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Contents

Sia $N > 1$ un intero fissato e $\mathbb{K}$ un campo algebricamente chiuso. Denotiamo con $S = \mathbb{K}[x_0, \ldots, x_N]$ l'anello delle coordinate omogenee su $\mathbb{P}^N$ e con $\mathfrak{m} = (x_0, \ldots, x_N)$ il suo ideale irrilevante.

Sia $Mod_S^{\mathbb{Z}}$ la categoria degli S-moduli $\mathbb{Z}$-graduati. Sono oggetti di tale categoria i moduli $S(h)$, dove $S(h)^i = S^{h+i}$ ed i moduli liberi $\bigoplus_i S(h_i)$.

1. La corrispondenza di Serre

Se $\mathcal{F}$ è un fascio quasi coerente su $\mathbb{P}^N$ denotiamo con:

(1) $\Gamma_*(\mathcal{F}) = \bigoplus_t \Gamma(\mathcal{F}(t)) \in Mod_S^{\mathbb{Z}}$,
(2) $H_*^i(\mathcal{F}) = \bigoplus_t H^i(\mathcal{F}(t)) \in Mod_S^{\mathbb{Z}}$.

Notiamo che $\Gamma_*(\mathcal{O}(h)) = H_*^0(\mathcal{O}(h)) = S(h)$ e $H_*^i(\mathcal{O}(h)) = 0$ per ogni i tale che $0 < i < N$.

Denotiamo con $Qcoh$ la categoria dei fasci quasi-coerenti su $\mathbb{P}^n$; il funtore di fascificazione

$$\widetilde{\cdot}\colon Mod_S \to Qcoh$$

è definito nel modo seguente: Se M è un S-modulo graduato e $p \in \mathbb{P}^n$, la spiga è ugualle alla localizzazione omogenea

$$\widetilde{M}_p = \{\frac{m}{f} \mid \deg(m) = \deg(f),\ f \in S,\ f(p) \neq 0\}.$$

Theorem 1.1. *Il funtore $\widetilde{\cdot}$ è esatto e per ogni fascio quasicoerente $\mathcal{F}$ vale $\mathcal{F} = \widetilde{\Gamma_*(\mathcal{F})}$.*

Proof. Vedi Hartshorne [1]. □

Per ogni $M \in Mod_S$ esiste un morfismo naturale $M \xrightarrow{\alpha} \Gamma_*(\widetilde{M})$ che in generale non è né iniettivo né surgettivo.

Definition 1.2. Per ogni $M \in Mod_S$ denotiamo

$$\Gamma_{\mathfrak{m}}(M) = \{a \in M \mid \mathfrak{m}^k a = 0 \text{ per } k >> 0\}.$$

Lemma 1.3. *Per ogni $M \in Mod_S$ esiste una successione esatta*

$$0 \to \Gamma_{\mathfrak{m}}(M) \to M \xrightarrow{\alpha} \Gamma_*(\widetilde{M}).$$

Inoltre, se M è iniettivo allora α è surgettiva.

Proof. Vale $\alpha(m) = 0$ se e solo se per ogni $p \in \mathbb{P}^n$ esiste $f_p \in S$ omogeneo tale che $f_p(p) \neq 0$ e $fm = 0$. Per il teorema degli zeri l'ideale generato dagli f_p contiene una potenza di $\mathfrak{m}$.

Supponiamo adesso M iniettivo e indichiamo con $\mathcal{M}$ il fascificato affine di M. Per quanto dimostrato in [1] $\mathcal{M}$ è un fascio fiacco su $\mathbb{A}^{n+1}$. D'altra parte ogni sezione di $\widetilde{M}(h)$ può essere interpretato come una sezione di $\mathcal{M}$ su $\mathbb{A}^{n+1} - \{0\}$ omogenea di grado h e tale sezione può essere estesa a tutto $\mathbb{A}^{n+1}$. □

I funtori $\Gamma_\mathfrak{m}$ e Γ_* sono esatti a sinistra. La categoria Mod_S possiede abbastanza iniettivi e proiettivi; esistono quindi i funtori derivati destri

$$R\Gamma_\mathfrak{m} \colon D^+(Mod) \to D^+(Mod), \qquad R\Gamma_* \colon D^+(Qcoh) \to D^+(Mod).$$

Poniamo per definizione $H^i_\mathfrak{m}(C^*) = R^i\Gamma_\mathfrak{m}(C^*)$.

Il Lemma implica che esiste un triangolo di funtori dalla categoria $D^+(Mod)$ in sé:

$$R\Gamma_\mathfrak{m} \to \mathrm{Id} \to R\Gamma_* \circ \widetilde{\cdot} \to R\Gamma_\mathfrak{m}[1]$$

Corollary 1.4. *Sia $P \in Mod_S$ libero. Allora:*

1. $H^i_\mathfrak{m}(P) = 0$ *per ogni* $i \neq N+1$.
2. $\{a \in H^{N+1}_\mathfrak{m}(P) \mid \mathfrak{m}a = 0\} \neq 0$.

