19 ξ is injective. On the other hand, ξ is surjective by 2.2, and so F is prorepresentable. □

Using 2.19 it is also possible to prove the following result ([1, thm. 6.6 and 6.11]), which justifies the name **Gdot**.

Proposition 2.21. *Let $F \in$* **Gdt**.

1. *If F has a complete linear obstruction theory V, then O_F is also linear and the natural map $O_F \to V$ is linear and injective.*
2. *O_F is linear if and only if F satisfies (L).*

Definition 2.22. *$G \in$* **Fun** *is called a* group functor *of Artin rings if it factors through the forgetful functor* **Group** $\to$ **Set**$_*$.

Given $F, G \in$ **Fun** *with G a group functor, an action of G on F is a morphism $\tau : G \times F \to F$ in* **Fun** *such that $\forall A \in$* **Art** *$\tau(A)$ is an action of $G(A)$ on $F(A)$.*

Given an action of a group functor G on $F \in$ **Fun**, there is an obvious way to define the quotient functor F/G, together with a (surjective) morphism $F \to F/G$ in **Fun**.

Proposition 2.23. *([1, prop. 7.5]) Let $F, G \in$* **Gdt** *with G a smooth group functor.*[3] *If G acts on F, then $F/G \in$* **Gdt***, the projection $\pi : F \to F/G$ is smooth and $\pi : O_F \to O_{F/G}$ is bijective. Moreover, if F satisfies (L) or $(H2')$, the same holds for F/G.*

[2] This is the correct version (communicated to us by M. Manetti) of the factorization theorem (in [1], instead of condition 1, it is required that $\mu : t_G \to t_H$ is surjective and $\xi : t_H \to t_F$ is injective); the proof is similar. Also, the other results of [1] which use the factorization theorem remain true with the same proofs, except for [1, cor. 6.3 (II)], stating that a functor is left–exact if and only if it satisfies $(H1)$, $(H2)$ and $(H4)$ (we don't know if this fact is true) and for [1, cor. 6.12], stating that a functor which satisfies $(H1)$, $(H2)$ and $(H4)$ is in **Gdot** (this is true, however, as it follows from 1.18 and 1.12).

[3] It can be proved that G is always smooth if the characteristic of $\mathbb{K}$ is zero ([1, thm. 7.19]).

3 Examples

Most of the functors coming from deformation problems satisfy conditions $(H1)$ and $(H2')$ (hence they are in **Gdot** by 1.12). In order to prove this in many common cases (for instance, deformations of algebras, of modules, of schemes, of sheaves, ...) the following results are useful.

Lemma 3.1. *([6, lemma 3.4]) Let $\pi_i : A_i \to A$ (for $i = 1, 2$) be morphisms in* **Art** *with π_2 a small extension, and let $B := A_1 \times_A A_2$. Let moreover M (respectively M_i) be a flat A–module (respectively A_i–module), and assume that $p_i : M_i \to M$ are morphisms (compatible with π_i) which induce isomorphisms $M_i \otimes_{A_i} A \xrightarrow{\sim} M$. Then $N := M_1 \times_M M_2$ is a flat B–module and $N \otimes_B A_i \cong M_i$.*

Lemma 3.2. *([6, lemma 3.3]) Let $f : M \to N$ be a morphism of A–modules ($A \in$* **Art***), and let $I \subset A$ be an ideal such that the morphism $M/IM \to N/IN$ induced by f is an isomorphism. If N is flat over A, then f is an isomorphism, too.*

We are going to determine the tangent space and a complete linear obstruction theory (it is in general much more difficult to find the universal one) for some morphisms of functors coming from deformation problems.

3.1 Deformations of submodules

Let R be a $\mathbb{K}$–algebra; $\forall\, A \in$ **Art** we denote by R_A the A–algebra $R \otimes A$ and, for every R–module P, by P_A the R_A–module $P \otimes A$. Given $M \subset N$ R–modules, we are going to compute tangent and obstruction spaces for the functor $G := Grass_{M/N}$ defined $\forall\, A \in$ **Art** by

$$G(A) := \{\mathcal{M} \subset N_A \ R_A\text{–submodule} \,|\, \mathcal{M}\ A\text{–flat},\ \mathcal{M} \otimes_A \mathbb{K} = M\}.$$

Observe that if $\mathcal{M} \subset N_A$ is A–flat, then the natural map $\mathcal{M} \otimes_A \mathbb{K} \to N_A \otimes_A \mathbb{K} = N$ is injective (and so the condition $\mathcal{M} \otimes_A \mathbb{K} = M$ makes sense): this is a consequence of the following lemma.

Lemma 3.3. *Let $0 \to Q \to P$ be an exact sequence of A–modules ($A \in$* **Art***) with P flat over A. Then Q is flat over A if and only if $0 \to Q \otimes_A \mathbb{K} \to P \otimes_A \mathbb{K}$ is exact.*

Using 3.1 and 3.2 it is easy to prove that G is a homogeneous functor. We will also need the following results on flatness.

Lemma 3.4. *Given $0 \to J \to B \to A \to 0$ in* **Smex***, let $\mathcal{M} \in G(A)$ and let $\mathcal{M}' \subset N_B$ be an R_B–submodule such that there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & M \otimes J \arrow[r] \arrow[d] & \mathcal{M}' \arrow[r, "\rho"] \arrow[d] & \mathcal{M} \arrow[d] & \\
0 \arrow[r] & N \otimes J \arrow[r] & N_B \arrow[r] & N_A \arrow[r] & 0
\end{tikzcd}$$

(where the vertical arrows are the inclusions). Then $\mathcal{M}' \in G(B)$ (i.e. $\mathcal{M}'$ is B–flat and $\mathcal{M}' \otimes_B \mathbb{K} = M$) if and only if ρ is surjective.

Lemma 3.5. *Given $0 \to J \to B \to A \to 0$ in* **Smex***, let $\mathcal{M}' \in G(B)$ be a lifting of $\mathcal{M} \in G(A)$, and let $f : N_A \to I_A$ be an R_A–morphism such that $\ker(f) = \mathcal{M}$, where I is an injective R–module. Then there exists an R_B–morphism $g : N_B \to I_B$ such that $\ker(g) = \mathcal{M}'$ and $g \otimes_B A = f$.*

Proposition 3.6. *$t_G \cong \mathrm{Hom}_R(M, N/M)$ and $\mathrm{Ext}^1_R(M, N/M)$ is a complete linear obstruction space for G.*

Proof. Given $\mathcal{M} \in t_G$, $\mathcal{M} \subset N_{\mathbb{K}[\epsilon]} = N \oplus \epsilon N$ is $\mathbb{K}[\epsilon]$–flat, hence tensoring with $\overline{\epsilon}$ yields a commutative diagram of $R_{\mathbb{K}[\epsilon]}$–modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "\cdot\epsilon"] & N \oplus \epsilon N \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

(where the vertical arrows are the inclusions). Given $x \in M \subset N$, a lifting of x in $\mathcal{M} \subset N \oplus \epsilon N$ is of the form $x + \epsilon y$ ($y \in N$). Since two such liftings differ by an element of the form ϵz with $z \in M$, the image $\overline{y}$ of y in N/M is independent of the choice made. So $\alpha(\mathcal{M})(x) := \overline{y}$ defines a map $\alpha(\mathcal{M}) : M \to N/M$, which is clearly R–linear. It is not difficult to prove that $\alpha : t_G \to \mathrm{Hom}_R(M, N/M)$ is a linear isomorphism.

As for the obstruction theory, for every $e : \ 0 \to J \to B \to A \to 0$ in **Smex** we have to define a map $v_e : G(A) \to \mathrm{Ext}^1_R(M, N/M) \otimes J$. Let's choose a morphism of R–modules $\phi : N \to I$ such that I is injective and $\ker(\phi) = M$. It follows from 3.5 and 1.8 that, given $\mathcal{M} \in G(A)$, we can find $f : N_A \to I_A$ such that $\ker(f) = \mathcal{M}$ and $f \otimes_A \mathbb{K} = \phi$. The choice of a lifting $g : N_B \to I_B$ of f (which can be made because $\mathrm{Ext}^1_{R_B}(N_B, I) \cong \mathrm{Ext}^1_R(N, I) = 0$) determines a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \otimes J \arrow[r] \arrow[d] & N \otimes J \arrow[r, "\phi\otimes J"] \arrow[d] & I \otimes J \arrow[r, "p"] \arrow[d, "i"] & P \otimes J \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{M}' \arrow[r] \arrow[d, "\rho"] & N_B \arrow[r, "g"] \arrow[d] & I_B \arrow[d] & & \\
0 \arrow[r] & \mathcal{M} \arrow[r] & N_A \arrow[r, "f"] \arrow[d] & I_A \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{M}' := \ker(g)$ and $P := \mathrm{coker}(\phi)$. By 3.5 and 3.4 it is clear that $\mathcal{M}$ can be lifted to $G(B)$ if and only if it is possible to choose g in such a way that ρ is surjective. Now, by snake lemma, the above diagram induces a map

$$\delta_g \in \mathrm{Hom}_{R_B}(\mathcal{M}, P \otimes J) \cong \mathrm{Hom}_R(M, P \otimes J)$$

such that $\mathcal{M}' \xrightarrow{\rho} \mathcal{M} \xrightarrow{\delta_g} P \otimes J$ is exact (whence $\mathcal{M}$ lifts to $G(B)$ if and only if there exists g such that $\delta_g = 0$). Moreover, if g' is another lifting of f, then $g' - g = i \circ \psi$ for some $\psi \in \mathrm{Hom}_{R_B}(N_B, I \otimes J) \cong \mathrm{Hom}_R(N, I \otimes J)$, and it is easy to see that $\delta_{g'} - \delta_g = p \circ \psi|_M$. Therefore, defining V by the exact sequence

$$\mathrm{Hom}_R(N, I) \xrightarrow{\sigma} \mathrm{Hom}_R(M, P) \longrightarrow V \longrightarrow 0,$$

we see that the image $\overline{\delta_g}$ of δ_g in $V \otimes J$ is independent of g. Since σ factors through the natural maps

$$\mathrm{Hom}_R(N, I) \xrightarrow{j} \mathrm{Hom}_R(M, I) \xrightarrow{l} \mathrm{Hom}_R(M, P)$$

and j is surjective (because I is injective), we obtain that $V \cong \operatorname{coker}(l)$. On the other hand, the short exact sequence of R–modules

$$0 \longrightarrow N/M \longrightarrow I \longrightarrow P \longrightarrow 0$$

yields an exact sequence

$$\operatorname{Hom}_R(M, I) \xrightarrow{l} \operatorname{Hom}_R(M, P) \to \operatorname{Ext}^1_R(M, N/M) \to \operatorname{Ext}^1_R(M, I) = 0\,,$$

so that $V \cong \operatorname{Ext}^1_R(M, N/M)$. One can prove that $v_e(\mathcal{M}) := \overline{\delta_g} \in V \otimes J$ is well defined (it doesn't depend on the choice of I, of $\phi : N \to I$ and of the lifting f), and that (V, v_e) is indeed a complete linear obstruction theory for G. $\square$

3.2 Deformations of an isolated singularity

Let X be a scheme over $\mathbb{K}$; $\forall\, A \in \mathbf{Art}$ X_A will be the scheme $X \times_{\operatorname{Spec}\mathbb{K}} \operatorname{Spec} A$. The functor Def_X of deformations of X is defined as follows: $\forall\, A \in \mathbf{Art}$

$$\mathit{Def}_X(A) := \{X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A \,|\, \pi \text{ flat},\ X \xrightarrow{\sim} \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec}\mathbb{K}\}/\sim$$

where $(X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A) \sim (X \xrightarrow{\iota'} \mathcal{X}' \xrightarrow{\pi'} \operatorname{Spec} A)$ if and only if there is an isomorphism $s : \mathcal{X} \to \mathcal{X}'$ such that $\pi' \circ s = \pi$ and $\iota' = s \circ \iota$ (we will call *compatible* such an isomorphism). In the following we will often denote simply by $\mathcal{X}$ an object of the form $X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A$. Using 3.1 and 3.2 it is easy to prove that Def_X satisfies $(H1)$ and $(H2')$ (but not $(H3)$ and $(H4)$ in general).

Assume that $\mathbb{K}$ is algebraically closed and that X is a variety over $\mathbb{K}$ which is nonsingular everywhere except (possibly) at a point p, and let Def_p be the functor of deformations of the singularity p. This can be defined to be Def_U, where $p \in U \subset X$ is an open affine subset (it can be proved that if $p \in V \subset U$ is another open affine subset, then the natural morphism $\mathit{Def}_U \to \mathit{Def}_V$ is an isomorphism, so that Def_p is well defined).

Proposition 3.7. *(see [7, prop. 6.4]) Let $\nu : \mathit{Def}_X \to \mathit{Def}_p$ be the natural morphism. Then $t_\nu \cong H^1(X, \Theta_X)$ and $H^2(X, \Theta_X)$ is a complete linear obstruction space for ν.*

Remark 3.8. *1. If $H^2(X, \Theta_X) = 0$, then ν is smooth and in particular surjective, whence every deformation of p can be extended to a deformation of X.*

2. If p is smooth, then $\mathit{Def}_p = \overline{*}$. Therefore we obtain in particular that if X is a nonsingular variety, then $t_{\mathit{Def}_X} \cong H^1(X, \Theta_X)$ and $O_{\mathit{Def}_X} \subset H^2(X, \Theta_X)$ (see [5]).

Proof. Let $\mathbf{U} = \{U^i \,|\, i \in I' = I \bigsqcup \{0\}\}$ be an open affine covering of X such that $p \in U^0$ and $p \notin U^i\ \forall\, i \in I$. We will consider ν as a morphism from Def_X to Def_{U^0}.

Let $\xi \in t_\nu$ be represented by $\mathcal{X}$. $\forall\, i \in I'$ there is a compatible isomorphism $\phi_i : \mathcal{X}|_{U^i} \xrightarrow{\sim} U^i_{\mathbb{K}[\epsilon]}$: this is true by hypothesis if $i = 0$, whereas if $i \in I$ it follows from the fact that every deformation of a nonsingular affine variety is trivial (see [5]). Then $\forall\, i, j \in I'$

$$\phi_{i,j} := \phi_i \circ \phi_j^{-1} : U^{i,j}_{\mathbb{K}[\epsilon]} \xrightarrow{\sim} U^{i,j}_{\mathbb{K}[\epsilon]}$$

is a lifting of $\mathrm{id}_{U^{i,j}}$ and clearly determines a cocycle ($\phi_{i,j} \circ \phi_{j,k} \circ \phi_{k,i} = \mathrm{id}_{U^{i,j,k}}$).[4] Moreover, each $\phi_{i,j}$ corresponds to a derivation (see [5]) $\theta_{i,j} \in \Gamma(U^{i,j}, \Theta_X)$ (which also satisfies the cocycle condition $\theta_{i,j} + \theta_{j,k} + \theta_{k,i} = 0$), hence it determines an element $\theta \in H^1(\mathbf{U}, \Theta_X) \cong H^1(X, \Theta_X)$. It is easy to see that θ is well defined (it does not depend on the choice of the

[4]Here and in the following we avoid to write explicitly that functions like ϕ_i and $\phi_{i,j}$ are restricted to a suitable open subset, which should always be clear from the context.

ϕ_i and of the covering $\mathbf{U}$) and that the map $t_\nu \to H^1(X, \Theta_X)$ thus obtained is a linear isomorphism.

Now we compute the obstructions. As usual, given $e:\ 0 \to J \to B \to A \to 0$ in **Smex**, we have to define a map

$$v_e : \widetilde{\nu}(e) = \mathit{Def}_X(A) \times_{\mathit{Def}_{U^0}(A)} \mathit{Def}_{U^0}(B) \longrightarrow H^2(X, \Theta_X) \otimes J.$$

Let $(\xi, \eta') \in \widetilde{\nu}(e)$ with ξ represented by $\mathcal{X}$ and η' by $\mathcal{U}'^0$. Then the common image $\eta \in \mathit{Def}_{U^0}(A)$ of ξ and η' is represented by $\mathcal{U}^0 := \mathcal{U}'^0 \times_{\operatorname{Spec} B} \operatorname{Spec} A$, and there is a compatible isomorphism $\phi_o : \mathcal{X}|_{U^0} \xrightarrow{\sim} \mathcal{U}^0$. Also, as before, $\forall\, i \in I$ there are compatible isomorphisms $\phi_i : \mathcal{X}|_{U^i} \xrightarrow{\sim} \mathcal{U}^i := U^i_A$. Again,

$$\phi_{i,j} := \phi_i \circ \phi_j^{-1} : \mathcal{U}^j|_{U^{i,j}} \xrightarrow{\sim} \mathcal{U}^i|_{U^{i,j}}$$

is a lifting of $\mathrm{id}_{U^{i,j}}$ and $\{\phi_{i,j}\}$ satisfies the cocycle condition. Clearly (ξ, η) can be lifted to some $\xi' \in \mathit{Def}_X(B)$ if and only if there are liftings

$$\phi'_{i,j} : \mathcal{U}'^j|_{U^{i,j}} \xrightarrow{\sim} \mathcal{U}'^i|_{U^{i,j}}$$

(where $\mathcal{U}'^i := U^i_B$ for $i \in I$) of $\phi_{i,j}$ such that $\{\phi'_{i,j}\}$ satisfies the cocycle condition.

Let's choose liftings $\phi'_{i,j}$ of $\phi_{i,j}$ (they exist because $U^{i,j}$ is affine and nonsingular) with $\phi'_{j,i} = (\phi'_{i,j})^{-1}$ and let's define $\forall\, i,j,k \in I'$

$$\phi'_{i,j,k} := \phi'_{i,j} \circ \phi'_{j,k} \circ \phi'_{k,i} : \mathcal{U}'^i|_{U^{i,j,k}} \longrightarrow \mathcal{U}'^i|_{U^{i,j,k}}\,.$$

Since $\phi'_{i,j,k}$ is a lifting of $\mathrm{id}_{\mathcal{U}^i|_{U^{i,j,k}}}$, it corresponds to a derivation $\theta_{i,j,k} \in \Gamma(U^{i,j,k}, \Theta_X) \otimes J$.

Lemma 3.9. $\{\theta_{i,j,k} \,|\, i,j,k \in I'\} \in Z^2(\mathbf{U}, \Theta_X) \otimes J$.

Proof. Given $i,j,k,l \in I'$, we have to show that

$$\theta_{j,k,l} - \theta_{i,k,l} + \theta_{i,j,l} - \theta_{i,j,k} = 0\,.$$

$\forall\, \alpha, \beta, \gamma \in \{i,j,k,l\}$ let $\sigma_\alpha : \mathcal{U}'^\alpha|_{U^{i,j,k,l}} \xrightarrow{\sim} U^{i,j,k,l}_B$ be a compatible isomorphism such that $\sigma_\alpha = \mathrm{id}_{U^{i,j,k,l}_B}$ if $\alpha \neq 0$. Then

$$\psi_{\alpha,\beta} := \sigma_\alpha \circ \phi'_{\alpha,\beta} \circ \sigma_\beta^{-1} : U^{i,j,k,l}_B \xrightarrow{\sim} U^{i,j,k,l}_B$$

is a lifting of $\mathrm{id}_{U^{i,j,k,l}}$ and

$$\psi_{\alpha,\beta,\gamma} := \psi_{\alpha,\beta} \circ \psi_{\beta,\gamma} \circ \psi_{\gamma,\alpha} = \sigma_\alpha \circ \phi'_{\alpha,\beta,\gamma} \circ \sigma_\alpha^{-1} : U^{i,j,k,l}_B \xrightarrow{\sim} U^{i,j,k,l}_B$$

is a lifting of $\mathrm{id}_{U^{i,j,k,l}_A}$. Therefore there are B–linear maps

$$d_{\alpha,\beta} := \psi^{\#}_{\alpha,\beta} - \mathrm{id} : \mathcal{O}_{U^{i,j,k,l}} \otimes B \longrightarrow \mathcal{O}_{U^{i,j,k,l}} \otimes \mathbf{m}_B\,,$$

$$d_{\alpha,\beta,\gamma} := \psi^{\#}_{\alpha,\beta,\gamma} - \mathrm{id} : \mathcal{O}_{U^{i,j,k,l}} \otimes B \longrightarrow \mathcal{O}_{U^{i,j,k,l}} \otimes J\,.$$

Obviously $d_{\alpha,\beta,\gamma}(\mathcal{O}_{U^{i,j,k,l}} \otimes \mathbf{m}_B) = 0$, and it is clear that the induced map $\mathcal{O}_{U^{i,j,k,l}} \to \mathcal{O}_{U^{i,j,k,l}} \otimes J$ is a derivation, which is just $\theta_{\alpha,\beta,\gamma}$. So it is enough to prove that

$$d := d_{j,k,l} - d_{i,k,l} + d_{i,j,l} - d_{i,j,k} = 0.$$

As $\psi^{\#}_{\alpha,\beta,\gamma} = \psi^{\#}_{\gamma,\alpha} \circ \psi^{\#}_{\beta,\gamma} \circ \psi^{\#}_{\alpha,\beta}$ and since $(\psi_{\gamma,\alpha})^{-1} = \psi_{\alpha,\gamma}$, we obtain

$$(\mathrm{id} + d_{\alpha,\gamma}) \circ (\mathrm{id} + d_{\alpha,\beta,\gamma}) = \psi^{\#}_{\alpha,\gamma} \circ \psi^{\#}_{\alpha,\beta,\gamma} = \psi^{\#}_{\beta,\gamma} \circ \psi^{\#}_{\alpha,\beta} = (\mathrm{id} + d_{\beta,\gamma}) \circ (\mathrm{id} + d_{\alpha,\beta}) \,,$$

which implies (taking into account that $d_{\alpha,\gamma} \circ d_{\alpha,\beta,\gamma} = 0$ because $\mathbf{m}_B J = (0)$)

$$d_{\alpha,\beta,\gamma} = d_{\alpha,\beta} + d_{\beta,\gamma} - d_{\alpha,\gamma} + d_{\beta,\gamma} \circ d_{\alpha,\beta} \,. \tag{2}$$

It follows that

$$\begin{aligned} d = {} & d_{k,l} \circ d_{j,k} - d_{k,l} \circ d_{i,k} + d_{j,l} \circ d_{i,j} - d_{j,k} \circ d_{i,j} = \\ & = d_{k,l} \circ (d_{j,k} - d_{i,k}) + (d_{j,l} - d_{j,k}) \circ d_{i,j} \end{aligned}$$

and using (2) again we obtain

$$\begin{aligned} d = {} & d_{k,l} \circ (d_{i,j,k} - d_{i,j} - d_{j,k} \circ d_{i,j}) + (d_{k,l} - d_{j,k,l} + d_{k,l} \circ d_{j,k}) \circ d_{i,j} = \\ & = d_{k,l} \circ d_{i,j,k} - d_{j,k,l} \circ d_{i,j} \,, \end{aligned}$$

whence $d = 0$ (again because $\mathbf{m}_B J = (0)$). □

End of proof of 3.7. If $\widetilde{\phi}'_{i,j}$ are other liftings of $\phi_{i,j}$, determining another cocycle $\{\widetilde{\theta}_{i,j,k}\}$, then $(\phi'_{i,j})^{-1} \circ \widetilde{\phi}'_{i,j}$ (being a lifting of $\mathrm{id}_{\mathcal{U}^j|_{U^{i,j}}}$) corresponds to a derivation $\delta_{i,j} \in \Gamma(U^{i,j}, \Theta_X) \otimes J$. It is easy to prove that

$$\widetilde{\theta}_{i,j,k} - \theta_{i,j,k} = \delta_{i,j} + \delta_{j,k} + \delta_{k,i} \,,$$

and so the element $\theta \in H^2(\mathbf{U}, \Theta_X) \otimes J \cong H^2(X, \Theta_X) \otimes J$ defined by $\{\theta_{i,j,k}\}$ does not depend on the chosen liftings of $\phi_{i,j}$. It is also not difficult to check that θ does not depend on the choice of the isomorphisms ϕ_i and of the covering $\mathbf{U}$. Hence $v_e(\xi, \eta) := \theta$ is well defined and it is easy to see that $(H^2(X, \Theta_X), v_e)$ is indeed a complete linear obstruction theory for ν. □

3.3 Deformations of maps with fixed smooth target

Let $f : X \to Y$ be a morphism of schemes over $\mathbb{K}$. The functor $\mathit{Def}_{f/Y}$ of deformations of f with fixed target Y is defined $\forall\, A \in \mathbf{Art}$ by

$$\begin{aligned} \mathit{Def}_{f/Y}(A) := \{ & (X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A, \varphi : \mathcal{X} \to Y) \mid \\ & \pi \text{ flat}, X \xrightarrow{\sim} \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{K}, \varphi \circ i = f \} / \sim \end{aligned}$$

where $(X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A, \varphi : \mathcal{X} \to Y) \sim (X \xrightarrow{\iota'} \mathcal{X}' \xrightarrow{\pi'} \operatorname{Spec} A, \varphi' : \mathcal{X}' \to Y)$ if and only if there is an isomorphism $s : \mathcal{X} \to \mathcal{X}'$ such that $\pi' \circ s = \pi$, $\iota' = s \circ \iota$ and $\varphi' \circ s = \varphi$. As usual, one can prove that $\mathit{Def}_{f/Y}$ satisfies $(H1)$ and $(H2')$.

Proposition 3.10. *(see [4, thm. 2]) Let $f : X \to Y$ be a morphism of schemes of finite type over an algebraically closed field $\mathbb{K}$, with X separated and Y smooth. Then the natural morphism $\nu : \mathit{Def}_{f/Y} \to \mathit{Def}_X$ has $t_\nu \cong \operatorname{coker}(H^0(X, \Theta_X) \to H^0(X, f^*\Theta_Y))$ and $H^1(X, f^*\Theta_Y)$ is a complete linear obstruction space for ν.*

Proof. Let $\xi \in t_\nu$ be represented by $\varphi : X_{\mathbb{K}[\epsilon]} \to Y$: clearly $|\varphi| = |f|$ and $\varphi^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_{X_{\mathbb{K}[\epsilon]}} = \mathcal{O}_X \oplus \epsilon \mathcal{O}_X$ can be written as $\varphi^{\#} = f^{\#} + \epsilon\delta$ with $\delta : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ a derivation, which can be considered as an element $\delta \in H^0(X, f^*\Theta_Y)$. If $\varphi' : X_{\mathbb{K}[\epsilon]} \to Y$ also represents ξ, there is a compatible isomorphism $s : X_{\mathbb{K}[\epsilon]} \to X_{\mathbb{K}[\epsilon]}$ such that $\varphi' = \varphi \circ s$, and since $s^{\#}$ is of the form $\mathrm{id} + \epsilon d$ for some $d \in H^0(X, \Theta_X)$, we obtain

$$\varphi'^{\#} = f^{\#} + \epsilon\delta' = s^{\#} \circ \varphi^{\#} = (\mathrm{id} + \epsilon d) \circ (f^{\#} + \epsilon\delta) \,,$$

whence $\delta' = d \circ f^{\#} + \delta$. The statement about t_ν follows easily.

$\forall\, e:\ 0 \to J \to B \to A \to 0$ in **Smex** we have to define an obstruction map

$$v_e : \widetilde{\nu}(e) = \mathit{Def}_{f/Y}(A) \times_{\mathit{Def}_X(A)} \mathit{Def}_X(B) \longrightarrow H^1(X, f^*\Theta_Y) \otimes J\,.$$

Let $(\xi, \eta') \in \widetilde{\nu}(e)$ and let $\eta \in \mathit{Def}_X(A)$ be the common image of ξ and η': if η' is represented by $\mathcal{X}'$, then η is represented by $\mathcal{X} := \mathcal{X}' \times_{\operatorname{Spec} B} \operatorname{Spec} A$, and we can assume that ξ is represented by some $\varphi : \mathcal{X} \to Y$. Clearly (ξ, η') lifts to some $\xi \in \mathit{Def}_{f/Y}(B)$ if and only if there is a lifting $\varphi' : \mathcal{X}' \to Y$ of φ, i.e. if and only if there is a lifting ${\varphi'}^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_{\mathcal{X}'}$ of $\varphi^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_{\mathcal{X}}$.

First, one can prove that if X is affine a lifting of $\varphi^{\#}$ always exists (here one uses the hypothesis Y smooth) and two liftings differ by an element of the form $\alpha \circ \theta$, where $\alpha : \mathcal{O}_X \otimes J \to \mathcal{O}_{\mathcal{X}'}$ is the inclusion induced by $J \to B$ and $\theta \in \Gamma(X, f^*\Theta_Y) \otimes J$. Then, in the general case, let $\mathbf{U} = \{U^i \,|\, i \in I\}$ and $\mathbf{V} = \{V^i \,|\, i \in I\}$ be open affine coverings of X and Y respectively, such that $f(U^i) \subset V^i$ $\forall\, i \in I$. We can choose liftings ρ_i of $\varphi^{\#}|_{U^i}$ and then $\forall\, i, j \in I$ there exists $\theta_{i,j} \in \Gamma(U^{i,j}, f^*\Theta_Y) \otimes J$ such that $\rho_i - \rho_j = \alpha \circ \theta_{i,j}$. Clearly $\{\theta_{i,j}\}$ is a cocycle and its class $\theta \in H^1(X, f^*\Theta_Y)$ is easily seen to be independent of the choice of the ρ_i and of the coverings. It is also straightforward to prove that $v_e(\xi, \eta') := \theta$ is well defined and that $(H^1(X, f^*\Theta_Y), v_e)$ is a complete linear obstruction theory for ν. $\square$

Given $p_1, \ldots, p_n \in X$ smooth points, it is possible to define the functor $\mathit{Def}_{X,\{p_i\}}$ of deformations of the n–pointed scheme $(X, \{p_i\})$: denoting by $\pi_A^* : \operatorname{Spec} \mathbb{K} \to \operatorname{Spec} A$ the natural map, $\forall\, A \in$ **Art**

$$\begin{aligned}\mathit{Def}_{X,\{p_i\}}(A) := \{X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A \xrightarrow{q_i} \mathcal{X} \,|\, \\ \pi \text{ flat},\ X \xrightarrow{\sim} \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{K},\ q_i \circ \pi_A^* = \iota \circ p_i\}/\approx\end{aligned}$$

where $(X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \operatorname{Spec} A \xrightarrow{q_i} \mathcal{X}) \approx (X \xrightarrow{\iota'} \mathcal{X}' \xrightarrow{\pi'} \operatorname{Spec} A \xrightarrow{q_i'} \mathcal{X}')$ if and only if there is an isomorphism $s : \mathcal{X} \to \mathcal{X}'$ such that $\pi' \circ s = \pi$, $\iota' = s \circ \iota$ and $q_i' = s \circ q_i$ for $i = 1, \ldots, n$. In a similar way one can define the functor $\mathit{Def}_{f/Y,\{p_i\}}$, and it is easy to see that 3.10 extends to this more general situation, so that we obtain, in particular, the following result.

Proposition 3.11. *If the pointed scheme $(X, \{p_i\})$ has no infinitesimal automorphisms and $H^1(X, f^*\Theta_X) = 0$, then the natural morphism $\nu : \mathit{Def}_{f/Y,\{p_i\}} \to \mathit{Def}_{X,\{p_i\}}$ is smooth and $t_\nu \cong H^0(X, f^*\Theta_X)$.*

From this one can prove (see [4, thm. 1]) the following theorem.

Theorem 3.12. *([2, thm. 2]) Let Y be a smooth projective convex variety over $\mathbb{C}$, let $(C, \{p_i\})$ be an n–pointed quasi–stable curve of arithmetic genus g with $N \geq 0$ nodes, let $\mu : (C, \{p_i\}) \to Y$ be a stable map (i.e., without infinitesimal automorphisms), and let $\beta := \mu_*[C] \in H_2(Y, \mathbb{Z})$. Then the base space B of the universal deformation of the data $(C, \{p_i\}, \mu)$ which leave Y fixed is smooth of dimension $\dim(Y) + \int_\beta c_1(\Theta_Y) + n - 3$.*

Moreover, denoting by $B_j \subset B$ the subvariety corresponding to curves with at least j nodes, B_1 is a normal crossing divisor (provided $N > 0$) and, $\forall\, j \leq n$, B_j is of pure codimension j.

3.4 Deformation functor associated to a DGLA

Here we assume that the characteristic of $\mathbb{K}$ is zero. Let

$$L = (\oplus_{i \in \mathbb{Z}} L^i, [\ ,\ ], d)$$

be a DGLA (= differential graded Lie algebra), and notice that if R is a commutative $\mathbb{K}$–algebra (possibly without unit), then $L \otimes R$ is also a DGLA in a natural way, and we will denote again by [,] and by d the bracket and the differential of $L \otimes R$. The Maurer–Cartan equation of L is

$$da + \frac{1}{2}[a,a] = 0\,, \qquad a \in L^1\,.$$

Then one can define the Maurer–Cartan functor of L by

$$MC_L(A) := \{x \in (L \otimes \mathbf{m}_A)^1 = L^1 \otimes \mathbf{m}_A \,|\, dx + \frac{1}{2}[x,x] = 0\}$$

$\forall\, A \in$ **Art**. MC_L is clearly left–exact. Denoting by **NLA** the category of nilpotent Lie algebras, there is a functor exp : **NLA** $\to$ **Group**, which, composed with the forgetful functor **Group** $\to$ $\mathbf{Set}_*$ is just the forgetful functor **NLA** $\to$ $\mathbf{Set}_*$.[5] If L is a DGLA such that every element of L^0 is ad–nilpotent in L, then $\exp(L^0)$ acts on L by

$$\exp(a) \cdot x := x + \sum_{n \geq 0} \frac{(ad\, a)^n}{(n+1)!}([a,x] - da)$$

$\forall\, a \in L^0$, $\forall\, x \in L$. It is not difficult to prove that this action preserves the solutions of the Maurer–Cartan equation. Therefore the gauge functor defined $\forall\, A \in$ **Art** by

$$G_L(A) := \exp(L^0 \otimes \mathbf{m}_A)$$

(which is clearly a smooth and left–exact group functor) acts on MC_L. The quotient functor $Def_L := MC_L/G_L$ is called the *deformation functor associated to L*. By 2.23 $Def_L \in$ **Gdot** and it also satisfies $(H2')$ (but in general not $(H4)$).

Proposition 3.13. *(see [3]) Let L be a DGLA. Then* $t_{Def_L} \cong H^1(L)$ *and* $H^2(L)$ *is a complete linear obstruction space for* Def_L.

Proof. By definition

$$t_{MC_L} = \{x \in L^1 \otimes (\epsilon) \,|\, dx + \frac{1}{2}[x,x] = 0\} = Z^1(L) \otimes (\epsilon) \cong Z^1(L)$$

and $t_{G_L} = \exp(L^0 \otimes (\epsilon)) \cong L^0$. Since the action $t_{G_L} \times t_{MC_L} \to t_{MC_L}$ is given by $(a,x) \mapsto x - da$, we obtain $t_{Def_L} \cong Z^1(L)/B^1(L) = H^1(L)$.

By 2.23 $O_{MC_L} \cong O_{Def_L}$, so that it is enough to prove that $H^2(L)$ is a complete linear obstruction space for MC_L. $\forall\, e :\ 0 \to J \to B \to A \to 0$ in **Smex** we have to define a map

$$v_e : MC_L(A) \longrightarrow H^2(L) \otimes J \cong H^2(L \otimes J)\,.$$

Given $x \in MC_L(A)$, let $\widetilde{x} \in L^1 \otimes \mathbf{m}_B$ be a lifting of x. Then

$$h := d\widetilde{x} + \frac{1}{2}[\widetilde{x}, \widetilde{x}] \in L^2 \otimes \mathbf{m}_B$$

is a lifting of $dx + 1/2[x,x] = 0$, and so $h \in L^2 \otimes J$. As

$$dh = d^2\widetilde{x} + [d\widetilde{x}, \widetilde{x}] = [h, \widetilde{x}] - \frac{1}{2}[[\widetilde{x}, \widetilde{x}], \widetilde{x}]$$

and since $[h, \widetilde{x}] = 0$ because $[L^2 \otimes J, L^1 \otimes \mathbf{m}_A] = 0$ and $[[\widetilde{x}, \widetilde{x}], \widetilde{x}] = 0$ by Jacobi identity, we see that $h \in Z^2(L \otimes J)$. Moreover, every other lifting of x is of the form $\widetilde{x}' = \widetilde{x} + y$, $y \in L^1 \otimes J$, whence

$$h' := d\widetilde{x}' + \frac{1}{2}[\widetilde{x}', \widetilde{x}'] = h + dy\,.$$

It follows that the class $\overline{h}$ of h in $H^2(L \otimes J)$ is well defined, and it is very easy to see that defining $v_e(x) := \overline{h}$, $(H^2(L), v_e)$ is a complete linear obstruction theory for MC_L. □

[5] Multiplication in $\exp(N)$ ($N \in$ **NLA**) is given by the Baker–Campbell–Hausdorff formula.

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The cotangent complex in characteristic 0

Marco Manetti

We use the same notation and conventions of [8]; in particular $\mathbb{K}$ will be a fixed field of characteristic 0.

1 Homotopy of differential graded algebras

Let A be a graded algebra, if $A \to B$ is a morphism of graded algebras then B has a natural structure of A-algebra. Given two A-algebras B, C it is defined their tensor product $B \otimes_A C$ as the quotient of $B \otimes_{\mathbb{K}} C = \oplus_{n,m} B_n \otimes_{\mathbb{K}} C_m$ by the ideal generated by $ba \otimes c - b \otimes ac$ for every $a \in A$, $b \in B$, $c \in C$. $B \otimes_A C$ has a natural structure of graded algebra with degrees $\overline{b \otimes c} = \overline{b} + \overline{c}$ and multiplication $(b \otimes c)(\beta \otimes \gamma) = (-1)^{\overline{c}\,\overline{\beta}} b\beta \otimes c\gamma$. Note in particular that $A[\{x_i\}] = A \otimes_{\mathbb{K}} \mathbb{K}[\{x_i\}]$.

Given a dg-algebra A and $h \in \mathbb{K}$ it is defined an evaluation morphism $e_h \colon A[t, dt] \to A$, $e_h(a \otimes p(t)) = ap(h)$, $e_h(a \otimes q(t)dt) = 0$.

Lemma 1.1. *For every dg-algebra A the evaluation map $e_h \colon A[t, dt] \to A$ induces an isomorphism $H(A[t, dt]) \to H(A)$ independent from $h \in \mathbb{K}$.*

Proof. Let $\imath \colon A \to A[t, dt]$ be the inclusion, since $e_h \imath = Id_A$ it is sufficient to prove that $\imath \colon H(A) \to H(A[t, dt])$ is bijective. For every $n > 0$ denote $B_n = At^n \oplus At^{n-1}dt$; since $d(B_n) \subset B_n$ and $A[t, dt] = \imath(A) \bigoplus_{n>0} B_n$ it is sufficient to prove that $H(B_n) = 0$ for every n. Let $z \in Z_i(B_n)$, $z = at^n + nbt^{n-1}dt$, then $0 = dz = dat^n + ((-1)^i a + db)nt^{n-1}dt$ which implies $a = (-1)^{i-1}db$ and then $z = (-1)^{i-1}d(bt^n)$. $\square$

Definition 1.2. *Given two morphisms of dg-algebras $f, g \colon A \to B$, a* homotopy *between f and g is a morphism $H \colon A \to B[t, dt]$ such that $H_0 := e_0 \circ H = f$, $H_1 := e_1 \circ H = g$. We denote by $[A, B]$ the quotient of $\mathrm{Hom}_{\mathbf{DGA}}(A, B)$ by the equivalence relation $\sim$ generated by homotopy. If $B \to C$ is a morphism of dg-algebras with kernel J, a homotopy $H \colon A \to B[t, dt]$ is called* constant on C *if the image of H is contained in $B \oplus_{j \geq 0} (Jt^{j+1} \oplus Jt^j dt)$. Two dg-algebras A, B are said to be* homotopically equivalent *if there exist morphisms $f \colon A \to B$, $g \colon B \to A$ such that $fg \sim Id_B$, $gf \sim Id_A$.*

According to Lemma 1.1 homotopic morphisms induce the same morphism in homology.

Lemma 1.3. *Given morphisms of dg-algebras,*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "l"'] & C \end{tikzcd},$$

if $f \sim g$ and $h \sim l$ then $hf \sim lg$.

Proof. It is obvious from the definitions that $hg \sim lg$. For every $a \in \mathbb{K}$ there exists a commutative diagram

$$\begin{tikzcd} B \otimes \mathbb{K}[t,dt] \arrow[r, "h\otimes Id"] \arrow[d, "e_a"] & C \otimes \mathbb{K}[t,dt] \arrow[d, "e_a"] \\ B \arrow[r, "h"] & C \end{tikzcd} .$$

If $F\colon A \to B[t,dt]$ is a homotopy between f and g, then, considering the composition of F with $h \otimes Id$, we get a homotopy between hf and hg. $\square$

Example 1.4. Let A be a dg-algebra, $\{x_i\}$ a set of indeterminates of integral degree and consider the dg-algebra $B = A[\{x_i, dx_i\}]$, where dx_i is an indeterminate of degree $\overline{dx_i} = \overline{x_i}+1$ and the differential d_B is the unique extension of d_A such that $d_B(x_i) = dx_i$, $d_B(dx_i) = 0$ for every i. The inclusion $i\colon A \to B$ and the projection $\pi\colon B \to A$, $\pi(x_i) = \pi(dx_i) = 0$ give a homotopy equivalence between A and B. In fact $\pi i = Id_A$; consider now the homotopy $H\colon B \to B[t,dt]$ given by

$$H(x_i) = x_i t, \quad H(dx_i) = dH(x_i) = dx_i t + (-1)^{\overline{x_i}} x_i dt, \quad H(a) = a, \forall a \in A.$$

Taking the evaluation at $t = 0, 1$ we get $H_0 = ip$, $H_1 = Id_B$.