Proof. Basta considerare il caso $P = S(h)$. Siccome $S(h) \to \Gamma_*(\mathcal{O}(h))$ è un isomorfismo, dal triangolo in $D^+(Mod)$

$$R\Gamma_\mathfrak{m}(S(h)) \to S(h) \to R\Gamma_*(\mathcal{O}(h)) \to R\Gamma_\mathfrak{m}[1](S(h))$$

si deduce che $H^i_*(\mathcal{O}(h)) \cong H^{i+1}_\mathfrak{m}(S(h))$. In particolare $H^{N+1}_\mathfrak{m}(S(h)) = \oplus_t H^N(\mathcal{O}(h+t))$ e quindi $\mathbb{K} = H^N(\mathcal{O}(-N-1)) \subset H^{N+1}_\mathfrak{m}(S(h))$ è annullato da $\mathfrak{m}$. □

2. Risoluzioni minimali

Chiameremo *complesso minimale* un complesso finito $P^* \in C^b(Mod_S)$ tale che ogni P^i è libero di rango finito e tale che per ogni i vale $\delta(P^i) \subset \mathfrak{m}P^{i+1}$.

Theorem 2.1 (Auslander-Buchsbaum). *Ogni $M \in Mod_S^\mathbb{Z}$ finitamente generato possiede una risuluzione*

$$0 \to P^{-N-1} \to \cdots \to P^0 \to M \to 0$$

dove P^* è un complesso minimale.

Proof. Vedi Matsumura. □

Lemma 2.2. *Sia*

$$P^* : \quad 0 \to P^m \to P^{m+1} \to \cdots \to P^{n-1} \to P^n \to 0$$

un complesso minimale. Allora:

1. $n = \max\{i \mid H^i(P^*) \neq 0\}$,
2. $m + N + 1 = \min\{i \mid H^i_\mathfrak{m}(P^*) \neq 0\}$.

Proof. Il punto (1) segue immediatamente da Nakayama. Per il punto 2 osserviamo che $H^i_\mathfrak{m}(P^j) = 0$ per $i \neq N+1$, mentre, se $P^j \neq 0$ allora esiste un sottomodulo $V \subset H^{N+1}_\mathfrak{m}(P^j) \neq 0$ tale che $V \neq 0$ e $\mathfrak{m}V = 0$. Segue dunque dalla successione spettrale del complesso doppio che $H^i_\mathfrak{m}(P*) = 0$ per ogni $i < m+N+1$ e che $H^{m+N+1}_\mathfrak{m}(P*)$ è uguale al nucleo del morfismo

$$H^{N+1}_\mathfrak{m}(P^m) \to H^{N+1}_\mathfrak{m}(P^{m+1})$$

che, se $P_m \neq 0$, abbiamo visto essere non banale. □

Lemma 2.3. *Sia $f \colon P \to Q$ un morfismo di S-moduli graduati finitamente generati con Q libero. Se*

$$\overline{f} \colon \frac{P}{\mathfrak{m}P} \to \frac{Q}{\mathfrak{m}Q}$$

è un isomorfismo, allora anche f è un isomorfismo.

Proof. Dato che $f(P)+\mathfrak{m}Q = Q$ segue da Nakayama che f è surgettivo. Sia $K = \ker f$. Siccome Q è libero esiste uno splitting $P = Q \oplus K$ e il fatto che $\overline{f}$ sia iniettivo implica che $K = \mathfrak{m}K$ e quindi $K = 0$. □

Lemma 2.4. *Sia $f\colon P^* \to Q^*$ un quasi-isomorfismo tra due complessi minimali. Allora f è un isomorfismo.*

Proof. Siccome i complessi sono limitati e formati da oggetti proiettivi, ogni quasi-isomorfismo è un'equivalenza omotopica. Il morfismo

$$\overline{f}\colon \frac{P^*}{\mathfrak{m}P^*} \to \frac{Q^*}{\mathfrak{m}Q^*}$$

è un'equivalenza omotopica di complessi a differenziale nullo, quindi è un isomorfismo e dunque anche f è un isomorfismo. □

Definition 2.5. Diremo che un morfismo $f\colon A \to B$ di S-moduli graduati finitamente generati è bigettivo modulo $\mathfrak{m}$ se

$$\overline{f}\colon \frac{A}{\mathfrak{m}A} \to \frac{B}{\mathfrak{m}B}$$

Abbiamo visto che se un morfismo è bigettivo modulo $\mathfrak{m}$ allora è surgettivo.

Lemma 2.6. *Per ogni S-modulo graduato finitamente generato A esiste un modulo libero P finitamente generato ed un morfismo $P \to A$ bigettivo modulo $\mathfrak{m}$.*

Proof. Basta scegliere P tale che $\frac{A}{\mathfrak{m}A} = \frac{P}{\mathfrak{m}P}$ e sollevare la proiezione $P \to \frac{A}{\mathfrak{m}A}$. □

Proposition 2.7. *Sia C^* un complesso limitato tale che $H^i(C^*)$ è finitamente generato per ogni i. Allora esiste un complesso minimale P^* ed un quasi-isomorfismo $P^* \to C^*$.*

Proof. Supponiamo di aver costruito, per un indice fissato n, un diagramma commutativo

$$\begin{array}{ccccccc}
 & & P^n & \xrightarrow{\delta^n} & P^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\
\cdots \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots
\end{array}$$

con le seguenti proprietà:

(1) Il complesso $P^n \to P^{n+1} \to \cdots$ è minimale.
(2) $f_i\colon H^i(P^*) \to H^i(C^*)$ è un isomorfismo per ogni $i > n$.
(3) Esiste una successione esatta

$$0 \to S \to \ker \delta^n \xrightarrow{f_n} H^n(C^*) \to 0$$

tale che $S \subset \mathfrak{m}P^n$.