Exercise 1.5. Let $f, g\colon A \to C$, $h\colon B \to C$ be morphisms of dg-algebras. If $f \sim g$ then $f \otimes h \sim g \otimes h\colon A \otimes_{\mathbb{K}} B \to C$. $\triangle$

Remark 1.6. *In view of future geometric applications, it seems reasonable to define the spectrum of a unitary dg-algebra A as the usual spectrum of the commutative ring $Z_0(A)$.*

If $S \subset Z_0(A)$ is a multiplicative part we can consider the localized dg-algebra $S^{-1}A$ with differential $d(a/s) = da/s$. Since the localization is an exact functor in the category of $Z_0(A)$ modules we have $H(S^{-1}A) = S^{-1}H(A)$. If $\phi\colon A \to C$ is a morphism of dg-algebras and $\phi(s)$ is invertible for every $s \in S$ then there is a unique morphism $\psi\colon S^{-1}A \to C$ extending ϕ. Moreover if ϕ is a quasiisomorphism then also ψ is a quasiisomorphism (easy exercise).

If $\mathcal{P} \subset Z_0(A)$ is a prime ideal, then we denote as usual $A_{\mathcal{P}} = S^{-1}A$, where $S = Z_0(A) - \mathcal{P}$. It is therefore natural to define Spec(A) *as the ringed space $(X, \tilde{A})$, where X is the spectrum of A and $\tilde{A}$ is the (quasi coherent) sheaf of dg-algebras with stalks $A_{\mathcal{P}}$, $\mathcal{P} \in X$.*

2 Differential graded modules

Let (A, s) be a fixed dg-algebra, by an A-dg-module we mean a differential graded vector space (M, s) together two associative distributive multiplication maps $A \times M \to M$, $M \times A \to M$ with the properties:

1. $A_i M_j \subset M_{i+j}$, $\quad M_i A_j \subset M_{i+j}$.
2. $am = (-1)^{\overline{a}\,\overline{m}} ma$, for homogeneous $a \in A$, $m \in M$.
3. $s(am) = s(a)m + (-1)^{\overline{a}} as(m)$.

If $A = A_0$ we recover the usual notion of complex of A-modules.

If M is an A-dg-module then $M[n] = \mathbb{K}[n] \otimes_{\mathbb{K}} M$ has a natural structure of A-dg-module with multiplication maps

$$(e \otimes m)a = e \otimes ma, \qquad a(e \otimes m) = (-1)^{n\overline{a}} e \otimes am, \qquad e \in \mathbb{K}[n],\, m \in M,\, a \in A.$$

The tensor product $N \otimes_A M$ is defined as the quotient of $N \otimes_{\mathbb{K}} M$ by the graded submodules generated by all the elements $na \otimes m - n \otimes am$.

Given two A-dg-modules $(M, d_M), (N, d_N)$ we denote by

$$\mathrm{Hom}^n_A(M,N) = \{\, f \in \mathrm{Hom}^n_{\mathbb{K}}(M,N) \mid f(ma) = f(m)a,\ m \in M, a \in A\}$$

$$\mathrm{Hom}^*_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_A(M,N).$$

The graded vector space $\mathrm{Hom}^*_A(M,N)$ has a natural structure of A-dg-module with left multiplication $(af)(m) = af(m)$ and differential

$$d\colon\ \mathrm{Hom}^n_A(M,N) \to \mathrm{Hom}^{n+1}_A(M,N), \qquad df = [d,f] = d_N \circ f - (-1)^n f \circ d_M.$$

Note that $f \in \mathrm{Hom}^0_A(M,N)$ is a morphism of A-dg-modules if and only if $df = 0$. A *homotopy* between two morphism of dg-modules $f, g\colon M \to N$ is a $h \in \mathrm{Hom}^{-1}_A(M,N)$ such that $f - g = dh = d_N h + h d_M$. Homotopically equivalent morphisms induce the same morphism in homology.

Morphisms of A-dg-modules $f\colon L \to M$, $h\colon N \to P$ induce, by composition, morphisms $f^*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(L,N)$, $h_*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(M,P)$;

Lemma 2.1. *In the above notation if f is homotopic to g and h is homotopic to l then f^* is homotopic to g^* and l_* is homotopic to h_*.*

Proof. Let $p \in \mathrm{Hom}^{-1}_A(L,M)$ be a homotopy between f and g, It is a straightforward verification to see that the composition with p is a homotopy between f^* and g^*. Similarly we prove that h_* is homotopic to l_*. $\square$

Lemma 2.2. *Let $A \to B$ be a morphism of unitary dg-algebras, M an A-dg-module, N a B-dg-modules. Then there exists a natural isomorphism of B-dg-modules*

$$\mathrm{Hom}^*_A(M,N) \simeq \mathrm{Hom}^*_B(M \otimes_A B, N).$$

Proof. Consider the natural maps:

$$\mathrm{Hom}^*_A(M,N) \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Hom}^*_B(M \otimes_A B, N)\ ,$$

$$Lf(m \otimes b) = f(m)b, \qquad Rg(m) = g(m \otimes 1).$$

We left as exercise the easy verification that $L, R = L^{-1}$ are isomorphism of B-dg-modules. $\square$

Given a morphism of dg-algebras $B \to A$ and an A-dg-module M we set:

$$\mathrm{Der}^n_B(A,M) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A,M) | \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}$$

$$\mathrm{Der}^*_B(A,M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n_B(A,M).$$

As in the case of Hom^*, there exists a structure of A-dg-module on $\mathrm{Der}^*_B(A,M)$ with product $(a\phi)(b) = a\phi(b)$ and differential

$$d\colon\ \mathrm{Der}^n_B(A,M) \to \mathrm{Der}^{n+1}_B(A,M), \qquad d\phi = [d,\phi] = d_M\phi - (-1)^n \phi d_A.$$

Given $\phi \in \mathrm{Der}^n_B(A,M)$ and $f \in \mathrm{Hom}^m_A(M,N)$ their composition $f\phi$ belongs to $\mathrm{Der}^{n+m}_B(A,N)$.

Proposition 2.3. *Let $B \to A$ be a morphisms of dg-algebras: there exists an A-dg-module $\Omega_{A/B}$ together a closed derivation $\delta\colon A \to \Omega_{A/B}$ of degree 0 such that, for every A-dg-module M, the composition with δ gives an isomorphism*

$$\mathrm{Hom}^*_A(\Omega_{A/B}, M) = \mathrm{Der}^*_B(A, M).$$

Proof. Consider the graded vector space

$$F_A = \bigoplus A\delta x, \quad x \in A \text{ homogeneous}, \qquad \overline{\delta x} = \overline{x}.$$

F_A is an A-dg-module with multiplication $a(b\delta x) = ab\delta x$ and differential

$$d(a\delta x) = da\delta x + (-1)^{\overline{a}} a\delta(dx).$$

Note in particular that $d(\delta x) = \delta(dx)$. Let $I \subset F_A$ be the homogeneous submodule generated by the elements

$$\delta(x+y) - \delta x - \delta y, \quad \delta(xy) - x(\delta y) - (-1)^{\overline{x}\,\overline{y}} y(\delta x), \quad \delta(b), b \in B,$$

Since $d(I) \subset I$ the quotient $\Omega_{A/B} = F_A/I$ is still an A-dg-module. By construction the map $\delta\colon A \to \Omega_{A/B}$ is a derivation of degree 0 such that $d\delta = d_\Omega\delta - \delta d_A = 0$. Let $\circ\delta\colon\ \mathrm{Hom}^*_A(\Omega_{A/B}, M) \to \mathrm{Der}^*_B(A, M)$ be the composition with δ:

a) L is a morphism of A-dg-modules. In fact $(af) \circ \delta = a(f \circ \delta)$ for every $a \in A$ and

$$\begin{aligned} d(f \circ \delta)(x) &= d_M(f(\delta x)) - (-1)^{\overline{f}} f\delta(dx) = \\ &= d_M(f(\delta x)) - (-1)^{\overline{f}} f(d(\delta x)) = df \circ \delta. \end{aligned}$$

b) $\circ\delta$ is surjective. Let $\phi \in \mathrm{Der}^n_B(A, M)$; define a morphism $f \in \mathrm{Hom}^n_A(F_A, M)$ by the rule $f(a\delta x) = (-1)^{n\overline{a}} a\phi(x)$; an easy computation shows that $f(I) = 0$ and then f factors to $f \in \mathrm{Hom}^n_A(\Omega_{A/B}, M)$: by construction $f \circ \delta = \phi$.

c) $\circ\delta$ is injective. In fact the image of δ generate $\Omega_{A/B}$.

□

When $B = \mathbb{K}$ we denote for notational simplicity $\mathrm{Der}^*(A, M) = \mathrm{Der}^*_{\mathbb{K}}(A, M)$, $\Omega_A = \Omega_{A/\mathbb{K}}$. Note that if $C \to B$ is a morphism of dg-algebras, then the natural map $\Omega_{A/C} \to \Omega_{A/B}$ is surjective and $\Omega_{A/C} = \Omega_{A/B}$ whenever $C \to B$ is surjective.

Definition 2.4. *The module $\Omega_{A/B}$ is called the module of relative* Kähler differentials *of A over B and δ the* universal derivation.

By the universal property, the module of differential and the universal derivation are unique up to isomorphism.

Example 2.5. If $A_\sharp = \mathbb{K}[\{x_i\}]$ is a polynomial algebra then $\Omega_A = \oplus_i A\delta x_i$ and $\delta\colon A \to \Omega_A$ is the unique derivation such that $\delta(x_i) = \delta x_i$.

Proposition 2.6. *Let $B \to A$ be a morphism of dg-algebras and $S \subset Z_0(A)$ a multiplicative part. Then there exists a natural isomorphism $S^{-1}\Omega_{A/B} = \Omega_{S^{-1}A/B}$.*

Proof. The closed derivation $\delta\colon A \to \Omega_{A/B}$ extends naturally to $\delta\colon S^{-1}A \to S^{-1}\Omega_{A/B}$, $\delta(a/s) = \delta a/s$, and by the universal property there exists a unique morphism of $S^{-1}A$ modules $f\colon \Omega_{S^{-1}A/B} \to S^{-1}\Omega_{A/B}$ and a unique morphism of A modules $g\colon \Omega_{A/B} \to \Omega_{S^{-1}A/B}$. The morphism g extends to a morphism of $S^{-1}A$ modules $g\colon S^{-1}\Omega_{A/B} \to \Omega_{S^{-1}A/B}$. Clearly these morphisms commute with the universal closed derivations and then $gf = Id$. On the other hand, by the universal property, the restriction of fg to $\Omega_{A/B}$ must be the natural inclusion $\Omega_{A/B} \to S^{-1}\Omega_{A/B}$ and then also $fg = Id$. □

3 Projective modules

Definition 3.1. *An A-dg-module P is called* projective *if for every surjective quasiisomorphism $f\colon M \to N$ and every $g\colon P \to N$ there exists $h\colon P \to M$ such that $fh = g$.*

$$\begin{tikzcd} & M \arrow[d, "f"', "qis"] \\ P \arrow[r, "g"'] & N \end{tikzcd} \quad \begin{array}{c}\Rightarrow \\ \\ \Rightarrow\end{array} \quad \begin{tikzcd} & M \arrow[d, "f"', "qis"] \\ P \arrow[ur, "h"] \arrow[r, "g"'] & N \end{tikzcd} .$$

Exercise 3.2. Prove that if $A = A_0$ and $P = P_0$ then P is projective in the sense of 3.1 if and only if P_0 is projective in the usual sense. △

Lemma 3.3. *Let P be a projective A-dg-module, $f\colon P \to M$ a morphism of A-dg-modules and $\phi\colon M \to N$ a surjective quasiisomorphism. If ϕf is homotopic to 0 then also f is homotopic to 0.*

Proof. We first note that there exist natural isomorphisms $\mathrm{Hom}^i_A(P, M[j]) = \mathrm{Hom}^{i+j}_A(P, M)$. Let $h\colon P \to N[-1]$ be a homotopy between ϕf and 0 and consider the A-dg-modules $M \oplus N[-1]$, $M \oplus M[-1]$ endowed with the differentials

$$d\colon M_n \oplus N_{n-1} \to M_{n+1} \oplus N_n, \quad d(m_1, n_2) = (dm_1, f(m_1) - dn_2),$$

$$d\colon M_n \oplus M_{n-1} \to M_{n+1} \oplus M_n, \quad d(m_1, m_2) = (dm_1, m_1 - dm_2).$$

The map $Id_M \oplus f\colon M \oplus M[-1] \to M \oplus N[-1]$ is a surjective quasiisomorphism and $(\phi, h)\colon P \to M \oplus N[-1]$ is morphism of A-dg-modules. If $(\phi, l)\colon P \to M \oplus M[-1]$ is a lifting of (ϕ, h) then l is a homotopy between ϕ and 0. □

Lemma 3.4. *Let $f\colon M \to N$ be a morphism of A-dg-modules, then there exist morphisms of A-dg-modules $\pi\colon L \to M$, $g\colon L \to N$ such that g is surjective, π is a homotopy equivalence and g is homotopically equivalent to $f\pi$.*

Proof. Consider $L = M \oplus N \oplus N[-1]$ with differential

$$d\colon M_n \oplus N_n \oplus N_{n-1} \to M_{n+1} \oplus N_{n+1} \oplus N_n, \quad d(m, n_1, n_2) = (dm, dn_1, n_1 - dn_2).$$

We define $g(m, n_1, n_2) = f(m) + n_1$, $\pi(m, n_1, n_2) = m$ and $s\colon M \to L$, $s(m) = (m, 0, 0)$. Since $gs = f$ and $\pi s = Id_M$ it is sufficient to prove that $s\pi$ is homotopic to Id_L. Take $h \in \mathrm{Hom}^{-1}_A(L, L)$, $h(m, n_1, n_2) = (0, n_2, 0)$; then

$$d(h(m, n_1, n_2)) + hd(m, n_1, n_2) = (0, n_1, n_2) = (Id_L - s\pi)(m, n_1, n_2).$$

□

Theorem 3.5. *Let P be a projective A-dg-module: For every quasiisomorphism $f\colon M \to N$ the induced map $\mathrm{Hom}^*_A(P, M) \to \mathrm{Hom}^*_A(P, N)$ is a quasiisomorphism.*

Proof. By Lemma 3.4 it is not restrictive to assume f surjective. For a fixed integer i we want to prove that $H^i(\mathrm{Hom}^*_A(P, M)) = H^i(\mathrm{Hom}^*_A(P, N))$. Replacing M and N with $M[i]$ and $N[i]$ it is not restrictive to assume $i = 0$. Since $Z^0(\mathrm{Hom}^*_A(P, N))$ is the set of morphisms of A-dg-modules and P is projective, the map

$$Z^0(\mathrm{Hom}^*_A(P, M)) \to Z^0(\mathrm{Hom}^*_A(P, N))$$

is surjective. If $\phi \in Z^0(\mathrm{Hom}^*_A(P, M))$ and $f\phi \in B^0(\mathrm{Hom}^*_A(P, N))$ then by Lemma 3.3 also ϕ is a coboundary. □

A projective resolution of an A-dg-module M is a surjective quasiisomorphism $P \to M$ with P projective. We will show in next section that projective resolutions always exist. This allows to define for every pair of of A-dg-modules M, N

$$\mathrm{Ext}^i(M, N) = H^i(\mathrm{Hom}^*_A(P, N)),$$

where $P \to M$ is a projective resolution.

Exercise 3.6. Prove that the definition of Ext's is independent from the choice of the projective resolution. $\triangle$

4 Semifree resolutions

From now on K is a fixed dg-algebra.

Definition 4.1. *A K-dg-algebra (R, s) is called* semifree *if:*

1. *The underlying graded algebra R is a polynomial algebra over K $K[\{x_i\}]$, $i \in I$.*
2. *There exists a filtration $\emptyset = I(0) \subset I(1) \subset \dots\ , \cup_{n\in\mathbb{N}} I(n) = I$, such that $s(x_i) \in R(n)$ for every $i \in I(n+1)$, where by definition $R(n) = K[\{x_i\}]$, $i \in I(n)$.*

Note that $R(0) = K$, $R(n)$ is a dg-subalgebra of R and $R = \cup R(n)$.

Let $R = K[\{x_i\}] = \cup R(n)$ be a semifree K-dg-algebra, S a K-dg-algebra; to give a morphism $f\colon R \to S$ is the same to give a sequence of morphisms $f_n\colon R(n) \to S$ such that f_{n+1} extends f_n for every n. Given a morphism $f_n\colon R(n) \to S$, the set of extensions $f_{n+1}\colon R(n+1) \to S$ is in bijection with the set of sequences $\{f_{n+1}(x_i)\}$, $i \in I(n+1) - I(n)$, such that $s(f_{n+1}(x_i)) = f_n(s(x_i))$, $\overline{f_{n+1}(x_i)} = \overline{x_i}$.

Example 4.2. $\mathbb{K}[t, dt]$ is semifree with filtration $\mathbb{K} \oplus \mathbb{K}\,dt \subset \mathbb{K}[t, dt]$. For every dg-algebra A and every $a \in A_0$ there exists a unique morphism $f\colon \mathbb{K}[t, dt] \to A$ such that $f(t) = a$.

Exercise 4.3. Let (V, s) be a complex of vector spaces, the differential s extends to a unique differential s on the symmetric algebra $\bigodot V$ such that $s(\bigodot^n V) \subset \bigodot^n V$ for every n. Prove that $(\bigodot V, s)$ is semifree. $\triangle$

Exercise 4.4. The tensor product (over K) of two semifree K-dg-algebras is semifree. $\triangle$

Proposition 4.5. *Let $(R = K[\{x_i\}], s)$, $i \in \cup I(n)$, be a semifree K-dg-algebra: for every surjective quasiisomorphism of K-dg-algebras $f\colon A \to B$ and every morphism $g\colon R \to B$ there exists a lifting $h\colon R \to A$ such that $fh = g$. Moreover any two of such liftings are homotopic by a homotopy constant on B.*

Proof. Assume by induction on n that it is defined a morphism $h_n\colon R(n) \to A$ such that fh_n equals the restriction of g to $R(n) = \mathbb{K}[\{x_i\}]$, $i \in I(n)$. Let $i \in I(n+1) - I(n)$, we need to define $h_{n+1}(x_i)$ with the properties $fh_{n+1}(x_i) = g(x_i)$, $dh_{n+1}(x_i) = h_n(dx_i)$ and $\overline{h_{n+1}(x_i)} = \overline{x_i}$. Since $dh_n(dx_i) = 0$ and $fh_n(dx_i) = g(dx_i) = dg(x_i)$ we have that $h_n(dx_i)$ is exact in A, say $h_n(dx_i) = da_i$; moreover $d(f(a_i) - g(x_i)) = f(da_i) - g(dx_i) = 0$ and, since $Z(A) \to Z(B)$ is surjective there exists $b_i \in A$ such that $f(a_i + b_i) = g(x_i)$ and then we may define $h_{n+1}(x_i) = a_i + b_i$. The inverse limit of h_n gives the required lifting.
Let $h, l\colon R \to A$ be liftings of g and denote by $J \subset A$ the kernel of f; by assumption J is acyclic and consider the dg-subalgebra $C \subset A[t, dt]$,

$$C = A \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt).$$

We construct by induction on n a coherent sequence of morphisms $H_n\colon R(n)\to C$ giving a homotopy between h and l. Denote by $N\subset \mathbb{K}[t,dt]$ the differential ideal generated by $t(t-1)$; there exists a direct sum decomposition $\mathbb{K}[t,dt]=\mathbb{K}\oplus\mathbb{K}t\oplus\mathbb{K}dt\oplus N$. We may write:

$$H_n(x)=h(x)+(l(x)-h(x))t+a_n(x)dt+b_n(x,t),$$

with $a_n(x)\in J$ and $b_n(x,t)\in J\otimes N$. Since $dH_n(x)=H_n(dx)$ we have for every $x\in R(n)$:

$$(-1)^{\overline{x}}(l(x)-h(x))+d(a_n(x))=a_n(dx),\quad d(b_n(x,t))=b_n(dx,t). \tag{1}$$

Let $i\in I(n+1)-I(n)$, we seek for $a_{n+1}(x_i)\in J$ and $b_{n+1}(x_i,t)\in J\otimes N$ such that, setting

$$H_{n+1}(x_i)=h(x_i)+(l(x_i)-h(x_i))t+a_{n+1}(x_i)dt+b_{n+1}(x_i,t),$$

we want to have

$$\begin{aligned}0&=dH_{n+1}(x_i)-H_n(dx_i)\\&=((-1)^{\overline{x_i}}(l(x_i)-h(x_i))+da_{n+1}(x_i)-a_n(dx_i))dt+db_{n+1}(x_i,t)-b_n(dx_i,t).\end{aligned}$$

Since both J and $J\otimes N$ are acyclic, such a choice of $a_{n+1}(x_i)$ and $b_{n+1}(x_i,t)$ is possible if and only if $(-1)^{\overline{dx_i}}(l(x_i)-h(x_i))+a_n(dx_i)$ and $b_n(dx_i,t)$ are closed.
According to Equation 1 we have

$$\begin{aligned}d((-1)^{\overline{dx_i}}(l(x_i)-h(x_i)+a_n(dx_i))&=(-1)^{\overline{dx_i}}(l(dx_i)-h(dx_i))+d(a_n(dx_i))\\&=a_n(d^2x_i)=0,\\db_n(dx_i,t)&=b_n(d^2x_i,t)=0.\end{aligned}$$

□

Definition 4.6. *A* K-semifree resolution *(also called resolvent) of a K-dg-algebra A is a surjective quasiisomorphism $R\to A$ with R semifree K-dg-algebra.*

By 4.5 if a semifree resolution exists then it is unique up to homotopy.

Theorem 4.7. *Every K-dg-algebra admits a K-semifree resolution.*

Proof. Let A be a K-dg-algebra, we show that there exists a sequence of K-dg-algebras $K=R(0)\subset R(1)\subset\ldots\subset R(n)\subset\ldots$ and morphisms $f_n\colon R(n)\to A$ such that:

1. $R(n+1)=R(n)[\{x_i\}]$, $dx_i\in R(n)$.
2. f_{n+1} extends f_n.
3. $f_1\colon Z(R(1))\to Z(A)$, $f_2\colon R(2)\to A$ are surjective.
4. $f_n^{-1}(B(A))\cap Z(R(n))\subset B(R(n+1))\cap R(n)$, for every $n>0$.

It is then clear that $R=\cup R(n)$ and $f=\varinjlim f_n$ give a semifree resolution. $Z(A)$ is a graded algebra and therefore there exists a polynomial graded algebra $R(1)=K[\{x_i\}]$ and a surjective morphism $f_1\colon R(1)\to Z(A)$; we set the trivial differential $d=0$ on $R(1)$. Let v_i be a set of homogeneous generators of the ideal $f_1^{-1}(B(A))$, if $f_1(v_i)=da_i$ it is not restrictive to assume that the a_i's generate A. We then define $R(2)=R(1)[\{x_i\}]$, $f_2(x_i)=a_i$ and $dx_i=v_i$. Assume now by induction that we have defined $f_n\colon R(n)\to A$ and let $\{v_j\}$ be a set of generators of $f_n^{-1}(B(A))\cap Z(R(n))$, considered as an ideal of $Z(R(n))$; If $f_n(v_j)=da_j$ we define $R(n+1)=R(n)[\{x_j\}]$, $dx_j=v_j$ and $f_{n+1}(x_j)=a_j$. □

Remark 4.8. *It follows from the above proof that if $K_i = A_i = 0$ for every $i > 0$ then there exists a semifree resolution $R \to A$ with $R_i = 0$ for every $i > 0$.*

Exercise 4.9. If, in the proof of Theorem 4.7 we choose at every step $\{v_i\} = f_n^{-1}(B(A)) \cap Z(R(n))$ we get a semifree resolution called *canonical*. Show that every morphism of dg-algebras has a natural lift to their canonical resolutions. △

Given two semifree resolutions $R \to A$, $S \to A$ we can consider a semifree resolution $P \to R \times_A S$ of the fibred product and we get a commutative diagram of semifree resolutions

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr] & R \arrow[d] \\ R \arrow[r] & A. \end{tikzcd}$$

Definition 4.10. *An A-dg-module F is called* semifree *if $F = \oplus_{i \in I} A m_i$, $\overline{m_i} \in \mathbb{Z}$ and there exists a filtration $\emptyset = I(0) \subset I(1) \subset \ldots \subset I(n) \subset \ldots$ such that*

$$i \in I(n+1) \Rightarrow dm_i \in F(n) = \oplus_{i \in I(n)} A m_i.$$

A semifree resolution *of an A-dg-module M is a surjective quasiisomorphism $F \to M$ with F semifree.*

The proof of the following two results is essentially the same of 4.5 and 4.7:

Proposition 4.11. *Every semifree module is projective.*

Theorem 4.12. *Every A-dg-module admits a semifree resolution.*

Exercise 4.13. An A-dg-module M is called *flat* if for every quasiisomorphism $f : N \to P$ the morphism $f \otimes Id : N \otimes M \to P \otimes M$ is a quasiisomorphism. Prove that every semifree module is flat. △

Example 4.14. If $R = K[\{x_i\}]$ is a K-semifree algebra then $\Omega_{R/K} = \oplus R\delta x_i$ is a semifree R-dg-module.

5 The cotangent complex

Proposition 5.1. *Assume it is given a commutative diagram of K-dg-algebras*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "p"'] & S \arrow[d] & R \arrow[l, "g"'] \arrow[dl, "p"] \\ & A & \end{tikzcd}$$

If there exists a homotopy between f and g, constant on A, then the induced morphisms of A-dg-modules

$$f, g : \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$$

are homotopic.

Proof. Let $J \subset S$ be the kernel of $S \to A$ and let $H\colon R \to S \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt)$ be a homotopy between f and g; the first terms of H are

$$H(x) = f(x) + t(g(x) - f(x)) + dt\, q(x) + \dots .$$

From $dH(x) = H(dx)$ we get $g(x) - f(x) = q(dx) + dq(x)$ and from $H(xy) = H(x)H(y)$ follows $q(xy) = q(x)f(y) + (-1)^{\overline{x}} f(x)q(y)$. Since $f(x) - g(x), q(x) \in J$ for every x, the map

$$q\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A, \quad q(\delta x \cdot r \otimes a) = \delta(q(x)) f(r) \otimes a,$$

is a well defined element of $\operatorname{Hom}^{-1}_A(\Omega_{R/K} \otimes_R A, \Omega_{S/K} \otimes_S A)$. By definition $f, g\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$ are defined by

$$f(\delta x \cdot r \otimes a) = \delta(f(x)) f(r) \otimes a, \quad g(\delta x \cdot r \otimes a) = \delta(g(x)) g(r) \otimes a = \delta(g(x)) f(r) \otimes a.$$

A straightforward verification shows that $dq = f - g$. □

Definition 5.2. *Let $R \to A$ be a K-semifree resolution, the A-dg-module $\mathbb{L}_{A/K} = \Omega_{R/K} \otimes_R A$ is called the* relative cotangent complex *of A over K. By 5.1 the homotopy class of $\mathbb{L}_{A/K}$ is independent from the choice of the resolution. For every A-dg-module M the vector spaces*

$$T^i(A/K, M) = H^i(\operatorname{Hom}^*_A(\mathbb{L}_{A/K}, M)) = \operatorname{Ext}^i_A(\mathbb{L}_{A/K}, M),$$

$$T_i(A/K, M) = H_i(\mathbb{L}_{A/K} \otimes M)) = \operatorname{Tor}^A_i(\mathbb{L}_{A/K}, M),$$

are called respectively the cotangent and tangent cohomolgy of the morphism $K \to A$ with coefficient on M.

Lemma 5.3. *Let $p\colon R \to S$ be a surjective quasiisomorphism of semifree dg-algebras: consider on S the structure of R-dg-module induced by p. Then:*

1. *$p_*\colon \operatorname{Der}^*(R, R) \to \operatorname{Der}^*(R, S)$, $f \to pf$, is a surjective quasiisomorphism.*
2. *$p^*\colon \operatorname{Der}^*(S, S) \to \operatorname{Der}^*(R, S)$, $f \to fp$, is an injective quasiisomorphism.*

Proof. A derivation on a semifree dg-algebra is uniquely determined by the values at its generators, in particular p_* is surjective and p^* is injective. Since Ω_R is semifree, by 3.5 the morphism $p_*\colon \operatorname{Hom}^*_R(\Omega_R, R) \to \operatorname{Hom}^*_R(\Omega_R, S)$ is a quasiisomorphism. By base change $\operatorname{Der}^*(R, S) = \operatorname{Hom}^*_S(\Omega_R \otimes_R S, S)$ and, since $p\colon \Omega_R \otimes_R S \to \Omega_S$ is a homotopy equivalence, also p^* is a quasiisomorphism. □

Every morphism $f\colon A \to B$ of dg-algebras induces a morphism of B modules $\mathbb{L}_A \otimes_A B \to \mathbb{L}_B$ unique up to homotopy. In fact if $R \to A$ and $P \to B$ are semifree resolution, then there exists a lifting of f, $R \to P$, unique up to homotopy constant on B. The morphism $\Omega_R \to \Omega_P$ induce a morphism $\Omega_R \otimes_R B = \mathbb{L}_A \otimes_A B \to \Omega_P \otimes_P B = \mathbb{L}_B$. If B is a localization of A we have the following

Theorem 5.4. *Let A be a dg-algebra, $S \subset Z_0(A)$ a multiplicative part: then the morphism*

$$\mathbb{L}_A \otimes_A S^{-1}A \to \mathbb{L}_{S^{-1}A}$$

is a quasiisomorphism of $S^{-1}A$ modules.

Proof. (sketch) Denote by $f\colon R\to A$, $g\colon P\to S^{-1}A$ two semifree resolutions and by

$$H=\{x\in Z_0(R)\,|\,f(x)\in S\},\qquad K=\{x\in Z_0(P)\,|\,g(x)\text{ is invertible }\}.$$

The natural morphisms $H^{-1}R\to S^{-1}A$, $K^{-1}P\to S^{-1}A$ are both surjective quasiisomorphisms. By the lifting property of semifree algebras we have a chain of morphisms

$$R\xrightarrow{\alpha}P\xrightarrow{\beta}H^{-1}R\xrightarrow{\gamma}K^{-1}P$$

with γ the localization of α. Since $\beta\alpha$ and $\gamma\beta$ are homotopic to the natural inclusions $R\to H^{-1}R$, $P\to K^{-1}P$, the composition of morphisms

$$\Omega_R\otimes_R S^{-1}A\xrightarrow{\alpha}\Omega_P\otimes_P S^{-1}A\xrightarrow{\beta}\Omega_{H^{-1}R}\otimes_{H^{-1}R}S^{-1}A=\Omega_R\otimes_R S^{-1}A,$$

$$\Omega_P\otimes_P S^{-1}A\xrightarrow{\beta}\Omega_{H^{-1}R}\otimes_{H^{-1}R}S^{-1}A\xrightarrow{\gamma}\Omega_{K^{-1}P}\otimes_{K^{-1}P}S^{-1}A=\Omega_P\otimes_P S^{-1}A$$

are homotopic to the identity and hence quasiisomorphisms. □

Example 5.5. Hypersurface singularities.
Let $X=V(f)\subset\mathbb{A}^n$, $f\in\mathbb{K}\,[x_1,\dots,x_n]$, be an affine hypersurface and denote by $A=\mathbb{K}\,[X]=\mathbb{K}\,[x_1,\dots,x_n]/(f)$ its structure ring. A DG-resolvent of A is given by $R=\mathbb{K}\,[x_1,\dots,x_n,y]$, where y has degree -1 and the differential is given by $s(y)=f$. The R-module Ω_R is semifreely generated by $dx_1,\dots,dx_n,dy$, with the differential

$$s(dy)=d(s(y))=df=\sum_i\frac{\partial f}{\partial x_i}dx_i.$$

The cotangent complex $\mathbb{L}_A$ is therefore

$$0\longrightarrow Ady\xrightarrow{s}\bigoplus_{i=1}^{n}Adx_i\longrightarrow 0.$$

In particular $T^i(A/\mathbb{K}\,,A)=\operatorname{Ext}^i(\mathbb{L}_A,A)=0$ for every $i\neq 0,1$. The cokernel of s is isomorphic to Ω_A and then $T^0(A/\mathbb{K}\,,A)=\operatorname{Ext}^0(\mathbb{L}_A,A)=\operatorname{Der}_{\mathbb{K}}(A,A)$. If f is reduced then s is injective, $\mathbb{L}_A$ is quasiisomorphic to Ω_A and then $T^1(A/\mathbb{K}\,,A)=\operatorname{Ext}^1(\Omega_A,A)$.

Exercise 5.6. In the set-up of Example 5, prove that the A-module $T^1(A/\mathbb{K}\,,A)$ is finitely generated and supported in the singular locus of X. △

6 The controlling differential graded Lie algebra

Let $p\colon R\to S$ be a surjective quasiisomorphism of semifree algebras and let $I=\ker p$. By the lifting property of S there exists a morphism of dg-algebras $e\colon S\to R$ such that $pe=Id_S$. Define

$$L_p=\{f\in\operatorname{Der}^*(R,R)\,|\,f(I)\subset I\}.$$

It is immediate to verify that L_p is a dg-Lie subalgebra of $\operatorname{Der}^*(R,R)$. We may define a map

$$\theta_p\colon L_p\to\operatorname{Der}^*(S,S),\qquad \theta_p(f)=p\circ f\circ e.$$

Since $pf(I)=0$ for every $f\in L_p$, the definition of θ_p is independent from the choice of e.

Lemma 6.1. *θ_p is a morphism of DGLA.*

Proof. For every $f, g \in L_p$, $s \in S$, we have:

$$d(\theta_p f)(s) = dpfe(s) - (-1)^{\overline{f}} pfe(ds) = pdfe(s) - (-1)^{\overline{f}} pfd(e(s)) = \theta_p(df)(s).$$

Since $pfep = pf$ and $pgep = pg$

$$[\theta_p f, \theta_p g] = pfepge - (-1)^{\overline{f}\,\overline{g}} pgepfe = p(fg - (-1)^{\overline{f}\,\overline{g}} gf)e = \theta_p([f, g]).$$

$\square$

Theorem 6.2. *The following is a cartesian diagram of quasiisomorphisms of DGLA*

$$\begin{tikzcd} L_p \arrow[r, hook, "\imath_p"] \arrow[d, "\theta_p"] & \mathrm{Der}^*(R,R) \arrow[d, "p_*"] \\ \mathrm{Der}^*(S,S) \arrow[r, "p^*"'] & \mathrm{Der}^*(R,S) \end{tikzcd},$$

where $\imath_p$ is the inclusion.

We recall that cartesian means that it is commutative and that L_p is isomorphic to the fibred product of p_* and p^*.

Proof. Since $pfep = pf$ for every $f \in L_p$ we have $p^*\theta_p(f) = pfep = pf = p_*f$ and the diagram is commutative. Let

$$K = \{(f, g) \in \mathrm{Der}^*(R, R) \times \mathrm{Der}^*(S, S) \,|\, pf = gp\}$$

be the fibred product; the map $L_p \to K$, $f \to (f, \theta_p(f))$, is clearly injective. Conversely take $(f, g) \in K$ and $x \in I$, since $pf(x) = gp(x) = 0$ we have $f(I) \subset I$, $f \in L_p$. Since p is surjective g is uniquely determined by f and then $g = \theta_p(f)$. This proves that the diagram is cartesian. By 5.3 p_* (resp.: p^*) is a surjective (resp.: injective) quasiisomorphism, by a standard argument in homological algebra also θ_p (resp.: $\imath_p$) is a surjective (resp.: injective) quasiisomorphism. $\square$

Corollary 6.3. *Let $P \to A$, $Q \to A$ be semifree resolutions of a dg-algebra. Then* $\mathrm{Der}^*(P, P)$ *and* $\mathrm{Der}^*(Q, Q)$ *are quasiisomorphic DGLA.*

Proof. There exists a third semifree resolution $R \to A$ and surjective quasiisomorphisms $p\colon R \to P$, $q\colon R \to Q$. Then there exists a sequence of quasiisomorphisms of DGLA

$$\begin{tikzcd} & L_p \arrow[dl, "\theta_p"'] \arrow[dr, "\imath_p"] & & L_q \arrow[dl, "\imath_q"'] \arrow[dr, "\theta_q"] & \\ \mathrm{Der}^*(P,P) & & \mathrm{Der}^*(R,R) & & \mathrm{Der}^*(Q,Q). \end{tikzcd}$$

$\square$

Remark 6.4. *If $R \to A$ is a semifree resolution then*

$$H^i(\mathrm{Der}^*(R, R)) = H^i(\mathrm{Hom}_R(\Omega_R, R)) = H^i(\mathrm{Hom}_R(\Omega_R, A)) =$$

$$= H^i(\mathrm{Hom}_A(\Omega_R \otimes_R A, A)) = \mathrm{Ext}^i(\mathbb{L}_A, A).$$

Unfortunately, contrarily to what happens to the cotangent complex, the application $R \to \mathrm{Der}^*(R,R)$ is quite far from being a functor: it only earns some functorial properties when composed with a suitable functor $\mathbf{DGLA} \to \mathbf{D}$.

Let $\mathbf{D}$ be a category and $\mathcal{F}\colon \mathbf{DGLA} \to \mathbf{D}$ be a functor which sends quasiisomorphisms into isomorphisms of $\mathbf{D}$[1]. By 6.3, if $P \to A$, $Q \to A$ are semifree resolutions then $\mathcal{F}(\mathrm{Der}^*(P,P)) \simeq \mathcal{F}(\mathrm{Der}^*(Q,Q))$; now we prove that the recipe of the proof of 6.3 gives a NATURAL isomorphism independent from the choice of P, p, q. For notational simplicity denote $\mathcal{F}(P) = \mathcal{F}(\mathrm{Der}^*(P,P))$ and for every surjective quasiisomorphism $p\colon R \to P$ of semifree dg-algebras, $\mathcal{F}(p) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}\colon \mathcal{F}(R) \to \mathcal{F}(P)$.

Lemma 6.5. *Let $p\colon R \to P$, $q\colon P \to Q$ be surjective quasiisomorphisms of semifree dg-algebras, then $\mathcal{F}(qp) = \mathcal{F}(q)\mathcal{F}(p)$.*

Proof. Let $I = \ker p$, $J = \ker q$, $H = \ker qp = p^{-1}(J)$, $e\colon P \to R$, $s\colon Q \to P$ sections. Note that $e(J) \subset H$. Let $L = L_q \times_{\mathrm{Der}^*(P,P)} L_p$, if $(f,g) \in L$ and $x \in H$ then $pg(x) = pg(ep(x)) = f(x) \in J$ and then $g(x) \in H$, $g \in L_{qp}$; denoting $\alpha\colon L \to L_{qp}$, $\alpha(f,g) = g$, we have a commutative diagram of quasiisomorphisms of DGLA

$$\begin{tikzcd}
L_{qp} \arrow[rrrd, "\imath_{qp}"] \arrow[dddr, "\theta_{qp}"'] & & & \\
& L \arrow[ul, "\alpha"'] \arrow[r, "\beta"] \arrow[d, "\gamma"'] & L_p \arrow[r, "\imath_p"'] \arrow[d, "\theta_p"] & \mathrm{Der}^*(R,R) \\
& L_q \arrow[r, "\imath_q"'] \arrow[d, "\theta_q"] & \mathrm{Der}^*(P,P) & \\
& \mathrm{Der}^*(Q,Q) & &
\end{tikzcd}$$

and then

$$\mathcal{F}(qp) = \mathcal{F}(\theta_{qp})\mathcal{F}(\imath_{qp})^{-1} = \mathcal{F}(\theta_q)\mathcal{F}(\gamma)\mathcal{F}(\alpha)^{-1}\mathcal{F}(\alpha)\mathcal{F}(\beta)^{-1}\mathcal{F}(\imath_p)^{-1} =$$

$$= \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1} = \mathcal{F}(q)\mathcal{F}(p).$$

□

Let P be a semifree dg-algebra $Q = P[\{x_i, dx_i\}] = P \otimes_{\mathbb{K}} \mathbb{K}[\{x_i, dx_i\}]$, $i\colon P \to Q$ the natural inclusion and $\pi\colon Q \to P$ the projection $\pi(x_i) = \pi(dx_i) = 0$: note that i, π are quasiisomorphisms. Since P, Q are semifree we can define a morphism of DGLA

$$\begin{aligned} i\colon \mathrm{Der}^*(P,P) &\longrightarrow \mathrm{Der}^*(Q,Q), \\ (if)(x_i) &= (if)(dx_i) = 0, \\ (if)(p) &= i(f(p)),\ p \in P. \end{aligned}$$

Since $\pi_* i = \pi^*\colon \mathrm{Der}^*(P,P) \to \mathrm{Der}^*(Q,P)$, according to 5.3 i is an injective quasiisomorphism.

Lemma 6.6. *Let P, Q as above, let $q\colon Q \to R$ a surjective quasiisomorphism of semifree algebras. If $p = qi\colon P \to R$ is surjective then $\mathcal{F}(p) = \mathcal{F}(q)\mathcal{F}(i)$.*

[1]The examples that we have in mind are the associated deformation functor and the homotopy class of the corresponding L_∞-algebra

Proof. Let $L = \mathrm{Der}^*(P,P) \times_{\mathrm{Der}^*(Q,Q)} L_q$ be the fibred product of i and $\imath_q$; if $(f,g) \in L$ then $g = if$ and for every $x \in \ker p$, $i(f(x)) = g(i(x)) \in \ker q \cap i(P) = i(\ker p)$. Denoting $\alpha\colon L \to L_p$, $\alpha(f,g) = f$, we have a commutative diagram of quasiisomorphisms

$$\begin{tikzcd}
& \mathrm{Der}^*(P,P) \arrow[rr, "i"] & & \mathrm{Der}^*(Q,Q) & \\
L_p \arrow[ur, "\imath_p"] \arrow[drr, "\theta_p"'] & & L \arrow[ll, "\alpha"'] \arrow[ul] \arrow[ur] \arrow[rr] & & L_q \arrow[ul, "\imath_q"'] \arrow[dll, "\theta_q"] \\
& & \mathrm{Der}^*(R,R) & &
\end{tikzcd}$$

and then $\mathcal{F}(q)\mathcal{F}(i) = \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(i) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}$. □

Lemma 6.7. *Let $p_0, p_1\colon P \to R$ be surjective quasiisomorphisms of semifree algebras. If p_0 is homotopic to p_1 then $\mathcal{F}(p_0) = \mathcal{F}(p_1)$.*

Proof. We prove first the case $P = R[t,dt]$ and $p_i = e_i$, $i = 0,1$, the evaluation maps. Denote by

$$L = \{f \in \mathrm{Der}^*(P,P) \mid f(R) \subset R, f(t) = f(dt) = 0\}.$$

Then $L \subset L_{e_\alpha}$ for every $\alpha = 0,1$, $\theta_{e_\alpha}\colon L \to \mathrm{Der}^*(P,P)$ is an isomorphism not depending from α and $L \subset L_{e_\alpha} \subset \mathrm{Der}^*(R,R)$ are quasiisomorphic DGLA. This proves that $\mathcal{F}(e_0) = \mathcal{F}(e_1)$. In the general case we can find commutative diagrams, $\alpha = 0,1$,

$$\begin{tikzcd}
P[\{x_j, dx_j\}] \arrow[r, "q"] \arrow[dr, "q_\alpha"] & R[t,dt] \arrow[d, "e_\alpha"] \\
P \arrow[u, "i"] \arrow[r, "p_\alpha"'] & R
\end{tikzcd}$$

with q surjective quasiisomorphism. We then have $\mathcal{F}(p_0) = \mathcal{F}(q_0)\mathcal{F}(i)^{-1} = \mathcal{F}(e_0)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(e_1)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(q_1)\mathcal{F}(i)^{-1} = \mathcal{F}(p_1)$. □

We are now able to prove the following

Theorem 6.8. *Let*

$$\begin{tikzcd}
R \arrow[r, "p"] \arrow[d, "q"'] & P \arrow[d] \\
Q \arrow[r] & A
\end{tikzcd}$$

be a commutative diagram of surjective quasiisomorphisms of dg-algebras with P, Q, R semifree. Then $\Psi = \mathcal{F}(p)\mathcal{F}(q)^{-1}\colon \mathcal{F}(Q) \to \mathcal{F}(P)$ does not depend from R, p, q.

Proof. Consider two diagrams as above

$$\begin{tikzcd}
R_0 \arrow[r, "p_0"] \arrow[d, "q_0"'] & P \arrow[d] \\
Q \arrow[r] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
R_1 \arrow[r, "p_1"] \arrow[d, "q_1"'] & P \arrow[d] \\
Q \arrow[r] & A.
\end{tikzcd}$$

There exists a commutative diagram of surjective quasiisomorphisms of semifree algebras

$$\begin{tikzcd} T \arrow[r, "t_1"] \arrow[d, "t_0"'] & R_1 \arrow[d, "q_1"] \\ R_0 \arrow[r, "q_0"'] & Q. \end{tikzcd}$$

By Lemma 6.5 $\mathcal{F}(q_0)\mathcal{F}(t_0) = \mathcal{F}(q_1)\mathcal{F}(t_1)$. According to 4.5 the morphisms $p_0t_0, p_1t_1 : T \to P$ are homotopic and then $\mathcal{F}(p_0)\mathcal{F}(t_0) = \mathcal{F}(p_1)\mathcal{F}(t_1)$. This implies that $\mathcal{F}(p_0)\mathcal{F}(q_0)^{-1} = \mathcal{F}(p_1)\mathcal{F}(q_1)^{-1}$. $\square$

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Rational homotopy and deformation theory

Michele Grassi

Introduction

In these notes we provide a brief introduction to some of the results contained in the papers [7] and [9]. In the firts of these two papers it is shown that for any differential graded algebra (DGA in what follows) with finite dimensional cohomology one can find an element in its *rational homotopy class* via a *deformation* of a *bigraded model* of its cohomology algebra. It is also explained what happens when two such deformations belong to the same rational homotopy class. In the second paper this result is recast in the language of differential graded Lie algebras and differential graded coalgebras (respectively DGLA and DGcoA in what follows), in order to obtain a deformation theoretic statement. The outcome is that the set of rational homotopy classes of DGA's with cohomology algebra isomorphic to H can be described as $V/\mathrm{Aut}(H)$, where V is the set of solutions to the *Maurer-Cartan* equation modulo *gauge equivalence* in an explicitely described DGLA. To prove this result the authors of [9] find it useful (and "philosophically" appealing) to pass tho the language of DGcoA's. In this notes, to provide a treatement as elementary as possible, we first state and prove (following the ideas and methods of [9]) the result in terms of DGLA's, and only afterwards we restate it in the language of coalgebras. With this language the main result says that the set of *augmented* rational homotopy classes of DGA's with a fixed cohomology algebra is described by the *path components* of an explicitely described DGcoA.

These notes are not intended as a replacement for the original papers (which are respectively 46 and 45 pages long, and contain much more material than we were able to cover). We have tried instead to go directly to the most central results, unfortunately omitting many interesting results and, more importantly, all the examples and applications. One possible use for our work is as a preliminary reading. One can then go on and read the original papers with a direct knowledge of the main difficulties that lie ahead, and of their solutions. A more exhaustive treatement of these (and other related) topics can be found in [5].

1 Background of rational homotopy theory

In this section we state without proof some results from rational homotopy theory, to put the papers [7] and [9] into perspective. This section can be omitted without affecting in any way the completeness of the following treatement. We refer to [10], [2], to [1] and to [5] for more details, notations and terminology.

Definition 1.1. *Two simply connected CW complexes X, Y have the same* rational homotopy type *if there exist a sequence of simply connected CW complexes $Z_0, ..., Z_n$ and morphisms $f_1, ..., f_n$ such that:*

1. *$Z_0 = X$ and $Z_n = Y$.*

2. *Either $f_i : Z_{i-1} \to Z_i$ or $f_i : Z_i \to Z_{i-1}$.*

3. *All the f_i induce isomorphisms in rational cohomology.*

Definition 1.2. *Two DGA's (A, d_A) and (B, d_B) have the same* rational homotopy type *if there exist a sequence of DGA's $(C_0, d_0), \dots, (C_n, d_n)$ and morphisms $\phi_1, ..., \phi_n$ such that:*

1. *$(C_0, d_0) = (A, d_A)$, $(C_n, d_n) = (B, d_B)$.*
2. *Either $\phi_i : (C_{i-1}, d_{i-1}) \to (C_i, d_i)$ or $\phi_i : (C_i, d_i) \to (C_{i-1}, d_{i-1})$.*
3. *All the ϕ_i are DGA morphisms which induce isomorphisms in cohomology.*

There is a functorial way to associate an isomorphism class of DGA's to a simply connected CW complex, in order to relate the two notions above of rational homotopy equivalence. The first method was described by Quillen (see [8]), but in a rather indirect way. Afterwards Sullivan described a more direct construction (see [10]). The following method (which we took from [1]) is less "computable" than Sullivan's one, but it has the advantage of being very direct, and of providing an explicit DGA and not, as in the case of Quillen's and Sullivan's methods, an isomorphism class of DGA's. It has also the advantage of being defined and making sense for all topological spaces (actually all simplicial sets) and not just simply connected CW complexes. Of course, also this construction is (at least by now) very standard, and in any case all three determine the same rational homotopy class of DGA's for a given simply connected CW complex.

Definition 1.3. *The category of finite ordinals Δ is defined by:*

$$\begin{aligned} \mathrm{Ob}(\Delta) &= \{[n] | n \in \mathbb{N}\} \\ \mathrm{Hom}_\Delta([n],[m]) &= \{\theta : \{0, \dots, n\} \to \{0, \dots, m\} | \theta \text{ is an order preserving map}\} \end{aligned}$$

Definition 1.4. *A* simplicial object *(resp.* cosimplicial object*) in the category $\mathcal{C}$ is a contravariant (resp. covariant) functor from the category Δ of finite ordinals to the category $\mathcal{C}$ (See [2]). A morphism of simplicial (or cosimplicial) objects is a natural transformation.*

Definition 1.5.

1. *The* standard simplices

$$\Delta^n = \{x \in \mathbb{R}^{n+1} | \forall i \, 0 \leq x_i \leq 1, \sum_i x_i = 1\}$$

give rise to a covariant functor

$$\Delta^- : \Delta \longrightarrow \top$$

(i.e. a cosimplicial topological space) defined as $\Delta^-([n]) = \Delta^n$, $\Delta^-(\theta) = \Delta^\theta$, with $\Delta^\theta(x)_i = \sum_{\theta(j)=i} x_j$.

2. *The simplicial differential graded algebra $\nabla = \bigoplus_{p,q} \nabla([p])^q$ is defined by (see [1], page 1) :*

   (a)

$$\nabla([p]) = S(t_0, ..., t_p, dt_0, ..., dt_p)/(t_0 + \cdots + t_p - 1, dt_0 + \cdots + dt_p)$$

   with $d(t_i) = dt_i$ for all i. We also indicate $t_i \in \nabla([p])$ with $t_i(p)$, and $dt_j \in \nabla([p])$ with $dt_j(p)$.

   (b) If $\theta : [m] \to [n]$, then $\nabla(\theta) : \nabla([n]) \to \nabla([m])$ is induced by the linear map $t_i(n) \to \sum_{\theta(j)=i} t_j(m)$, $dt_i(n) \to \sum_{\theta(j)=i} dt_j(m)$.

If we think of Δ^θ as a smooth map, and of $\nabla([n])$ as a set of smooth differential forms, then $\nabla(\theta) = (\Delta^\theta)^*$.

Definition 1.6. *Let X be a topological space.*

1. $\mathrm{Sing}(X)$ *is a simplicial set, defined as*

$$\mathrm{Sing}(X) \; = \; \mathrm{Hom}_{\top}(\; - \; , X) \circ \Delta^{-}$$

2. *(See [1], page 7)* $A_{\mathrm{PL}}(X)$ *is a DGA, defined as*

$$A_{\mathrm{PL}}(X) \; = \; \mathrm{Hom}_{\mathbf{SimpSet}}(\mathrm{Sing}(X), \nabla)$$

Remark 1.7.

1. $\mathrm{Sing}(X)([n]) \; = \; \mathrm{Hom}_{\top}(\Delta^n, X)$ is the set of continuous maps from the standard n-symplex into X. If $\theta : [m] \to [n]$, then $\mathrm{Sing}(X)(\theta)$ is the map induced by the continuous map $\Delta^{\theta} : \Delta^m \to \Delta^n$.

2. $A_{\mathrm{PL}}(X)$ is the set of "polynomial differential forms on the simplices in X", with wedge multiplication and "deRham" differential.

Theorem 1.8. *Let X be a simply connected CW complex.*

1. *There is a natural morphism (induced by integration) of graded complexes*

$$A_{\mathrm{PL}}(X) \longrightarrow C^*_{\mathrm{Sing}}(X, \mathbb{Q})$$

inducing a graded algebra isomorphism in cohomology .

2. *If X is a smooth manifold, $A_{\mathrm{PL}}(X) \otimes \mathbb{R}$ has the same (real) homotopy type as $\Omega^*_{DR}(X)$.*

3. *If $X = \Sigma$ is a simplicial complex, and S_Σ is the naturally associated simplicial set, then there is a natural inclusion of simplicial sets $S_\Sigma \subset \mathrm{Sing}(S)$, and the induced restriction map $A_{\mathrm{PL}}(\Sigma) \to A^*(\Sigma)$ to the "picewise linear forms" on Σ is a DGA map inducing an isomorphism in cohomology.*

References for the Proof. 1. This is the *PL deRham Theorem*, see [1].

2. This follows from the PL and the smooth de Rham theorems.

3. See [6] and [1].

□

Theorem 1.9. *Let X, Y be two simply connected CW complexes Then the following two facts are equivalent:*

1. *X has the same rational homotopy type as Y.*

2. *$A_{\mathrm{PL}}(X)$ has the same rational homotopy type as $A_{\mathrm{PL}}(Y)$.*

Moreover, any simply connected DGA is in the rational homotopy class of some $A_{\mathrm{PL}}(X)$, for a simply connected CW complex X.

Sketch of Proof. It is clear that a morphism of topological spaces $X_1 \to X_2$ induces a (simplicial) morphism between the associated singular sets $\mathrm{Sing}(X_1) \to \mathrm{Sing}(X_2)$, and therefore a DGA morphism $f^* : A_{\mathrm{PL}}(X_2) \to A_{\mathrm{PL}}(X_1)$. If moreover f induces an isomorphism in rational cohomology, from the PL deRham theorem and the naturality of integration if follows that f^* induces an isomorphism in cohomology. This proves that 1 implies 2.

Assume conversely 2. Then from point 3 of the previous theorem and the theory developed in [1] and [10] the minimal models of X and Y are isomorphic. However, the spatial realization of the minimal model has the same rational homotopy type of the space (see [10]), and therefore 1 follows. □

The previous theorem provides a strong motivation for the study of the set of rational homotopy classes of DG algebras with a given cohomology algebra, at least in the simply connected case. Indeed, such a set will then describe all simply connected CW complexes with given singular cohomology algebra, up to rational homotopy equivalence.

2 DGA's as deformations of the model of their cohomology

In the previous section we have seen that to study the rational homotopy classes of simply connected CW complexes it is enough to study the rational homotopy classes of (simply connected) DGA's. Here we describe how Halperin and Stasheff in [7] provide a method for obtaining all rational homotopy classes of DGA's with a given cohomology algebra. Very briefly, the idea is that one first constructs a minimal model of the cohomology algebra, with an extra gradation (the *bigraded model*). All the rational homotopy classes are then obtained modifying the differential of this minimal model (in a prescribed way, determined by the second gradation). The models that one obtains by this method are called *filtered models*, and are no longer bigraded or minimal. With this procedure one does not obtain a completely satisfactory description, because to different deformations can correspond the same rational homotopy class. This problem is solved in the paper [9], which will be the topic of the next section.

As mentioned above, the starting point of the considerations of [7] is the construction of the Tate-Jozefiak (or *bigraded*) model of a graded algebra (thought of as a DGA whith zero differential). In this section, whenever we are dealing with a DGA C with a lower gradation $C = \bigoplus_i C_i$, we will use the notation $C_{(n)}$ to indicate the DG subalgebra of C generated by the elements in C of degree smaller or equal to n.

Theorem 2.1 (Bigraded model, [7], pages 242-244). *Let $H = H^*$ be a graded commutative connected algebra over the field k. Think of H as a DGA with zero differential. There exist then a bigraded vector space $Z = Z^*_*$ together with a differential d on the free (graded commutative) algebra $S(Z)$ of lower degree -1 and upper degree $+1$, and a morphism of DGA's*

$$\rho : (S(Z), d) \longrightarrow (H, 0)$$

such that:

1. $\rho(S(Z)_+) = 0$;
2. $\rho^* : H_0(S(Z), d) \to H$ *is an isomorphism of graded algebras;*
3. $H_+(S(Z), d) = 0$;
4. $(S(Z), d)$ *is minimal (forgetting the lower gradation).*

Moreover, any other bigraded differential commutative algebra (C, δ) with a differential δ of lower degree -1 and upper degree $+1$, and a morphism of DGA's $\sigma : (C, \delta) \to (H, 0)$ satisfying properties $1-4$ must be isomorphic with $(S(Z), d)$ with an isomorphism which commutes with ρ.

Proof. In this proof we follow the ideas outlined in [7], with a few formal simplifications here and there. We first construct Z, d, ρ and afterwards we prove their properties. We will use the notation $Z_{(n)} = \bigoplus_{k \leq n} Z_k$. Set

$$Z_0 = H^+ / H^+ \cdot H^+,$$

set $d = 0$ on $S(Z_0)$, and define $\rho : Z_0 \to H^+ \subset H$ so that it provides a (homogeneous) left inverse to the projection $H^+ \to H^+/H^+ \cdot H^+$. We extend ρ to all of $S(Z_0)$ in the only possible way. We then have that the ρ thus built is a surjective DGA morphism from $S(Z_0)$ to H. Let $K = \ker(\rho)$. Then by connectedness $K^0 = 0$, and $K^1 = 0$ because $H^+ \cdot H^+ \subset H^{\geq 2}$. Set

$$Z_1 = \big(K/K \cdot S^+(Z_0)\big)\,[1]$$

(with the shift affecting the upper gradation only). As K is concentrated in (upper) degrees greater or equal to 2, Z_1 will be concentrated in (upper) degrees greater or equal to 1. Extend d to Z_1 so that it induces a linear homogeneous map $Z_1[-1] \to K$ splitting the projection. Extend then d to all of $S(Z_{(1)})$ in the only possible way so that it is a differential. Extend ρ so that it is zero on Z_1. Assume inductively that we have constructed $Z_{(n)}$ and d, ρ on $S(Z_{(n)})$, for $n \geq 1$. Define

$$Z_{n+1} = \big(H_n(S(Z_{(n)}), d)/H_n(S(Z_{(n)}), d) \cdot H_0^+(S(Z_{(n)}), d)\big)\,[1].$$

Extend d so that it induces a splitting $Z_{n+1}[-1] \to (S(Z_{(n)}))_n \cap \ker(d)$ of the projection, and extend ρ to be zero on Z_{n+1}. Both d and ρ are then to be extended to all of $S(Z_{(n+1)})$ in the only possible way.
Finally, define $Z = \bigoplus_{n\geq 0} Z_n$, and define ρ and d on $S(Z)$ as the limits of their restrictions to the various $S(Z_{(n)})$. It is clear that the construction is well defined. We have now to prove that it satisfies the three conditions requested. Observe first that

$$x \in (\ker(d))_1 \Longrightarrow x = u_0 + \sum_{i=1}^{k} u_i v_i,$$

with the u_i in Z_1 and the v_j in $S^+(Z_0)$. From $dx = 0$ we deduce then that $du_0 = -\sum_{i=1}^k du_i v_i$. The right hand side is however in $K \cdot S^+(Z_0)$, and is therefore zero in $K/K \cdot S^+(Z_0)$. As d is constructed to be bijective when seen from Z_1 to $K/K \cdot S^+(Z_0)$, this implies that $u_0 = 0$, and therefore $x \in Z_1 \cdot S^+(Z_0)$. For general $n \geq 2$,

$$x \in (\ker(d))_n \Longrightarrow x = u_0 + \sum_{i=1}^{k} u_i v_i + w,$$

with the u_i in Z_n, the v_j in $S^+(Z_0)$ and $w \in S(Z_{(n)})_+ \cdot S(Z_{(n)})_+ \subset S(Z_{(n-1)})$. From $dx = 0$ we deduce then that $du_0 = -\sum_{i=1}^k du_i v_i - dw$. Therefore du_0 represents a class in $H_{n-1}(S(Z_{(n-1)}), d) \cdot H_0(S(Z_{(n-1)}))^+$. As d is constructed to be bijective once seen as a map

$$Z_n \longrightarrow H_{n-1}(S(Z_{(n-1)}), d)/H_{n-1}(S(Z_{(n-1)}), d) \cdot H_0(S(Z_{(n-1)}))^+,$$

this implies that $u_0 = 0$, and hence $x \in Z_n \cdot S^+(Z_0) + S(Z_{(n)})_+ \cdot S(Z_{(n)})_+$. The above considerations prove that

$$\forall n \geq 1 \quad (\ker(d))_n \subset S^+(Z) \cdot S^+(Z)$$

Because $d(Z_{n+1}) \subset (\ker(d))_n$, from the above it follows that $(S(Z), d)$ is minimal.
Take now $x \in (\ker(d))_n$, with $n \geq 1$. We clearly have $x \in S(Z_{(n)})$. From the construction of Z_{n+1}, we may therefore write

$$[x] = [\sum_i d(s_i) t_i], \quad \textit{with } s_i \in Z_{n+1},\ t_i \in S(Z_0).$$

This follows from the fact that $S(Z_0)^+$ is concentrated in positive degrees, and the following inductive argument: write $[x] = [d(s^0)] + \sum_i [x_i^1][z_i^1]$, with $[x_i^1] \in H_n(S(Z_{(n)}), d)$ and $z_i^1 \in$

$H_0(S(Z_{(n)}), d)^+$. Proceding inductively, and using the fact that the (upper) degree of all the $[x_i^1]$ is lower than that of $[x]$, we may assume that $x_i^1 == [\sum_j d(s_j^i)t_j^i]$, with $s_j^i \in Z_{n+1}$ and $t_j^i \in S(Z_0)$, which proves the thesis. However, $\sum_i d(s_i)t_i = d(\sum_i s_i t_i)$, and therefore $H_n(S(Z), d) = 0$ for $n \geq 1$.
The map $H_0(S(Z), d) \to H$ is surjective by construction, and because H^+ is concentrated in positive degrees (the argument is the same as the one above showing $H_+(S(Z), d) = 0$). Moreover, still the same argument shows that any element of the kernel of the map can be written as $\sum_i p_i d(z_i)$, with $p_i \in S(Z_0)$ and $z_i \in Z_1$. However, $\sum_i p_i d(z_i) = d(\sum_i p_i z_i)$, and therefore the map $H_0(S(Z), d) \to H$ is also injective.
We have therefore proved the properties $(i) - (iv)$ for the DGA $(S(Z), d)$. Assume now that we have another DGA (C, δ) together with a morphism of DGA's

$$\sigma : (C, \delta) \to (H, 0)$$

with $C = \bigoplus_{i,j} C_j^i$ satisfying properties $(i) - (iv)$. It is immediate to verify that there is a map $\phi : Z_0 \to C_0$ such that $\sigma\phi = \rho$, which implies that there is a morphism $S(Z_0) \to C_0$ commuting with the maps into H. There is therefore an induced map from $\ker(\rho)$ to $\ker(\sigma)$, which can be used to extend ϕ to Z_1 so that $\phi d = \delta\phi$ on Z_1, as $\ker(\sigma) \subset \delta(C_1)$. We have therefore a morphism of DGA's $\phi : (S(Z_{(1)}), d) \to (C_{(1)}, \delta)$, inducing an isomorphism at the level of H_0 (because $H_0(C, \delta)$ is mapped isomorphically onto H by σ). Assume inductively that we have defined the map

$$(S(Z_{(n)}), d) \longrightarrow (C_{(n)}, \delta),$$

for $n \geq 1$, such that it induces an isomorphism in cohomology at level 0. The elements of Z_{n+1} are mapped by d to elements of $S(Z_{(n)})$ which are mapped by ϕ to closed elements in C_n. As σ induces an isomorphism in cohomology, these elements must be exact. This guarantees that we can find elements $\phi(z)$ for all $z \in Z_{n+1}$ such that $\delta\phi(z) = \phi d(z)$. From the algebraic freeness of $S(Z)$ we see that we may extend this ϕ to a morphism of DGA's $(S(Z_{(n+1)}), d) \to (C, \delta)$. This extended ϕ will still commute with ρ and σ, as they are zero on the parts of positive lower degree. Continuing this way, we can extend ϕ to all of $S(Z)$. By construction, this ϕ induces an isomorphism in cohomology, and therefore must be an isomorphism, because the two DGA's are minimal. □

Theorem 2.2. *Let (A, d_A) be a connected DGA and let*

$$\rho : (S(Z), d) \longrightarrow (H, 0)$$

be a bigraded model for $H(A)$. Then there are a DGA $(S(Z), D)$ and a morphism $\pi : (S(Z), D) \to (A, d_A)$ such that:

- (E_1) $(D - d) : Z_n \to F_{n-2}(S(Z))$ *for* $n \geq 0$*;*
- (E_2) $cl(\pi(z)) = \rho(z)$ *for* $z \in S(Z_0)$*;*
- (E_3) π *induces an isomorphism in cohomology.*

Any such $(S(Z), D)$ is called a filtered model *of (A, d_A) relative to the bigraded model $(S(Z), d)$ of its cohomology. Moreover, suppose that $\pi' : (S(Z), D') \to (A, d_A)$ satisfies the same properties $E_1 - E_3$. Then there is an isomorphism $\phi : (S(Z), D) \to (S(Z), D')$ such that:*

- (U_1) $\phi - \mathrm{Id}$ *is filtration-decreasing;*
- (U_2) $H(\pi'\phi) = H(\pi)$.

Proof. Fix a linear map

$$\eta : H(A) \longrightarrow S(Z_0) \text{ such that } \rho\eta = \mathrm{Id}.$$

We define D and π inductively on $Z_0, Z_1, \ldots$. Set $F_{-1}(S(Z)) = \{0\}$. For $n = 0$, define $D = 0$, and π so that $d_A(\pi(Z_0)) = 0$ and $\mathrm{cl}(\pi(z)) = \rho(z)$ for $z \in Z_0$. Here cl indicates the class in $H(A)$. Both D and π are then extended to all of $S(Z)_{(0)}$ in the only possible way. It is immediate to verify that E_1 and E_2 hold, and that $D^2 = 0$. Assume defined π and D on $S(Z)_{(n)}$ with $n \geq 0$, so that E_1 holds, E_2 holds, $D^2 = 0$ and the induced maps $F_{n-1}(H(S(Z)_{(n)}, D)) \to H(A, d_A)$ and $F_0(H(S(Z)_{(n)}, D)) \to H(A, d_A)$ are injective. Take $z \in Z_{n+1}$. Then

$$Dd(z) = ((D - d) + d)(dz) = (D - d)(dz) \in F_{n-2}(S(Z)) \subset S(Z)_{(n-2)}$$

is a D-cocycle (for $n = 0, 1$ we are just saying $Dd(z) = 0$). From the inductive hypothesis there exist

$$w \in S(Z)_{(n-1)}, \alpha \in H \text{ such that } Ddz - \eta(\alpha) = D(w).$$

Applying π to this equation gives $d_A\pi dz - \pi\eta\alpha = d_A\pi w$, hence $0 = cl(\pi\eta\alpha) = \alpha$. This implies that $D(dz - w) = 0$. Clearly w can be chosen to depend linearly on z, and to be 0 when $Ddz = 0$ (e.g. by fixing a basis for Z_{n+1} adapted to the subspace $\ker(Dd)$). Extend D to Z_{n+1} by defining

$$Dz = dz - w - \eta(\mathrm{cl}(\pi(dz - w))) \; for \; z \in Z_{n+1}$$

As by construction $\mathrm{cl}(\pi(Dz)) = 0$, we define π on Z_{n+1} so that $d_A\pi z = \pi Dz$ (in a linear manner). We then extend π and D to all of $S(Z)_{(n+1)}$ in the only possible way. For $z \in Z_{n+1}$ we have that

$$D^2(z) = D(dz - w - \eta(\mathrm{cl}(\pi(dz - w)))) = D(dz - w) - d(\eta(\mathrm{cl}(\pi(dz - w))))$$

and therefore $D^2 = 0$ on all of $S(Z)_{(n+1)}$. Notice that by construction $D = d$ on $S(Z)_{(1)}$, and therefore we have that the induced map $\pi^* : \; H_0(S(Z)_{(n+1)}, D) \to H(A, d_A)$ is an isomorphism. To complete the proof that the induced map in cohomology is an isomorphism in lower degree $\leq n$ it is enough to prove that any cocycle in $u \in F_n(S(Z)_{(n+1)})$ with $\mathrm{cl}(\pi u) = 0$ is a coboundary with respect to the D just defined (in the case $n \geq 1$). Take therefore such an u, with $Du = 0$. Write

$$u = \sum_{j=0}^{n} u_j, \text{ with } u_j \in (S(Z)_{(n+1)})_j$$

Because $(d - D)(Z_n) \subset F_{n-2}(S(Z)_{(n+1)})$, we have that $du_n = 0$. Hence there is v_{n+1} such that $u_n = dv_{n+1}$. We have therefore that

$$u - Dv_n \in F_{n-1}(S(Z)_{(n+1)}) = F_{n-1}(S(Z)_{(n)})$$

satisfies $\mathrm{cl}(\pi(u - Dv_n)) = \mathrm{cl}(\pi u) = 0$ and hence, by the induction hypothesis, there exists $v \in S(Z)_{(n)}$ such that $u - Dv_n = Dv$. We skip the proof of the uniqueness part, which the reader can find in [7]. $\square$

3 Deformations and DGLA's

As we mentioned at the beginning of the previous section, the bigraded/filtered model construciton is not completely satisfactory, as to different filtered models can (and indeed do) correspond the same rational homotopy type. In this section we will show how one can describe the lack of injectivity of the map from filtered models to rational homotopy types using the language of differential graded Lie algebras and the Maurer-Cartan equation. In a nutshell, what Shlessinger and Stasheff show in [9] is that the "acceptable" deformations of the bigraded model are solutions of the Maurer-Cartan equation in a suitably chosen DGLA, and moreover that two such solutions describe the same rational homotopy type if and only if they are in the same orbit of the so-called Gauge group of the same DGLA. For the terminology and the notation concerning DGLA's, the Maurer-Cartan equation and the Gauge group the reader can consult [5] (or any one of the many sources present in the literature). The following theorem is obtained combining and adapting results from [9], in the spirit of Theorem 4.1 on page 11 of that paper. Give a DGLA L we indicate with $MC(L)$ the set of elements p of L_1 (or of L_{-1} if L is negatively graded) satisfying the Maurer-Cartan equation $d_L(p) + \frac{1}{2}[p,p] = 0$.

Theorem 3.1. *Let (A, d_A) be a DGA, and let $(S(Z), d)$ be a bigraded model of $H = H(A)$, with bigraded quasi-isomorphism $\rho : (S(Z), d) \to H(A)$. Let L be the DGLA of derivations of $(S(Z), d)$ which decrease the sum of upper degree and lower degree ("weight"), graded with respect to the shift that they induce in the upper degree. Then L is complete with respect to the filtration $\cdots [L_0, [L_0, L]] \subset [L_0, L] \subset L$. Moreover, there is a canonical bijection from the first of the following two sets to the second one:*

1. $\mathrm{Def}_L = \mathrm{MC}(L)/\sim$*, where "$\sim$" is $\exp(L_0)$-equivalence.*
2. $\{(B, d_B, \sigma) | \sigma : H(B) \to H(A)$ is a GA isomorphism$\} / \sim$*, where we have $(B_1, d_{B_1}, \sigma_1) \sim (B_2, d_{B_2}, \sigma_2)$ if they have the same augmented rational homotopy type.*

The bijection is induced by the map

$$p \in \mathrm{MC}(L) \longrightarrow (S(Z), d + p, \sigma_p),$$

where σ_p is the unique GA isomorphism from $H(S(Z), d+p)$ to H such that $\sigma_p([z]) = \rho([z])$ for all $z \in Z_0$.

Proof. The map at the level of $\mathrm{MC}(L)$ is well defined because if p satisfies the Maurer-Cartan equation

$$d_L p + \frac{1}{2}[p,p] = 0 \Longrightarrow (d+p)^2 = 0,$$

and therefore $d+p$ is a differential on $S(Z)$. To see that σ_p exists, first observe that there are natural GA morphisms $S(Z_0) \to H$ and $S(Z_0) \to H(S(Z), d+p)$, induced respectively by ρ and by the identity map followed by the quotient by the image of $d+p$. The kernel of the first map is $d(S(Z)_1)$, as ρ induces an isomorphism in cohomology at the level of H_0. The kernel of the secon map is $D(S(Z)) \cap S(Z_0) = d(S(Z)_1)$ because $H_+(S(Z), d) = 0$. Therefore the two maps have the same kernel. Moreover, they are both easily seen to be surjective, therefore there are induced GA isomorphisms $S(Z_0)/d(S(Z)_1) \to H$ and $S(Z_0)/d(S(Z)_1) \to H(S(Z), d+p)$. The induced isomorphism $H(S(Z), d+p) \to H$ is what we call σ_p, and it clearly satisfies the condition $\sigma_p([z]) = \rho([z])$ for all $z \in Z_0$. The argument above shows also that the previous condition characterizes uniquely σ_p. Assume that p_1 and p_2 are $\exp(L_0)$-equivalent. Then by definition

$$p_2 = \phi(p_1),\ \phi = \exp(x), x \in L_0$$

It then follows that if we let ϕ act on $S(Z)$ in the natural way (induced by the action of UL_0), ϕ becomes a DGA isomorphism of $(S(Z), d+p_1)$ onto $(S(Z), d+p_2)$. Moreover, $\sigma_{p_2} H(\phi) = \sigma_{p_1}$, because ϕ is the identity on Z_0. This proves that

$$\phi : (S(Z), d + p_1, \sigma_{p_1}) \longrightarrow (S(Z), d + p_2, \sigma_{p_2})$$

is an equivalence, and there is therefore a well defined natural map from the set Def_L to the set described in 2). We have to prove that this map is a bijection. The fact that the map is surjective follows from the existence part of theorem 2.2. Indeed, let (B, d_B) be a DGA with $\sigma : H(B) \to H(A)$ a GA isomorphism. We then have that

$$\sigma^{-1}\rho : (S(Z), d) \longrightarrow H(B)$$

is a filtered model. From the existence part of theorem 2.2 there exist then a derivation $p \in MC(L)$ and a DGA morphism

$$\pi : (S(Z), d + p) \longrightarrow (B, d_B)$$

such that $cl(\pi z) = \sigma^{-1}\rho z$ for $z \in S(Z)_{(0)}$, and $H(\pi)$ is an isomorphism. We then have that $\sigma H(\pi) = \sigma_p$, and therefore π determines an identification of augmented homotopy types between $(S(Z), d + p, \sigma_p)$ and (B, d_B, σ). For the uniqueness part, assume that there exists a morphism

$$\pi : (S(Z), d + p_1, \sigma_{p_1}) \longrightarrow (S(Z), d + p_2, \sigma_{p_2})$$

inducing an isomorphism in cohomology. Then if you call $(B, d_B) = (S(Z), d + p_2)$, we have that

$$\sigma_{p_2}^{-1}\rho : (S(Z), d) \longrightarrow H(B, d_B)$$

is a bigraded model. Moreover, from the construction of σ_{p_2} it follows that $\sigma_{p_2}^{-1}\rho([z]) = [z]$ for $z \in Z_0$, and from the definition of σ_{p_1} and the properties of π it follows that

$$H(\pi)([z]) = \sigma_{p_2}^{-1}\sigma_{p_1}([z]) = \sigma_{p_2}^{-1}\rho([z])$$

for all $z \in Z_0$. From this we have that both

$$\mathrm{Id} : (S(Z), d + p_2) \longrightarrow (S(Z), d + p_2)$$

and

$$\pi : (S(Z), d + p_1) \longrightarrow (S(Z), d + p_2)$$

are filtered models of (B, d_B) relative to the bigraded model $\sigma_{p_2}^{-1}\rho : (S(Z), d) \to H(B, d_B)$, and they satisfy the hypotheses of the uniqueness part of theorem 2.2. It follows that there exists

$$\phi : (S(Z), d + p_1) \longrightarrow (S(Z), d + p_2)$$

inducing an isomorphism in cohomology and such that $\phi - \mathrm{Id}$ lowers filtration degree. It is clear that $\phi = \mathrm{Id}$ on $S(Z_0)$, and therefore $\sigma_{p_2} H(\phi) = \sigma_{p_1}$. This implies that

$$\phi : (S(Z), d + p_1, \sigma_{p_1}) \longrightarrow (S(Z), d + p_2, \sigma_{p_2})$$

Take $b = \log(\phi - \mathrm{Id})$, where log is the standard power series expression, and the series converges once applied to any element of $S(Z)$ due to the fact that $\phi - \mathrm{Id}$ lowers (strictly) filtration degree. Then $b \in L_0$, and $\phi = \exp(b)$, and moreover from the fact that ϕ is a DGA morphism it follows that $\exp(b)p_1 = p_2$ (using the adjoing action of L_0 on L, and the completeness to give a meaning to the infinite sum) as desired. □

Corollary 3.2. *Let (A, d_A) be a DGA, and let $(S(Z), d)$ be a bigraded model of $H = H(A)$, with bigraded quasi-isomorphism*

$$\rho : (S(Z), d) \longrightarrow H(A).$$

The bijection of the previous theorem is equivariant with respect to the natural action of $\mathrm{Aut}(H)$ on the two sets. It follows that $M_H = \mathrm{Def}_L/\mathrm{Aut}(H)$ has a natural bijection to the set of rational homotopy classes of DGA's with cohomology isomorphic to H.

4 "Classifying maps" into coalgebras

We now show, following [9], how to translate the results of the previous section in the language of coalgebras and coalgebra maps. Although this step is not necessary to understand the results of the previous section, it provides a slightly different viewpoint. Moreover, the language of coalgebras is used in many recent papers on rational homotopy and deformation theory, and it may therefore be useful to see it in action on a classical and well understood problem. As in the previous section, we used the standard notation and terminology, which is described in the original paper and also in [5].

Definition 4.1. *Let L be a DGLA. With the symbol $C(L)$ we describe the free cocommutative differential graded coalgebra over $L[1]$, endowed with the coalgebra differential cogenerated as a coderivation by d_L and $[\cdot\,,\cdot]_L$. If L if filtered, with $\hat{C}(L)$ we describe the completion of $C(L)$ with respect to the naturally induced filtration.*

The idea here is that the set of solutions of the Maurer-Cartan equation in a DGLA L is in a natural way the set of "points" of the geometric object associated to the coalgebra $C(L)$. In the spirit of algebraic geometry, a point is an embedding of the geometric point. i.e. In this case this corresponds to a (nonzero) coalgebra map from the base field k seen as a coagebra to $C(L)$, as coagebras are covariant objects (not contravariant, as algebras). This construction is valid independently of the current context, and applies to any DGLA in which one considers the Maurer-Cartan equation. For technical reasons which we will not discuss here, it will be necessary to use a completion $\hat{C}(L)$ of the coalgebra $C(L)$, instead of $C(L)$ itself.

Definition 4.2. *Let L be a DG Lie algebra, and let $C = C(L)$ be the associated DG coalgebra. Let $x \in L_1$ be an element satistying $dx + \frac{1}{2}[x, x] = 0$. Let $\hat{C}$ be the completion of C with respect to the tensor degree filtration, and let*

$$\chi(x) : k \longrightarrow \hat{C}$$

the DG coalgebra map determined by the linear homogeneous map

$$k \ni q1_k \longrightarrow q1_{k[1]} \otimes x \in L[1]$$

The map $\chi(x)$ is the classifying map *associated to the element x.*

The following proposition and theorem are proved in [9], page 21.

Proposition 4.3. *In the hypotheses of the previous definition, and identifying for simplicity in the notation x with $1_{k[1]} \otimes x \in L[1]$,*

$$\chi(x)(1_k) = 1 + x + x \otimes x + x \otimes x \otimes x + \cdots$$

There is also a converse to the previous results, which says that whenever one has a filtered coalgebra map $k \to \hat{C}(L)$, then the projection to L_1 of the image of 1_k is a solution of the Maurer-Cartan equation in L. To fix the terminology, we note that the geometric object associated to a DG coalgebra is often called a "formal DG manifold" or "formal Q-manifold". Having established that to any deformation of the bigraded model (or equivalently to any solution to the Maurer-Cartan equation in the DGLA L) correspond bijectively a point of the geometric object associated to the coalgebra $\hat{C}(L)$, we come to the question of a coalgebra-theoretic description of when two such points correspond to the same rational homotopy class. For that, we need the notion of homotopy for coalgebras, which is less intuitive than the corresponding one for algebras.

Definition 4.4. *The differential graded coalgebra I_* has as (additive) basis $\{t_i, t_i s | i = 0, 1, ...\}$ and comultiplication and differential defined as*

$$\Delta t_n = \sum_{i+j=n} t_i \otimes t_j, \Delta(t_n s) = \sum_{i+j=n} (t_i \otimes t_j s + t_i s \otimes t_j), \, d(t_n s) = (n+1)t_{n+1}.$$

We denote also $t_0 = 1$. The coalgebra J is the completion of I_* with respect to the obvious filtration. J is also called the coalgebra of formal copower series.

Notice that J could also be defined as the dual DG coalgebra to the (algebraically free graded commutative) algebra $S(t, dt)$, where t has degree 0 and dt has degree one, and the differential sends t to dt.

Definition 4.5 ([9], Definition 4.5 page 21). *Given two complete filtered cocommutative differential graded coalgebras C, D, two filtration preserving morphisms $f_i : C \to D$ are* homotopic *if there is a filtration preserving morphism $h : C \hat{\otimes} J \to D$ such that $f_0(c) = h(c \otimes 1)$ and $f_1(c) = \sum_i h(c \otimes t_i)$ for all $c \in C$.*

Theorem 4.6 ([9], page 21). *Let L be a DGLA. If $\lambda : J \to \hat{C}(L)$ is a homotopy or* path, *and we associate to it the elements $\eta(t)$ and $\zeta(t)$ in $L[[t]]$, defined as*

$$\eta(t) = \sum_i \pi\lambda(t_i)t^i, \quad \zeta(t) = \sum_i \pi\lambda(t_i s)t^i,$$

where π is the projection $\hat{C}(L) \to L$, we have that these formal power series satisfy the following system of differential equations:

$$\begin{cases} d_L\eta + \frac{1}{2}[\eta, \eta] = 0 \\ \dot{\eta} + d_L\zeta + [\eta, \zeta] = 0 \end{cases}$$

Conversely, for any such pair the coalgebra morphism induced by the graded morphismg $J \to L$ associated to them is a differential coalgebra morphism, i.e. a path in $\hat{C}(L)$.