Vogliamo trovare un modulo P^{n-1} e due morfismi δ^{n-1}, f_{n-1} in modo tale che le precedenti proprietà continuino a valere (con $n-1$ al posto di n).

Per prima cosa prendiamo un modulo libero finitamente generato Q ed un morfismo $\alpha\colon Q \to S$ bigettivo modulo $\mathfrak{m}$ (in particolare $\ker \alpha \subset \mathfrak{m}Q$). Siccome $f_n\alpha(Q) \subset d^{n-1}(C^{n-1})$ possiamo trovare un sollevamento

$$\begin{array}{ccc}
Q & \xrightarrow{\alpha} & P^n \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_n} \\
C^{n-1} & \xrightarrow{d^{n-1}} & C^n
\end{array}$$

Prendiamo adesso un'altro modulo libero finitamente generato R ed un morfismo

$$\beta\colon R \to \frac{H^{n-1}(C^*)}{g(\ker \alpha)}$$

bigettivo modulo $\mathfrak{m}$ (in particolare $\ker\beta \subset \mathfrak{m}R$). Scegliamo infine un morfismo $\gamma\colon R \to Z^{n-1}(C^*)$ che solleva β e definiamo

$$P^{n-1} = R \oplus Q, \qquad \delta^{n-1}(r,q) = \alpha(q), \qquad f_{n-1}(r,q) = \gamma(r) + g(q).$$

Si ha $\ker\delta^{n-1} = R \oplus \ker\alpha$, per costruzione l'applicazione $\overline{f_{n-1}}\colon \ker\delta^{n-1} \to H^{n-1}(C^*)$ è suriettiva e e se $\overline{f_{n-1}}(r,q) = 0$ allora $\beta(r) = 0$ e quindi

$$\ker\overline{f_{n-1}} \subset \ker\beta \oplus \ker\alpha \subset \mathfrak{m}R \oplus \mathfrak{m}Q = \mathfrak{m}P^{n-1}.$$

Chiaramente, se $C^i = 0$ per ogni $i > n$, si ha $P^i = 0$ per ogni $i > n$, mentre se $C^i = 0$ per ogni $i < m$, allora per Ausl.-Buch. si ha $P^i = 0$ per ogni $i < m - N - 1$.

□

3. Il teorema di Horrocks

Theorem 3.1. *Sia $\mathcal{E}$ un fibrato di rango finito su $\mathbb{P}^N$ e k un intero. Se $H^i_*(\mathcal{E}) = 0$ per ogni i tale che $k \le i < N$ (per $k = N$ tale condizione è automaticamente soddisfatta), allora esiste una successione esatta di fasci*

$$0 \to \mathcal{E} \to \mathcal{P}^0 \to \cdots \to \mathcal{P}^{k-1} \to 0$$

dove ogni $\mathcal{P}^i$ è somma diretta di line bundles. (Il viceversa è quasi ovvio).

Per uso futuro è utile premettere alla dimostrazione alcuni lemmi.

Lemma 3.2. *Sia $\mathcal{E}$ un fibrato di rango finito su $\mathbb{P}^N$. Allora il modulo $H^i_*(\mathcal{E})$ ha lunghezza finita per ogni i tale che $0 < i < N$.*

Proof. Vanishing e dualità di Serre. □

Example 3.3. Supponiamo $N > 2$ e sia $\mathbb{K}$ il fascio grattacielo sul punto $x_1 = x_2 = \cdots = x_N = 0$ e si consideri la successione esatta

$$0 \to \mathcal{F} \to \mathcal{O}(-1)^N \xrightarrow{(x_1,\ldots,x_N)} \mathcal{O} \to \mathbb{K} \to 0.$$

Allora, $H^2(\mathcal{F}(t)) = \mathbb{K}$ per ogni $t < 0$ e quindi $\mathcal{F}$ non è localmente libero.

Lemma 3.4. *Sia $\mathcal{E}$ un fibrato di rango finito su $\mathbb{P}^N$ e $C^* \in C^b(Mod)$ un rappresentante di $\tau_{<N}R\Gamma_*(\mathcal{E})$. Allora $\mathcal{E}$ è quasi-isomorfo a $\widetilde{C^*}$, ossia $\mathcal{E}$ è isomorfo a $\widetilde{C^*}$ nella categoria derivata $D^b(Qcoh)$.*

Proof. Prendendo una risoluzione iniettiva di $\mathcal{E}$ e poi troncando, possiamo scegliere come rappresentante di $\tau_{<N}R\Gamma_*(\mathcal{E})$ un complesso del tipo

$$C^*: \qquad 0 \to C^0 \xrightarrow{\delta^0} C^1 \to \cdots \to C^{N-1} \to 0.$$

Siccome $H^i(C^*) = H^i_*(\mathcal{E})$ ha lunghezza finita per ogni i con $0 < i < N$, la successione di fasci

$$\tilde{C^0} \xrightarrow{\delta^0} \tilde{C^1} \to \cdots \to \tilde{C^{N-1}} \to 0.$$

Inoltre Γ_* è esatto a sinistra e quindi $\Gamma_*(\mathcal{E}) = \ker\delta^0$ e quindi $\mathcal{E} = \widetilde{\Gamma_*(\mathcal{E})} \to \widetilde{C^*}$ è un quasi-isomorfismo. □

Lemma 3.5. *Sia $\mathcal{E}$ un fibrato di rango finito su $\mathbb{P}^N$. Allora la risoluzione minimale di $\tau_{<N}R\Gamma_*(\mathcal{E})$ ha la forma*

$$P^*: \qquad 0 \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0$$

e vale $\mathcal{E} \simeq \widetilde{H^0(P^*)}$.