In the second part of this section we cover material contained in section 5 of [9]. There, the authors prove that the notion of gauge equivalence obtained by imposing a differential equation and the one obtained by using simply the adjoint action of the Gauge group $\exp(L_0)$ coincide. The result is proved for $\mathrm{ad}(L_0)$-complete DGLA's, which happens for example when L_0 is a nilpotent Lie algebra.

Theorem 4.7 ([9], pages 23-24). *Let L be a DGLA which is complete with respect to the filtration $\cdots [L_0, [L_0, L]] \subset [L_0, L] \subset L$. Assume that there is an element $\mu(t)$ of $L_1[[t]]$ satisfying*

$$[\mu(t), \mu(t)] = 0, , \dot{\mu}(t) + [\mu(t), \zeta(t)] = 0$$

for some $\zeta(t) = \sum_k z_k t^k \in L_0 \hat{\otimes} k[[t]]$. Then $\mu(1)$ is well defined, and there is an element $b \in L_0$ such that $\mu(1) = exp(b)\mu(0)$

Proof. We follow the ideas and (almost completely) the method of proof of [9]. For the purpouses of the proof, we first fix some notation. If we use the notation $F_1(L) = L, F_{k+1}(L) = [L_0, F_k(L)]$, for any element $\sum_k \alpha_k t^k \in L[[t]]$ we define $f_n(\sum_k \alpha_k t^k) = \sup\{k | \alpha_n \in F_k\}$. Then f_n can also assume the value $+\infty$ (exactly when $\alpha_n = 0$, by completeness), and the condition $\mathrm{Lim}_n f_n(\sum_k \alpha_k t^k) = +\infty$ is equivalent to $\sum_k \alpha_k t^k \in L \hat{\otimes} k[[t]]$ (where the completion on L is with respect to the filtration F_*). For $n = 1$, define $\theta_1 = z_0$, and $\zeta_1(t) = z_0$.

We then have $\phi_1(t) = exp(tz_0)$, and $\mu_1(t) = \exp(tz_0)\mu(0)$. It can be easily proved that $f_k(\mu_1(t)) \geq \min\{k, f_k(\zeta(t))\}$ and that $\dot{\mu}_1(t) = \mathrm{ad}(z_0)\,(\mu_1(t))$. Define

$$c_m = \inf\left\{f_k(\zeta(t))|k \geq m\right\}, \, d_m = \inf\left\{c_m, c_{m-1}+1, ..., c_0+m\right\}$$

Notice that $\{c_m\}$ is a non decreasing sequence with $\mathrm{Lim}_m c_m = +\infty$, and a rough estimate gives $d_m \geq \min\{c_{[\frac{m}{3}]}, [\frac{m}{3}]\}$, so that $\mathrm{Lim}_m d_m = +\infty$ too. For $n > 1$, asume inductively that we have $\theta_n \in L_0$ and $\zeta_n(t)$ such that:

1. $\zeta_n(t)$ equals $\zeta(t)$ in degree strictly smaller than n;
2. If $\phi_n(t) = \exp(t^n\theta_k)\cdots\exp(t\theta_1)$, then $\mu_n(t) = \phi_n(t)\mu(0)$ satisfies
$$\dot{\mu}_n(t) + [\mu_n(t), \zeta_n(t)] = 0;$$
3. $\forall k \geq n \ \ f_k(\zeta_n(t) - \zeta(t)) \geq d_n$.

Define

$$\zeta_{n+1}(t) = \exp(\theta_{n+1}t^{n+1})\zeta_n(t) + (n+1)\theta_{n+1}t^n.$$

We then have immediately that $\zeta_{n+1}(t)$ coincides with $\zeta(t)$ up to degree n included. Moreover, the differences between $\zeta_{n+1}(t)$ and $\zeta_n(t)$ (which are in degree greater or equal to $n+1$) are by construction obtained via (possibly iterated) adjunction of coefficients of $\zeta_n(t)$ with θ_{n+1}. Therefore, as $[L_0, F_k(L)] \subset F_{k+1}(F)$, and $\theta_{n+1} \in F_{d_n}(F)$ by construction, we must have that the above differences lie in $F_{d_n+1}(F)$. This shows therefore that for $k \geq n+1$

$$\begin{aligned} &f_k(\zeta_{n+1}(t) - \zeta(t)) = f_k((\zeta_{n+1}(t) - \zeta_n(t)) + (\zeta_n(t) - \zeta(t))) \geq \\ &\quad \geq \inf\{f_k(\zeta_{n+1}(t) - \zeta_n(t)), f_k(\zeta_n(t) - \zeta(t))\} = \inf\{d_n+1, c_{n+1}\} = d_{n+1} \end{aligned}$$

as required by the inductive process. Let

$$\phi_{n+1}(t) = \exp(\theta_{n+1}t^{n+1})\phi_n(t), \; \mu_{n+1} = \phi_{n+1}\mu_0$$

It remains to be shown that μ_{n+1} satisfies the required "differential equation". Therefore, we must compute:

$$\begin{aligned} &\frac{d}{dt}\left(\exp(\theta_{n+1}t^{n+1})\phi_n(t)\mu(0)\right) = \\ &=\frac{d}{dt}\left(\exp(\theta_{n+1}t^{n+1})\phi_n(s)\mu(0) + \exp(\theta_{n+1}s^{n+1})\phi_n(t)\mu(0)\right)|_{s=t} = \\ &= \big(-[\exp(\theta_{n+1}t^{n+1})\phi_n(s)\mu(0), (n+1)\theta_{n+1}t^n] - \\ &\quad \exp(\theta_{n+1}s^{n+1})[\phi_n(t)\mu(0), \zeta_n(t)]\big)|_{s=t} = \\ &= -\,[\phi_{n+1}(t)\mu(0), (n+1)\theta_{n+1}t^n] - [\phi_{n+1}(t)\mu(0), \exp(\theta_{n+1}t^{n+1})\zeta_n(t)]. \end{aligned}$$

The computations above follow if we observe that $\exp(A)B$ (using the adjoint action) translates into $\exp(A)B(\exp(A))^{-1}$ once we work in the (completed) universal enveloping algebra $\mathcal{U}(L)\hat{\otimes}k[[t]]$. These details can be safely left to the interest reader, who is advised to compute first $\frac{d}{dt}(A(t)^{-1})$ using (only) the fact that

$$\frac{d}{dt}(A(t)B(t)) = \frac{d}{dt}(A(t))B(t) + A(t)\frac{d}{dt}(B(t)).$$

Now that we have built the ϕ_k for all k, it is easy to prove, using the fact that $\theta_k \in F_{d_{k-1}}(L)$, that there is a well defined element $\phi = \mathrm{Lim}_k\phi_k$ which converges for all $t \in \mathbb{Q}$, and that

$\nu(t) = \phi(t)\mu(0)$ satisfies the differential equation $\dot{\nu}(t) = [\nu(t), \zeta(t)]$, with $\nu(0) = \mu(0)$. It follows that

$$(\dot{\nu} - \dot{\mu})(t) = [\nu(t) - \mu(t), \zeta(t)],\ (\nu - \mu)(0) = 0$$

An easy inductive argument shows that all the coefficients of $\nu(t) - \mu(t)$ must then be zero, and therefore $\mu(t) = \phi(t)\mu(0)$. We finally observe that there is an element $b \in L_0$ such that $\phi(1) = exp(b)$, and this concludes the proof. □

Corollary 4.8. *Let p_1, p_2 be elements of L_1 such that $d_L p_i + \frac{1}{2}[p_1, p_i] = 0$ for $i = 1, 2$. Then the following two facts are equivalent:*

1. *There is a filtered homotopy from $\chi(p_1) : \mathbb{Q} \to \hat{C}(L)$ to $\chi(p_2) : \mathbb{Q} \to \hat{C}(L)$.*
2. *There is an element $\phi \in exp(L_0)$ such that $\phi(p_1) = p_2$.*

Proof. Assume 1. Then from the theorem 4.6 there are η, ζ satisfying the differential equations described in that theorem. If we define $\mu(t) = d + \eta(t)$, we have that

$$[\mu(t), \mu(t)] = 0,\ \dot{\mu}(t) + [\mu(t), \zeta(t)] = 0$$

Then from theorem 4.7 we obtain that there is an element $\phi \in \exp(L_0)$ such that $\phi(p_1) = p_2$. Conversely, assume 2. Then if we define $\zeta(t) = b$ (with $\exp(b) = \phi$) and $\eta(t) = \exp(tb)(d + p_1) - d$, we obtain that η, ζ satisfy the differential equations of theorem 4.6, and therefore from the second part of that theorem they define a homotopy from $\chi(p_1)$ to $\chi(p_2)$. □

While the use of the gauge group is the natural thing to do when using the language of DGLA's and the Maurer-Cartan equation, the differential equation approach is natural when using the language of DGcoA's and classifying maps. With the above result therefore the authors prove that the DGLA and the DGcoA approaches to deformation theory are equivalent, at least for $ad(L_0) - complete$ Lie algebras. As a final corollary, we restate the "Main theorem" of [9] (which is our theorem 3.1) in the language of coalgebras, which is the one preferred by the authors.

Theorem 4.9 (Main Homotopy Theorem,[9] page 22). *Let (A, d_A) be a DGA, and let $(S(Z), d)$ be a filtered model of $H = H(A)$, with filtered quasi-isomorphism $\rho : (S(Z), d) \to H(A)$. Let L be the DGLA of derivations of $(S(Z), d)$ which decrease the sum of upper degree and filtration degree ("weight"), graded with respect to the shift that they induce in the upper degree. Then L is complete with respect to the filtration $\cdots [L_0, [L_0, L]] \subset [L_0, L] \subset L$. Moreover, there is a canonical bijection from the first of the following two sets to the second one:*

1. $\{\chi : \mathbb{Q} \to \hat{C}(L) : \chi$ morphism of DGcoA's$\}$/filtered homotopy *(i.e. the path components of $\hat{C}(L)$).*
2. $\{(B, d_B, \sigma) | \sigma : H(B) \to H(A)$ is a GA isomorphism$\}$ / $\equiv$*, where we have $(B_1, d_{B_1}\sigma_1) \equiv (B_2, d_{B_2}, \sigma_2)$ if they have the same augmented rational homotopy type.*

The bijection is induced by the map

$$(\chi : \mathbb{Q} \to \hat{C}(L)) \longrightarrow (S(Z), d + \pi\chi(1), \sigma_{\pi\chi(1)}),$$

where $\sigma_{\pi\chi(1)}$ is the unique GA isomorphism from $H(S(Z), d + \pi\chi(1))$ to H such that $\sigma_{\pi\chi(1)}([z]) = \rho([z])$ for all $z \in Z_0$.

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An Introduction to the Language of Operads

Domenico Fiorenza

"Solo dopo aver conosciuto la superficie delle cose
ci si può spingere a cercare quel che c'è sotto.
Ma la superficie delle cose è inesauribile."
— *Italo Calvino - Palomar*

1 Introduction

The aim of this note is to give an informal introduction to the language of operads, which has lately become ubiquous in both modern abstract algebra and theoretical physics. The basic idea is that an operad $\mathcal{O}$ is a collection $\mathcal{O}(n)$ of "spaces" of n-ary operations, where by the generic word "space" we mean a set or a vector space or a topological space or a variety (topological, differential, analytic, algebraic), or, in the maximal generality, an object in a suitable tensor category. An n-ary operation has n inputs and just one output; this asymmetry makes the combinatorial definition of operads quite obscure at a first sight. To remedy such an asymmetry, we consider operations with n inputs and m outputs; this leads to the definition of MacLane's PROPs[1], which are collections $\mathcal{P}(n, m)$ of spaces of operations with n inputs and m outputs. In this more general context, an operad is simply the n-to-1 part of a PROP.

2 What is a PROP?

By definition, a PROP is simply a tensor category whose Hom-spaces have some additional structure, i.e. are objects of some other category. Before making the somehow colloquial definition above completely rigourous, let us make some example to show that objects like these are extremely common in mathematics:

A, B	$\mathrm{Hom}(A, B)$
Finite sets	Finite set
$\mathbb{K}$-vector spaces	$\mathbb{K}$-vector space
Topological spaces	Topological space
Metric spaces	Metric space
$\mathbb{K}$-algebras	$\mathbb{K}$-vector space
Hilbert spaces	Banach space
Differential manifolds	Frechét manifold

Now observe that for a generic category, Hom is a functor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Sets}$$

So we can formalize the concept of Hom-sets with structure by saying that a PROP is a couple of categories $\mathcal{C}_{\mathrm{ob}}$ (the category of the objects) and $\mathcal{C}_{\mathrm{Hom}}$ (the category of the morphisms)

[1] PROP is an acronym for PROducts and Permutations category

together with a functor

$$\mathcal{P} : (\mathcal{C}_{\mathrm{ob}})^{\mathrm{op}} \times \mathcal{C}_{\mathrm{ob}} \to \mathcal{C}_{\mathrm{Hom}}$$

To look at $\mathcal{P}(A, B)$ as a space of morphisms between the object A and B of $\mathcal{C}_{\mathrm{ob}}$, we need some more axioms, generalizing what happens for Hom-spaces of generic categories.

Composition

The composition map

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

$$(f, g) \mapsto f \circ g$$

can be seen as a natural transformation

$$\circ_{A,B,C} : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

It satisfies the following associativity condition: the diagrams

$$\begin{tikzcd}
(\mathrm{Hom}(C, D) \times \mathrm{Hom}(B, C)) \times \mathrm{Hom}(A, B) \arrow[r, "\circ\times\mathrm{Id}"] \arrow[d, "\sim"'] & \mathrm{Hom}(B, D) \times \mathrm{Hom}(A, B) \arrow[d, "\circ"] \\
\mathrm{Hom}(C, D) \times (\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B)) \arrow[d, "\mathrm{Id}\times\circ"'] & \mathrm{Hom}(A, D) \\
\mathrm{Hom}(C, D) \times \mathrm{Hom}(A, C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

commute for every choice of A, B, C, D. A blind generalization would be requiring the existence of a natural transformation

$$\circ_{A,B,C} : \mathcal{P}(B, C) \times \mathcal{P}(A, B) \to \mathcal{P}(A, C)$$

such the all diagrams as the one above would commute. But it would not be the right thing to do. For instance, if A, B, C are $\mathbb{K}$-vector spaces (one of the basic examples above), the composition

$$\circ_{A,B,C} : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

is *not* linear. So $\circ_{A,B,C}$ is not a morphism of vector spaces between $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A, C)$. But it is bilinear i.e. it is a morphism of vector spaces

$$\circ_{A,B,C} : \mathrm{Hom}(B, C) \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

So what we really have to ask is that $\mathcal{C}_{\mathrm{Hom}}$ be a tensor category and that $\circ$ be a natural transformation

$$\circ_{A,B,C} : \mathcal{P}(B, C) \otimes \mathcal{P}(A, B) \to \mathcal{P}(A, C)$$

such that all the diagrams

$$\begin{tikzcd}
(\mathcal{P}(C, D) \otimes \mathcal{P}(B, C)) \otimes \mathcal{P}(A, B) \arrow[r, "\circ\otimes\mathrm{Id}"] \arrow[d, "a"'] & \mathcal{P}(B, D) \otimes \mathcal{P}(A, B) \arrow[d, "\circ"] \\
\mathcal{P}(C, D) \otimes (\mathcal{P}(B, C) \otimes \mathcal{P}(A, B)) \arrow[d, "\mathrm{Id}\otimes\circ"'] & \mathcal{P}(A, D) \\
\mathcal{P}(C, D) \otimes \mathcal{P}(A, C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

commute, where a denotes the associativity isomorphism of the tensor category $\mathcal{C}_{\mathrm{Hom}}$. Note that we are not requiring $\mathcal{C}_{\mathrm{Hom}}$ to have products.

Identity element

The set $\mathrm{Hom}(A, A)$ has a distinguished element Id_A that is the neutral element respect to composition. Generalizing this concept requires a generalization of the concept of element so that we could speak of "elements" of $\mathcal{P}(A, A)$ and of the composition of them. A first attemp could consist in requiring $\mathcal{C}_{\mathrm{Hom}}$ to be a category of sets, i.e. a category endowed with a faithful functor

$$\pi : \mathcal{C}_{\mathrm{Hom}} \to \mathsf{Sets}$$

("forget the structure" functor), but then, to define the composition of $f \in \pi(\mathcal{P}(B, C))$ with $g \in \pi(\mathcal{P}(A, B))$ via $\circ$ we would need a natural map

$$\pi(\mathcal{P}(B, C)) \times \pi(\mathcal{P}(A, B)) \to \pi(\mathcal{P}(B, C) \otimes \mathcal{P}(A, B)) .$$

This works very well in many situations (sets, groups, topological spaces, vector spaces ...), but consider the category Top_X of topological spaces over a fixed space X (with fibre product as tensor product). What are the "elements" of $\varphi : Y \to X$? There is no natural answer and, though $\pi(\varphi : Y \to X) \mapsto Y$ as a set is a faithful functor from Top_X to Sets, there is no natural map

$$Y_1 \times Y_2 \to Y_1 \times_X Y_2 .$$

We must seek for an alternative definition. Note that, if S is a set, giving an element of S is the same thing as giving a morphism $j : * \to S$, where $*$ denotes the set with only one element, i.e. the functor "elements" is representable. This happens in all the basic examples we are trying to generalize. We have, in fact we have:

Category of morphisms	**Elements**
Finite sets	$\mathrm{Hom}(*, -)$
$\mathbb{K}$-vector spaces	$\mathrm{Hom}(\mathbb{K}, -)$
Topological spaces	$\mathrm{Hom}(*, -)$
Metric spaces	$\mathrm{Hom}(*, -)$
Banach spaces	$\mathrm{Hom}(\mathbf{C}, -)$
Frechét manifolds	$\mathrm{Hom}(*, -)$

This happens also in other tensor categories we could decide to use as categories of morphisms. Note that in these cases there is no "natural" category having such categories of morphisms. Anyway these somehow "artificial" PROPs will be important in what follows.

Category of morphisms	**Elements**
Finite sets	$\mathrm{Hom}(*, -)$
$\mathbb{K}$-algebras	$\mathrm{Hom}(\mathbb{K}[x], -)$
Hilbert spaces	$\mathrm{Hom}(*, -)$
Differential manifolds	$\mathrm{Hom}(*, -)$
$\mathrm{Schemes}/_{\mathbb{K}}$	$\mathrm{Hom}(\mathrm{Spec}(\mathbb{K}), -)$

Denote by E the object representing the functor "elements". In all the examples above, E is a co-associative co-algebra in $\mathcal{C}_{\mathrm{Hom}}$ i.e. there is a map $\Delta : E \to E \otimes E$ such that the diagram

$$\begin{tikzcd}
 & E \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
E \otimes E \arrow[d, "\Delta\otimes\mathrm{Id}_E"] & & E \otimes E \arrow[d, "\mathrm{Id}_E\otimes\Delta"'] \\
(E \otimes E) \otimes E \arrow[rr, "a"] & & E \otimes (E \otimes E)
\end{tikzcd}$$

commutes (the only non completely trivial co-associative co-algebra structure is the one on $\mathbb{K}[x]$ in the category of $\mathbb{K}$-algebras; it is induced by the natural algebraic group structure on $\mathrm{Spec}(\mathbb{K}[x]) = \mathbf{A}^1_{\mathbb{K}}$). Having such an E at our disposal we can define the composition of two elements $f : E \to \mathcal{P}(B, C)$ and $g : E \to \mathcal{P}(A, B)$ via $\circ$ as the composition

$$E \xrightarrow{\Delta} E \otimes E \xrightarrow{f\otimes g} \mathcal{P}(B, C) \otimes \mathcal{P}(A, B) \xrightarrow{\circ} \mathcal{P}(A, C)$$

We write $f \circ g$ to denote the composition of f and g.

Finally we can give a meaning to the identity element of $\mathcal{P}(A, A)$. It will be a natural map

$$j_A : E \to \mathcal{P}(A, A)$$

which is the neutral element respect to the composition. Due to the co-asociativity of Δ, the composition defined above is associative.

This means that we can define a new category structure on $\mathcal{C}_{\mathrm{ob}}$ setting:

$$\mathrm{Hom}^{\mathrm{new}}(A, B) := \mathrm{Hom}(E, \mathcal{P}(A, B))$$

This new category is called the category "underlying" the PROP $\mathcal{P}$ (of course the underlying category depends on E and Δ, but these are data of the PROP). To ease notations wi will write $f \in \mathcal{P}(A, B)$ to say that f is an element of $\mathcal{P}(A, B)$.
Another completely natural request is that the elements of $\mathcal{P}(B, C)$ acts as left operators on $\mathcal{P}(A, B)$ and that the elements of $\mathcal{P}(A, B)$ acts as right operators on $\mathcal{P}(B, C)$, i.e. that there are maps

$$\mathrm{Hom}(E, \mathcal{P}(B, C)) \to \mathrm{Hom}(\mathcal{P}(A, B), \mathcal{P}(A, C))$$

and

$$\mathrm{Hom}(E, \mathcal{P}(A, B)) \to \mathrm{Hom}(\mathcal{P}(B, C), \mathcal{P}(A, C))$$

To have this, we just need two natural trasformations

$$\varphi^l_X : X \to E \otimes X$$

$$\varphi^r_X : X \to X \otimes E$$

inducing an E-co-bi-module structure on X. We further require

$$\phi^l_E = \phi^r_E = \Delta$$

and that for any morphism $\zeta\colon X \to Y$, it results

$$\phi^l_\zeta = \mathrm{Id}_E \otimes \zeta\,; \quad \phi^r_\zeta = \zeta \otimes \mathrm{Id}_E\,.$$

The maps φ^l's allow to assign to each element of $\mathcal{P}(B,C)$ a morphism between $\mathcal{P}(A,B)$ and $\mathcal{P}(A,C)$; since the objects of $\mathcal{C}_{\mathrm{Hom}}$ are left E-co-modules, this defines an action. The same considerations are valid for the φ^r's. A further request we make is that j_A acts trivially on both $\mathcal{P}(X,A)$ and $\mathcal{P}(A,X)$ for every X. Note that in any tensor category $\mathcal{C}$, one can take the unit object $\mathbf{1}_{\mathcal{C}}$ as E. Unless it is explicitely said, we will always assume $E = \mathbf{1}_{\mathcal{C}}$ in the following sections.

Tensor products

In a generic tensor category $\mathcal{C}$, the tensor product is a bi-functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

satisfying some associativity constrain. The functoriality of the tensor product means that there is a natural transform

$$\otimes_{A,B,C,D} : \mathrm{Hom}(A,B) \times \mathrm{Hom}(C,D) \to \mathrm{Hom}(A \otimes C, B \otimes D)$$

Transalting this in the setting of PROPs, in which Hom-spaces have some structure, we are going to ask for a natural transform

$$\otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \to \mathcal{P}(A \otimes C, B \otimes D)\,,$$

satysfying associativity constrains that generalize those of ordianry tensor products. It is important to remark that, in order to generalize the diagram

$$\begin{tikzcd}
(\mathrm{Hom}(A,B) \otimes \mathrm{Hom}(F,A)) \otimes (\mathrm{Hom}(C,D) \otimes \mathrm{Hom}(G,C)) \arrow[r, "\circ\otimes\circ"] \arrow[d, leftrightarrow, "\text{shuffle}"'] & \mathrm{Hom}(F,B) \otimes \mathrm{Hom}(G,D) \arrow[d, "\otimes"] \\
(\mathrm{Hom}(A,B) \otimes \mathrm{Hom}(C,D)) \otimes (\mathrm{Hom}(F,A) \otimes \mathrm{Hom}(G,C)) \arrow[d, "(-\otimes-)\otimes(-\otimes-)"'] & \mathrm{Hom}(F \otimes G, B \otimes D) \\
\mathrm{Hom}(A \otimes C, B \otimes D) \otimes \mathrm{Hom}(F \otimes G, A \otimes C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

to

$$\begin{tikzcd}
(\mathcal{P}(A,B) \otimes \mathcal{P}(F,A)) \otimes (\mathcal{P}(C,D) \otimes \mathcal{P}(G,C)) \arrow[r, "\circ\otimes\circ"] \arrow[d, leftrightarrow, "\text{shuffle}"'] & \mathcal{P}(F,B) \otimes \mathcal{P}(G,D) \arrow[d, "\otimes"] \\
(\mathcal{P}(A,B) \otimes \mathcal{P}(C,D)) \otimes (\mathcal{P}(F,A) \otimes \mathcal{P}(G,C)) \arrow[d, "(-\otimes-)\otimes(-\otimes-)"'] & \mathcal{P}(F \otimes G, B \otimes D) \\
\mathcal{P}(A \otimes C, B \otimes D) \otimes \mathcal{P}(F \otimes G, A \otimes C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

we must require the category $\mathcal{C}_{\mathrm{Hom}}$ to be *symmetric*. We are finally ready to give the formal definition of a PROP.

A formal definition

A PROP is $(\mathcal{C}_{\mathrm{ob}}, \mathcal{C}_{\mathrm{Hom}}, \mathcal{P}, \circ, E, \Delta, j, \varphi, \otimes)$ where

1. $\mathcal{C}_{\mathrm{ob}}$ is a tensor category, called the *category of objects*;
2. $\mathcal{C}_{\mathrm{Hom}}$ is a symmetric tensor category, called the *category of morphisms*;
3. $\mathcal{P}\colon (\mathcal{C}_{\mathrm{ob}})^{op} \times \mathcal{C}_{\mathrm{ob}} \to \mathcal{C}_{\mathrm{Hom}}$ is a functor, called the Hom-*space* functor;
4. $\circ_{A,B,C} : \mathcal{P}(B,C) \otimes \mathcal{P}(A,B) \to \mathcal{P}(A,C)$ is a natural transform, called the *composition map*;
5. (E, Δ) is a co-associative co-algebra in $\mathcal{C}_{\mathrm{Hom}}$;
6. j is a functorial map $j_A : E \to \mathcal{P}(A,A)$, called the *identity element*;
7. $\otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \to \mathcal{P}(A \otimes C, B \otimes D)$ is a natural transform inducing a tensor category structure on the underlying category of the PROP.

Braided and symmetric operads are defined adding the following axiom

8. $\sigma_{A,B} : E \to \mathcal{P}(A \otimes B, B \otimes A)$ is a natural map inducing a braided (symmetric) tensor category structure on the category underlying the operad.

3 PROPs over $\mathbb{N}$ and Operads

This section is dedicated to PROPs having the natural numbers as category of objects. They were actually the first structures to be called “PROPs”. For this reason we will completely restate the definition given above for such PROPs to find back the definition of operad one commonly finds in literature (see, for instance [Ad]). Let $\{\mathsf{N}, +\}$ be the discrete tensor category having the natural numbers (0 included) as objects and the sum as tensor product, and let $\mathcal{C}$ be a symmetric tensor category. Then the data for a PROP with N as category of objects and $\mathcal{C}$ as category of morphisms (a $\mathcal{C}$-PROP over $\mathbb{N}$ for short) reduce to:

1. A map $\mathcal{P} : \mathbb{N} \times \mathbb{N} \to \mathcal{C}$;
2. For each $m, n, k \in \mathbb{N}$, composition maps
$$\circ_{mnk} : \mathcal{P}(n,k) \otimes \mathcal{P}(m,n) \to \mathcal{P}(m,k)$$
3. For each $m, n, k, l \in \mathbb{N}$, a tensor product
$$\otimes_{mnkl} : \mathcal{P}(m,n) \otimes \mathcal{P}(k,l) \to \mathcal{P}(m+k, n+l)$$
4. For each $n \in \mathbb{N}$ a distinguished element (identity) $j_n \in \mathcal{P}(n,n)$;

such that all the natural diagrams commute.

Braided and symmetric PROPs are defined by adding

5. For each $m, n \in \mathbb{N}$, a distinguished element $\sigma_{m,n} \in \mathcal{P}(m+n, n+m)$ inducing a braided (symmetric) tensor category structure on the category underlying the PROP.

Note that the commutativity of diagrams required above implies

$$j_n = j_1 \otimes \cdots \otimes j_1 = j_1^{\otimes n}$$

and that the braid relation tells that the $\sigma_{m,n}$ are generated by the elementary braidings $\sigma_i^{(n+m)}$defined by

$$\sigma_i^{(n)} := j_1 \otimes \cdots \otimes j_1 \otimes \sigma_{1,1} \otimes j_1 \otimes \cdots \otimes j_1 = j_1^{\otimes(i-1)} \otimes \sigma 1, 1 \otimes j_1^{\otimes(n-i-1)}$$

Moreover, the σ's induce natural actions of the braid (symmetric) group B_n (S_n) on the spaces $\mathcal{P}(n,m)$ and $\mathcal{P}(m,n)$ for each m.

Operads

An operad is simply the n-to-1 part of a symmetric PROP having N as category of the objects: assume that $\mathcal{P}$ be a symmetric $\mathcal{C}$-PROP, and set

$$\mathcal{O}(n) := \mathcal{P}(n,1), \qquad n \geq 1$$

Then we have a collection $\{\mathcal{O}(n), n \geq 1\}$ of objects of $\mathcal{C}$ equipped with the following set of data:

1. An action of the symmetric group S_n on $\mathcal{O}(n)$ for each $n \geq 1$.
2. Morphisms (called compositions)

$$\gamma_{m_1,\ldots,m_l} : \mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \mathcal{O}(m_2) \otimes \cdots \otimes \mathcal{O}(m_l) \to \mathcal{O}(m_1 + m_2 + \cdots + m_l)$$

   defined as the composition

$$\mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l) := \mathcal{P}(l,1) \otimes \mathcal{P}(m_1,1) \otimes \cdots \otimes \mathcal{P}(m_l,1) \xrightarrow{\mathrm{Id}_{\mathcal{P}(l,1)} \otimes \bigotimes}$$

$$\xrightarrow{\mathrm{Id}_{\mathcal{P}(l,1)} \otimes \bigotimes} \mathcal{P}(l,1) \otimes \mathcal{P}(m_1 + m_2 + \cdots m_l, l) \xrightarrow{\circ}$$

$$\xrightarrow{\circ} \mathcal{P}(m_1 + m_2 + \cdots m_l, 1)$$

$$=: \mathcal{O}(m_1 + m_2 + \cdots m_l)$$

3. An element j_1 of $\mathcal{O}(1)$, called the unit, such that $\gamma_{1,1,\ldots,1}(\mu, j_1, j_1, \ldots, j_1) = \mu$ for any l and any $\mu \in \mathcal{O}(l)$.

These data satisfy associativity and equivariance respect to the symmetric groups conditions, so that the collection $\{\mathcal{O}(n), n \geq 1\}$ is an operad according to the original definition given by May in [Ma]. Vice versa, if $\{\mathcal{O}(n), n \geq 1\}$ is an operad, we can define a PROP by the following construction. Start setting

$$\mathcal{P}(n,1) := \mathcal{O}(n), \qquad n \geq 1$$

The spaces $\mathcal{P}(m,n)$ have to be constructed in such a way that we have natural maps

$$\otimes : \mathcal{P}(m_1,n_1) \otimes \mathcal{P}(m_2,n_2) \otimes \cdots \otimes \mathcal{P}(m_l,n_l) \to \mathcal{P}(\sum_i m_i, \sum_i n_i)$$

In particular, we must have

$$\otimes : \mathcal{P}(m_1,1) \otimes \mathcal{P}(m_2,1) \otimes \cdots \otimes \mathcal{P}(m_l,1) \to \mathcal{P}(\sum_i m_i, l)$$

i.e. for every $(m_1, m_2, \dots, m_l)$ with $\sum_l m_l = m$, a map

$$\otimes : \mathcal{P}(m_1, 1) \otimes \mathcal{P}(m_2, 1) \otimes \cdots \otimes \mathcal{P}(m_l, 1) \to \mathcal{P}(m, l)$$

Then set

$$\begin{aligned}
\mathcal{P}(m, l) &:= \bigoplus_{(m_1, m_2, \dots, m_l) | \sum_i m_i = m} \mathcal{P}(m_1, 1) \otimes \cdots \otimes \mathcal{P}(m_l, 1) \\
&= \bigoplus_{(m_1, m_2, \dots, m_l) | \sum_i m_i = m} \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l)
\end{aligned}$$

(we are now assuming that $\mathcal{C}$ has direct sums). Then the tensor products

$$\otimes_{mnkl} : \mathcal{P}(m, n) \otimes \mathcal{P}(k, l) \to \mathcal{P}(m + k, n + l)$$

are defined by the inclusion

$$\begin{aligned}
&\left\{ (m_1, m_2, \dots, m_n) | \sum_i m_i = m \right\} \times \left\{ (k_1, k_2, \dots, k_l) | \sum_j k_j = k \right\} \hookrightarrow \\
&\qquad\qquad \hookrightarrow \left\{ (s_1, s_2, \dots, s_{n+l}) | \sum_i s_i = m + k \right\}
\end{aligned}$$

$$((m_1, m_2, \dots, m_n), (k_1, k_2, \dots, k_l)) \mapsto (m_1, m_2, \dots, m_n, k_1, k_2, \dots, k_l)$$

while the compositions

$$\circ_{mnk} : \mathcal{P}(n, k) \otimes \mathcal{P}(m, n) \to \mathcal{P}(m, k)$$

are defined component-wise as

$$\begin{aligned}
&\big(\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k)\big) \otimes \big(\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)\big) \\
&\quad \simeq (\mathcal{O}(n_1) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_{n_1})) \otimes \cdots \otimes \\
&\qquad \otimes \big(\mathcal{O}(n_k) \otimes \mathcal{O}(m_{(n_1 + \cdots + n_{k-1}) + 1}) \otimes \cdots \otimes \mathcal{O}(m_n)\big) \xrightarrow{\gamma \otimes \cdots \otimes \gamma} \\
&\quad \xrightarrow{\gamma \otimes \cdots \otimes \gamma} \mathcal{O}(m_1 + \cdots + m_{n_1}) \otimes \cdots \otimes \mathcal{O}(m_{(n_1 + \cdots + n_{k-1}) + 1} + \cdots + m_n)
\end{aligned}$$

We refer to the PROP defined above as to the PROP generated by the operad $\mathcal{O}$, and denote it by the symbol $\mathcal{P}_{\mathcal{O}}$.

A basic example: the endomorphisms PROP and Operad

The basic example of a PROP is the following. Let V be a vector space over some fixed base field $\mathbb{K}$. It's *endomorphisms PROP* is by definition the symmetric $\mathbb{K}$-vector spaces PROP given by

$$\underline{\mathrm{End}}(V)(m, n) := \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$$

According to the definitions above, the *endomorphisms operad* of V is the operad of vector spaces

$$\underline{\mathrm{End}}(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)\,.$$

4 Representations of PROPs (algebras over operads)

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two operads with the same category of morphisms $\mathcal{C}_{\mathrm{Hom}}$. Then a morphism

$$\rho : \mathcal{P}_1 \to \mathcal{P}_2$$

is simply a tensor functor

$$\rho_{\mathrm{ob}} : \mathcal{C}_{\mathrm{ob},1} \to \mathcal{C}_{\mathrm{ob},2}$$

together with natural trasformations

$$\rho_{\mathrm{Hom},A,B} : \mathcal{P}_1(A,B) \to \mathcal{P}_2(\rho_{\mathrm{ob}}A, \rho_{\mathrm{ob}}B)$$

satisfying natural conditions so to generalize the concept of tensor funcor between tensor categories to this setting of "tensor categories with structure on the Hom-spaces". For instance, we require that the identity element (and the braiding, if the PROPs $\mathcal{P}_1$ and $\mathcal{P}_2$ are braided) go to the identity element and to the braiding respectively. Morphisms between operads are defined as morphisms between the PROPs they generate. Now let us stop with the abstract nonsense an go on concrete examples. Take as $\mathcal{P}_2$ the vector space PROP given by the category of vector spaces over some fixed base field $\mathbb{K}$, and let $\mathcal{P}$ be any PROP of vector spaces over $\mathbb{N}$. Since $\mathbb{N}$ is generated as a tensor category by 1, the image of ρ_{ob} for a morphism

$$\rho : \mathcal{P} \to \mathsf{Vect}_{\mathbb{K}}$$

will be generated by the vector space $V = \rho_{\mathrm{ob}}(1)$. So the data of our morphism will be a family of morphisms

$$\rho_{m,n} : \mathcal{P}(m,n) \to \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}) \,,$$

satisfying some comatibility axioms. Note that these data are equivalent to a morphism

$$\rho : \mathcal{P} \to \underline{\mathrm{End}}(V)$$

Elements of $\mathcal{P}(m,n)$ acts as multilinear operators on V. A vector space V together with a morphism $\rho\colon \mathcal{P} \to \underline{\mathrm{End}}(V)$ is called a $\mathcal{P}$-algebra. If $\mathcal{O}$ is an operad of vector spaces, an $\mathcal{O}$-algebra is ,by definition, a $\mathcal{P}_{\mathcal{O}}$-algebra. It can be seen as the datum of a collection of morphisms

$$\rho_n : \mathcal{O}(n) \to \mathrm{Hom}(V^{\otimes m}, V) \,,$$

i.e. as an operad morphism $\mathcal{O} \to \underline{\mathrm{End}}(V)$. Elements of $\mathcal{O}(n)$ acts as n-ary products on V.

5 Free Operads

¿From now on, we will work with PROPs and operads of vector spaces, i.e. such that their category of morphisms is the category of $\mathbb{K}$-vector spaces, for some fixed base field $\mathbb{K}$. As usual, we will use the same symbol to denote a vector space and its underlying set. The problem we are going to solve in this section is the following: given a collection of sets $X = \{X(n)\}$, is there an operad of vector spaces $\mathcal{O}$ such that for any operad of vector spaces $\tilde{\mathcal{O}}$ and any family of maps of sets

$$\rho_n : X(n) \to \tilde{\mathcal{O}}$$

there is a unique lifting of ρ to an operad morphism $\mathcal{O} \to \tilde{\mathcal{O}}$? It will come out that this operad exists (and is clearly unique up to isomorphism); it will be called the *free operad* generated by the collection X and will be denoted by the symbol $\mathcal{F}(X)$. In particular, we obtain that for any vector space V, any set $X(n)$ of n-ary multiplications on V defines an $\mathcal{F}(X)$-algebra structure on V.

Trees and forests

A Reshetikhin-Turaev's tree is a directed tree embedded in $\mathbb{R} \times [0,1]$ in such a way that all the imputs lie on $\mathbb{R} \times \{0\}$, its only output lies on $\mathbb{R} \times \{1\}$, and the height function is strictly increasing on each edge in the direction given by the orientation on the edge. Define the tensor product of two trees T_1 and T_2 as the object obtained drawing T_2 on the right of T_1. Such an object is not a tree anymore (is not connected). Since it's a disjoint union of trees it is called a "forest". Extending the tensor product from trees to forests we get a tensor category, called the category of forests (the unit object being the void forest). We denote it by the symbol $\mathcal{F}$. Let $\mathcal{F}(m,n)$ be the set of all forests with m inputs and n outputs. Defining the composition of a forest F_1 in $\mathcal{F}(n,k)$ with a forest F_2 in $\mathcal{F}(m,n)$ as the forest $F_1 \circ F_2$ in $\mathcal{F}(m,k)$ obtained putting F_1 over F_2 one gets a PROP of sets having N as category of the objects; such a PROP is called the Reshetikhin-Turaev's forest PROP. The identity element is represented by the stright line in $\mathcal{F}(1,1)$. It is a symmetric PROP, the braiding given by two crossing lines in $\mathcal{F}(2,2)$, with an identification of undercrossings and overcrossings. Call *vertex of type* $(n,1)$ a tree with only one $n+1$ valent vertex and containing no braiding. Any forest can be built by composing and taking tensor products of vertices of type $(n,1)$ with $n \in \mathbb{N}$, the identity element and the braiding. A fundamental result by Reshetikhin and Turaev (see [RT]) is that $\mathcal{F}$ is freely generated as a symmetric PROP by these elementary pieces. Being $\mathcal{F}$ a PROP of sets, we can immediately get from it a PROP of vector spaces, simply taking the free vector spaces generated by the $\mathcal{F}(m,n)$ on some fixed base field $\mathbb{K}$. When it causes no confusion we will denote the PROP of vector spaces obtained in this way by the same symbol $\mathcal{F}$.

Coloured forests

Now consider the elements of the set $X(n)$ as a set of colours for the $n+1$-valent vertices of the trees in the forests; this way we obtain a new PROP, called the PROP of X-coloured forests and denoted by $\mathcal{F}(X)$. Note that non-coloured forests can therefore be thought as being coloured with just one colour for each valency. If $X(n) = \emptyset$ we mean that there are no vertices of valence $n+1$ in the forests of $\mathcal{F}(X)$. The PROP $\mathcal{F}(X)$ is freely generated as a symmetric PROP by vertices of type $(n,1)$ (coloured with all its possible colorations), the identity element and the braiding. Again we can consider the PROP of vector spaces generated by $\mathcal{F}(X)$ over $\mathbb{K}$ (and we will denote it by the same symbol). Since $\mathcal{F}(X)$ is freely generated by the vertices, the braidings and the identity element, the datum of a morphism of symmetric PROPs of vector spaces

$$\rho : \mathcal{F}(X) \to \tilde{\mathcal{P}}$$

reduces to the datum of a family of maps of sets

$$\rho_n : \left\{ \text{Vertices of type } (n,1) \right\} \to \tilde{\mathcal{P}}(n,1)$$

(for any morphism of symmetric PROPs, the images of the identity element and of the braiding of the source PROP are prescribed to be the identity element and the braiding of the target PROP). But, by construction, there is a bijection

$$X(n) \leftrightarrow \left\{ \text{Vertices of type } (n,1) \right\}.$$

This shows that the n-to-1 part of the PROP $\mathcal{F}(X)$ is the free operad we were looking for; we will denote this opeard by the same symbol $\mathcal{F}(X)$. More explicitely, an element of the free operad $\mathcal{F}(X)$ is a linear combination of Reshetikhin-Turaev's trees with the vertices decorated by the elements of X.