Proof. Sia $P^* \in C^b(Mod)$ la risoluzione minimale di $\tau_{<N} R\Gamma_*(\mathcal{E})$. Siccome $H^i(P^*) = 0$ per ogni $i \geq N$ ne segue che $P^i = 0$ per ogni $i \geq N$.

Per il lemma precedente $\tilde{P^*}$ è quasiismorfo a $\mathcal{E}$ e quindi il triangolo

$$R\Gamma_{\mathfrak{m}}(P^*) \to P^* \to R\Gamma_*(\widetilde{P^*}) \to R\Gamma_{\mathfrak{m}}(P^*)[1]$$

è isomorfo a

$$R\Gamma_{\mathfrak{m}}(P^*) \to P^* \xrightarrow{\beta} R\Gamma_*(\mathcal{E}) \to R\Gamma_{\mathfrak{m}}(P^*)[1]$$

Per costruzione $\beta \colon H^i(P^*) \to H^i_*(\mathcal{E})$ è un isomorfismo per $i < N$ e $P^i = 0$ per ogni $i \geq N$ e quindi $H^i_{\mathfrak{m}}(P^*) = 0$ per ogni $i \leq N$. Siccome P^* è minimale ne segue che $P^i = 0$ per ogni $i < 0$.

Siccome $H^0(P^*) = \Gamma_*(\mathcal{E})$ segue dalla corrispondenza di Serre che

$$\widetilde{H^0(P^*)} = \widetilde{\Gamma_*(\mathcal{E})} = \mathcal{E}.$$

□

Lemma 3.6. *Sia*

$$P^* : \qquad 0 \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0$$

un complesso minimale tale che $H^i(P^*)$ *ha lunghezza finita per ogni* $i > 0$. *Allora* $\mathcal{E} = \widetilde{H^0(P^*)}$ *è un fibrato e* P^* *è quasiisomorfo a* $\tau_{<N} R\Gamma_*(\mathcal{E})$.

Proof. Il complesso di fasci localmemte liberi

$$0 \to \mathcal{E} \to \tilde{P^0} \to \tilde{P^1} \to \cdots \to \widetilde{P^{N-1}} \to 0$$

è esatto e quindi $\mathcal{E}$ è localmente libero. Inoltre $\mathcal{E}$ è quasi-isomorfo a $\tilde{P^*}$ ed abbiamo un triangolo

$$R\Gamma_{\mathfrak{m}}(P^*) \to P^* \xrightarrow{\beta} R\Gamma_*(\mathcal{E}) \to R\Gamma_{\mathfrak{m}}(P^*)[1]$$

da cui segue che $H^i(P^*) \to H^i_*(\mathcal{E})$ è un isomorfismo per ogni $i < N$ e quindi

$$\tau_{<N}(P^*) = P^* \xrightarrow{\tau_{<N}\beta} \tau_{<N} R\Gamma_*(\mathcal{E})$$

è un quasi-isomorfismo.

□

Dimostrazione del Teorema. Sia $\mathcal{E}$ un fibrato e supponiamo che $H^i_*(\mathcal{E}) = 0$ per $i = k, k+1, \ldots N-1$. Allora la risoluzione minimale di $\tau_{<N} R\Gamma_*(\mathcal{E})$ ha la forma

$$P^* : \qquad 0 \to P^0 \to P^1 \to \cdots \to P^{k-1} \to 0$$

ed esiste una successione esatta di fasci

$$0 \to \mathcal{E} \to \tilde{P^0} \to \tilde{P^1} \to \cdots \to \widetilde{P^{k-1}} \to 0$$

□

Denotiamo con $\mathfrak{B}$ la categoria dei fibrati di rango finito su $\mathbb{P}^n$ e con $\mathfrak{Z} \subset K^b(Mod_S)$ la sottocategoria piena dei complessi minimali del tipo

$$P^* : \qquad 0 \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0$$

con $H^i(P^*)$ di lunghezza finita. I risultati precedenti mostrano che tali categorie sono equivalenti, con l'equivalenza data dai funtori:

$$\mathfrak{B} \to \mathfrak{Z}, \qquad \mathcal{E} \mapsto \text{risoluzione minimale di } \tau_{<N} R\Gamma_*(\mathcal{E}),$$

$$\mathfrak{Z} \to \mathfrak{B}, \qquad P^* \mapsto \widetilde{H^0(P^*)}.$$

Abbiamo già visto che tali funtori sono uno l'inverso dell'altro sulle classi di isomorfismo degli oggetti. Siccome ogni morfismo di fibrati si solleva ad un morfismo dei rispettivi $\tau_{<N} R\Gamma_*$, la composizione $\mathfrak{B} \to \mathfrak{Z} \to \mathfrak{B}$ è l'identità sui morfismi.