6 Ideals and quotients

Let $\mathcal{P}$ be a PROP of vector spaces. An ideal of $\mathcal{P}$ is a collection $\mathcal{I} = \{\mathcal{I}(n,m)\}$ of subspaces of $\mathcal{P}(n,m)$ which is closed with respect to composition and tensor products with $\mathcal{P}$. It is immediate to check that, if $\mathcal{I}$ is an ideal of $\mathcal{P}$, then the collection of vector spaces

$$(\mathcal{P}/\mathcal{I})((n,m) := \mathcal{P}(m,n)/\mathcal{I}(n,m)$$

defines a PROP $\mathcal{P}/\mathcal{I}$, that will be called the *qouotient* PROP of $\mathcal{P}$ by $\mathcal{I}$. If $\mathcal{O}$ is an operad, and $\mathcal{I}$ is an ideal of the PROP $\mathcal{P}_{\mathcal{O}}$, then the n-to-1 part of the quotient PROP $\mathcal{P}_{\mathcal{O}}/\mathcal{I}$ will be called the quotient operad of $\mathcal{O}$ by $\mathcal{I}$ and will be denoted by $mathcalO/\mathcal{I}$. If $I = \{I(n,m)\}$ is a collection of subsets of $\mathcal{P}(n,m)$, the smallest ideal of $\mathcal{P}$ containing I is called the ideal generated by I.

As in standard algebra representation theory, if $\mathcal{I}$ is an ideal of the PROP $\mathcal{P}$ there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{PROP}}(\mathcal{P}/\mathcal{I}, \tilde{\mathcal{P}}) \simeq \left\{\rho \in \mathrm{Hom}_{\mathrm{PROP}}(\mathcal{P}, \tilde{\mathcal{P}}) \text{ such that } \rho(\mathcal{I}) = 0\right\}$$

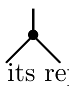
In particular, if $\mathcal{I}$ is an ideal of $\mathcal{F}(X)$, a representation

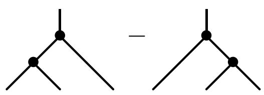
$$\mathcal{F}(X)/\mathcal{I} \to \underline{\mathrm{End}}(V)$$

is the same thing as defining on V a set of multiplications which satisfy the relations prescribed by $\mathcal{I}$.

7 Examples

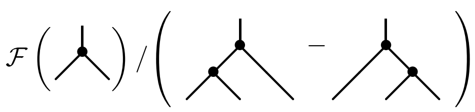
The operad Assoc

Let $X(2) = \{$  $\}$ and $X(n) = \emptyset$ for $n \neq 2$. The operad $\mathcal{F}(X)$ is simply the operad of trivalent trees; its representations are algebras with one binary operation. Let now $\mathcal{I}$ be the ideal of $\mathcal{F}(X)$ generated by



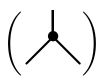
(associativity)

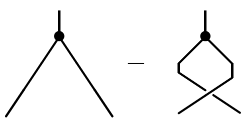
and let **Assoc** be the quotient operad



Then **Assoc**-algebras are exactly associative algebras.

The operad Comm

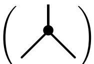
It is the quotient operad of $\mathcal{F}$  by the ideal generated by

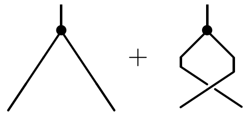


(commutativity)

Clearly, **Comm**-algebras are exactly commutative associative algebras.

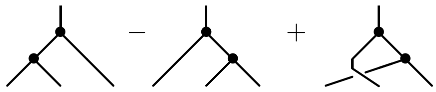
The operad Lie

It is the quotient operad of $\mathcal{F}$ () by the ideal generated by



(anticommutativity)

and



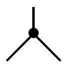
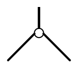
(Jacobi)

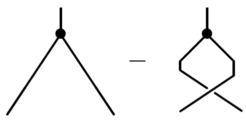
As the name says, **Lie**-algebras are exactly Lie algebras.

The operad Poisson

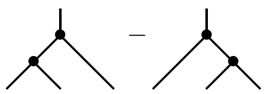
We end this short introduction to the language of operads with an example of an operad describing Poisson algebras. These are algebras with two binary operations: a commutative associative multiplication $\cdot$ and a Lie braket $\{\, ,\, \}$ (Poisson bracket) which satysfy the compatibility realtion

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

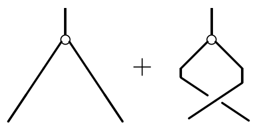
Since there are two binary operations, let $X(2) = \{$  ,  $\}$ and $X(n) = \emptyset$ for $n \neq 2$. Let $\mathcal{I}$ be the ideal of $\mathcal{F}$ generated by



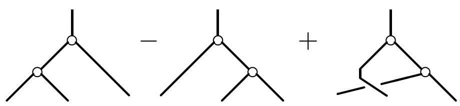
(commutativity)



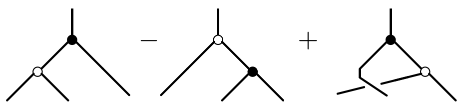
(associativity)



(anticommutativity)



(Jacobi)



(compatibility)

and let **Poisson** be the quotient operad of $\mathcal{F}(X)$ by $\mathcal{I}$. Then **Poisson**-algebras are exactly Poisson algebras.

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Introduction to A_∞ and L_∞ algebras

Riccardo Murri

March 15, 2002

"Ti accorgi di come vola bassa la mia mente?
È colpa dei pensieri associativi
se non riesco a stare adesso qui"
— *F. Battiato*

These notes aim to be an introduction to the A_∞ and L_∞ structures, through their operadic formulations. The accent is on generalizing the well-known results for associative and Lie algebras; there is not much in here, save for definitions and a few easy propositions: the presentation is a diluted version of sections 1–3 in [HS93].

1 Preliminaries on notation

For the most part of this note we shall be concerned with objects and morphisms from the category of differential graded modules over a fixed associative differential graded $\mathbb{k}$-algebra R.

The degree i component of an object M will be denoted M^i; the grading group will always be $\mathbb{Z}$. Elements of a graded object appearing as exponents to a number will stand for their degree, i.e., $(-1)^a = (-1)^{\deg a}$; therefore, $(-1)^{ab} = (-1)^{\deg a + \deg b}$.

Precise definitions of objects in this category are as follows.

Definition 1.1. A dg-algebra R is the data of a graded vector space $R^\#$ over $\mathbb{k}$, a $\mathbb{k}$-bilinear associative product, and a $\mathbb{k}$-linear map $\mathrm{d} : R \to R$ such that:

1. $R^i \cdot R^j \subset R^{i+j}$;
2. $\mathrm{d}(R^i) \subset R^{i+1}$;

and, for all homogeneous a, b:

3. $ab = (-1)^{ab} ba$ (graded commutativity);
4. $\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^a a \cdot \mathrm{d}(b)$ (graded Leibniz rule).

Furthermore, we shall always assume that a dg-algebra has a unit $1 \in R^0$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

Any dg-algebra is (graded) associative commutative algebra; write $\#$ for the underlying functor that "forgets the differential".

Definition 1.2. An R-dg-module M is an R-bimodule such that:

1. $R^i M^j \subset M^{i+j}$, $M^i R^j \subset M^{i+j}$;
2. $\lambda x = (-1)^{\lambda x} x\lambda$, for all homogeneous $x \in M$, and $\lambda \in R$;
3. $\mathrm{d}(\lambda x) = \mathrm{d}(\lambda)x + (-1)^\lambda \lambda \mathrm{d}(x)$.

There is an obvious underlying functor # from the category of R-dg-modules to the category of R-modules.

Definition 1.3. For every two R-dg-modules M, N we define the module of degree p morphisms

$$\operatorname{Hom}_R^p(M,N) := \left\{ f \in \prod_i \operatorname{Hom}_{\Bbbk}(M^i, N^{i+p}) : f(x\lambda) = f(x)\lambda \quad \forall x \in M, \forall \lambda \in R \right\},$$

and the graded module of morphisms

$$\operatorname{Hom}_R^* := \bigoplus_{p\in\mathbb{Z}} \operatorname{Hom}_R^p(M,N)[-p],$$

that is, $\operatorname{Hom}_R^p(M,N)$ is the degree p component of $\operatorname{Hom}_R^*(M,N)$.

$\operatorname{Hom}_R^*(M,N)$ is made into an R-dg-module by the differential:

$$\mathrm{d}f := \mathrm{d}_N \circ f + (-1)^{p+1} f \circ \mathrm{d}_M, \quad f \in \operatorname{Hom}_R^p(M,N)$$

The term "morphism" will denote any element of Hom_R^*, while "dg-morphism" will be applied only to those such that $\mathrm{d}f = 0$.

1.1 Signs

Recall that the symmetric monoidal category of graded modules has a non-trivial (yet involutive) twisting isomorphism $T_{(12)} : M_1 \otimes M_2 \ni x_1 \otimes x_2 \mapsto (-1)^{x_1x_2} x_2 \otimes x_1 \in M_2 \otimes M_1$, for all objects M_1, M_2. For any $\sigma \in \mathfrak{S}_n$, there's an isomorphism T_σ between $M_1 \otimes \cdots \otimes M_n$ and $M_{\sigma_1} \otimes \cdots \otimes M_{\sigma_n}$ made up of twists $T_{(ij)}$; since the category of graded modules is *symmetric*, any two such compositions are equal, that is, T_σ depends only on $\sigma \in \mathfrak{S}_n$.

Now, suppose $M_1 = \cdots = M_n = M$, a fixed object: $x_1 \otimes \cdots \otimes x_n$ and $X_\sigma(x_1 \otimes \cdots \otimes x_n)$ differ only by the sign; therefore we define $\epsilon(\sigma; x_1, \ldots, x_n)$ so that the following holds:

$$\epsilon(\sigma; x_1, \ldots, x_n) x_1 \otimes \cdots \otimes x_n = T_\sigma(x_1 \otimes \cdots \otimes x_n), \qquad \sigma \in \mathfrak{S}_n. \tag{1.1}$$

Also define $\chi(\sigma; x_1, \ldots, x_n)$ by:

$$\chi(\sigma; x_1, \ldots, x_n) x_1 \otimes \cdots \otimes x_n = (-1)^\sigma T_\sigma(x_1 \otimes \cdots \otimes x_n), \qquad \sigma \in \mathfrak{S}_n. \tag{1.2}$$

We shall omit $x_1, \ldots, x_n$ from the above when it will be clear from the context which elements ϵ or χ are being applied to.

Definition 1.4. A map $f : M^{\otimes n} \to N$ is graded symmetric iff

$$f(x_1, \ldots, x_n) = \epsilon(\sigma) f(x_{\sigma_1}, \ldots, x_{\sigma_n}) \qquad \forall x_1, \ldots, x_n \in M$$

It is graded antisymmetric iff

$$f(x_1, \ldots, x_n) = \chi(\sigma) f(x_{\sigma_1}, \ldots, x_{\sigma_n}) \qquad \forall x_1, \ldots, x_n \in M.$$

Since permutation of factors in a tensor product may change the sign, the contraction map

$$c : \bigotimes_i \operatorname{Hom}(M_i, N_i) \otimes \bigotimes M_i \to \bigotimes_i N_i$$

is well-defined only up to a sign. In search of a remedy, we fix an ordering of the factors which gives "unsigned" contraction, and then define contraction of any permutation of the factors

$\{\mathrm{Hom}(M_i, N_i), M_i\}_{i\in I}$ by first applying a twist X_σ to get the fixed "standard" ordering, and then the contraction before. Formally, let:

$$\mathrm{ev}_{MN} : \mathrm{Hom}(M,N)\otimes M \to N, \qquad (1.3)$$
$$f\otimes x \mapsto f(x);$$
$$\textstyle\bigotimes_i \mathrm{ev}_{M_iN_i} : \bigotimes_i\big(\mathrm{Hom}(M_i,N_i)\otimes M_i\big) \to \bigotimes N_i, \qquad (1.4)$$
$$(f_1\otimes x_1)\otimes\cdots\otimes(f_k\otimes x_k) \mapsto f_1(x_1)\otimes\cdots\otimes f_k(x_k);$$

then let $W_{2i-1} := \mathrm{Hom}(M_i, N_i)$, $W_{2i} := M_i$, for $i = 1, \dots, k$; define the contraction

$$\textstyle\bigotimes_{j=1}^{2k} W_{\sigma_j} \to \bigotimes_{i=1}^{k} N_i$$

by the composite

$$\textstyle\bigotimes_{j=1}^{2k} W_{\sigma_j} \xrightarrow{T_\sigma} \bigotimes_{j=1}^{2k} W_j \xrightarrow{\bigotimes_i \mathrm{ev}} \bigotimes_{i=1}^{k} N_i.$$

Infact, this boils down to the rule: "change the sign by $(-1)^{pq}$ when interchanging objects of degree p and q".

2 A_∞-algebras

A_∞-algebras are a generalization of homotopy associative algebras, that is, algebras where $a(bc) - (ab)c = \ell(a, b, c)$ for some "homotopy" ℓ; for this reason they are also called "*strongly* homotopy associative algebras."

Let A be a $\Bbbk$-vector space.

Definition 2.1. A structure of A_p-algebra on A is given by a collection of $\Bbbk$-linear maps $\{m_n \in \mathrm{Hom}^{n-2}_{\Bbbk}(A^{\otimes n}, A)\}_{1\leqslant n<p+1}$ that satisfy the following set of relations ($1 \leqslant n < p + 1$):

$$\sum_{i+j=n+1}\sum_{s=0}^{i-1} \pm m_i(a_1,\dots,a_s, m_j(a_{s+1},\dots,a_{s+j}), a_{s+j+1},\dots,a_n) = 0, \qquad (A_{n+1})$$

the $\pm$ sign being:

$$\pm = (-1)^{j+s+js+j(a_1+\cdots+a_s)}.$$

An A_∞-algebra structure on A is given by $\Bbbk$-linear maps $\{m_n\}_{n\geqslant 1}$ such that $\{m_n\}_{1\leqslant n<p+1}$ defines a structure of A_p-algebra on A for every $p > 1$.

Let us spell out the first three (A_{n+1}) relations:

$$0 = m_1^2 \qquad (A_2)$$
$$0 = m_1(m_2(a_1, a_2)) - m_2(m_1(a_1), a_2) - (-1)^{a_1} m_2(a_1, m_1(a_2)) \qquad (A_3)$$
$$\begin{aligned} 0 = &-m_1(m_3(a_1,a_2,a_3)) + m_2(m_2(a_1,a_2),a_3) - m_2(a_1, m_2(a_2,a_3)) \qquad (A_4)\\ &- m_3(m_1(a_1),a_2,a_3) - (-1)^{a_1} m_3(a_1, m_1(a_2), a_3)\\ &- (-1)^{a_1+a_2} m_3(a_1,a_2,m_1(a_3)) \end{aligned}$$

The first one tells us that $\partial := m_1$ is a differential on A, compatible with the multiplication $m_2 : A^{\otimes 2} \to A$ by (A_3). If we rewrite (A_4) as

$$\begin{aligned} m_2(m_2(a_1,a_2),a_3) - m_2(a_1,m_2(a_2,a_3)) = &\\ \partial\circ m_3(a_1,a_2,a_3) + m_3(\partial a_1,a_2,a_3) + (-1)^{a_1} m_3(a_1,\partial a_2,a_3) + &\\ + (-1)^{a_1+a_2} m_3(a_1,a_2,\partial a_3), &\quad (A_4') \end{aligned}$$

then we see that m_2 is associative up to an homotopy given by m_3; we get a truly associative algebra if $m_3 \equiv 0$, but the converse needs not be true: i.e., the right-hand side of A'_4 can be 0 without m_3 being trivial—see for instance the example in section 2.1.

Summing up, an A_1 algebra is just a $\Bbbk$-dg-module, an A_2-algebra is a $\Bbbk$-dg-algebra (not necessarily associative), an A_3-algebra is a homotopy associative $\Bbbk$-dg-algebra.

Example 2.1. Any differential graded associative algebra is an A_3-algebra with $m_3 = 0$. Any graded associative algebra is an A_3-algebra with $m_1 = m_3 = 0$.

Example 2.2 ([Kha, p. 5]). Given an A_∞-algebra (A, m_*) and an associative algebra B (concentrated in degree 0) define a new A_∞-algebra $(A \otimes B, m')$ by $(A \otimes B)_p := A_p \otimes B$ and

$$m'_n(a_1 \otimes b_1, \ldots, a_n \otimes b_n) := m_n(a_1, \ldots, a_n) \otimes (b_1 \cdots b_n).$$

Example 2.3. Given an associative $\Bbbk$-algebra A, graded over $\mathbb{Z}/2\mathbb{Z}$ and concentrated in degree 0, pick $\mu_2, \mu_4, \ldots, \mu_{2t} \in \Bbbk$ and define:

$$\begin{aligned} m_{2i+1}(a_1, \ldots, a_{2q+1}) &:= 0 & q &= 0, 1, \ldots; \\ m_{2r}(a_1, \ldots, a_{2r}) &:= \mu_{2r} \cdot a_1 \cdots a_{2r} & 1 &\leqslant r \leqslant t; \\ m_{2s}(a_1, \ldots, a_{2s}) &:= 0 & s &> t; \end{aligned}$$

Then (A, m'_*) is an A_∞-algebra.

Note that we may rewrite A_{n+1} as

$$\sum_{\substack{i+j=n+1 \\ i,j \geqslant 2}} \sum_{s=0}^{i-1} \pm m_i(a_1, \ldots, a_s, m_j(a_{s+1}, \ldots, a_{s+j}), a_{s+j+1}, \ldots, a_n) \\ = [m_n, \partial](a_1, \ldots, a_n), \quad (A'_{n+1})$$

where $\partial = m_1$ and $[\cdot, \cdot]$ is the Lie bracket on the Hochschild complex:

$$[m_n, \partial](a_1, \ldots, a_n) := \sum_{s=0}^{n-1} (-1)^{a_1 + \cdots + a_s} m_n(a_1, \ldots, a_s, \partial a_{s+1}, a_{s+2}, \ldots, a_n) + \\ + (-1)^{n+1} \partial m_n(a_1, \ldots, a_n).$$

The right-hand side of (A'_{n+1}) vanishes on passing to ∂-cohomology: therefore, $H_\partial(A)$ inherits a structure of A_p-algebra with trivial differential.

2.1 A less trivial example

In [Zho00], J. Zhou adapted a construction by S. A. Merkulov to give an A_∞-structure on the space of harmonic forms on a Riemannian manifold. Here's a sketch of their construction.

Let V be a DGA over $\Bbbk$ with differential d; let W be a vector subspace of V such that $\mathrm{d}W \subset V$; let $Q : V \to V$ be an *odd* operator such that $P := (\mathrm{Id} - [\mathrm{d}, Q])$ has range lying in W. Define $\Bbbk$-linear maps $\lambda_n : V^{\otimes n} \to V$ by

$$\lambda_2(v_1, v_2) := v_1 \cdot v_2$$

where $\cdot$ is the ordinary multiplication in V, and then, recursively, for $n \geqslant 3$:

$$\lambda_n(v_1, \ldots, v_n) := (-1)^{n-1} \big(Q\lambda_{n-1}(v_1, \ldots, v_n)\big) \cdot v_n + \\ - \sum_{\substack{k+l=n+1 \\ k,l \geqslant 2}} (-1)^{k+(l-1)(v_1+\cdots+v_k)} Q\lambda_k(v_1, \ldots, v_k) \cdot Q\lambda_l(v_{k+1}, \ldots, v_n) + \\ - (-1)^{n v_1} v_1 \cdot \big(Q\lambda_{n-1}(v_2, \ldots, v_n)\big)$$

Then a longish, yet direct, computation proves the following statement.

Proposition 1 (Merkulov, [Mer99]). *The $\Bbbk$-linear maps $m_n : A^{\otimes n} \to A$ defined by*

$$m_1 := \mathrm{d},$$
$$m_n := P \circ \lambda_n, \quad n \geqslant 2,$$

define a structure of A_∞-algebra on W.

As an immediate corollary, we get an A_∞-algebra structure on the space of harmonic forms on a Riemannian manifold X: take $V = \mathcal{E}^*(X)$, the DGA of differential forms with multiplication given by the wedge product, $W = \mathcal{H}^*(X)$ the space of harmonic forms, and $Q = G\mathrm{d}^*$ where $\mathrm{d}^* = - * \circ \mathrm{d} \circ *$ and G is the Green operator. Then $P = \mathrm{Id} - G\mathrm{d}^*\mathrm{d} - \mathrm{d}G\mathrm{d}^*$ is the projector on the space $\mathcal{H}^*(X)$, and the multiplication $m_2 : \mathcal{H}^*(X) \to \mathcal{H}^*(X)$ takes two forms to the harmonic part of their wedge product. It can be shown (see [Zho00]) that m_2 is associative; therefore, by (A'_4), $[d, m_3] = 0$, yet this does not imply that $m_3 = 0$.

Passing to homology with respect to d, we get an A_∞-structure on

$$H_{\mathrm{d}}(\mathcal{H}^*(X)) = H^*_{\mathrm{dR}}(X)$$

with trivial differential; so the left-hand side of (A'_{n+1}) vanishes, in particular, m_2 is an associative multiplication—indeed it is the usual cup product.

Similarly, one can define A_∞-algebra structures on the Dolbeault cohomology of any complex manifold X; for this and other examples, see [Zho00, Mer99].

3 L_∞-algebras

L_∞-algebras arise as a generalization of homotopy Lie algebras, i.e., algebras which satisfy the Jacobi identity only up to an homotopy: this homotopy is in turn required to satisfy "higher" Jacobi identities up to "higher" homotopies, etc. Hence, L_∞-algebras are also called "*strongly* homotopy Lie algebras".

Let L be a graded $\Bbbk$-vector space. Say that $\sigma \in \mathfrak{S}_n$ is a $(j, n-j)$-shuffle iff $\sigma_1 < \cdots < \sigma_j$ and $\sigma_{j+1} < \cdots < \sigma_n$; name $\mathfrak{S}_{j,n-j}$ the set of $(j, n-j)$-shuffles.

Definition 3.1. An L_∞-algebra structure on L is a system $\{l_k\}_{k\geqslant 1}$ of graded antisymmetric $\Bbbk$-linear maps $l_k : L^{\otimes k} \to L$ with $\deg l_k = k-2$, that satisfy the following set of identities:

$$\sum_{i+j=n+1} \sum_{\sigma \in \mathfrak{S}_{j,n-j}} (-1)^{j(i-1)} \chi(\sigma) l_i(l_j(a_{\sigma_1}, \ldots, a_{\sigma_j}), a_{\sigma_{j+1}}, \ldots, a_{\sigma_n}) = 0 \qquad (L_{n+1})$$

An L_∞-algebra structure on L is given by $\Bbbk$-linear maps $\{l_n\}_{n\geqslant 1}$ such that $\{l_n\}_{1\leqslant n<p+1}$ defines a structure of L_p-algebra on L for every $p > 1$.

Let us spell out the first three (L_n) relations:

$$0 = l_1^2, \qquad (L_2)$$
$$0 = l_1(l_2(a_1, a_2)) - l_2(l_1(a_1), a_2) - (-1)^{a_1} l_2(l_1(a_2), a_1), \qquad (L_3)$$
$$\begin{aligned} 0 = {} & l_1(l_3(a_1, a_2, a_3)) - l_2(l_2(a_1, a_2), a_3) - (-1)^{a_2 a_3 + 1} l_2(l_2(a_1, a_3), a_2) + \qquad (L_4) \\ & - (-1)^{a_2(a_1+a_3)} l_2(l_2(a_2, a_3), a_1) + l_3(l_1(a_1), a_2, a_3) + \\ & + (-1)^{a_1 a_2 + 1} l_3(l_1(a_2), a_1, a_3) + (-1)^{a_3(a_1+a_2)} l_3(l_2(a_3), a_1, a_2), \end{aligned}$$

The first one tells us that $\partial := l_1$ is a differential on L, compatible with the bracket $[\cdot,\cdot] := l_2$ by L_3. Rewriting (L_4) as:

$$\begin{aligned} l_2(l_2(a_1, a_2), a_3) + (-1)^{a_1 a_2 + 1} l_2(a_2, l_2(a_1, a_3)) + (-1)^{a_1(a_2+a_3)} l_2(a_1, l_2(a_2, a_3)) \\ = l_3(l_1(a_1), a_2, a_3) + (-1)^{a_1} l_3(a_1, l_1(a_2), a_3) \\ + (-1)^{a_1+a_2} l_3(a_1, a_2, l_2(a_3)) - l_1(l_3(a_1, a_2, a_3)) \quad (L'_4) \end{aligned}$$

we see that l_2 satisfies the Jacobi identity up to an homotopy l_3; if $l_3 = 0$, we recover a true Lie algebra.

Example 3.1. Any DGLA L is an L_∞-algebra with $l_3 = l_4 = \cdots = 0$. Any Lie algebra is an L_∞-algebra with $l_1 = l_3 = \cdots = 0$.

Note that we may rewrite L_{n+1} as

$$\sum_{\substack{i+j=n+1\\ i,j\geqslant 2}} \sum_{\sigma\in\mathfrak{S}_{j,n-j}} (-1)^{j(i-1)}\chi(\sigma)l_i(l_j(a_{\sigma_1},\dots,a_{\sigma_j}),a_{\sigma_{j+1}},\dots,a_{\sigma_n}) \\ = (-1)^n[l_n,\partial](a_1,\dots,a_n), \quad (L'_{n+1})$$

where $\partial = l_1$ and $[\cdot,\cdot]$ is the bracket:

$$[l_n,\partial](a_1,\dots,a_n) := \sum_{0\leqslant s\leqslant n-1} (-1)^{a_1+\dots+a_s} l_n(a_1,\dots,a_s,\partial a_{s+1},a_{s+2},\dots,a_n) + \\ -(-1)^n\partial l_n(a_1,\dots,a_n).$$

The right-hand side of L'_{n+1} vanishes if l_n commutes with the differential ∂; this happens in particular when l_n is a dg-morphism or when $\partial = 0$. Therefore, any L_∞-algebra induces a Lie algebra structure on its ∂-homology.

Every A_∞-algebra structure $\{m_n\}$ on a $\Bbbk$-vector space X induces an L_∞-structure $\{l_n\}$ on X by antisymmetrization:

$$l_n(x_1,\dots,x_n) := \sum_{\sigma\in\mathfrak{S}_n} \chi(\sigma)m_n(x_1,\dots,x_n),$$

similarly to an associative algebra inducing a Lie structure through the bracket $[x,y] := xy - yx$.

Example 3.2. Recall that, given an A_∞-algebra X and an associative algebra Y, we can form a new A_∞-algebra $X\otimes Y$: in particular, when Y is the algebra of $n\times n$ matrices with entries in $\Bbbk$, we get an A_∞-algebra $M_n(X)$. By antisymmetrization, we define an L_∞-algebra structure $\mathfrak{gl}_n(X)$ on $M_n(X)$.

Other constructions of ordinary Lie algebra theory generalize to the L_∞ case: tensor algebra, enveloping algebra, homology. These will be better described using the language of operads.

4 Operads

Fix a differential graded $\Bbbk$-algebra R; the category of R-dg-modules and their morphisms (not necessarily commuting with the differential) is Abelian and symmetric monoidal with the tensor product $\otimes := -\otimes_R -$. In this section, we mainly follow [KM95] for the definition of an operad.

Definition 4.1. An operad $\mathscr{O}$ is a collection of R-modules $\{\mathscr{O}(n)\}_{n\geqslant 0}$ together with

i) a unit map $\eta : R \to \mathscr{O}(1)$;

ii) a right action of the symmetric group $\mathfrak{S}_n$ on $\mathscr{O}(n)$, for all $n \geqslant 0$;

iii) composition maps $\mathscr{O}(n)\otimes\mathscr{O}(k_1)\otimes\cdots\otimes\mathscr{O}(k_n) \xrightarrow{\gamma_{n;k_1,\dots,k_n}} \mathscr{O}(k)$ for all $n\geqslant 0$ and $k = \sum k_s$; $\deg\gamma = 0$ as a map of graded modules.

These data are required to satisfy the following compatibility relations.

1. The following *associativity* diagrams commute, for all $n \geqslant 0$, $k_s \geqslant 0$, and $j_{sr} \geqslant 0$:

$$\begin{tikzcd}
\mathscr{O}(n) \otimes \big(\bigotimes_{s=1}^{n} \mathscr{O}(k_n)\big) \otimes \big(\bigotimes_{s=1}^{n} \bigotimes_{r=1}^{k_s} \mathscr{O}(j_{sr})\big) \arrow[dr, "\gamma_{n;k_1,\dots,k_n} \otimes \mathrm{id}"] \arrow[dddd, leftrightarrow, "\text{signed reordering}"'] & \\
& \mathscr{O}(k) \otimes \big(\bigotimes_{s=1}^{n} \bigotimes_{r=1}^{k_s} \mathscr{O}(j_{sr})\big) \arrow[d, "\gamma_{k;j_{11},\dots,j_{nk_n}}"'] \\
& \mathscr{O}(j) \\
& \mathscr{O}(n) \otimes \big(\bigotimes_{s=1}^{n} \mathscr{O}(j_s)\big) \arrow[u, "\gamma_{n;j_1,\dots,j_n}"'] \\
\mathscr{O}(n) \otimes \Big(\bigotimes_{s=1}^{n} \big(\mathscr{O}(k_s) \otimes \bigotimes_{r=1}^{k_s} \mathscr{O}(j_{sr})\big)\Big) \arrow[ur, "\mathrm{id} \otimes \bigotimes_{s=1}^{n} \gamma_{k_s;j_{s1},\dots,j_{sk_s}}"'] &
\end{tikzcd} \tag{4.1}$$

   where
   $$k := \sum_s k_s, \qquad j_r := \sum_{r=1}^{k_s} j_{sr},$$
   and the "signed reordering" acts as the appropriate composition of commutators in the symmetric category.

2. The following *unit* diagrams commute.

$$\begin{tikzcd}
\mathscr{O}(n) \otimes R^{\otimes n} \arrow[dr] \arrow[dd, "\mathrm{id} \otimes \eta^{\otimes n}"'] & \\
& \mathscr{O}(n) \\
\mathscr{O}(n) \otimes \mathscr{O}(1)^{\otimes n} \arrow[ur, "\gamma_{n;1,\dots,1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes \mathscr{O}(n) \arrow[dr] \arrow[dd, "\eta \otimes \mathrm{id}"'] & \\
& \mathscr{O}(n) \\
\mathscr{O}(1) \otimes \mathscr{O}(n) \arrow[ur, "\gamma_{1;n}"'] &
\end{tikzcd} \tag{4.2}$$

3. The following *equivariance* diagrams commute, for $\sigma \in \mathfrak{S}_n$ and $\tau_s \in \mathfrak{S}_{k_s}$, the permutation $\sigma(k_1, \dots, k_n) \in \mathfrak{S}_k$ permutes blocks of lengths $k_1, \dots, k_n$ like σ permutes letters; $\tau_1 \oplus \cdots \oplus \tau_n$ permutes letters in the s-th block as τ_s does:

$$\begin{tikzcd}
\mathscr{O}(n) \otimes \mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_n) \arrow[r, "\sigma \otimes T_{\sigma^{-1}}"] \arrow[d, "\gamma_{n;k_1,\dots,k_n}"'] & \mathscr{O}(n) \otimes \mathscr{O}(k_{\sigma_1}) \otimes \cdots \otimes \mathscr{O}(k_{\sigma_n}) \arrow[d, "\gamma_{n;k_{\tau_1},\dots,k_{\tau_n}}"] \\
\mathscr{O}(k) \arrow[r, "\sigma(k_{\sigma_1},\dots,k_{\sigma_n})"] & \mathscr{O}(k)
\end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd}
\mathscr{O}(n) \otimes \mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_n) \arrow[r, "\gamma_{n;k_1,\dots,k_n}"] \arrow[d, "\mathrm{id} \oplus (\tau_1 \otimes \cdots \otimes \tau_n)"'] & \mathscr{O}(k) \arrow[d, "\tau_1 \oplus \cdots \oplus \tau_n"] \\
\mathscr{O}(n) \otimes \mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_n) \arrow[r, "\gamma_{n;k_1,\dots,k_n}"'] & \mathscr{O}(k)
\end{tikzcd} \tag{4.4}$$

Operads may be defined in any symmetric tensor category. For our purposes, it will suffice to always restrict to the category of R-dg-modules.

The prototypical operad is the *endomorphism operad* $\mathscr{E}_M$ of some R-dg-module M, which is defined by:

$$\mathscr{E}_M(n) := \mathrm{Hom}(M^{\otimes n}, M).$$

The unit is given by the identity map $\mathrm{id}_M \in \mathrm{Hom}(M, M) = \mathscr{E}_M(1)$; the $\mathfrak{S}_n$-action is the (signed) permutation of tensor product factors, and, finally, the maps $\gamma_{n;k_1,\dots,k_n}$ are given by

$$\begin{tikzcd}
\mathrm{Hom}(M^{\otimes n}, M) \otimes \mathrm{Hom}(M^{\otimes k_1}, M) \otimes \cdots \otimes \mathrm{Hom}(M^{\otimes k_n}, M) \arrow[d, "\mathrm{id} \otimes (n\text{-fold tensor product of maps})"] \\
\mathrm{Hom}(M^{\otimes n}, M) \otimes \mathrm{Hom}(M^{\otimes k}, M^{\otimes n}) \arrow[d, "\circ"] \\
\mathrm{Hom}(M^{\otimes k}, M)
\end{tikzcd}$$

It is trivial to verify that $\mathscr{E}_M$ is an operad.

A *morphism of operads* $\phi : \mathscr{O}' \to \mathscr{O}''$ is a collection of R-dg-modules morphisms $\{\phi_n : \mathscr{O}'(n) \to \mathscr{O}''(n)\}$ satisfying the obvious compatibility conditions coming from diagrams (4.1)–(4.4).

4.0.1 Signs in operads

The "signed reordering" of diagram (4.1) may appear as to introduce an unnatural sign: infact it is not so, the reason being that a Koszul sign is hidden into the γ's, as the following example shows.

Let M be an R-dg-module, $A \in \mathrm{Hom}(M^{\otimes 2}, M)$, $B_1 \in \mathrm{Hom}(M, M)$ and $B_2 \in \mathrm{Hom}(M^{\otimes 2}, M)$. According to the definition of operad, $\gamma_{2;1,2}(A \otimes B_1 \otimes B_2) \in \mathrm{Hom}(M^{\otimes 3}, M)$, so we pick $x_1 \otimes x_2 \otimes x_3 \in M^{\otimes 3}$ and reckon:

$$y := (A \otimes B_1 \otimes B_2)(x_1 \otimes x_2 \otimes x_3) = A \otimes \big((B_1 \otimes B_2)(x_1 \otimes x_2 \otimes x_3)\big),$$

so, by the Koszul sign convention,

$$\begin{aligned}
y = A \otimes \big((-1)^{x_1 B_2} B_1(x_1) \otimes B_2(x_2 \otimes x_3)\big) \\
= (-1)^{x_1 B_2} A\big((-1)^{x_1 B_2} B_1(x_1) \otimes B_2(x_2 \otimes x_3)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(A \otimes B_1 \otimes B_2)(x_1 \otimes x_2 \otimes x_3) \\
= (-1)^{x_1 B_2} A\big((-1)^{x_1 B_2} B_1(x_1) \otimes B_2(x_2 \otimes x_3)\big).
\end{aligned} \tag{4.5}$$

Let us apply this to a particular case of (4.1): pick $C_1, C_2, C_3 \in \mathrm{Hom}(M, M)$, then use (4.5) to walk (4.1) in two ways. First, top-down:

$$\begin{aligned}
&A\big(B_1\big(C_1(x_1)\big), B_2\big(C_2(x_2), C_3(x_3)\big)\big) \\
&\qquad = (-1)^{(x_1+C_1)B_2}\gamma(A \otimes B_1 \otimes B_2)\big(C_1(x_1) \otimes C_2(x_2) \otimes C_3(x_3)\big) \\
&\qquad\qquad = (-1)^{C_1B_2+x_1B_2+x_1(C_2+C_3)+x_2C_3} \times \\
&\qquad\qquad\qquad \gamma\big(\gamma(A \otimes B_1 \otimes B_2) \otimes C_1 \otimes C_2 \otimes C_3\big)(x_1 \otimes x_2 \otimes x_3);
\end{aligned}$$

then, bottom-up:

$$\begin{aligned}
&A\Big(B_1\big(C_1(x_1)\big), B_2\big(C_2(x_2), C_3(x_3)\big)\Big) \\
&\qquad = (-1)^{x_2C_3}A\big(\gamma(B_1 \otimes C_1)(x_1), \gamma(B_2 \otimes C_2 \otimes C_3)(x_2 \otimes x_3)\big) \\
&\qquad\qquad = (-1)^{x_2C_3+x_1(B_2+C_2+C_3)} \times \\
&\qquad\qquad\qquad \gamma\big(A \otimes \gamma(B_1 \otimes C_1) \otimes \gamma(B_2 \otimes C_2 \otimes C_3)\big)(x_1 \otimes x_2 \otimes x_3).
\end{aligned}$$

Now we see that the sign $(-1)^{B_2C_1}$ coming from the reordering

$$T_{(34)} \colon A \otimes B_1 \otimes B_2 \otimes C_1 \otimes C_2 \otimes C_3 \mapsto (-1)^{B_2C_1} A \otimes B_1 \otimes C_1 \otimes B_2 \otimes C_2 \otimes C_3$$

is the one needed to make diagram (4.1) commute.

4.1 Algebras over operads

The concept of "algebra over an operad" is the right one needed to introduce operads into our discussion of A_∞ and L_∞ algebras.

Let X be an R-dg-module, and $\mathscr{O}$ a fixed operad.

Definition 4.2. A structure of $\mathscr{O}$-algebra on X is an operad morphism $\mathscr{O} \to \mathscr{E}_X$ of $\mathscr{O}$ into the endomorphism operad of X.

So, elements in $\mathscr{O}(n)$ are interpreted as maps $X^{\otimes n} \to X$, that is, they are n-ary operations on X. In the sequel we shall see how to give a operadic formulation of common algebraic structures.

The following proposition encodes many well-known structure transfer theorems.

Proposition 2. *Any morphism $\mathscr{O} \to \mathscr{O}'$ induces an $\mathscr{O}$-algebra structure on every $\mathscr{O}'$-algebra X, by means of the composition $\mathscr{O} \to \mathscr{O}' \to \mathscr{E}_X$.*

In an equivalent manner, an $\mathscr{O}$-algebra structure on X, is given by maps

$$\phi_n : \mathscr{O}(n) \otimes X^{\otimes n} \to X,$$

which satisfy associativity, unit and equivariance conditions coming from diagrams (4.1)–

(4.4), that we can express by requiring the following diagrams to commute:

$$
\begin{tikzcd}
\mathscr{O}(n) \otimes \mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_n) \otimes X^{\otimes k} \arrow[r, "\gamma \otimes \mathrm{id}"] \arrow[dd, leftrightarrow, "\text{signed reordering}"'] & \mathscr{O}(k) \otimes X^{\otimes k} \arrow[d, "\phi_k"] \\
 & X \\
\mathscr{O}(n) \otimes \big(\mathscr{O}(k_1) \otimes X^{\otimes k_1}\big) \otimes \cdots \otimes \big(\mathscr{O}(k_n) \otimes X^{\otimes k_n}\big) \arrow[r, "\mathrm{id} \otimes \phi^{\otimes k}"'] & \mathscr{O}(n) \otimes X^{\otimes n} \arrow[u, "\phi_n"']
\end{tikzcd}
\tag{4.6}
$$

$$
\begin{tikzcd}
\mathscr{O}(n) \otimes X^{\otimes n} \arrow[r, "\mathrm{id} \otimes \varepsilon(\sigma)"] \arrow[d, "\sigma \otimes \mathrm{id}^{\otimes n}"'] & \mathscr{O}(n) \otimes X^{\otimes n} \arrow[d, "\phi_n"] \\
\mathscr{O}(n) \otimes X^{\otimes n} \arrow[r, "\phi_n"'] & X
\end{tikzcd}
\tag{4.7}
$$

$$
\begin{tikzcd}
X \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}_X"'] & \mathscr{O}(1) \otimes X \arrow[r, "\phi_1"] & X
\end{tikzcd}
\tag{4.8}
$$

Example 4.1 (Free algebras over an operad). Fix an operad $\mathscr{O}$, and an R-dg-module V. Define the free $\mathscr{O}$-algebra $X := F_{\mathscr{O}}(V)$ by:

$$X^p := \mathscr{O}(p) \otimes V.$$

In order to define the structure maps ϕ, observe that

$$
\begin{aligned}
\big(\mathscr{O}(n) \otimes X^{\otimes n}\big)^p &= \mathscr{O}(p) \otimes \bigoplus_{p_1+\cdots+p_n=p} \big((\mathscr{O}(p_1) \otimes X^{p_1}) \otimes \cdots \otimes (\mathscr{O}(p_n) \otimes X^{p_n})\big) \\
&\simeq \big(\mathscr{O}(p) \otimes \bigoplus_{p_1+\cdots+p_n=p} (\mathscr{O}(p_1) \otimes \cdots \otimes \mathscr{O}(p_n))\big) \otimes (X^{\otimes n})^p,
\end{aligned}
$$

so a direct sum of the maps $\gamma_{n;p_1,\dots,p_n}$ will do the job.