Gli elementi di $\mathfrak{Z}$ sono complessi finiti di proiettivi, e quindi il morfismo naturale $\mathfrak{Z} \to D^b(Mod_S)$ è un'equivalenza sulla sottocategoria piena dell'immagine.

Ogni morfismo $P^* \to Q^*$ in $\mathfrak{Z}$ induce in $D^b(Mod_S)$ un diagramma commutativo

$$\begin{array}{ccc} P^* & \to & \tau_{<N}R\Gamma_*(\widetilde{P^*}) \\ \big\downarrow & & \big\downarrow \\ Q^* & \to & \tau_{<N}R\Gamma_*(\widetilde{Q^*}) \end{array}$$

con le frecce orizzontali quasiisomorfismi. Ne segue che la composizione $\mathfrak{Z} \to \mathfrak{B} \to \mathfrak{Z}$ è iniettiva sui morfismi. Questo prova che tali categorie sono equivalenti.

4. La corrispondenza di Horroks

Lemma 4.1. *Sia $P \in Mod_S$ libero di rango finito e sia $M \subset P$ un sottomodulo. Se M non è contenuto in $\mathfrak{m}P$, allora esiste un isomorfismo*

$$\phi\colon P \to Q \oplus S(h)$$

con Q libero e $N \subset Q$ tale che $\phi(M) = N \oplus S(h)$.

Proof. Sia $P = \bigoplus_{i=1}^n S(h_i)$, per ipotesi esiste un indice i tale che la restrizione ad M della proiezione $P \to S(h_i)$ è surgettiva ed esiste usa sezione $s\colon S(h_i) \to M$. A meno di scambiare gli indici supponiamo $i = 1$; basta definire $Q = \bigoplus_{i=2}^n S(h_i)$ e ϕ come l'inverso dell'isomorfismo

$$Q \oplus S(h_1) \to P, \qquad (a, b) \mapsto a + s(b).$$

□

Definition 4.2. Diremo che un fibrato è *minimale* se non ha line bundles come addendi diretti.

Corollary 4.3. *Sia $\mathcal{E}$ un fibrato minimale e sia*

$$P^* : \qquad 0 \to P^0 \xrightarrow{\delta^0} P^1 \to \cdots \to P^{N-1} \to 0$$

la risoluzione minimale di $\tau_{<N}R\Gamma_*(\mathcal{E})$. Allora $\ker\delta^0 \subset \mathfrak{m}P^0$.

Proof. Se $\ker\delta^0$ non è contenuto in $\mathfrak{m}P^0$, possiamo scrivere $P^0 = Q^0 \oplus S(h)$ con $S(h) \subset \ker\delta^0$ e quindi

$$\mathcal{E} = \widetilde{H^0(P^*)} = \mathcal{O}(h) \oplus \widetilde{H^0(Q^*)}$$

dove

$$Q^* : \qquad 0 \to Q^0 \to P^1 \to \cdots \to P^{N-1} \to 0$$

□

Theorem 4.4. *Sia $\mathcal{E}$ fibrato e sia Q^* la risoluzione minimale di $\tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{E})$. Allora vale $\mathcal{E} \cong \widetilde{Z^0(Q^*)}$ se e solo se $\mathcal{E}$ è minimale.*

Proof. Supponiamo $\mathcal{E}$ minimale e sia

$$P^* : \qquad 0 \to P^0 \xrightarrow{\delta^0} P^1 \to \cdots \to P^{N-1} \to 0$$

la risoluzione minimale di $\tau_{<N}R\Gamma_*(\mathcal{E})$. Allora

$$0 \to \frac{P^0}{\ker\delta^0} \xrightarrow{\delta^0} P^1 \to \cdots \to P^{N-1} \to 0$$

è quasi-isomorfo a $\tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{E})$. Sia

$$\cdots \to Q^{-2} \to Q^{-1} \to \ker\delta^0 \to 0$$

una risoluzione minimale di $\ker\delta^0$; siccome $\ker\delta^0 \subset \mathfrak{m}P^0$ il complesso

$$Q^* : \qquad \cdots Q^{-2} \to Q^{-1} \to P^0 \xrightarrow{\delta^0} P^1 \to \cdots \to P^{N-1} \to 0$$

è la risoluzione minimale di $\tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{E})$ e $Z^0(Q^*) = \ker\delta^0$.

Supponiamo adesso che $\mathcal{E}$ non sia minimale, possiamo scrivere $\mathcal{E} = \mathcal{H} \oplus \mathcal{G}$ con $\mathcal{H}$ minimale e $\mathcal{G}$ somma diretta di line bundles. Siccome $\tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{G}) = 0$ si ha $\tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{E}) = \tau_{>0}\tau_{<N}R\Gamma_*(\mathcal{H})$ e quindi $\mathcal{H} \cong \widetilde{Z^0(Q^*)}$.