4.1.1 The operad of associative algebras

If X is an associative algebra, then a map $m : X^{\otimes 2} \to X$ is defined, which satisfies the constraint

$$m(x_1, m(x_2, x3)) = m(m(x_1, x_2), x_3). \tag{4.9}$$

Let us define an operad $\mathscr{A}$ which governs associative algebras; we need an element $\mu \in \mathscr{A}(n)$ for every n-ary operation on X. Now, the product of two elements $x_1 \otimes x_2 \mapsto m(x_1, x_2) =: x_1 x_2$ is a binary operation, but also the product in reverse order $(x_1 \otimes x_2 \mapsto m(x_2, x_1) =: x_2 x_1)$ is; moreover, we can form n-ary operations by composing product maps (e.g., $x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto m(m(x_1, x_2), m(x_4, x_3)) = (x_1 x_2)(x_3 x_4)$). The associativity relation (4.9) tells us that any monomial $x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_n}$ defines a n-ary operation by composition of binary products, independently of the way we put parentheses in it. Therefore, we can state that $\mathscr{A}(n)$ is the R-module freely generated by the set of all possible products of symbols $x_1, \dots, x_n$ with every x_j appearing once and once only; for example,

$$\mathscr{A}(1) := R, \quad \mathscr{A}(2) := \langle x_1 x_2, x_2 x_1 \rangle_R, \quad \mathscr{A}(3) := \langle x_{\sigma_1} x_{\sigma_2} x_{\sigma_3} : \sigma \in \mathfrak{S}_3 \rangle_R.$$

The $\mathfrak{S}_n$-action on $\mathscr{A}(n)$ consists in permuting the x_j's. The structure maps $\gamma_{n;k_1,\dots,k_n} : \mathscr{A}(n) \otimes \mathscr{A}(k_1) \otimes \cdots \otimes \mathscr{A}(k_n) \to \mathscr{A}(k_1 + \cdots + k_n)$ replace the x_j in $\mu \in \mathscr{A}(n)$ by a monomial $\mu_j \in \mathscr{A}(k_j)$, simultaneously replacing x_l in μ_j with $x_{k_1+\cdots+k_{j-1}+l}$.

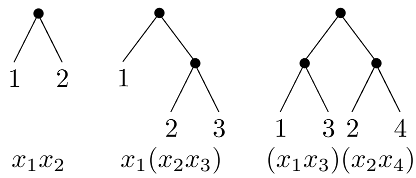
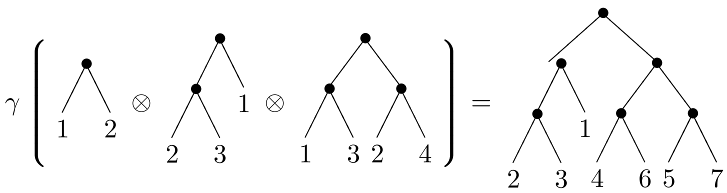


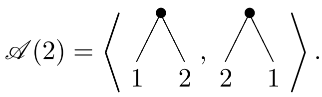
Figure 1: Binary trees correspond to meaningful ways of inserting parentheses into a product.

The operad $\mathscr{A}$ is most easily described using a graphical notation. Depict an element of $\mathscr{A}(n)$ as a *binary* tree having n leaves (inputs) and one root (the output); the leaves are numbered, and $\mathfrak{S}_n$ acts by permuting numbers on leaves. Such trees can be easily seen to correspond to the meaningful insertion of $n-2$ pairs of parentheses into a monomial $\mu \in \mathscr{A}(n)$ (see Figure 4.1.1 on page 91). The structure map $\gamma_{n;k_1,\dots,k_n}$ simply grafts the tree $t_j \in \mathscr{O}(k_j)$ onto the j-th input of $t \in \mathscr{O}(n)$ and renumbers the leaves of t_j; for instance,

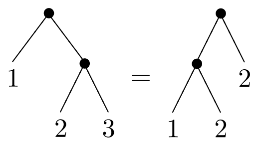


(4.10)

The whole family of numbered binary trees is generated by elements in $\mathscr{A}(2)$ via the composition maps γ; infact,

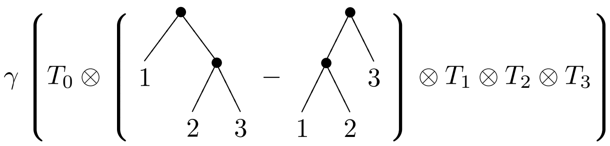


What is more, the associativity relation (4.9) can be rewritten as:



(4.11)

So, the operad $\mathscr{A}$ is the quotient of the family of binary trees by relations of the form:



where T_0, T_1, T_2 and T_3 are arbitrary binary trees. It is easy to check that these relations imply that any tree in $\mathscr{A}$ has a representative such that left-wing branches have no ramification; such trees will be called *regular trees* in the sequel.

Definition 4.3. The space $\mathscr{A}(n)$ is the R-linear span of the set of regular binary trees. It is a dg-module with the trivial differential $D = 0$.

The collection $\{\mathscr{A}(n)\}$ forms an operad with the structure maps given by the grafting operation γ and the obvious $\mathfrak{S}_n$ action.

It is an easy exercise to check the following.

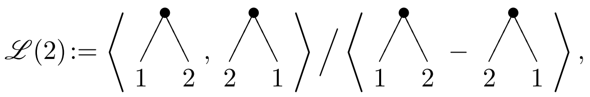
Proposition 3. *Any associative algebra is an algebra over $\mathscr{A}$, and vice-versa.*

4.1.2 The operad of Lie algebras

In a similar way, one can construct an operad governing Lie algebras. This turn, we have binary operations $x_1 \otimes x_2 \mapsto [x_1, x_2]$ and $x_1 \otimes x_2 \mapsto [x_2, x_1]$ which are related by

$$[x_1, x_2] = -[x_1, x_2].$$

So, we have the graphical presentation

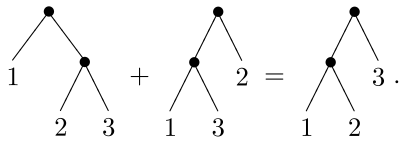


that is to say, $\mathscr{L}(2)$ is a 1-dimensional R-module endowed with the sign representation of $\mathfrak{S}_2$.

Higher-order operations in a Lie algebra are compositions of binary operations. However, since the binary operation $[-,-]$ is alternating, the action of $\mathfrak{S}_n$ on $\mathscr{L}(n)$ is not as easy as in the case of associative algebras. More over, the Jacobi equation,

$$[x_1, [x_2, x_3]] + [[x_1, x_3], x_2] = [[x_1, x_2], x_3],$$

which expresses relations among ternary operations $[-, [-,-]]$ translates into:



(4.12)

So we have yet one more relation to take into account for $\mathscr{L}(n)$. We quote the following from [GK94].

Lemma 4.1. *The space $\mathscr{L}(n)$ is (isomorphic to) the target space of the representation induced on $\mathfrak{S}_n$ by any non-trivial character of the cyclic group C_n.*

When describing the operad structure, it is easier to picture the space $\mathscr{L}(n)$ as the space of all binary trees with n leaves, modulo the relations given by (4.12) and the alternation of signs when swapping two branches of a tree. $\mathscr{L}(n)$ is a dg-module with the trivial differential $D = 0$.

Definition 4.4. The collection $\{\mathscr{L}(n)\}$ forms an operad with the structure maps given by the grafting operation γ.

Hinich and Schechtman [HS93] have called $\mathscr{L}$ the "trivial Lie operad".

Proposition 4. *Any DGLA is an algebra over $\mathscr{L}$, and vice-versa.*

4.2 Modules over an operad algebra

Fix an operad $\mathscr{O}$ and an $\mathscr{O}$-algebra X, and an R-module M.

Definition 4.5. A structure of $(\mathscr{O}, X)$-module on M is given by a collection of maps $\psi_n : \mathscr{O}(n+1) \otimes X^{\otimes n} \otimes M \to M$ which satisfy the following compatibility relations.

$$
\begin{tikzcd}
\mathscr{O}(n) \otimes \mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_n) \otimes X^{\otimes k-1} \otimes M \arrow[r, "\gamma \otimes \mathrm{id}_X^{\otimes k-1} \otimes \mathrm{id}_M"] \arrow[d, leftrightarrow, "\substack{\text{signed}\\ \text{reordering}}"'] & \mathscr{O}(k) \otimes X^{\otimes k-1} \otimes M \arrow[dd, "\psi_k"] \\
\mathscr{O}(n) \otimes \bigotimes_{j=1}^{n-1} \big(\mathscr{O}(k_j) \otimes X^{\otimes k_j}\big) \otimes \mathscr{O}(k_n) \otimes X^{\otimes k_n - 1} \otimes M \arrow[d, "\mathrm{id} \otimes \phi^{\otimes k-1} \otimes \psi_{k_n - 1}"'] & \\
\mathscr{O}(n) \otimes X^{\otimes n-1} \otimes M \arrow[r, "\psi_n"'] & X
\end{tikzcd}
\tag{4.13}
$$

$$
\begin{tikzcd}
\mathscr{O}(n) \otimes X^{\otimes n-1} \otimes M \arrow[r, "\mathrm{id}_{\mathscr{O}} \otimes \varepsilon(\sigma) \otimes \mathrm{id}_M"] \arrow[d, "\sigma \otimes \mathrm{id}_X^{\otimes n-1} \otimes \mathrm{id}_M"'] & \mathscr{O}(n) \otimes X^{\otimes n-1} \otimes M \arrow[d, "\psi_n"] \\
\mathscr{O}(n) \otimes X^{\otimes n-1} \otimes M \arrow[r, "\psi_n"'] & X
\end{tikzcd}
\tag{4.14}
$$

$$
\begin{tikzcd}
M \arrow[r, "\eta \otimes \mathrm{id}_M"] \arrow[rr, bend right, "\mathrm{id}_M"'] & \mathscr{O}(1) \otimes M \arrow[r, "\psi_1"] & M
\end{tikzcd}
\tag{4.15}
$$

It is easy to check that modules over an $\mathscr{A}$-algebra are the usual modules over an associative algebra, and modules over an $\mathscr{L}$-algebra are Lie modules.

Furthermore, one can define a notion of "universal enveloping algebra" in an operadic context; the category of modules over an $\mathscr{O}$-algebra X is naturally equivalent to that of modules over the universal enveloping algebra $U_{\mathscr{O}}(X)$. As one would expect, for $\mathscr{A}$ and $\mathscr{L}$ algebras one gets the same standard notions of enveloping algebras.

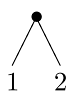
5 Operads of A_∞ and L_∞ algebras

The operadic approach to infinity algebras was started by Hinich and Schechtman in [HS93]; it became very popular and has thenceforth been adopted by many authors; let us just mention [GK94], [Mara], [Mar00], which most directly relate with the contents of this notes.

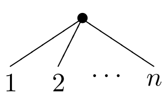
5.1 The $\mathscr{A}_\infty$ operad

We can build an operad $\mathscr{A}_\infty$ which governs A_∞-algebras, paralleling the construction of the $\mathscr{A}$ operad of associative algebras.

Recall from Section 2 that an A_∞-algebra has multiplications m_i related by (A_{n+1}). Let us use the same graphical notation of Section 4.1.1; we still depict the multiplication $m_2(x_1, x_2)$ by the two-branched tree

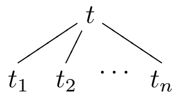


However, higher-order operations are no longer compositions of binary multiplications: we have primitive n-ary maps, for any $n \geqslant 1$; therefore, it is natural to associate the homotopy m_n with the n-corolla



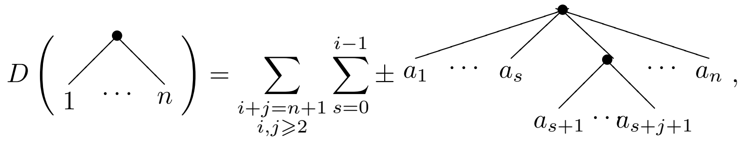
Define the composition $\gamma(t \otimes t_1 \otimes \ldots \otimes t_n)$ of trees t, with n leaves, and t_j, with k_j leaves each, to be the tree in which t_j has been grafted onto the j-th leaf of t:

$$\gamma(t \otimes t_1 \otimes \ldots \otimes t_n) =$$



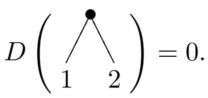
By repeated composition of corollas, we can form *any* tree. These trees have numbered leaves; define the action of $\mathfrak{S}_n$ on the set of n-leaved trees by permutation of numbers on leaves.

We know from (A_2) that m_1 is a differential on any A_∞-algebra X; therefore, a standard calculation shows that $\mathrm{ad}(m_1) := [m_1, -]$ is a differential in the Hochschild complex of X. Since $\mathscr{A}_\infty$ ought to be an operad in the category of R-dg-modules, we define a differential D on $\mathscr{A}_\infty(n)$ by $D := \mathrm{ad}(m_1)$; the fundamental relation (A'_{n+1}) translates into:



(5.1)

for all $n \geqslant 3$. The Leibniz rule (A_3) now becomes:



(5.2)

Summing up, we can give the following definition.

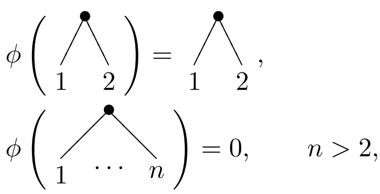
Definition 5.1. The $\mathscr{A}_\infty$-operad is defined by the collection $\{\mathscr{A}_\infty(n)\}$, where $\mathscr{A}_\infty(n)$ is the R-dg-module freely generated by set of all trees with n leaves, numbered from 1 to n, equipped with the differential defined on generators by (5.1) and (5.2). The structure maps are R-multilinear extensions of the grafting operation γ (see Definition 4.1 and Equation (4.10)). Finally, $\mathfrak{S}_n$ acts on $\mathscr{A}_\infty(n)$ by permuting the numbers on leaves.

It is important to note that, in contrast with the associative operad $\mathscr{A}$, there no longer is any relation among the trees in $\mathscr{A}_\infty(n)$, that is, $\mathscr{A}_\infty$ is a *free* operad (generated by the set of corollas) and all relations are encoded into the differential. (This has been singled out by Markl as a characteristics of "homotopy algebras" operads, see [Mar00, Marb].)

The relation between the associative and the A_∞ operad is summarized in the following.

Proposition 5. *There is a surjective morphism $\mathscr{A}_\infty \to \mathscr{A}$ in the category of differential graded R-operads.*

Proof. Define a morphism $\phi : \mathscr{A}_\infty \to \mathscr{A}$ on the generators by



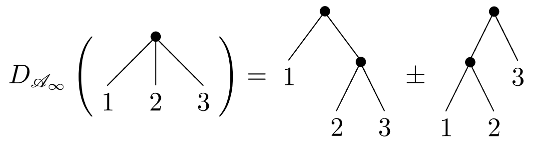
and extend it to be R-linear and a morphism of operads.

Compatibility with the differential D requires

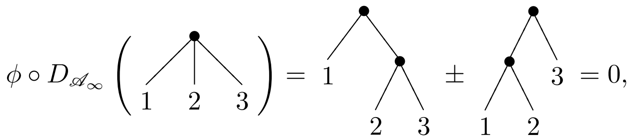
$$\phi \circ D = D \circ \phi,$$

which again needs only be checked on generators. Now, D of a n-corolla is given by (5.1); observe that, for $n > 3$ on RHS we always see the 2-corolla paired with some other (> 2)-corolla — since ϕ is an operad morphism, this dooms it to be zero.

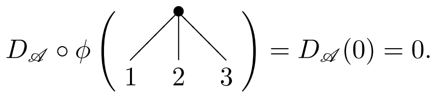
So we are left with verifying compatibility only for 2- and 3- corollas. Apply ϕ to both sides of



to get



because the 2-corolla satisfies the associativity relation (4.11). On the other hand,



The remaining verification is trivial. □

As an immediate corollary to the above and the structure transfer theorem 2, we get the following.

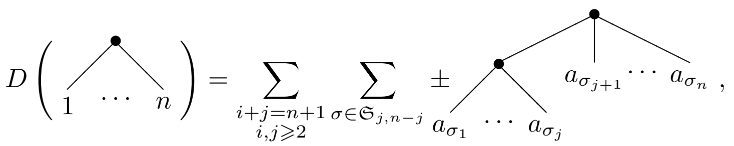
Corollary 1. *Every associative algebra is an A_∞-algebra.*

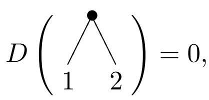
5.2 The $\mathscr{L}_\infty$ operad

Imitate the construction of the $\mathscr{A}_\infty$ operad: an L_∞-algebra has a differential l_1 and multilinear brackets $l_2, l_3, l_4, \ldots$; assign to the n-ary bracket l_n the n-corolla, define the structure maps γ by means of tree-grafting, define the unit η by the identification $\mathscr{L}(1) = R$, define the action of $\mathfrak{S}_n$ on $\mathscr{L}(n)$ by requiring it to be the sign representation on the subspace spanned by the n-corolla (i.e., the n-corolla stands for an antisymmetric operation).

There is a unique minimal operad satisfying all these requirements; name it $\mathscr{L}_\infty$. The space $\mathscr{L}(n)$ is the R-module freely generated by trees with n leaves, numbered from 1 to n; as in the $\mathscr{L}$ operad, it is difficult to describe explicitly the action of $\mathfrak{S}_n$ on $\mathscr{L}(n)$.

Again, $\operatorname{ad}(l_1)$ is a differential on the endomorphism operad of any L_∞ algebra (and, a fortiori, on its Hochschild complex), so we require the differential D on $\mathscr{L}_\infty$ to satisfy the same relations. These are given by equations (L'_{n+1}) and (L_3):


(5.3)


(5.4)

where the sign in (5.3) is given by $\pm = (-1)^{n+j(i-1)}\chi(\sigma)$.

Also in this case, it is important to point out that $\mathscr{L}_\infty$ is a *free* operad: all relations defining the structure of a L_∞-algebra are encoded in the differential and the action of the permutation group. The relation of $\mathscr{L}_\infty$ to $\mathscr{L}$ is given by the following theorem, whose proof is analogous to 5.

Proposition 6. *There is a surjective morphism $\mathscr{L}_\infty \to \mathscr{L}$ in the category of R-dg-operads.*

5.3 From A_∞ to L_∞

If X is an associative algebra, $[x, y] := xy - yx$ defines a Lie bracket on X; a parallel result holds with respect to A_∞ and L_∞ algebras, which seems to have first been proved in [LM95].

Proposition 7. *There is an embedding of operads $\mathscr{L}_\infty \hookrightarrow \mathscr{A}_\infty$.*

Proof. Since $\mathscr{L}_\infty$ is a free operad, we need only define the embedding map on generators; let $l_n \in \mathscr{L}_\infty(n)$ be the n-corolla corresponding to the n-place bracket, and $m_n \in \mathscr{A}_\infty(n)$ be the n-corolla corresponding to the n-ary multiplication. Define a map $\phi : \mathscr{L}_\infty \to \mathscr{A}_\infty$ by:

$$\phi : l_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} \chi(\sigma) \cdot m_n \circ T_\sigma,$$

and then extend it to be a morphism of operads.

Since $\mathscr{A}_\infty$ is freely generated by the m_n's and $\mathscr{L}_\infty$ is freely generated by the l_n's, then ϕ is an injective map.

In order to conclude, we only need to show that ϕ commutes with the differential D. From relation (5.1) we get:

$$\phi \circ D(l_n) = \phi \left(\sum_{i+j=n+1} \sum_{\sigma \in \mathfrak{S}_{j,n-j}} (-1)^{j(i-1)} \chi(\sigma) \quad \begin{array}{ccccc} & & l_i & & \\ & \swarrow & \downarrow & \searrow & \\ & l_j & \sigma_{j+1} & \cdots & \sigma_n \\ \swarrow & \cdots & \searrow & & \\ \sigma_1 & \cdots & \sigma_j & & \end{array} \right)$$

$$= \sum_{i+j=n+1} \sum_{\substack{\sigma \in \mathfrak{S}_{j,n-j} \\ \rho \in \mathfrak{S}_j \\ \tau \in \mathfrak{S}_i}} (-1)^{j(i-1)} \chi(\sigma)\chi(\rho)\chi(\tau) \quad \begin{array}{ccccc} & & m_i & & \\ & \swarrow & \downarrow & \searrow & \\ & m_j & (\tau\sigma)_{j+1} & \cdots & (\tau\sigma)_n \\ \swarrow & \cdots & \searrow & & \\ (\rho\sigma)_1 & \cdots & (\rho\sigma)_j & & \end{array} .$$

The real cumbersome part here is getting the sign right. Let us focus on the map $\mathfrak{S}_{j,n-j} \times \mathfrak{S}'_i \times \mathfrak{S}_j \ni (\sigma, \tau, \rho) \mapsto \sigma \circ (\tau \circ_1 \rho) \in \mathfrak{S}_n$, where:

i) $\mathfrak{S}'_i$ is the subgroup of all $\tau \in \mathfrak{S}_i$ such that $\tau_1 = 1$;

ii) the composition $(\tau \circ_1 \rho) \in \mathfrak{S}_n$ is defined by:

$$(\tau \circ_1 \rho)(p) := \begin{cases} \rho_p & 1 \leqslant p \leqslant j, \\ j + \tau_{p-j} & j < p \leqslant n. \end{cases}$$

Given any $\eta \in \mathfrak{S}_n$, there exists one (and one only) $\rho \in \mathfrak{S}_j$ such that $(\eta \circ \rho)_1 < (\eta \circ \rho)_2 < \ldots < (\eta \circ \rho)_j$, and, likewise, there exists one (and only one) $\check{\tau} \in \mathfrak{S}_n$ such that $(\eta \circ \check{\tau})_{j+1} < \ldots < (\eta \circ \check{\tau})_n$ and $\check{\tau}_h = h$ for all $h = 1, \ldots, j$. Then, $\rho \circ \check{\tau} = \check{\tau} \circ \rho = \tau \circ_1 \rho$ for some $\tau \in \mathfrak{S}_{n-j}$ with $\tau_1 = 1$, so $\eta \circ (\tau \circ_1 \rho) = \sigma \in \mathfrak{S}_{j,n-j}$, therefore, $\eta = \sigma \circ (\tau \circ_1 \rho)^{-1} = \sigma \circ (\tau^{-1} \circ_{\tau_1} \rho^{-1})$. Thus, $(\sigma, \tau, \rho) \mapsto \sigma \circ (\tau \circ_1 \rho)$ is a bijection between $\mathfrak{S}_{j,n-j} \times \mathfrak{S}_i \times \mathfrak{S}_j$ and $\mathfrak{S}_n$.

Now, let us show that $\chi(\sigma)\chi(\rho)\chi(t) = (-1)^{j\tau_1}\chi(\eta)$. From the definition (1.2), it is clear that χ is multiplicative, so

$$\chi(\eta) = \chi(\sigma) \cdot \chi(\tau^{-1} \circ_{\tau_1} \rho^{-1}).$$

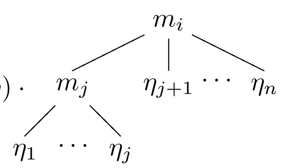
If $\tau_1 = 1$, take $\alpha, \beta \in \mathfrak{S}_n$ to be such that α leaves $1, \dots, j$ pointwise fixed and permutes $j+1, \dots, n$ like τ^{-1} does on $1, \dots, n-j$; similarly, β permutes $1, \dots, j$ in the same way as ρ^{-1} and leaves $j+1, \dots, n$ pointwise fixed. Then,

$$\chi(\tau^{-1} \circ_{\tau_1} \rho^{-1}) = \chi(\alpha \circ \beta) = \chi(\alpha) \circ \chi(\beta) = \chi(\tau) \cdot \chi(\rho).$$

If $\tau_1 \neq 1$, then we get $\tau^{-1} \circ_{\tau_1} \rho^{-1}$ from some ${\tau'}^{-1} \circ_1 \rho$ by τ_1 transpositions of the whole (j elements long) block on which τ^{-1} acts nontrivially — these $j\tau_1$ transpositions contribute a $(-1)^{j\tau_1}$ sign.

Summing up,

$$\phi \circ D(l_n) = \sum_{i+j=n+1} \sum_{s=0}^{i-1} \sum_{\substack{\eta \in \mathfrak{S}_n, \\ \eta_1 = s+1}} (-1)^{j(n-j)+j(s+1)} \chi(\eta) \cdot$$



$$= \sum_{\eta \in \mathfrak{S}_n} \chi(\eta) D(m_n) = D\big(\phi(m_n)\big)$$

□

By Proposition 2, we get the following.

Corollary 2. *Let X be an $\mathscr{A}_\infty$-algebra with multiplications m_n. The maps $l_n := \sum_\sigma \chi(\sigma) m_n \circ X_\sigma$ define a structure of L_∞-algebra on X.*

6 Enveloping algebras

Let $\mathscr{O}$ be an operad and X an R-module.

Definition 6.1 (cf. [HS93]). The tensor algebra $T_{\mathscr{O}}(X)$ is the graded algebra over R defined by

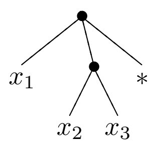
$$T_{\mathscr{O}}(X)^p := \big(\mathscr{O}(p+1) \otimes X^{\otimes p}\big)_{\mathfrak{S}_p}, \qquad n \geqslant 0.$$

The action of $\mathfrak{S}_p$ on $\mathscr{O}(p+1) \otimes X^{\otimes p}$ is given by $\sigma \otimes \epsilon(\sigma^{-1})$, that is, it acts on $\mathscr{O}(p+1)$ by the inclusion $\mathfrak{S}_p \hookrightarrow \mathfrak{S}_{p+1}$ and on $X^{\otimes p}$ by the Koszul sign.

The multiplication in $T_{\mathscr{O}}(X)$ is induced by the composition

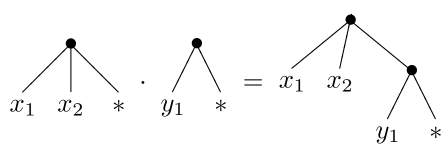
$$\big(\mathscr{O}(p+1) \otimes X^{\otimes p}\big) \otimes \big(\mathscr{O}(q+1) \otimes X^{\otimes q}\big) \xrightarrow{\eta^{\otimes p} \otimes \mathrm{sign}}$$
$$\mathscr{O}(p+1) \otimes \mathscr{O}(1)^{\otimes p} \otimes \mathscr{O}(q+1) \otimes X^{\otimes p+q} \xrightarrow{\gamma} \mathscr{O}(p+q+1) \otimes X^{\otimes p+q}.$$

A graphical description is more suggestive. Elements in $\mathscr{O}(p+1)$ can be depicted as trees with $p+1$ numbered leaves, therefore, elements of $T_{\mathscr{O}}(X)^p$ can be depicted as trees with all leaves except the last one decorated by $x_i \in X$:



$\in T_{\mathscr{O}}(X)^p.$

Such elements multiply by grafting the second tree onto the last branch of the first:



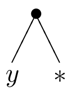
.

It is now evident that this multiplication is well-defined and associative.

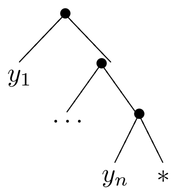
Of course, in the well-known cases of Lie and associative algebras, we get the same notion.

Proposition 8. *If Y is a Lie algebra, then $T_{\mathscr{L}}(Y)$ is the tensor algebra (free associative algebra) of Y.*

Proof. For any $y \in Y$, put

$d_y :=$  .

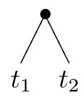
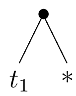
Call "regular trees" the products

$d_{y_1} \cdots d_{y_n} =$  .

The subalgebra of $T_{\mathscr{L}}(Y)$ spanned by $\{d_y\}$ is free associative; let us prove that any element of $T_{\mathscr{L}}(Y)$ is a linear combination of d_y's.

Any $t \in T_{\mathscr{L}}(Y)$ is a linear combination of binary trees with the rightmost leaf marked by $*$ and the other decorated by $y_1, \ldots, y_n \in Y$ — call such trees "almost regular". It clearly suffices to prove our claim only for almost regular trees.

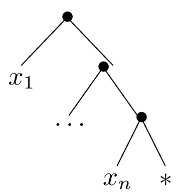
By induction on the number k of leaves, we prove that any almost regular tree t is a linear combination of regular trees: the Jacobi identity (4.12) settles the case $k = 3$; for $k > 3$ we can write

$t =$  $=$  $\cdot t_2,$

so we conclude, by applying the induction hypothesis to t_1 and t_2. $\square$

Proposition 9. *If X is an associative algebra, $T_{\mathscr{A}}(X)$ is the graded symmetric algebra on X.*

Proof. Every element in $\mathscr{A}(n)$ has a representative which is a combination of regular binary trees, by the associative relation (4.11). On any regular tree,



the action of $\sigma \in \mathfrak{S}_n$ will merely change the sign by $\epsilon(\sigma; x_1, \ldots, x_n)$, which is exactly the sign rule characterizing the graded symmetric algebra SX. $\square$

6.1 The enveloping algebra

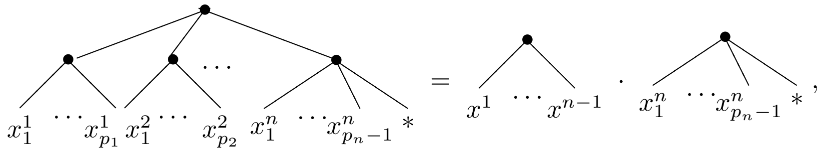
Let $\mathscr{O}$ be an operad, and X an $\mathscr{O}$-algebra.

Definition 6.2 (cf. [HS93]). The enveloping algebra $U_{\mathscr{O}}(X)$ is the quotient of the tensor algebra $T_{\mathscr{O}}(X)$ by the ideal generated by the relations

$$\begin{aligned}\gamma(\omega \otimes \omega_1 \otimes \cdots \otimes \omega_n) \otimes x_1^1 \otimes \cdots x_{p_1}^1 \otimes x_1^2 \otimes \cdots \otimes x_1^n \otimes \cdots \otimes x_{p_n-1}^n \\ = \big(\omega \otimes \psi_{p_1}(\omega_1 \otimes x_1^1 \otimes \cdots \otimes x_{p_1}^1) \otimes \cdots \otimes \\ \otimes \psi_{p_{n-1}}(\omega_{n-1} \otimes x_1^{n-1} \otimes \cdots \otimes x_{p_{n-1}}^{n-1})\big) \times \\ \times (\omega_n \otimes x_1^n \otimes \cdots \otimes x_{p_n-1}^n),\end{aligned}$$

where $\omega \in \mathscr{O}(n)$, $\omega_i \in \mathscr{O}(p_i)$, $p = p_1 + \cdots + p_n$, γ is the composition map of $\mathscr{O}$ and ψ is the $\mathscr{O}$-algebra structure on X.

Once again, a graphical representation may clarify things: according to Definition 6.1, $T_{\mathscr{O}}(X)$ is generated by trees having all leaves decorated by some element of X, save for the rightmost, which is "an empty plug" (therefore marked by a "$*$" sign). The enveloping algebra $U_{\mathscr{O}}(X)$ is the quotient of $T_{\mathscr{O}}(X)$ by the relations:

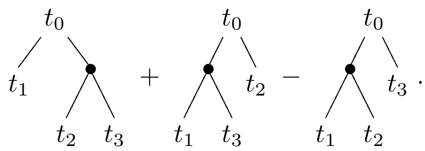


where $x^i := \psi_{p_i}(\omega_i \otimes x_1^i \otimes \cdots \otimes x_{p_i}^i) \in X$. The associativity of the product in $T_{\mathscr{O}}(X)$ implies that $U_{\mathscr{O}}(X)$ is associative too.

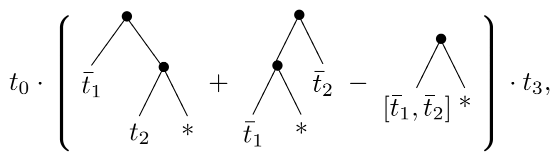
Proposition 10. *If Y is a Lie algebra, $U_{\mathscr{L}}(Y)$ is the usual enveloping algebra.*

Proof. By Proposition 8 we already now that $T_{\mathscr{L}}(Y)$ is the free associative algebra on Y; we only need to check that relations (6.1) together with the Jacobi equation generate the ideal $(\{xy - yx - [x,y] : x, y \in Y\})$ in $T_{\mathscr{L}}(Y)$.

Now, $\mathscr{L}$ is the quotient of a free operad by the (operadic) ideal $\mathfrak{J}$ generated by the Jacobi equation (4.12); its elements share the form:



In $U_{\mathscr{L}}(Y)$, because of relation (6.1), this can be rewritten as:



where $\bar{t}_1$, $\bar{t}_2$ are the elements of Y gotten by evaluation of trees t_1, t_2. Therefore, the image of $\mathfrak{J}$ in $U_{\mathscr{L}}(Y)$ is exactly $(\{xy - yx - [x,y] : x, y \in Y\})$. $\square$

Using similar methods, one can prove the following.

Proposition 11. *If X is an associative algebra, then $U_{\mathscr{A}}(X)$ is isomorphic to the algebra $X^{\mathrm{op}} \otimes X$.*

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DG-schemes and Derived Parameter Spaces (after Ciocan-Fontanine and Kapranov)

Andrea Bruno

1 Introduction

In recent times the following two ideas have received particular attention:

- Every reasonable deformation problem in characteristic zero is governed by a differential graded Lie algebra;
- Moduli spaces are always smooth in an appropriate sense, i.e. obstructions should be considered as tangent vectors of some objects more complicated than schemes.

While the first of the above ideas, somehow of a local nature, is rapidly finding solid foundations and is getting used in many contexts, the scope of this lecture is to show the state of the art on the second, and especially on its global part, as the local one is now understood (see [8]). A typical problem in algebraic geometry is to overcome phenomena like nonsmoothness of naturally defined schemes or non transverse intersection of cycles; as said above, a way to solve this kind of problems is suggested by the language of dg-manifolds. The idea of using these objects, already suggested by Deligne, Drinfeld, Feigin and others, has the first published appearance in the notes [7] of Kontsevich and in the paper [6] of the same author (where the author says: "It was recently spelled out clearly in a letter of P. Deligne to H. Esnault, together with a proposal to apply it to the algebro-geometric formulation of Mirror Symmetry.") and has found applications in works on quantization of Poisson structures and in Generalized Mirror Symmetry by Barannikov. The prehistory of the subject is in [9] (see also [4]), where supermanifolds are studied, which are manifolds together with a $\mathbb{Z}_2$-grading of the structure sheaf.

Kapranov ([5]), and later Ciocan-Fontanine and Kapranov ([1, 2]), started the project to put on firm basis Kontsevich's suggestion of systematically studying moduli spaces together with additional structures which give smoothness. Roughly speaking the datum of a dg-manifold is the datum of a smooth scheme X together with a sheaf of differential graded commutative algebras which, as a sheaf of graded algebras, is free. For these objects it makes perfect sense to extend the usual constructions of schemes, like tangent and cotangent sheaves, fiber products, etc., if one agrees to work in the derived category, where everything is up to homotopy and where quasisomorphisms are inverted.

In this paper we will almost faithfully follow [1] and [2], to describe their construction of the derived version of the Quot and Hilbert schemes: these schemes are the basic ingredients in all constructions of moduli spaces in geometry. First we will give the basic definitions, examples and features of dg-schemes and of the category $\mathcal{DS}ch$, the *right derived category of schemes*. The spaces we construct are in fact smooth spaces in this category with two properties: their restriction to the ordinary category of schemes is given by the classical Quot and Hilbert schemes in the scheme-theoretic sense, their tangent spaces are complexes whose homology in low degrees consists of tangent vectors and obstructions to the classical Quot and Hilbert schemes; by virtue of a "Whitehead Theorem" (see proposition 3.7), in the derived category two objects are isomorphic if there is a map which induces isomorphism on the ordinary underlying schemes and on the "tangent spaces" (see paragraph 5.5), so that the spaces constructed by Ciocan-Fontanine and Kapranov seem quite natural objects to study. An important feature of these objects is that they are almost functorial (see paragraph 5.6), in the sense that they represent functors from affine objects to sets but only up to quasiisomorphism, while in order to have them representing a functor in $\mathcal{DS}ch$ the authors suggest to construct an even larger category in which objects have affine charts given by dg-schemes which are glued together via quasiisomorphisms.

2 Differential graded schemes

Definition 2.1. *A* dg-scheme *is a pair $X = (X^0, \mathcal{O}_X^{\cdot})$, where X^0 is a scheme and $\mathcal{O}_X^{\cdot}$ is a sheaf of ($\mathbb{Z}_-$)-graded commutative dg-algebras on X^0 such that $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$ and such that $\mathcal{O}_X^{-n}$, for every $n \geq 0$, is quasicoherent as an $\mathcal{O}_{X^0}$-module. With the above notations we will say that X^0 is the support of X. A morphism $f : X \to Y$ of dg-schemes is a morphism of the underlying supports together with a morphism of sheaves of dg-algebras $f_0^* : \mathcal{O}_Y^{\cdot} \to \mathcal{O}_X^{\cdot}$. A dg-scheme $X = (X^0, \mathcal{O}_X^{\cdot})$ is said* of finite type *if its support is of finite type and if each $\mathcal{O}_X^{-i}$ is coherent on X^0. We will denote by dgSch the category of dg-schemes.*

If X is any scheme, X has a natural realization as a dg-scheme $\widehat{X}$ with support X and with trivial differential and grading, i.e. as a (trivially) graded scheme. Conversely, to a dg-scheme $(X^0, \mathcal{O}_X^{\cdot})$ we usually associate two graded schemes:

$$X_\sharp = (X^0, \mathcal{O}_{X_\sharp}^{\cdot}) \qquad \text{and} \qquad X_h = (\pi_0(X), \mathcal{H}^{\cdot}(\mathcal{O}_X^{\cdot})),$$

where the scheme $\pi_0(X)$ is defined as $\mathrm{Spec}_{\mathcal{O}_{X^0}} \mathcal{H}^0(\mathcal{O}_X^{\cdot})$. The ordinary scheme $\pi_0(X)$ associated to X is the *degree 0 truncation of X* and has the following property:

$$\text{for } S \in \mathsf{Ob}(\mathcal{S}ch), \quad \mathrm{Hom}_{\mathcal{S}ch}(S, \pi_0(X)) = \mathrm{Hom}_{dg\mathcal{S}ch}(\widehat{S}, X).$$

By definition, if $f : X \to Y$ is a morphism of dg-schemes, f commutes with the differentials, so that it induces a map $f_h : X_h \to Y_h$ of the associated graded schemes.

Definition 2.2. *A morphism $f : X \to Y$ of dg-schemes is a* quasiisomorphism *if the induced morphism of graded schemes $f_h : X_h \to Y_h$ is an isomorphism. The* (right) derived category of schemes $\mathcal{DS}ch$ *is the category obtained from dgSch by inverting all quasiisomorphisms.*

Examples

1. If $A^{\cdot}$ is a $\mathbb{Z}_-$graded commutative dg-algebra, the associated *affine* dg-scheme is the scheme $\mathrm{Spec}\, A^{\cdot}$, whose support is the affine scheme $\mathrm{Spec}\, A^0$ and whose structure sheaf is $\tilde{A}^{\cdot}$ (i.e. $\tilde{A}^{-i}$ is the the quasicoherent sheaf associated to A^{-i} for each $i \geq 0$). Note that in this example, coordinates in even degree commute and coordinates in odd degres anticommute.

2. In particular, if $E^{\cdot}$ is a $\mathbb{Z}_+$-graded dg-vector space, $E^{\cdot}$ supports the affine scheme structure given by $\mathrm{Spec}\,\mathrm{Sym}(E^{\cdot})^\vee$, where the symmetric algebra is taken in the graded sense.

3. If X is an ordinary scheme and if X has an embedding into a scheme of finite type X^0 as a complete intersection, the dg-scheme $\widehat{X}$ is isomorphic in $\mathcal{DS}ch$ to the dg-scheme Y defined as follows:

   (a) the support of Y is X^0;

(b) if $\mathcal{E}$ is the locally free $\mathcal{O}_{X^0}$-module of generators of the ideal of the embedding of X into X^0, we define $\mathcal{O}_Y^{-i} := \wedge^i \mathcal{E}^\vee$, for $i \leq 0$. The graded sheaf $\mathcal{O}_Y^{\cdot}$ so defined inherits a structure of (graded-)commutative algebra from the exterior algebra structure and the differential is given by the Koszul differential, which makes the complex
$$\mathcal{O}_{X^0} \longleftarrow \mathcal{O}_Y^{-1} \longleftarrow \mathcal{O}_Y^{-2} \longleftarrow \dots$$
into the Koszul resolution of the ideal of X.

Definition 2.3. *A* quasicoherent dg-sheaf on a dg-scheme X *is a sheaf $\mathcal{F}^{\cdot}$ of dg-modules over $\mathcal{O}_X^{\cdot}$ such that each $\mathcal{F}^{-i}$ is quasicoherent on X^0.*
A dg-bundle *on X is a locally free dg-sheaf on X, i.e. a dg-sheaf whose associated graded sheaf $\mathcal{F}_\sharp^{\cdot}$ is locally isomorphic to a sheaf of the form $E^{\cdot} \otimes \mathcal{O}_X^{\cdot}$, for $E^{\cdot}$ a graded vector space.*

A useful fact is the following standard:

Lemma 2.4. *Let X be a quasiprojective dg-scheme (its support is quasiprojective). If $\mathcal{F}^{\cdot}$ is quasicoherent (resp. coherent), it is quasiisomorphic to a flat (locally free) dg-sheaf $\mathcal{E}^{\cdot}$.*

Proof. The lemma follows by induction arguments based on the case of ordinary sheaves in which it is easily true that a quasicoherent (resp. coherent) sheaf on a quasiprojective scheme is (by definition) the quotient of a flat (resp. locally free) sheaf. □

The above lemma allows us to define the tensor product of two quasicoherent sheaves on a dg-scheme by means of flat resolutions. The result is unique in the derived category $\mathcal{DC}oh(X)$ of quasicoherent sheaves where morphisms are taken up to homotopy and quasiisomorphisms are inverted.

3 Differential graded manifolds

Definition 3.1. *A dg-scheme $X = (X^0, \mathcal{O}_X^{\cdot})$ is said smooth (or a* dg-manifold*) if its support X^0 is a smooth algebraic variety and if its structure sheaf $\mathcal{O}_X^{\cdot}$ is isomorphic as a* graded algebra *to* $\mathrm{Sym}(\mathcal{Q}_X^{\cdot})$*, where $\mathcal{Q}_X^{\cdot} = \bigoplus_{i\geq 1} Q_X^{-i}$ is a graded vector bundle. The dimension of a dg-manifold is the sequence $\{d_i(X) | i \geq 0\}$, where*
$$d_0(X) = \dim X^0 \quad \text{and} \quad d_i(X) = \mathrm{rk}\, Q_X^{-i}.$$

Remark 3.2. A dg-manifold is then an algebraic variety which is a closed subscheme of a smooth algebraic variety embedded as the zero section of a "skew" vector bundle. This notion is quite subtle and is analogous to smoothness for formal schemes, even though here there is the differential which is not taken into consideration. This notion is not even invariant for quasiisomorphisms: consider for instance the dg-scheme Y of example 3 above: considering the vector bundle $\mathcal{E}$ on X^0 as a trivial $\mathbb{Z}_+$ graded dg-bundle concentrated in degree zero, we have that Y has for structure sheaf $\mathcal{O}_Y^{\cdot} = \mathrm{Sym}(\mathcal{E}^\vee[-1])$ so that *every scheme X* (not necessarily nonsingular, think about a very singular projective hypersurface) which can be embedded into a smooth scheme as a complete intersection has an associated graded scheme $\widehat{X}$ which *is quasiisomorphic to a dg-manifold* . More generally, every quasiprojective scheme has an associated graded scheme which is quasiisomorphic to a dg-manifold: it suffices to embed the scheme in projective space, to consider the vector bundle on projective space of its generators and then a quasifree resolution of it. We will see a generalization of this construction in Lemma 3.3

If $f : X \to Y$ is a morphism of dg-schemes we will say that f is a *closed embedding* if it so at the level of supports and if the induced morphism $f^* : \mathcal{O}_Y^{\cdot} \to \mathcal{O}_X^{\cdot}$ is surjective.

The flexibility of the above notion is contained also in the following:

Lemma 3.3. *Any quasiprojective dg-scheme X is a quasiisomorphic closed subscheme of a dg-manifold M and any two closed embeddings $X \to M$, $X \to N$ can be extended with commutative embeddings $M \to L$ and $N \to L$ to a dg-manifold L.*

Proof. For the second part, once we have $X \to M$ and $X \to N$, it will be enough to embed $M \cup_X N$ into a dg-manifold. The first part is done by considering first an embedding into projective space and then resolving the sheaf $\mathcal{O}^{\cdot}_X$ by a quasi-free dg-algebra, by the technique of "imitating the procedure of attaching cells to kill homotopy groups". We will leave details to [**?**], but the starting point is to first consider the open part $Y \subset \mathbb{P}$ where X is closed and then construct the symmetric algebra over the bundle of generators of the ideal of the embedding $X \to Y$. We then consider a bundle of cycles representing the -1^{st} homology of this algebra and we consider the symmetric algebra over this; this does not affect homology on larger degrees and we go on. □

The tangent space to a dg-manifold X is defined as in the ordinary sense: there exists a dg-sheaf $\Omega^{1\cdot}_X$ and a derivation $\delta : \mathcal{O}^{\cdot}_X \to \Omega^{1\cdot}_X$ which are universal.

Lemma 3.4. $\Omega^{1\cdot}_X$ *is a dg-vector bundle of rank $\{d_i(X)|i \leq 0\}$.*

Definition 3.5. *The* tangent bundle *$T^{\cdot}X$ of a dg-manifold X is the $\mathbb{Z}_+$graded dg-bundle $\mathcal{H}om(\Omega^{1\cdot}_X, \mathcal{O}^{\cdot}_X)$. If $x \in X$ the* tangent space to X at x *is the* complex of vector spaces *$T^{\cdot}_x X = T^{\cdot}X \otimes K_x$, where K_x is the dg-algebra of the point x.*

Let us consider for instance the example of an ordinary scheme X which is a complete intersection in projective space $\mathbb{P}^N$ and let us consider the quasiisomorphic dg-manifold $\widehat{X}$ considered in example 3. Let $\mathcal{E}$ be the bundle on $\mathbb{P}^N$ such that $\mathcal{E}|_X = \mathcal{N}_X$ and that $\mathcal{O}^{\cdot}_{\widehat{X}} = \mathrm{Sym}(\mathcal{E}^\vee[-1])$. Then $T^{\cdot}\widehat{X}$ is the dg-bundle

$$T_{\mathbb{P}^N} \longrightarrow E,$$

where the differential restricted to X is given by the *Jacobian matrix* . This shows that even if X is singular, X can be thought of as "smooth in an appropriate sense". Notice that in this example the kernel of the restriction (after truncation) of the above map is intrinsic (it is T_X) and does not depend on the chosen embedding.

For the cohomology groups of the tangent space at a point of a dg-manifold X the following suggestive "topological" notion is used:

$$\pi_i(X, x) := H^{-i}(T^{\cdot}_x X), i < 0.$$

The notation reminds us of two topological theorems:

Proposition 3.6. *Given a dg-manifold X there are natural bilinear maps (the* Whitehead products*)*

$$\pi_i(X, x) \otimes \pi_j(X, x) \longrightarrow \pi_{i+j-1}(X, x)$$

which make $\pi_{\cdot+1}(X, x)$ into a graded Lie algebra. For any morphism $f : X \to Y$, the induced morphism $\pi_{\cdot}(X, x) \to \pi_{\cdot}(Y, f(x))$ is a Lie algebra homomorphism.

Proposition 3.7. *Let $f : X \to Y$ be a morphism of dg-manifolds. Then f is a quasiisomorphism if and only if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism and the differential induces isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ for all $x \in X$ and $i < 0$.*

Sketch of proofs. For both statements we look at the completion $\widehat{\mathcal{O}}^{\cdot}_{X,x}$ which is in a natural way the completion of a graded free algebra; moreover, we have by definition that any choice of a coordinate system at x corresponds to an isomorphism between $\widehat{\mathcal{O}}^{\cdot}_{X,x}$ and the completion of the Symmetric algebra on the vector space $\Omega^{1\cdot}_{X,x}$. As for proposition 3.6, the differential on the completion $\widehat{\mathcal{O}}^{\cdot}_{X,x}$ induces a differential on the completed symmetric algebra $\widehat{\mathrm{Sym}}(\Omega^{1\cdot}_{X,x}) = \widehat{\mathrm{Sym}}(T^{\cdot}_x X[-1])^{\vee}[-1]$. Now, (see [10]), if $\mathfrak{g}$ is a graded vector space, a differential on $\mathrm{Sym}(\mathfrak{g}^{\vee}[-1])$ corresponds to an L_∞ structure on $\mathfrak{g}$ and the cohomology of an L_∞ algebra is a differential graded Lie algebra. As for proposition 3.7, isomorphisms as those indicated induce isomorphisms on the symmetric powers of the cotangent spaces, which are the quotients of the filtrations of the completions of the local rings. They induce isomorphisms of the completions because the sheaves are bounded above. □

Definition 3.8. *A morphism $f : X \to Y$ is* smooth *if the following hold:*

1. *the underlying morphism $f^0 : X^0 \to Y^0$ of supports is smooth;*
2. *locally on X we have an isomoprphism $\mathcal{O}^{\cdot}_{X^\sharp} \simeq f^{0*}\mathcal{O}^{\cdot}_{Y^\sharp} \otimes \mathrm{Sym}(\mathcal{Q}^{\cdot})$, where $\mathrm{Sym}(\mathcal{Q}^{\cdot})$ is free as a commutative graded algebra.*

We have a relative analog of lemma 3.3:

Proposition 3.9. *Let $f : X \to Y$ be a morphism of quasiprojective dg-schemes. Then there exists a factorization $X \to \tilde{X} \to Y$ where $X \to \tilde{X}$ is a quasiisomorphic closed embedding and $\tilde{f} : \tilde{X} \to Y$ is smooth.*

Proof. The proof is as in 3.3, but here we embed X into $Y \times \mathbb{P}$. □

The above proposition makes it easy to give the following

Proposition-Definition 3.10. *Let $f : X \to Y$ be a morphism of quasiprojective dg-schemes and let $i : X \to \tilde{X}$ be a quasiisomorphic closed embedding which factorizes f. The* relative cotangent sheaf $\mathbb{L}^{\cdot}_{X/Y} := \Omega^{1\cdot}_{\tilde{X}/Y}$ *is the sheaf of relative differentials of the smooth morphism $\tilde{f}$ and it does not depend on $\tilde{f}$ up to quasisomorphism.*

Especially important is the following

Definition 3.11. *If $f_i : X_i \to S$ are morphisms of quasiprojective schemes the* derived fiber product *is defined as follows: we fix a factorization $\tilde{f}_1 : \tilde{X}_1 \to Y$ and we consider the dg-scheme with support $\tilde{X}^0_1 \times_{Y^0} X^0_2$ and structure sheaf $(\tilde{g}^0_1)^{-1}\mathcal{O}_{\tilde{X}^{\cdot}_1} \otimes_{(f_1 g_1)^{-1}\mathcal{O}^{\cdot}_Y} (g_2{}^0)^{-1}\mathcal{O}_{X_2{}^{\cdot}}$, where maps are as in the diagram:*

$$\begin{tikzcd} \tilde{X}_1 \times_S X_2 \arrow[r, "g_2"] \arrow[d, "\tilde{g}_1"'] & X_2 \arrow[d, "f_2"] \\ \tilde{X}_1 \arrow[r, "\tilde{f}_1"'] & S \end{tikzcd}$$

The construction does not depend on any choice.

Notice that via derived fiber products it is possible to define fibers of morphisms, intersections and many other constructions which are ubiquous.

Examples

1. If X and Y are quasiprojective ordinary schemes contained in Z, and if we consider them as graded schemes we can compute their derived intersection $X \cap^D Y$ by taking the fiber product with respect to the embeddings $X \to Z$ and $Y \to Z$. The resulting scheme has for cohomology of the structure sheaf the sheaves $\mathcal{T}or_i^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y)$ (see the introduction).

2. If $f : X \to Y$ is a smooth morphism, we can consider the Kodaira-Spencer map

$$R\mathfrak{k} : T_y^{\cdot} Y \longrightarrow R\Gamma(Rf^{-1}, T^{\cdot}(X/Y)).$$

In this setting this is also a map of graded Lie algebras; moreover the cohomology groups of the base act on the hypercohomology of the fiber in a way remindful of monodromy.

4 Ordinary Quot and Hilbert schemes

Let X be a projective scheme, let $\mathcal{O}_X(1)$ be a very ample sheaf on X, let $\mathcal{F}$ be a coherent sheaf on X and let $h^{\mathcal{F}}(t) = \chi(\mathcal{F}(t)) \in \mathbb{Q}[t]$. For any fixed $h \in \mathbb{Q}[t]$, and for any scheme S, the $\mathrm{Quot}_h(\mathcal{F})$ scheme is defined as the scheme representing the functor

$$\begin{aligned}&\mathcal{Q}uot_h(\mathcal{F})(S) = \\ &\qquad = \{\mathcal{O}_S \text{ flat subsheaves of } \pi_X^*\mathcal{F} \text{ with relative Hilbert polynomial } h(t)\}.\end{aligned}$$

In particular, if $\mathcal{F} = \mathcal{O}_X$, $\mathrm{Quot}_h(\mathcal{O}_X) = \mathrm{Hilb}_h(X)$. The infinitesimal study of the Quot-scheme shows that if $[\mathcal{K}]$ is a closed point of $\mathrm{Quot}_h(\mathcal{F})$ (i.e $\mathcal{K} \subset \mathcal{F}$ and $h^{\mathcal{K}} = h$), then

$$T_{[\mathcal{K}]}\mathrm{Quot}_h(\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}}) \text{ and obstructions live in } \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}}).$$

Notice that in the case of the Hilbert scheme, where $\mathcal{K}$ is the sheaf of ideals of a subscheme of X with given Hilbert polynomial, that, if the subscheme associated to $\mathcal{K}$ is locally a complete intersection, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\frac{\mathcal{K}}{\mathcal{K}^2}, \mathcal{O}_X) = \mathcal{N}$ is locally free (the normal bundle) and we have identifications

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}}) = H^0(X, \mathcal{N}), \quad Ext^1_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}}) = H^1(X, \mathcal{N}).$$

The basic idea of the construction of the Quot scheme is that for $r \gg 0$, any subsheaf of $\mathcal{F}$ with Hilbert polynomial h is r-regular in the sense of Castelnuovo-Mumford. In particular, the map that associates $\mathcal{K}$ to the linear subspace $H^0(X, \mathcal{K}(r)) \subset H^0(X, \mathcal{F}(r))$ determines an embedding of $\mathrm{Quot}_h(\mathcal{F})$ into the Grassmannian $G(h(r), H^0(X, \mathcal{F}(r)))$.

Let us now consider the following data:

1. An associative algebra A.
2. A finite-dimensional (left) A-module M.
3. A positive integer $k \leq \dim M$.

Definition 4.1. *The A-*Grassmannian *is the subscheme $G_A(k,M)$ of $G(k,M)$ parametrizing subspaces of M which are in fact (left) A-submodules of M.*

One can define $G_A(k,M)$ as a scheme in the following way: if $[V] \in G_A(k,M)$ the natural action $A \otimes M \to M$, restricts to an action $A \otimes V$ with values in V. This means that we can describe $G_A(k,M)$ as the *zero scheme of the natural section* $s \in H^0(G(k,M), \mathcal{H}om(\tilde{V}, (M \otimes \mathcal{O}_G)/\tilde{V})$ *induced by the A-module structure on M* , where $\tilde{V} \subset M \otimes \mathcal{O}_G$ is the universal inclusion over the Grassmannian. From this description it is easy to check that

$$T_{[V]}G_A(k,M) = \mathrm{Hom}^0_A(V, \frac{M}{V}).$$

In the formula above Hom^0 stands for degree zero homomorphisms in case A and M are graded. Ciocan Fontanine and Kapranov apply the above construction to the following data:

1. $A = \bigoplus_n H^0(X, \mathcal{O}_X(n))$.
2. $M_{[p,q]} := \dfrac{M_{\geq p}}{M_{\geq q}}$, where $M = \bigoplus_n H^0(X, \mathcal{F}(n))$.
3. $k = h(i)$ for $i = p, \dots, q$, with $0 \ll p \ll q$.

We define $G_A(k, M_{[p,q]})$ as $\prod_i G_A(h(i), M_i)$ and we have the following

Theorem 4.2. *The natural morphism $\mathrm{Quot}_h(\mathcal{F}) \to G_A(k, M_{[p,q]})$ is an isomorphism.*

The idea of the proof is quite basic and elementary, since in order to reconstruct a sheaf of $\mathcal{O}_X$-modules it is enough to determine finitely many graded pieces of the associated graded modules. In [2] the theorem is extended in the following way: if $M = A$ we denote $G_A(k,A)$ by $J(k,A)$, the *scheme of (left) ideals of A* of dimension k.

Theorem 4.3. *The natural morphism $\mathrm{Hilb}_h(X) \to J(k, A_{[p,q]})$ is an isomorphism.*

The non trivial part of the above statement is to invert the obvious inclusion $G_A(k, A_{[p,q]}) \to J(k, A_{[p,q]})$. In order to understand the dg-generalisation of the construction we have to understand the following two constructions:

Definition 4.4. *Let V be a finite-dimensional vector space and let A be an associative algebra. The* space of actions $\mathrm{Act}(A,V)$ *is the subscheme of the affine scheme* $\mathrm{Hom}(A \otimes V, V)$ *consisting of all A-actions.*

Proposition 4.5. *If $f \in \mathrm{Act}(A,V)$, then $T_f\mathrm{Act}(A,V) = Z^1(C^\cdot)$, where $C^\cdot$ is the complex $\mathrm{Hom}^\cdot(B_A(V), V)$ computing $\mathrm{Ext}^\cdot_A(V,V)$ and $B_A(V)$ is the bar resolution of V as an A-module.*

Proof. If $f \in \mathrm{Hom}(A \otimes V, V)$ then f is in $\mathrm{Act}(A,V)$ if and only if the associativity rule $f(a_1 \cdot (a_2 \cdot v)) = f(a_1 a_2 \cdot v)$. Differentiating the expression we get that for $f \in \mathrm{Act}(A,V)$ the map $\delta f : A \otimes A \otimes V \to V$ defined by

$$\delta f(a_1 \otimes a_2 \otimes v) = f(a_1 a_2 \otimes v) - f(a_1 \otimes f(a_2 \otimes v))$$

vanishes, i.e if an donly if $f \in Z^1(C^\cdot)$. □

Remark 4.6. If we consider the quotient stack $\mathrm{Act}(A,V)/GL(V)$, we have as tangent spaces the first two spaces of cohomology $\mathrm{Ext}^i_A(V,V)$ for $i=0,-1$.

Definition 4.7. *Let S be a scheme, let $\mathcal{M}$ and $\mathcal{N}$ be two vector bundles with fiberwise (left) A-action (they are $\mathcal{O}_S \otimes A$-modules which are locally $\mathcal{O}_S$-sheaves and let $\phi : \mathcal{M} \to \mathcal{N}$ be an $(\mathcal{O}_S)$-morphism. The* **linearity locus** $\mathrm{Lin}_A(\phi)$ *is the scheme*

$$\mathrm{Lin}_A(\phi) := \{s \in S \,|\, \phi_s : \mathcal{M} \otimes k(s) \to \mathcal{N} \otimes k(s) \text{ is } A\text{-linear}\}.$$

Proposition 4.8. $\mathrm{Lin}_A(\phi)$ *is the fiber product*

$$\begin{tikzcd}
\mathrm{Lin}_A(\phi) \arrow[r] \arrow[d] & S \arrow[d, "\phi"] \\
|\mathrm{Hom}_{A\otimes\mathcal{O}_S}(\mathcal{M},\mathcal{N})| \arrow[r] & |\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{M},\mathcal{N})|
\end{tikzcd}$$

Proof. Obvious. □

An important construction

Let $\tilde{V} \subset M \otimes \mathcal{O}_G$ be the universal inclusion over $G(h,M)$. We construct the scheme $\mathrm{Act}(A,\tilde{V})$ fiberwise as the subscheme of $|\mathrm{Hom}(A \otimes \tilde{V}, \tilde{V})|$ fiberwise satisfying associativity equations. We have the projection $q : \mathrm{Act}(A,\tilde{V}) \to G(h,M)$ and a natural morphism $\phi : q^*\tilde{V} \to M \otimes \mathcal{O}$. Then

Theorem 4.9. *$G_A(h,M)$ is isomorphic to the linearity locus of the morphism ϕ constructed above.*

5 Derived Quot and Hilbert schemes

We will construct a dg-manifold $\mathrm{RQuot}_h(\mathcal{F})$ such that:

1. $\pi_0(\mathrm{RQuot}_h(\mathcal{F})) = \mathrm{Quot}_h(\mathcal{F})$;
2. if $[\mathcal{K}] \in \mathrm{RQuot}_h(\mathcal{F})$, $\pi_i(\mathrm{RQuot}_h(\mathcal{F}),[\mathcal{K}]) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}})$, for $i > 0$.

5.1 The bar resolution

In this section we recall the principal definitions on the bar resolution of an associative algebra; this is in fact a very special case of a much more general construction: if $\mathcal{O}$ is an operad which satisfies some finiteness conditions (admissibility, see [3]), the bar-construction of the cobar construction of the operad $\mathcal{O}$ is a free operad $\mathcal{D}(O)$ which is quasiisomorphic to $\mathcal{O}$. The operads $\mathcal{A}ss$ and $\mathcal{C}omm$ satisfy this property so that for algebras over these operads, i.e. associative and commutative algebras there is a natural quasifree resolution given by the bar resolution. For associative algebras this is constructed in the following way: if A is an associative $\mathbb{Z}_-$-graded algebra, let V be the graded vector space

$$F = \bigoplus_{n=1}^{\infty} A^{\otimes n}[n-1].$$

Then the bar resolution of A is the free associative algebra $\mathrm{Bar}_A(A)$ on F. Let us denote $*$ the multiplication on $\mathrm{Bar}_A(A)$; we put a differential structure on $Bar_A(A)$ in the following way:

$$d = d' + d'',$$

where d' is the natural tensor differential on $A^{\otimes n}$ inherited by the differential on A and d'' is defined by

$$\begin{aligned} d''(a_0,\dots,a_n) = &\sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n - \\ &-\sum_{i=0}^{n-1}(-1)^i(a_0\otimes\ldots\otimes a_i)*(a_{i+1}\otimes\ldots\otimes a_n). \end{aligned}$$

Proposition 5.1. *1. The map d defined above is a differential ($d^2=0$).*

2. The projection $\mathrm{Bar}_A(A)\to F\to A$ *, where the last map is given by inner multiplication on the summands, is a quasiisomorphism of associative algebras.*

Analogously, if V is an A-module, we consider the free left $A_\sharp-$module

$$\mathrm{Bar}_A(V) = \bigoplus_{n=1}^{\infty} A^{\otimes n}\otimes V[n-1].$$

Let $\mu_n : A^{\otimes n}\otimes V\to V$ be the n^{th} component of the projection $\mathrm{Bar}_A(V)\to V$. We put on $\mathrm{Bar}_A(V)$ a structure of dg-module over A by defining the differential:

$$\begin{aligned} d(a_0\otimes\ldots\otimes a_n\otimes m) = &\\ &\sum_{i=0}^{n}(-1)^{i-1}a_0\otimes\ldots\otimes da_i\otimes\ldots\otimes a_n\otimes m + \\ &+\sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n\otimes m + \\ &+\sum_{p=0}^{n}(-1)^{p(n-p)}a_0\otimes\ldots\otimes a_p\otimes\mu_{n-p}(a_{p+1}\otimes\ldots\otimes a_n\otimes m). \end{aligned}$$

Proposition 5.2. *1. The map d defined above is a differential ($d^2=0$) and makes* $\mathrm{Bar}_A(V)$ *a free left dg-module over A.*

2. The projection $\mathrm{Bar}_A(V)\to V$ *is a quasiisomorphism of dg-modules over A.*

5.2 The derived space of actions

We start with a finite dimensional K-vector space V (considered as a graded vector space in degree zero) and with a $\mathbb{Z}_-$-graded associative K-algebra A. We construct, as in the non-graded case, the space of actions in the following way: the complex $\mathrm{Hom}_K(A\otimes V,V)$ is a $\mathbb{Z}_+$-graded complex and we have already defined the affine scheme $|\mathrm{Hom}_K(A\otimes V,V)| = \mathrm{SpecSym}(\mathrm{Hom}_K(V,A\otimes V))$; in the scheme $|\mathrm{Hom}_K(A\otimes V,V)|$ we consider the subscheme $\mathrm{Act}(A,V)$ as the one defined by the dg-ideal of $\mathrm{Sym}(\mathrm{Hom}_K(V,A\otimes V))$ generated by the associativity conditions. Consider next the case in which $A=F(E)$ is a tensor algebra on a $\mathbb{Z}_-$-graded vector space E. Then, by definition, to give an action of A on V is the same as to specify the action of generators so that $\mathrm{Act}(A,V) = |\mathrm{Hom}_K(E\otimes V,V)|$. The algebras of our interest will be quasifree associative algebras, i.e. dg-algebras A such that $A_\sharp = F(E)$ is a free algebra; in this case the scheme $\mathrm{Act}(A,V)$ has the obvious property that $\mathrm{Act}(A,V)_\sharp$ is an affine dg-scheme, identified with $|\mathrm{Hom}_K(E\otimes V,V)|$. The definition and the construction of $\mathrm{Act}(A,V)$ is trivially functorial in A in each of the above cases:

if $f : A_1 \to A_2$ is a morphism of algebras it is then defined a morphism of dg-schemes $f^*Act(A_2, V) \to \mathrm{Act}(A_1, V)$, satisfying functoriality conditions.

Let now A be a *finite dimensional* ungraded associative algebra and let V be a finite dimensional K-vector space; if needed, we will consider A and V as graded with trivial grading.

Definition 5.3. *The* derived space of actions of A on V, $\mathrm{RAct}(A, V)$ *is defined as* $\mathrm{Act}(B, V)$ *where* $B \to A$ *is any quasifree resolution.*

In order to show that the definition is well posed, Kapranov and Ciocan-Fontanine prove:

Theorem 5.4. *If* $f : B_1 \to B_2$ *is a quasiisomorphism of quasifree associative* $\mathbb{Z}_-$*-graded algebras the induced morphism* $f^*\mathrm{Act}(B_2, V) \to \mathrm{Act}(B_1, V)$ *is a quasiisomorphism of dg-schemes.*

As a corollary we have that:

Corollary 5.5. *Let* A *be an associative algebra.*

1. *The scheme* $\mathrm{RAct}(A, V)$ *is well defined up to quasiisomorphism.*
2. *If* $A = F(E)$ *is a free associative dg-algebra (with trivial differential) then* $\mathrm{RAct}(A, V)$ *is quasiisomorphic to* $\mathrm{Act}(A, V) = |\mathrm{Hom}_K(E \otimes V, V)|$.
3. *If every graded piece of* A *is finite dimensional,* $\mathrm{RAct}(A, V)_\sharp$ *is quasiisomorphic to an affine linear dg-manifold.*
4. *If* A *is concentrated in degree zero* $\pi_0(\mathrm{RAct}(A, V)) = \mathrm{Act}(A, V)$.

Sketch of proof. The first two parts are corollaries 3.7.1 and 3.7.2 of [2]. We will discuss the last two points, which are the relevant ones for our purposes. Since the construction of $\mathrm{RAct}(A, V)$ is, up to quasiisomorphism, independent of the resolution we choose we will use the bar resolution $\mathrm{Bar}_A(A)$ to compute $\mathrm{RAct}(A, V)$; the n^{th} graded piece of the bar resolution $\mathrm{Bar}_A(A)$ is $A^{\otimes n}$ which is finite dimensional because A is finite dimensional, so that the coordinate ring of $\mathrm{RAct}(A, V)$ is the symmetric algebra on the matrix elements of indeterminate linear operators $\phi_n : A^{\otimes n} \otimes V \to V$, which is quasifree and with finitely many generators in each degree. As for the last point, if A has trivial grading, always considering the model of $\mathrm{RAct}(A, V)$ built upon the bar resolution $\mathrm{Bar}_A(A)$, $\pi_0(\mathrm{RAct}(A, V))$ is the closed subscheme of the affine ordinary scheme $|\mathrm{Hom}_K(A \otimes V, V)|$ whose defining ideal in $\mathrm{Sym}\mathrm{Hom}_K(V, A \otimes V)$ is the image of the differential

$$\begin{aligned}\delta :&\mathrm{Hom}_K(V, \mathrm{Bar}_A(V)_{-1}) = \mathrm{Hom}_K(V, A \otimes A \otimes V) \longrightarrow \\ &\longrightarrow \mathrm{Hom}_K(V, Bar_A(V)_0) = \mathrm{Hom}_K(V, A \otimes V).\end{aligned}$$

Now, by construction, the image of the above map is dual to the kernel of the map $\delta^\vee : \mathrm{Hom}_K(A \otimes V, V) \to \mathrm{Hom}_K(A \otimes A \otimes V, V)$ defined by $\delta^\vee(\phi)(a_1 \otimes a_2 \otimes v) = \phi(a_1 a_2 \otimes v) - \phi(a_1 \otimes (\phi(a_2)v))$. □

5.3 The derived linearity locus

Let S be a $\mathbb{Z}_-$-graded dg-scheme, let $\mathcal{M}$ and $\mathcal{N}^\vee$ be two quasicoherent dg-sheaves such that $\mathcal{M}_\sharp$ and $\mathcal{N}^\vee_\sharp$ are $\mathcal{O}_{S_\sharp}$ locally free; we assume there is a fiberwise (left) A-action on $\mathcal{M}$ and $\mathcal{N}^\vee$. Let $\phi : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}_S$-morphism. In order to construct the derived version of the linearity locus of the morphism ϕ, we proceed as follows:

1. we choose a resolution $\mathcal{P} \to \mathcal{M}$ such that $\mathcal{P}$ is quasicoherent and $\mathcal{P}_\sharp$ is $\mathcal{O}_{S_\sharp} \otimes A_\sharp$-locally projective;

2. we define:

Definition 5.6. *The* derived linearity locus $\mathrm{RLin}_A(\phi)$ *is the derived fiber product*

$$\begin{tikzcd}
\mathrm{RLin}_A(\phi) \arrow[rr] \arrow[d] & & S \arrow[d, "\phi"] \\
|\mathrm{Hom}_{A\otimes\mathcal{O}_S}(\mathcal{P},\mathcal{N})| \arrow[rr, "\psi"'] & & |\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{M},\mathcal{N})|
\end{tikzcd}$$

Proposition 5.7. *If $\mathcal{P}$ is a resolution of $\mathcal{M}$ satisfying property 1 above and such that the morphism ψ in definition 5.6 above is flat, the scheme $\mathrm{RLin}_A(\phi)$ is independent, up to quasi-isomorphism on the choice of $\mathcal{P}$. The bar resolution $\mathrm{Bar}_A(\mathcal{M})$ satisfies the above properties and in particular the derived product in definition 5.6 coincides with the usual fiber product.*

Proposition 5.8. *The model of $\mathrm{RLin}_A(\phi)$ built on the bar resolution $\mathrm{Bar}_A(\mathcal{M})$ is the total space of a vector bundle and $\pi_0(\mathrm{RLin}_A(\phi)) = \mathrm{Lin}_A(\phi)$.*

Sketch of proof. The proposition is a consequence of the fact that to give an $\mathcal{O}_S \times A$-morphism of $\mathrm{Bar}_A(\mathcal{M})$ into $\mathcal{N}$ is equivalent to give an $\mathcal{O}_S$ morphism of $\mathrm{Bar}_A(\mathcal{M})$ into $\mathcal{N}$ with one shift in degrees, so that, if $\mathrm{RLin}_A(\phi, bar)$ denotes this particular model for $\mathrm{RLin}_A(\phi)$,

$$\mathrm{RLin}_A(\phi, bar) = |\mathrm{Cone}\{f : \mathcal{O}_S \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Bar}_A(\mathcal{M}), \mathcal{N})\}[1]|,$$

where the degree zero part of f defines in $\mathrm{Hom}_{\mathcal{O}_S}(A \otimes \mathcal{M}, \mathcal{N})$ the map $a \otimes m \mapsto \phi(a \otimes m) - a\phi(m)$. This way a point of $\mathrm{RLin}_A(\phi, bar)$ is identified with a point $s \in S$ together with a collection of maps $(\phi, f_1, f_2, f_n, \dots)$ at the point $s \in S$, with $f_i : A^{\otimes i} \otimes \mathcal{M}|_s \to \mathcal{N}|_s$, such that $(\phi, f_1, f_2, f_n, \dots)$ is an A_∞ morphism (see below). The degree zero truncation of this is by definition $\mathrm{Lin}_A(\phi)$. □

5.4 The derived A-Grassmannian

Let us be given a finite dimensional associative K-algebra, a finite dimensional $A-$module and a positive integer k. The scope of this paragraph is to define a dg-scheme $\mathrm{R}G_A(k, M)$ with $\pi_0(\mathrm{R}G_A(k, M)) = G_A(k, M)$ as defined in section 4. Having already constructed the derived version of the schemes $\mathrm{Act}(A, V)$ and $\mathrm{Lin}_A(\phi)$, we will just parallel the construction above. Let $G = G(k, M)$ be the Grassmannian of $k-$dimensional subspaces of M and let $\mathcal{V} \subset M \otimes \mathcal{O}_G$ be the universal subbundle on G; we can easily construct $\mathrm{RAct}(A, \mathcal{V})$ which is a flat family of schemes $\mathrm{RAct}(A, V)$ on the fibers $V \subset \mathcal{V}$. We can choose a model for $\mathrm{RAct}(A, \mathcal{V})$ such that $\mathrm{RAct}(A, \mathcal{V})$ is a dg-manifold and such that the natural projection $p : \mathrm{RAct}(A, \mathcal{V}) \to G$ is a smooth morphism: it suffices to choose a resolution $B \to A$ with finitely many generators in each degree. Consider next the pullback morphism

$$\phi : p^*\mathcal{V} \to M \otimes p^*\mathcal{O}_G.$$

The morphism ϕ is an $\mathcal{O}_{\mathrm{RAct}(A,V)}$-morphism of locally free sheaves with a B-action on the fibers. It then applies the construction of paragraph 5.3: we choose, as we can, a good resolution $\mathcal{P} \to \mathcal{V}$ and we construct $\mathrm{RLin}_B(\phi)$.

Definition 5.9. *The* derived A-Grassmannian $\mathrm{R}G_A(k, M)$ *is the derived linearity locus* $\mathrm{RLin}_B(\phi)$.

Theorem 5.10. *The dg-scheme $\mathrm{R}G_A(k, M)$ is independent, up to a quasiisomorphism, of the resolutions $B \to A$ and $\mathcal{P} \to \mathcal{V}$, satisfying the hypotheses of 5.7. There is a model of it which is a dg-manifold and $\pi_0(\mathrm{R}G_A(k, M)) = G_A(k, M)$.*

Proof. The result follows from 5.5 and 5.7. □

5.5 Infinitesimal computations

In this paragraph we will compute tangent complexes for the spaces $\mathrm{RAct}(A, V)$, $\mathrm{RLin}_A(\phi)$ and $\mathrm{R}G_A(k, M)$ constructed above; we will do so up to quasiisomorphism, so that we will in fact perform all our computations using the particular models obtained using bar resolutions. The complete result for this section is:

Theorem 5.11. *We have the following:*

1. *If* $\mu \in \mathrm{Act}(A, V)$*, then* $H^i T^{\cdot}\mathrm{RAct}(A, V) = \mathrm{Ext}^{i+1}_A(V, V)$ *if* $i > 1$ *and* $H^0 T^{\cdot}\mathrm{RAct}(A, V) = T_\mu \mathrm{Act}(A, V)$.

2. *If* $s \in ST^{\cdot}_s\mathrm{RLin}_A(\phi) = \mathrm{Cone}\{T^{\cdot}_s S \to T^{\cdot}_s|\mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Bar}_A(\mathcal{M}), \mathcal{N})|\}[1]$ *up to quasiisomorphism.*

3. *If* $V \subset M$ *is any* A*-submodule,* $H^i T^{\cdot}_V \mathrm{R}G_A(k, M) = \mathrm{Ext}^i_A(V, \frac{M}{V})$.

Proof. As for the first result, we go back to paragraph 5.2 for notations and in particular we use the bar resolution for A in order to make computations. In this case, we have, by definition:

$$T_\mu \mathrm{RAct}(A, V, bar) = \mathrm{Hom}_A(\mathrm{Bar}^{\leq -1}_A(V), V)[1]$$

and the cohomology of this complex computes the Ext's for $i \geq 1$; for $i = 0$ we have already shown that $\pi_0(\mathrm{RAct}(A, V) = \mathrm{Act}(A, V)$. The second result is trivial, considering the result of Proposition 5.8. As for the third result, from the exact sequence $0 \to V \to M \to \frac{M}{V} \to 0$ it follows that in the derived category we have an identification

$$\mathrm{RHom}_A(V, \frac{M}{V}) = \mathrm{Cone}\{\mathrm{RHom}_A(V, V) \longrightarrow \mathrm{RHom}_A(V, M)\}[1].$$

Using the model for $\mathrm{R}G_A(k, M)$ with bar resolutions, we get:

$$\begin{aligned} T^{\cdot}_V \mathrm{R}G_A(k, M) =& \mathrm{Cone}\{T^{\cdot}_{(V,\mu)}\mathrm{RAct}(A, \mathcal{V}) \longrightarrow \\ & \longrightarrow T^{\cdot}_{(V,\mu)}|Hom_{\mathcal{O}_{RAct(A,V)}}(Bar_A(p^*\mathcal{V}), M \otimes \mathcal{O}_{RAct(A,V)})|. \end{aligned}$$

Since $\mathrm{RAct}(A, V)$ and $|\mathrm{Hom}_{\mathcal{O}_{\mathrm{RAct}(A,V)}}(\mathrm{Bar}_A(p^*\mathcal{V}), M \otimes \mathcal{O}_{\mathrm{RAct}(A,V)})|$ are fibrations over $G(k, M)$ we have two exact sequences:

$$0 \longrightarrow \mathrm{Hom}_A(\mathrm{Bar}_A(V)^{\leq -1}(V), V) \longrightarrow T^{\cdot}_{(V,\mu)}\mathrm{RAct}(A, \mathcal{V}) \longrightarrow T_V G(k, M) \longrightarrow 0$$

and

$$\begin{aligned} & 0 \longrightarrow \mathrm{Hom}_A(\mathrm{Bar}_A(V), M) \longrightarrow \\ & \longrightarrow T^{\cdot}_{(V,\mu)}|_{\mathrm{Hom}_{\mathcal{O}_{\mathrm{RAct}(A,V)}}}(\mathrm{Bar}_A(p^*\mathcal{V}), M \otimes \mathcal{O}_{\mathrm{RAct}(A,V)}) \longrightarrow T_V G(k, M) \to 0. \end{aligned}$$

Taking the cone the last part in the two exact sequences cancel out and, taking into consideration the first observation we made, we get

$$T^{\cdot}_V \mathrm{R}G_A(k, M) = \mathrm{RHom}_A(V, \frac{M}{V}),$$

which means that the cohomology of the first complex is as stated. $\square$

5.6 Functorial properties

We recall the principal definitions regarding A_∞-structures.

Definition 5.12. *Let A be an associative dg-algebra. A left A-module over A is a graded vector space V together with a collection of linear maps*

$$\mu_n : A^{\otimes n+1} \otimes V \to V, \quad n \geq 0, \quad \deg \mu_n = 1 - n,$$

satisfying the conditions:

$$\begin{aligned} &\sum_{i=1}^{n} (-1)^{\overline{a}_1 + \ldots + \overline{a}_{i-1}} \mu_n(a_1, \ldots, da_i, \ldots, a_n, m) = \\ &\quad = \sum_{i=1}^{n-1} (-1)^i \mu_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_n, m) - \\ &\quad - \sum_{\substack{p+q=n \\ p+q \geq 0}} (-1)^{q(\overline{a}_1 + \ldots + \overline{a}_p) + p(q-1) + q(p-1)} \mu_p(a_1, \ldots, a_p, \mu_q(a_{p+1}, \ldots, a_n, m)). \end{aligned}$$

In particular $d_M = \mu_0$ is a differential and , if $\mu_n = 0$ for $n \geq 2$, an A_∞- module over A is the same as an ordinary dg A-module.

Definition 5.13. *Let A be an associative dg-algebra, let M be an A_∞-module over A and let N be an A-module. An A_∞ morphism $\phi : M \to N$ is a collection of linear maps*

$$\phi_n : A^{\otimes n} \otimes M \to N, \quad n \geq 0, \quad \deg \phi_n = -n,$$

satisfying the conditions:

$$\begin{aligned} d\phi_n(a_0, \ldots, a_n, m) - \sum_{i=1}^{n} (-1)^i \phi_n(a_1, \ldots, da_i, \ldots, a_n, m) &= \\ = \sum_{i=0}^{n-1} (-1)^i \phi_{n-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_n, m)& \\ + \sum_{p=0}^{n} (-1)^{p(n-p)} \phi_p(a_0, \ldots, a_p, \mu_{n-p}(a_{p+1}, \ldots, a_n, m)).& \end{aligned}$$

As before, ϕ_0 is a morphism of complexes and, if M is in fact an A-module and if $\phi_n = 0$ for $n \geq 2$, the morphism ϕ is an A-morphism.

As a consequence of the definitions we have:

Proposition 5.14. *Let A be an associative algebra and let V be a graded vector space.*

1. *An A_∞−action of A on V is the same as a dg-action of $Bar_A(A)$ on V.*
2. *Let N be any dg-module over A. An A_∞−morphism $\phi : M \to N$ is the same as a morphism of dg-modules $Bar_A(M) \to N$.*

For all the rest of this section, we will consider the model for $\mathrm{R}G_A(k, M)$ obtained by taking the bar resolution of A for the construction of $\mathrm{RAct}(A, V)$ and the bar resolution of M in the construction of $\mathrm{RLin}(\phi)$, always with notations of paragraphs 5.1, 5.2 and 5.3.

Proposition 5.15. *Let R be any commutative dg-algebra. We have:*

1. $\mathrm{Hom}_{dg-alg}(K[\mathrm{RAct}(A, V, \mathrm{bar})], R)$ *is naturally identified with the set of R-multilinear* A_∞*-actions of* $A \otimes R$ *into* $V \otimes R$.