□

References

[1] Hartshorne: *Algebraic geometry.*
[2] Hartshorne: *Residues and duality.*
[3] Walter: *Pfaffian subschemes.*

PFAFFIAN SUBSCHEMES, SECTION 2

CHARLES H. WALTER

1. The Horrocks Correspondence

In this section we give a modern description of the Horrocks correspondence of [Ho] using derived categories. We include a full proof of the principal properties of the correspondence from this point of view (Theorem 1.4). Taking advantage of the greater flexibility of the derived category viewpoint, we develop a technique which allows us to transfer a prescribed portion of the cohomology of $\mathcal{O}_X$ to prescribed parts of a locally free resolution (Proposition 1.8).

Notation and Generalities. We first recall some generalities about complexes. If $\mathfrak{A}$ is an abelian category, let $C(\mathfrak{A})$ (resp. $K(\mathfrak{A})$, $D(\mathfrak{A})$) denote the category (resp. homotopy category, derived category) of complexes of objects of $\mathfrak{A}$, and let $C^b(\mathfrak{A})$, $C^-(\mathfrak{A})$, $C^+(\mathfrak{A})$, etc., denote the corresponding complexes of bounded (resp. bounded above, bounded below) complexes of objects of $\mathfrak{A}$. When speaking of complexes, we will generally reserve the word "isomorphism" for isomorphisms in $C(\mathfrak{A})$. Isomorphisms in $K(\mathfrak{A})$ (resp. $D(\mathfrak{A})$) are referred to as homotopy equivalences (resp. quasi-isomorphisms).

If r is an integer, then any complex C^* of objects of $\mathfrak{A}$ has two *canonical truncations* at r and a *naive truncation*:

$$\begin{array}{llllllll} \tau_{\le r}(C^*): & \cdots \to C^{r-2} \to C^{r-1} \to & \ker(\delta^r) & \to & 0 & \to & 0 & \to \cdots, \\ \tau_{>r}(C^*): & \cdots \to \quad 0 \quad \to \quad 0 & \to C^r/\ker(\delta^r) & \to & C^{r+1} & \to & C^{r+2} & \to \cdots. \\ \sigma_{\ge r}(C^*): & \cdots \to \quad 0 \quad \to \quad 0 & \to \quad C^r & \to & C^{r+1} & \to & C^{r+2} & \to \cdots. \end{array}$$

All the truncations are functorial in $C(\mathfrak{A})$. The canonical truncations are functorial in $K(\mathfrak{A})$ and $D(\mathfrak{A})$ as well. We will often find it more convenient to write $\tau_{<r+1}$ instead of $\tau_{\le r}$.

Suppose now that $\mathfrak{A}$ has enough projectives. Every bounded above complex C^* of objects in $\mathfrak{A}$ admits a *projective resolution*, i.e. a quasi-isomorphism $P^* \to C^*$ with P^* a complex of projectives ([Ha] Proposition I.4.6). The projective resolution of a complex is unique up to homotopy equivalence. If C^* and E^* are bounded above complexes of objects in $\mathfrak{A}$, and if $P^* \to C^*$ is a projective resolution of C^*, then there is a natural isomorphism $\mathrm{Hom}_{D^-(\mathfrak{A})}(C^*, E^*) \cong \mathrm{Hom}_{K^-(\mathfrak{A})}(P^*, E^*)$. In particular if $\mathfrak{P}$ denotes the full subcategory of projective objects of $\mathfrak{A}$, then the natural functor $K^-(\mathfrak{P}) \to \mathfrak{D}^-(\mathfrak{A})$ is an equivalence of categories ([Ha] Proposition I.4.7). This can be refined to the following statement:

Lemma 1.1. *Let $\mathfrak{A}$ be an abelian category with enough projectives, and let $\mathfrak{P}$ be the full subcategory of projective objects of $\mathfrak{A}$. Suppose $A \subset D^-(\mathfrak{A})$ and $P \subset K^-(\mathfrak{P})$ are full subcategories such that $\mathrm{ob}(P) \subset \mathrm{ob}(A)$ and every object of A has a projective resolution belonging to P. Then the natural functor $P \to A$ is an equivalence of categories.*

Let $S = k[X_0, \dots, X_N]$ be the homogeneous coordinate ring of $\mathbb{P}^N$, and let $\mathfrak{m} = (X_0, \dots, X_N)$ be its irrelevant ideal. Let $\mathrm{Mod}_{S,\mathrm{gr}}$ be the category of graded S-modules. Then $\mathrm{Mod}_{S,\mathrm{gr}}$ has enough projectives, namely the free modules. We will call a complex P^* of projectives in $\mathrm{Mod}_{S,\mathrm{gr}}$ a *minimal* if all its objects P^i are free of finite rank and its differential δ^* satisfies $\delta^i(P^i) \subset \mathfrak{m}P^{i+1}$ for all i. If C^* is a bounded above complex of objects in $\mathrm{Mod}_{S,\mathrm{gr}}$ whose cohomology

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modules $H^i(C^*)$ are all finitely generated, then C^* has a *minimal projective resolution*, i.e. a projective resolution by a minimal complex of projectives. The next lemma, which is a well known consequence of Nakayama's lemma, says that minimal projective resolutions are unique up to isomorphism and not merely up to homotopy equivalence:

Lemma 1.2. *Let $\phi : P^* \to Q^*$ be a homotopy equivalence between* minimal *complexes of free graded S-modules of finite rank. Then ϕ is an isomorphism.*

Let $\mathrm{Mod}_{\mathcal{O}}$ be the category of sheaves of $\mathcal{O}_{\mathbb{P}^N}$-modules. For $\mathcal{E}$ a sheaf of $\mathcal{O}_{\mathbb{P}^N}$-modules, let $\Gamma_*(\mathcal{E}) = \bigoplus_{t\in\mathbb{Z}} \Gamma(\mathcal{E}(t))$. Then Γ_* defines a left exact functor from $\mathrm{Mod}_{\mathcal{O}}$ to $\mathrm{Mod}_{S,\mathrm{gr}}$. It has a right derived functor $\mathbf{R}\Gamma_* : D^b(\mathrm{Mod}_{\mathcal{O}}) \to D^b(\mathrm{Mod}_{S,\mathrm{gr}})$ whose cohomology functors we denote $H^i_*(\mathcal{E}) = \bigoplus_{t\in\mathbb{Z}} H^i(\mathcal{E}(t))$. The functor Γ_* has an exact left adjoint $\tilde{}$, the functor of associated sheaves.

Let $\Gamma_{\mathfrak{m}} : \mathrm{Mod}_{S,\mathrm{gr}} \to \mathrm{Mod}_{S,\mathrm{gr}}$ be the functor associating to a graded S-module M the maximal submodule $\Gamma_{\mathfrak{m}}(M) \subset M$ supported at the origin 0 of $\mathbb{A}^{N+1}$. This functor is also left exact and has a right derived functor $\mathbf{R}\Gamma_{\mathfrak{m}} : D^b(\mathrm{Mod}_{S,\mathrm{gr}}) \to D^b(\mathrm{Mod}_{S,\mathrm{gr}})$. Its cohomology functors are denoted $H^i_{\mathfrak{m}}$.

Lemma 1.3. *Let P^* be a bounded complex of free graded S-modules of finite rank where $S = k[X_0, \dots, X_N]$. If P^* is minimal, then*

$$\max\{i \mid P^i \neq 0\} = \max\{i \mid H^i(P^*) \neq 0\},$$
$$\min\{i \mid P^i \neq 0\} = \min\{i \mid H^i_{\mathfrak{m}}(P^*) \neq 0\} - N - 1.$$

Proof. The assertion about maxima is a simple and well-known application of the minimality condition and Nakayama's Lemma. The assertion about minima, which is essentially the Auslander-Buchsbaum theorem, reduces to the assertion about maxima by Serre duality. □

The Horrocks Correspondence. We now begin to describe the components of the Horrocks correspondence. Let $\mathfrak{B}$ be the full subcategory of $\mathrm{Mod}_{\mathcal{O}}$ of locally free sheaves of finite rank, and let $\mathfrak{Z}$ denote the full category of $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes C^* such that $H^i(C^*)$ is of finite length for $0 < i < N$ and $H^i(C^*)$ vanishes for all other i.

The Horrocks correspondence consists of a functor $\zeta : \mathfrak{B} \to \mathfrak{Z}$ and a map $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ in the opposite direction. The functor ζ is simply $\tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*$. For $\mathcal{E}$ a vector bundle on $\mathbb{P}^N$, the cohomology of $\zeta(\mathcal{E})$ is of course:

$$H^i(\zeta(\mathcal{E})) = \begin{cases} H^i_*(\mathcal{E}) & \text{if } 0 < i < N, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{E}$ is locally free of finite rank, $H^i_*(\mathcal{E})$ is of finite length for $0 < i < N$. So $\zeta(\mathcal{E}) \in \mathrm{ob}(\mathfrak{Z})$.

We now define $\mathcal{H}$. Any $C^* \in \mathrm{ob}(\mathfrak{Z})$ has a minimal projective resolution $P^* \to C^*$. We define $\mathcal{H}(C^*)$ to be the kernel of the differential $\widetilde{\delta}^0 : \widetilde{P}^0 \to \widetilde{P}^1$. Then $\mathcal{H}(C^*)$ is a vector bundle because it fits into an exact complex of vector bundles

$$\cdots \to 0 \to \mathcal{H}(C^*) \to \widetilde{P}^0 \to \widetilde{P}^1 \to \cdots \to \widetilde{P}^{N-1} \to 0 \to \cdots . \tag{1}$$

Note that $\mathcal{H}(C^*)$ is well-defined up to isomorphism because the minimal projective resolution P^* of C^* is unique up to isomorphism because of Lemma 1.2. However, $\mathcal{H}$ is not a functor.

The principal results of Horrocks' paper [Ho] can be described in the following way:

Theorem 1.4 (Horrocks). *Let $\mathfrak{B}$ be the category of locally free sheaves of finite rank on $\mathbb{P}^N$, and let $\mathfrak{Z}$ be the full subcategory of $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes C^* such that $H^i(C^*)$ is of finite length if $0 < i < N$, and $H^i(C^*) = 0$ for all other i. Let $\zeta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \to \mathfrak{Z}$, and let $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ be the map defined as in (1) above.*

(*a*) *If $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$, then $\mathcal{E} \cong \mathcal{H}\zeta(\mathcal{E}) \oplus \bigoplus_i \mathcal{O}_{\mathbb{P}^N}(n_i)$ for some integers n_i.*

(*b*) *If $C^* \in \mathrm{ob}(\mathfrak{Z})$, then $\zeta\mathcal{H}(C^*) \simeq C^*$.*