2. $\mathrm{Hom}_{dg-alg}(K[\mathrm{RLin}(\phi, \mathrm{bar})], R)$ *is naturally identified with the set of data* $(g, h_1, \ldots, h_n, \ldots)$ *where* $g : \mathrm{Spec}(R) \to S$ *is a morphism of dg-schemes and* $h_n : A^{\otimes n} \otimes g^*M \to g^*N$ *are such that* $(g^*\phi, h_1, \ldots)$ *is an* A_∞*-morphism* $g^*M \to g^*N$.

3. $\mathrm{Hom}_{dg-alg}(K[\mathrm{R}G_A(k, M, \mathrm{bar})], R)$ *is naturally identified with the union of the above sets of data.*

5.7 From the A-Grassmannian to the derived Quot scheme

In previous paragraphs we constructed, for a given finite dimensional K-algebra and for a given finite dimensional A-module a dg-manifold $\mathrm{R}G_A(k, M)$. In this paragraph we will show how to use this construction in order to construct derived Quot schemes. The case of interest in geometry is the following:

1. X is a projective scheme;

2. $\mathcal{F}$ is a coherent sheaf on X;

3. $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i))$;

4. $M = \bigoplus_{i=p}^{q} H^0(X, \mathcal{F}(i))$.

So the main difference with the case we have analysed in the last paragraphs is that A is no more finite dimensional, although each graded piece of A is.

Convention 5.16. *Given two graded left A-modules M and N, we define the* $\mathrm{Ext}^{i,0}_A(M, N)$ *as the derived functors of* $\mathrm{Hom}^0_A(M, N)$*, which consists of degree zero A-morphisms from M into N.*

Now *the construction of paragraph 5.4 can be repeated verbatim if we consider everywhere morphisms of degree zero with respect to this new (projective) grading and we replace the functor* Hom *with the functor* Hom^0; the resulting dg-scheme will be indicated by $\mathrm{R}G^0_A(k, M)$.

Proposition 5.17. *Let A be an infinite dimensional* ($\mathbb{Z}_+$)*-graded associative algebra with* $\dim_K A_i < \infty$ *for every* $i \geq 0$ *and let M be a* finite dimensional *graded A-module. The (graded version of the) derived A-Grassmannian* $\mathrm{R}G^0_A(k, M)$ *is a dg-manifold with:*

1. $\pi_0(\mathrm{R}G^0_A(k, M)) = G^0_A(k, M)$;

2. $H^i T^\cdot_V \mathrm{R}G^0_A(k, M) = \mathrm{Ext}^{i,0}_A(V, \frac{M}{V})$.

Proof. The only fact to be proved is that $\mathrm{R}G^0_A(k, M)$ is a dg-manifold. In order to prove this it is enough to prove that the corresponding graded version $\mathrm{RAct}^0(A, V)$ of $\mathrm{RAct}(A, V)$ is a dg-manifold, i.e. that its coordinate sheaf has finitely many generators in each degree; for $\mathrm{RAct}(A, V)$ this is a consequence of the finiteness of A which implies finitely many generators in each degree for the bar resolution. But this holds also in our case: $\mathrm{RAct}^0(A, V, bar)$, by its very definitions has generators of given degree t given by morphisms *of degree zero* $A^{\otimes n} \otimes V \to V$ and since V is finite dimensional, there are only finitely many possibilities for maps

$$A_{i_1} \otimes \ldots \otimes A_{i_n} \ V_j \to V_{i_1 + \ldots + i_n + j}.$$

□

Definition 5.18. *Let X be a projective scheme, $\mathcal{F}$ be a coherent sheaf on X and let $h(t) \in \mathbb{Q}[t]$ be a polynomial. Let $A = \oplus_{i\geq 0} H^0(X, \mathcal{O}_X(i))$ and let $M = \oplus_{i=p}^{q} H^0(X, \mathcal{F}(i))$ with $0 \ll p \ll q$. The derived Quot scheme is defined as*

$$\mathrm{RQuot}_h(\mathcal{F}) := \mathrm{R}G_A^0(h, M_{[p,q]}).$$

This definition is well posed in the derived category of dg-schemes: in order to prove this we finally only need to prove independence up to quasiisomorphism on the choice of p and q.

Theorem 5.19. *The following hold true:*

1. *For $0 \ll p \ll p' \ll q' \ll q$ the natural projection $\mathrm{R}G_A^0(h, M_{[p,q]}) \to \mathrm{R}G_A^0(h, M_{[p,q]})$ is a quasiisomorphism of dg-manifolds.*
2. *$\pi_0(\mathrm{RQuot}_h(\mathcal{F})) = \mathrm{Quot}_h(\mathcal{F})$;*
3. *if $\mathcal{K} \subset \mathcal{F}$ has Hilbert polynomial h, then $H^i T_V^{\cdot} \mathrm{RQuot}_h(\mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}})$.*

Sketch of proof. The second result is a consequence of theorem 4.2. The first part is a consequence of the other two, because of the "Whitehead Theorem" (proposition 3.7). So we need to prove the third part. This is a consequence of the next proposition and of standard relations between sheaves and associated garded modules, given by Serre's correspondence. $\square$

Proposition 5.20. *Let X be a projective scheme, let $\mathcal{F}$, $\mathcal{G}, \mathcal{K}$ be coherent sheaves on X and let $A = \bigoplus_{i\geq 0} H^0(X, \mathcal{O}_X(i))$, $M = \bigoplus_{i\geq 0} H^0(X, \mathcal{F}(i))$, $N = \bigoplus_{i\geq 0} H^0(X, \mathcal{G}(i))$, $V = \bigoplus_{i\geq 0} H^0(X, \mathcal{K}(i))$.*

1. *$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \lim_{\to} \mathrm{Ext}^{i,0}_A(M_{\geq p}, N_{\geq p})$;*
2. *there exists p such that*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \frac{\mathcal{F}}{\mathcal{K}}) = \mathrm{Ext}^{i,0}_A(V_{\geq p}, \frac{M_{\geq p}}{V_{\geq p}}).$$

5.8 Derived Hilbert schemes

In the category of schemes, if we choose on a projective as coherent sheaf $\mathcal{F} = \mathcal{O}_X$, as we have already noticed $\mathrm{Quot}_h(\mathcal{O}_X) = \mathrm{Hilb}_h(X)$, where $\mathrm{Hilb}_h(X)$ is the scheme parametrizing subschemes of X with Hilbert polynomial h. In fact, as we sketched in section 4, we can construct $\mathrm{Hilb}_h(X)$ as a Quot scheme, parametrizing subspaces of $A_{[p,q]}$ of given dimensions which are also A-modules, see theorem 4.3 , where this scheme is defined as $J(k, A_{[p,q]})$. The point here is that *A-submodules of A are in fact ideals of A*: we can also construct a derivation of the Hilbert scheme in the derived category, whose end is to derive the property to be an ideal in a commutative algebra, i.e. the property, for a subspace of a given vector space to be a commutative algebra. If we perform this construction in the derived category we end up with a dg-manifold whose degree zero truncation is $\mathrm{Hilb}_h(X)$, but which is *not quasiisomorphic* to the derived Quot scheme. In this paragraph we give the main ingredients of the construction of the derived Hilbert scheme. We give it in steps:

First step: reduction to a finite-dimensional model

Exactly as in paragraph 5.6 we reduce the problem to construct $\mathrm{Hilb}_h(X)$ to the following problem:

1. A is a finite dimensional commutative algebra (possibly without unit) and h is an integer.

2. Construct a dg-manifold $\mathrm{R}J(k,A)$ such that $\pi_0(\mathrm{R}J(k,A) = J(k,A)$, where $J(k,A)$ parametrizes ideals in A.

As in the case of the Quot scheme, we subdivide the problem into two steps, since giving an ideal structure to a subspace V of A is equivalent to giving a commutative algebra structure to V and to making the natural inclusion a homomorphism of commutative algebras. Then the problem is in fact reduced to construct, for every $V \subset A$ of desired dimension, the following:

1. A dg-manifold $\mathrm{RCA}(V)$ such that $\pi_0(\mathrm{RCA}(V) = \mathrm{CA}(V)$, the space of commutative algebra structures on V.

2. A dg-manifold $\mathrm{RHom}(\phi)$ such that $\pi_0(\mathrm{RHom}(\phi) = \mathrm{Hom}(\phi)$, the homomorphicity locus of a morphism between two commutative algebras.

As in the Quot case, once one has the above dg-schemes, one constructs $\mathrm{R}J(k,A)$ by considering the relative construction $\mathrm{RCA}(\mathcal{V})$ for the universal subsheaf of a Grassmannian G, and considering the scheme $\mathrm{RHom}(\phi)$ over $p : \mathrm{RCA}(\mathcal{V}) \to G$, referred to the natural $p^*\mathcal{V} \to A \otimes \mathcal{O}_{\mathrm{RCA}}(\mathcal{V})$.

Second step: construction of the derived space of commutative algebra structures

In order to do so, as in the case of the Quot-scheme, we consider the bar resolution $\mathcal{C}om(A) \to A$ and we perform a parallel construction to the one given in 5.2: we derive the space of commutative algebra structures on V by considering the space of algebra structures on V given by the structure of an operad over $\mathcal{C}om$, i.e " $\mathcal{C}om_\infty$"-algebras. One finds a model for the space so constructed which is a dg-manifold and has functorial properties analogous to those in paragraph 5.5.

Third step: construction of the derived space of homomorphicity

This construction is the same as in paragraph 5.3 and it is a derived fiber product. There are two differences with the case of the Quot-scheme:

1. it is not possible to prove a theorem like theorem 5.20 in the general case, so that one has independence of the result on p and q only in special cases, like for instance if we choose to parametrise only locally complete intersection (LCI) subschemes of X.

2. if $h = 1$ we have $\mathrm{RQuot}_h(\mathcal{O}_X) \neq X$, as one can see by computing tangent spaces in Theorem 5.19, while $\mathrm{RHilb}_h(X) = X$. We have in fact:

Theorem 5.21. *There exists a dg-manifold* $\mathrm{RHilb}_h^{\mathrm{LCI}}(X)$ *such that:*

1. $\pi_0(\mathrm{RHilb}_h^{\mathrm{LCI}}(X)) = \mathrm{Hilb}_h^{\mathrm{LCI}}(X)$,

2. for any LCI subscheme $Z \subset X$ *of Hilbert polynomial* h $H^iT^{\cdot}_Z\mathrm{RHilb}_h(X) = H^i(Z, \mathcal{N}_{Z/X})$.

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Deformations of algebras and cohomology

Domenico Fiorenza

"forse di Irene ho già parlato sotto altri nomi;
forse non ho parlato che di Irene."
— *Italo Calvino - Le città invisibili*

1 Introduction

In this short note we show how each algebra deformation problem is described by a suitable cohomology theory. The language of *operads* allows us to describe such a cohomology theory in an universal way (i.e. independent of the particular algebra we are studying). As an example we will show how this general theory specializes, in the case of associative algebras, to the well known Hochshild cohomology. All ideas and tecniques used in this note are taken from the beautiful paper *"Cotangent cohomology of a category and deformations"* by Martin Markl.

Let us start in a *naive* way. Let V be a $\mathbb{K}$-vector space and let

$$\varphi : V^{\otimes n} \to V^{\otimes m}$$

A *formal deformation* of φ will be

$$\varphi_\hbar = \varphi + \hbar\varphi_1 + \hbar^2\varphi_2 + \cdots$$

where

$$\varphi_i : V^{\otimes n} \to V^{\otimes m}$$

and the *deformation parameter* $\hbar$ is to be thought so little to ensure the convergence of the series. Actually, we are not interested at all in the convergence of the series defining $\varphi_\hbar$, but we treat it simply as a formal series (this is the reason why $\varphi_\hbar$ was called formal). If we now write

$$\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \otimes \mathbb{K}[[\hbar]]$$

and by

$$\bmod \hbar : \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \to \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$$

the canonical projection, we can restate what we have written above by sayng that the formal deformations of an element $\varphi \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ are

$$\left\{\varphi_\hbar \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \text{ such that } \varphi_\hbar = \varphi \mod \hbar\right\}$$

We can make one more step further: if $\langle *_{n,m} \rangle$ denotes the one-dimensional $\mathbb{K}$-vector space generated by the element $*_{n,m}$, then we have a canonical isomorphism

$$\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \tilde{=} \mathrm{Hom}(\langle *_{n,m} \rangle, \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}))$$

so that we can think of φ as

$$\varphi : \langle *_{n,m} \rangle \to \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$$

Thus, the deformations of φ can be seen as liftings:

$$\begin{tikzcd}
 & \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \arrow[dd, "\mathrm{mod}\ \hbar"] \\
\langle *_{n,m} \rangle \arrow[ur, "\varphi_\hbar"] \arrow[dr, "\varphi"'] & \\
 & \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})
\end{tikzcd}$$

2 $\mathcal{O}$-algebras and their deformations.

Recall that, if $\mathcal{O}$ is an operad of vector spaces, then an $\mathcal{O}$-algebra is simply an operad morphism

$$A : \mathcal{O} \to \underline{\mathrm{End}}(V)$$

where V is some vector space on a fixed base field $\mathbb{K}$ and $\underline{\mathrm{End}}(V)$ is its endomorphisms operad. Let now $\mathcal{C}$ be any PROP of $\mathbb{K}$-vector spaces. By $\mathcal{C}[[\hbar]]$ we denote the PROP defined as follows:

1. The objects of $\mathcal{C}[[\hbar]]$ are the same as the objects of $\mathcal{C}$

2. $\mathrm{Hom}_{\mathcal{C}[[\hbar]]}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathbb{K}[[\hbar]]$

Note that the essential here is that the *morphisms* are $\mathbb{K}$-vector spaces (whatever the objects are). In particular $\underline{\mathrm{End}}(V)$ is an operad of vector spaces, so we can consider the operad $\underline{\mathrm{End}}(V)[[\hbar]]$. The reduction modulus $\hbar$ is a functor

$$\mathrm{mod}\ \hbar : \mathcal{C}[[\hbar]] \to \mathcal{C}$$

Then, *formal deformations* of an algebra A are the liftings

$$\begin{tikzcd}
 & \underline{\mathrm{End}}(V)[[\hbar]] \arrow[dd, "\mathrm{mod}\ \hbar"] \\
\mathcal{O} \arrow[ur, "A_\hbar"] \arrow[dr, "A"'] & \\
 & \underline{\mathrm{End}}(V)
\end{tikzcd}$$

Since

$$\mathbb{K}[[\hbar]] = \lim_{\substack{\leftarrow \\ n}} \mathbb{K}[[\hbar]]/(\hbar^n) = \lim_{\substack{\leftarrow \\ n}} \mathbb{K}[\hbar]/(\hbar^n)$$

A formal deformation of A can be thought as a projective limit

$$\begin{tikzcd}
& \underline{\mathrm{End}}(V)[[\hbar]] \arrow[d] \\
& \vdots \arrow[d] \\
& \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^3) \arrow[d, "\mathrm{mod}\ \hbar^2"] \\
\mathcal{O} \arrow[uuur, "A_\hbar"] \arrow[ur, "A_2"] \arrow[r, "A_1"] \arrow[dr, "A"'] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2) \arrow[d, "\mathrm{mod}\ \hbar"] \\
& \underline{\mathrm{End}}(V)
\end{tikzcd}$$

The deformations A_n are called *infinitesimal deformations* of A. In particular, A_n is said to be a n^{th} order infinitesimal deformation of A. Then we can try to solve the problem of describing formal deformations of A by a *step-by-step* approach: starting from A, describe the 1^{st} order deformations of A; given a 1^{st} order deformation of A, try to lift it to a 2^{nd} order deformation, and so on (notice that the problem of the *existence* of formal deformations is trivial: the *trivial* deformation $A_\hbar = A + \hbar \cdot 0 + \hbar^2 \cdot 0) + \dots$ works. What we are interested in is describing the space of *all* formal deformations of A.). For any n, given an n^{th} order deformation A_n, there are two questions we have to answer:

1. Does A_n lift to a $(n+1)^{th}$ order deformation A_{n+1}?
2. If yes, which are these liftings?

In the next section we will show how the *obstruction* to the answer *yes* in the first question is given by an element in the second cohomology group of a certain complex, and that, when this obstruction vanishes, the liftings are parametrized by elements in the first cohomology group of that complex. Such a cohomological theory is called the *cotangent cohomology* of A. Describing it is the aim of the next section.

3 Resolutions of operads and cotangent cohomology.

As we have seen in [Fi], any operad over $\mathbb{N}$ can be realized as a quotient of a free operad:

$$0 \longleftarrow \mathcal{O} \xleftarrow{\pi} \mathcal{F}(X) \longleftarrow \mathcal{I} \longleftarrow 0$$

The kernel $\mathcal{I}$ of the natural projection $\pi : \mathcal{F}(X) \to \mathcal{O}$ is an ideal of $\mathcal{F}$. In particular it is an $\mathcal{F}(X)$-bi-module. Since any $\mathcal{F}(X)$-bi-module can be realized as the quotient of a free $\mathcal{F}(X)$-bi-module, the short exact sequence above can be extended to a resolution

$$\mathcal{R}: \ 0 \longleftarrow \mathcal{O} \xleftarrow{\pi} \mathcal{F}(X) \xleftarrow{\alpha} \mathcal{F}(X)\langle Y \rangle \xleftarrow{\beta} \mathcal{F}(X)\langle Z \rangle \longleftarrow \cdots$$

of $\mathcal{F}(X)$-bi-modules ($\mathcal{O}$ is a $\mathcal{F}(X)$-bi-module via π). Let now M be an $\mathcal{O}$-bi-module. Then M is an $\mathcal{F}(X)$-bi-module via π. Denote by

$$\mathrm{Der}(\mathcal{F}(X), M)$$

the space of *derivations* of $\mathcal{F}(X)$ with values in M, i.e. the set of all linear maps $\theta : \mathcal{F}(X) \to M$ such that

$$\begin{aligned} \theta(\varphi \circ \psi) &= \theta(\varphi) \cdot \psi + \varphi \cdot \theta(\psi) \\ \theta(\varphi \otimes \psi) &= \theta(\varphi) \otimes \psi + \varphi \otimes \theta(\psi) \end{aligned}$$

Since $\mathrm{Der}(\mathcal{F}(X), M)$ is a subspace of the space $\mathrm{Hom}(\mathcal{F}(X), M)$ of all linear maps from $\mathcal{F}(X)$ to M, the composition with α defines a morphism

$$\alpha^* : \mathrm{Der}(\mathcal{F}(X), M) \to \mathrm{Hom}_{\mathsf{Vect}}(\mathcal{F}(X)\langle Y\rangle, M)$$

What is crucial here is that the image of α^* is actually contained in the subspace $\mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle, M)$ of $\mathcal{F}(X)$-bi-module homomorfisms from $\mathcal{F}(X)\langle Y\rangle$ to M. In fact, if $\theta \in \mathrm{Der}(\mathcal{F}(X), M)$, $\varphi \in \mathcal{F}(X)$ and $\psi \in \mathcal{F}(X)\langle Y\rangle$, we have

$$(\alpha^*\theta)(\varphi \circ \psi) = \theta\left(\alpha(\varphi \circ \psi)\right) =$$

by $\mathcal{F}(X)$-linearity of α

$$\begin{aligned} &= \theta\left(\varphi \circ \alpha(\psi)\right) = \\ &\quad \theta(\varphi) \cdot \alpha(\psi) + \varphi \cdot \theta\left(\alpha(\psi)\right) = \end{aligned}$$

since $\mathcal{F}(X)$ acts on M via π

$$\begin{aligned} &= \theta(\varphi) \cdot \pi\left(\alpha(\psi)\right) + \varphi \cdot (\alpha^*\theta)(\psi) = \\ &\quad \varphi \cdot (\alpha^*\theta)(\psi) \end{aligned}$$

(the proof for the tensor product is completely analogue). Since β is an homomorphism of $\mathcal{F}(X)$-bi-modules, it induces (by composition) a map

$$\beta^* : \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Y\rangle, M) \to \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Z\rangle, M)$$

Finally, being $\mathcal{R}$ a resolution, is $\beta^* \circ \alpha^* = 0$, so the sequence

$$0 \to \mathrm{Der}(\mathcal{F}(X), M) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Y\rangle, M) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Z\rangle, M) \tag{1}$$

is a cohomological complex. We set

$$\begin{aligned} \mathcal{E}^1(\mathcal{R}, M) &:= \mathrm{Der}(\mathcal{F}(X), M) \\ \mathcal{E}^2(\mathcal{R}, M) &:= \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Y\rangle, M) \\ \mathcal{E}^3(\mathcal{R}, M) &:= \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Z\rangle, M) \\ \delta^1 := \alpha^*; &\quad \delta^2 := \beta^* \end{aligned}$$

so that we can rewrite (1) as

$$0 \longrightarrow \mathcal{E}^1(\mathcal{R}, M) \xrightarrow{\delta^1} \mathcal{E}^2(\mathcal{R}, M) \xrightarrow{\delta^2} \mathcal{E}^3(\mathcal{R}, M) \tag{2}$$

The relosution of the operad $\mathcal{O}$ is not unique (it depends, among other things, on a choiche of the set of generators X). But the usual arguments show that the cohomology of $\{\mathcal{E}^*(\mathcal{R}, M), \delta^*\}$ does not depend on the particular resolution chosen. So we can define

$$\mathcal{T}^*(\mathcal{O}, M) := H^*(\mathcal{E}^*(\mathcal{R}, M), \delta^*)$$

It is called the *cotangent cohomology* of $\mathcal{O}$ with coefficients in the $\mathcal{O}$-module M. Directly from the definition of the complex, it follows that $\mathcal{T}^1(\mathcal{O}, M) = \mathrm{Der}(\mathcal{O}, M)$. Notice that, if $A : \mathcal{O} \to \underline{\mathrm{End}}(V)$ is an algebra, then $\underline{\mathrm{End}}(V)$ is in a natural way an $\mathcal{O}$-module via A, so the cohomology groups

$$\mathcal{T}^*(\mathcal{O}, \underline{\mathrm{End}}(V))$$

are defined.

4 First order deformations.

The algebra A clearly lifts to a first order deformation, since the trivial deformation provides such a lifting. Since A is the unique 0^{th} order deformation of A, the answer to the first of our questions is trivially *yes*. To describe all the possible first order deformation, notice that, if A_1 is a first order deformation of A, we can write

$$A_1 = A + \hbar A_{(1)} \mod \hbar^2$$

The condition "$A_1 : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2)$ is a morphism" is equivalent to

$$\begin{aligned}(A + \hbar A_{(1)})(\varphi \circ \psi) &= (A + \hbar A_{(1)})(\varphi) \circ (A + \hbar A_{(1)})(\psi) \mod \hbar^2,\\ (A + \hbar A_{(1)})(\varphi \otimes \psi) &= (A + \hbar A_{(1)})(\varphi) \otimes (A + \hbar A_{(1)})(\psi) \mod \hbar^2, \quad \forall \varphi, \psi \in \mathcal{O}\end{aligned}$$

We can rewrite this as

$$\begin{aligned}A(\varphi \circ \psi) + \hbar A_{(1)}(\varphi \circ \psi) &= \big(A(\varphi) + \hbar A_{(1)}(\varphi)\big) \circ \big(A(\psi) + \hbar A_{(1)}(\psi)\big) \mod \hbar^2\\ &= A(\varphi \circ \psi) + \hbar \left(A_{(1)}(\varphi) \circ \psi + \varphi \circ A_{(1)}(\psi)\right) \mod \hbar^2\end{aligned}$$

This gives

$$A_{(1)}(\varphi \circ \psi) = A_{(1)}(\varphi) \circ \psi + \varphi \circ A_{(1)}(\psi)$$

(exactly the same argument works for $\otimes$ in place of $\circ$), so $A_{(1)}$ is a *derivation* of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$. Vice versa, the argument above shows that if $A_{(1)}$ is a derivation of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$, the map $A_1 := A + \hbar A_{(1)}$ is an homomorphism $A_1 : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2)$, and is clearly a lifting of A. Then we have proved

$$\{\text{First order deformations of } A\} \leftrightarrow \mathrm{Der}(\mathcal{O}, \underline{\mathrm{End}}(V)) = \mathcal{T}^1(\mathcal{O}, \underline{\mathrm{End}}(V))$$

5 Higher order deformations.

Assume now $A_n : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})$ is a n^{th} order deformation. We wonder if it lifts to a $(n+1)^{th}$ order deformation A_{n+1}. This time there is no section

$$\underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$$

so we have no trivial answer as in the case $n = 0$. Here we use the power of free objects. Being $\mathcal{F}(X)$ free, we can lift A_n to $\tilde{A}_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. This way we have

$$\begin{tikzcd}
\mathcal{F}(X)\langle Y\rangle \arrow[d, "\alpha"] & \\
\mathcal{F}(X) \arrow[r, "\tilde{A}_{n+1}"] \arrow[d, "\pi"] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \arrow[d, "\mathrm{mod}\ \hbar^{n+1}"] \\
\mathcal{O} \arrow[r, "A_n"] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})
\end{tikzcd}$$

In particular $\tilde{A}_{n+1} \circ \alpha$ maps all $\mathcal{F}(X)\langle Y\rangle$ to $0 \mod \hbar^{n+1}$. This means

$$\begin{aligned}
\alpha^* \tilde{A}_{n+1} : \mathcal{F}(X)\langle Y\rangle \to & \ker\{\mathrm{mod}\hbar^{n+1} : \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})\} \\
& \simeq \underline{\mathrm{End}}(V)
\end{aligned}$$

We denote this composition as

$$\Omega(\tilde{A}_{n+1}) : \mathcal{F}(X)\langle Y\rangle \to \underline{\mathrm{End}}(V)$$

The map $\Omega(\tilde{(}A_{n+1})$ is an homomorphism of $\mathcal{F}(X)$-modules ($\tilde{A}_{n+1}$ and α are), so $\Omega(\tilde{(}A_{n+1})$ is an element of $\mathcal{E}^2(\mathcal{R}, \underline{\mathrm{End}}(V))$. We have $\delta^2\Omega(\tilde{(}A_{n+1}) = \iota \circ \tilde{A}_{n+1} \circ \alpha \circ \beta = 0$ (where ι denotes the isomorphism between $\ker\{\mod \hbar^{n+1} : \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})\}$ and $\underline{\mathrm{End}}(V)$); so $\Omega(\tilde{A}_{n+1})$ is a 2-cocycle. The element $\Omega(\tilde{A}_{n+1})$ depends on the lifting $\tilde{A}_{n+1}$. But, if we chose another lifting $\tilde{A}'_{n+1}$, we have

$$\tilde{A}'_{n+1} = \tilde{A}_{n+1} + \hbar^{n+1} A_{(n+1)} \mod \hbar^{n+2}$$

Since both $\tilde{A}'_{n+1}$ and $\tilde{A}_{n+1}$ are homomorphisms $\mod \hbar^{n+2}$, $A_{(n+1)}$ is a derivation of $\mathcal{F}(X)$ with values in $\underline{\mathrm{End}}(V)$. We have

$$\begin{aligned}
\Omega(\tilde{A}'_{n+1}) &= \iota\alpha^* \tilde{A}_{n+1} + \iota\left(\hbar^{n+1}\alpha^* A_{(n+1)}\right) \\
&= \Omega(\tilde{A}_{n+1}) + \alpha^* A_{(n+1)} \\
&= \Omega(\tilde{A}_{n+1}) + \delta^1 A_{(n+1)}
\end{aligned}$$

i.e. $\Omega(\tilde{A}'_{n+1})$ and $\Omega(\tilde{A}_{n+1})$ differ by a coboundary. This means that the cohomology class $[\Omega(\tilde{A}_{n+1})]$ is independent of the particular lifting chosen. We write

$$[\Omega(A_n)]$$

to denote it. Assume now that A_n has a lifting $A_{n+1} : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. Then $\pi^* A_{n+1}$ is a lifting of A_n from $\mathcal{F}(X)$ to $\underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. We have that $\alpha^*(\pi^* A_{n+1}) = 0$, so $[\Omega(A_n)] = [\Omega(\pi^* A_{n+1})] = 0$. This means that, if A_n lifts to a $(n+1)^{th}$ deformation, then the *obstruction class* $[\Omega(A_n)]$ vanishes. The converse is true: assume $[\Omega(A_n)] = 0$. Then there exists a lifting $\tilde{A}_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$ and a derivation $A_{(n+1)} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)$ such that

$$\Omega(\tilde{A}_{n+1}) = \delta^1 A_{(n+1)}$$

We can rewrite this equation as

$$\iota\alpha^*\tilde{A}_{n+1} = \iota\hbar^{n+1}\delta^1 A_{(n+1)} = \iota\hbar^{n+1}\alpha^* A_{(n+1)}$$

that is

$$\iota\left(\alpha^*(\tilde{A}_{n+1} - \hbar^* A_{(n+1)})\right) = 0$$

Since ι is an isomorphism, this gives

$$\alpha^*(\tilde{A}_{n+1} - \hbar^* A_{(n+1)}) = 0$$

Let now $A_{n+1} := \tilde{A}_{n+1} - \hbar^{n+1}A_{(n+1)} \mod \hbar^{n+2}$. Since $\tilde{A}_{n+1}$ is an homomorphism $\mod \hbar^{n+2}$ and $A_{(n+1)}$ is a derivation, A_{n+1} is an homomorphism

$$A_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$$

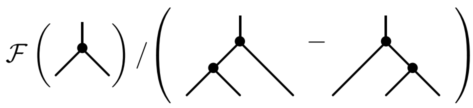
and lifts A_n. Moreover $\alpha^* A_{n+1} = 0$, so A_{n+1} defines an homomorphism $A_{n+1} : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$ which lifts A_n. This way we have proved that the cohomology class $[\Omega(A_n)]$ is a complete obstruction to the lifting of A_n to an $(n+1)^{th}$ order deformation, i.e. the answer to the first of our two questions is given by an element in the secon cohomology group

$$\mathcal{T}^2(\mathcal{O}, \underline{\mathrm{End}}(V))$$

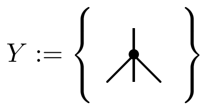
Assume now that a lift A_{n+1} exists. Then all the others are of the form $A_{n+1} + \hbar^{n+1}A_{(n+1)} \mod \hbar^{n+2}$, with $A_{(n+1)}$ derivation of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$ (the proof of this statement is completely analogue to the one given for the case $n = 0$). This means that, when the answer to the first question is yes, the answer to the second is: $\mathcal{T}^1(\mathcal{O}, \underline{\mathrm{End}}(V))$.

6 Deformations of associative algebras.

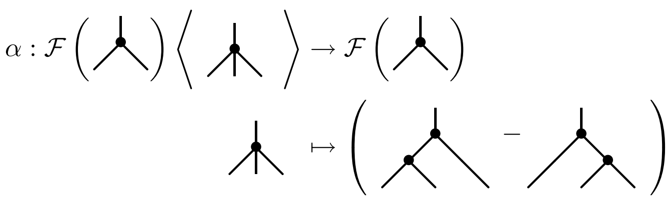
We must find a resolution (at least the first three terms) of the operad **Assoc** governing associative algebras. We already know that **Assoc** can be presented as the quotient



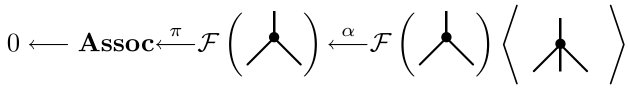
So the kernel of the canonical projection can be generated by a single element, which is an element with 3 inputs and one output. Thus, if we set



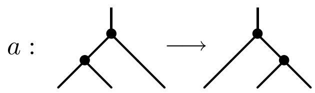
and define $\alpha : \mathcal{F}(X)\langle Y\rangle \to \mathcal{F}(X)$ as



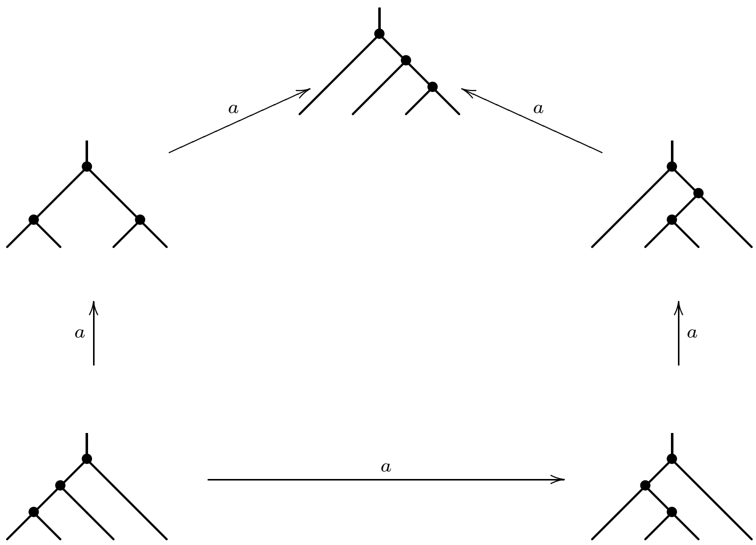
the sequence



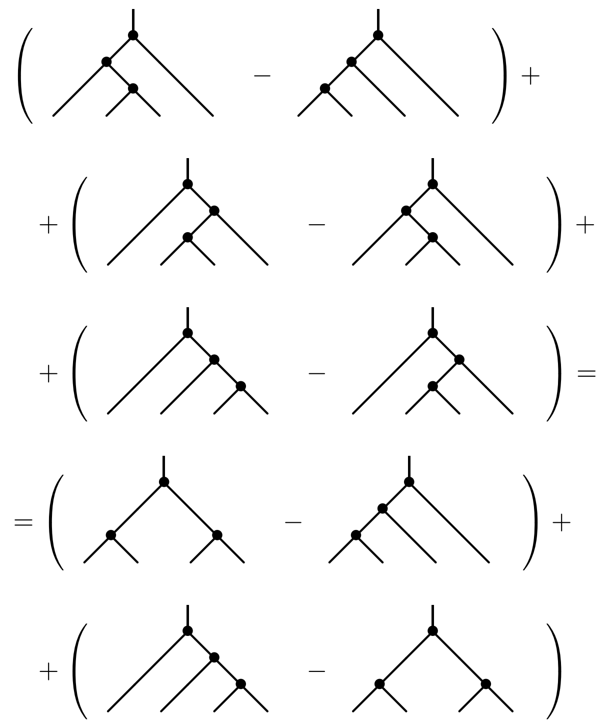
is a resolution of **Assoc**. We need one more term. To describe the kernel of α we use McLane coherence theorem. If we denote by a the *move*



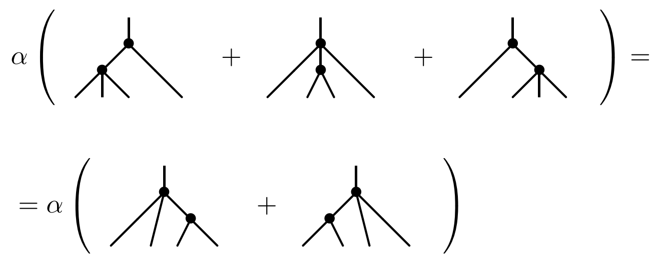
then we have the well known pentagon relation:



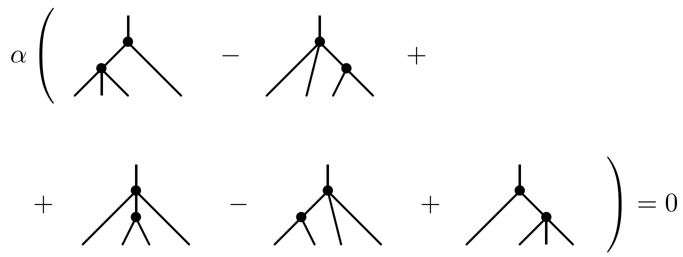
This gives:



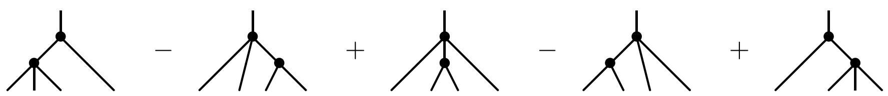
that can be rewritten as:



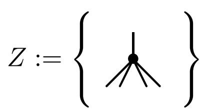
that is



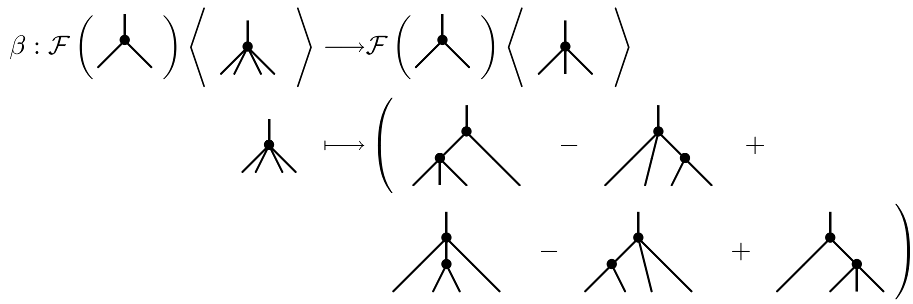
The McLane coherence theorem tells us that the pentagon generates all possible relations among the a moves. This implies that the element



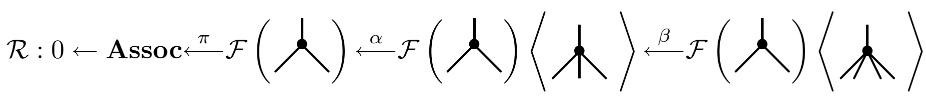
generates $\ker \alpha$. Thus, if we set

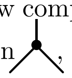


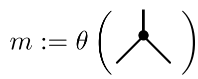
and define



then



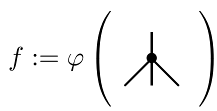
is a resolution of **Assoc**. We can now compute the complex $\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V))$ for an associative algebra $\{V, \mu\}$. Since $F(X)$ is free on , to assign a derivation θ, we just have to assign the value



Then θ is completely determined by the Leibnitz rule. Since m can be arbitrarly chosen in $\mathrm{Hom}(V^{\otimes 2}, V)$, we have a canonical isomorphism

$$\mathcal{E}^1(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 2}, V)$$

In the same way, $\mathcal{F}(X)$-module maps $\varphi : F(X)\langle Y\rangle \to \underline{\mathrm{End}}(V)$ are completely detrmined by the image



which is an element of $\mathrm{Hom}(V^{\otimes 3}, V)$ (the element  is forced to act as μ since φ is an $\mathcal{F}(X)$-modules map, and $\mathcal{F}(X)$ acts on $\underline{\mathrm{End}}(V)$ via π). Thus we have a canonical isomorphism

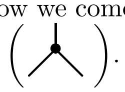
$$\mathcal{E}^2(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 3}, V)$$

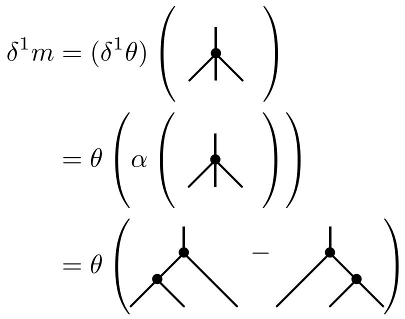
A completely analogue argument gives

$$\mathcal{E}^3(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 4}, V)$$

This means that, at the level of cochain spaces, we have

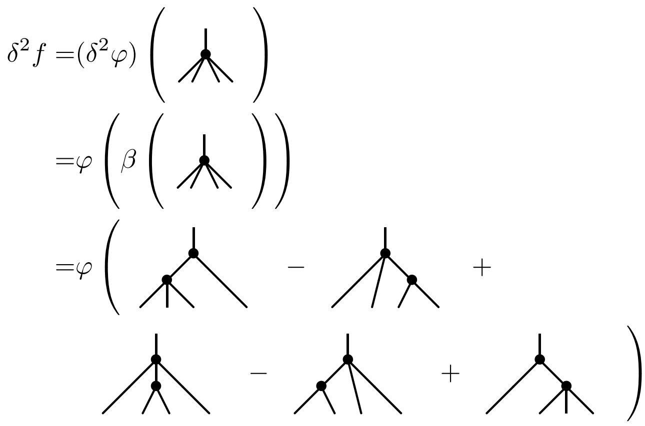
$$\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)) = C^*(V, V)[1]$$

where the latter is the cochain space of the well known Hochshild complex. Now we come to the differentials. Let $m \in \mathrm{Hom}(V^{\otimes 2}, V)$. Then there exists θ such that $m = \theta$ . We have



$$= \mu(m \otimes Id) - \mu(Id \otimes Id) + m(\mu \otimes Id) - m(Id \otimes \mu)$$
$$= d^1_{Hoch}\ m$$

For $f \in \mathrm{Hom}(V^{\otimes 3}, V)$, results



$$= \mu(f \otimes Id) - f(Id^{\otimes 2} \otimes \mu) + f(Id \otimes \mu \otimes Id) - f(\mu \otimes Id^{\otimes 2}) + \mu(Id \otimes f)$$
$$= d^2_{Hoch}\ f$$

Then also the differentials coincide with the Hochshild ones, and we have proved that the complex $\{\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)), \delta^*\}$ is canonically isomorphic to the Hochshild complex:

$$\{\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)), \delta^*\} = \{C^*(V, V)[1], d^*_{Hoch}\}$$

In particular we have found again the well known result that the obstruction to the lifting for an infinitesimal deformation of an associative algebra (V, μ) lies in $H^2_{Hoch}(V, V)$ and that, when this obstruction vanishes, the liftings are parametrized by $H^1_{Hoch}(V, V)$.

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