(*c*) *If $\mathcal{E}, \mathcal{F} \in \mathrm{ob}(\mathfrak{B})$, then* $\mathrm{Hom}_{\mathfrak{Z}}(\zeta(\mathcal{E}), \zeta(\mathcal{F})) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{F}) / \mathrm{Hom}_{\Phi}(\mathcal{E}, \mathcal{F})$ *where* $\mathrm{Hom}_{\Phi}(\mathcal{E}, \mathcal{F})$ *is the set of all morphisms which factor through a direct sum of line bundles.*

The theorem may be read as saying the following. Call two vector bundles $\mathcal{E}$ and $\mathcal{F}$ *stably equivalent* if there exist sets of integers $\{n_i\}$ and $\{m_j\}$ such that $\mathcal{E} \oplus \bigoplus_i \mathcal{O}_{\mathbb{P}^N}(n_i) \cong \mathcal{F} \oplus \bigoplus_j \mathcal{O}_{\mathbb{P}^N}(m_j)$. Then the theorem says that ζ and $\mathcal{H}$ induce a one-to-one correspondence between stable equivalence classes of vector bundles on $\mathbb{P}^N$ and quasi-isomorphism classes of complexes in $\mathfrak{Z}$.

For Horrocks' proof of the theorem, see [Ho] Lemma 7.1 and Theorem 7.2 and the discussion between them. However, Horrocks' definition of the category $\mathfrak{Z}$ and the functor ζ are different from ours, and demonstrating the equivalence of the definitions is somewhat tedious. So instead of referring the reader to Horrocks' paper, we give a new proof. The first step is the following lemma:

Lemma 1.5. (a) *Suppose*

$$P^* : \qquad \cdots \longrightarrow 0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^{N-1} \longrightarrow 0 \longrightarrow \cdots$$

is a complex of free graded S-modules of finite rank such that $H^i(P^*)$ is a module of finite length for $0 < i < N$. Let $\mathcal{E} = H^0(P^*)^\sim$. Then P^* is quasi-isomorphic to $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$.

(b) *Conversely, if $\mathcal{E}$ is a vector bundle on $\mathbb{P}^N$, then the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is of the above form.*

Proof. (a) Note that the complex $\widetilde{P}^*$ of coherent sheaves on $\mathbb{P}^N$ has vanishing cohomology in degrees different from 0. So it is quasi-isomorphic to $H^0(\widetilde{P}^*) = \mathcal{E}$. Hence the triangle of functors of [W] Proposition 1.1:

$$\mathbf{R}\Gamma_{\mathfrak{m}} \longrightarrow \mathrm{Id} \longrightarrow \mathbf{R}\Gamma_* \circ ^\sim \longrightarrow \mathbf{R}\Gamma_{\mathfrak{m}}[1],$$

when applied to P^*, yields a triangle

$$\mathbf{R}\Gamma_{\mathfrak{m}}(P^*) \longrightarrow P^* \xrightarrow{\beta} \mathbf{R}\Gamma_*(\mathcal{E}) \longrightarrow \mathbf{R}\Gamma_{\mathfrak{m}}(P^*)[1]. \qquad (2)$$

By Lemma 1.3, we have $H^i_{\mathfrak{m}}(P^*) = 0$ for $i \le N$. So $H^i(\beta) : H^i(P^*) \longrightarrow H^i_*(\mathcal{E})$ is an isomorphism for $i < N$. Therefore β induces a quasi-isomorphism of P^* onto $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$.

(b) Conversely, if $\mathcal{E}$ is a vector bundle on $\mathbb{P}^N$, then $H^i_*(\mathcal{E})$ is finitely generated for $i < N$. Hence $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ has a minimal projective resolution P^*. For $0 < i < N$ the module $H^i(P^*) = H^i_*(\mathcal{E})$ is of finite length because $\mathcal{E}$ is locally free. By construction $H^i(P^*) = H^i(\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})) = 0$ for $i \ge N$. So we have $P^i = 0$ for $i \ge N$ by Lemma 1.3. Looking again at the triangle (2), we see by the construction of P^* that $H^i(\beta)$ is an isomorphism for $i < N$ and an injection for $i = N$. So $H^i_{\mathfrak{m}}(P^*) = 0$ for $i \le N$. So by Lemma 1.3 we see that $P^i = 0$ for $i \le -1$. Thus P^* has the form asserted by the lemma. □

We now wish to functorialize the previous lemma. Let $B \subset K^b(\mathrm{Mod}_{S,\mathrm{gr}})$ be the full subcategory of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^{N-1} \longrightarrow 0 \longrightarrow \cdots \qquad (3)$$

such that the P^i are free of finite rank for all i, the modules $H^i(P^*)$ are of finite length for $0 < i < N$ and the differentials satisfy $\delta^i(P^i) \subset \mathfrak{m}P^{i+1}$ for all i. For any vector bundle $\mathcal{E}$ on $\mathbb{P}^N$ we now define $P^*(\mathcal{E})$ as the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$. By Lemma 1.5, $P^*(\mathcal{E})$ is always an object of B.

Lemma 1.6. *The functor $P^* : \mathfrak{B} \longrightarrow B$ which associates to an $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$ the minimal projective resolution of $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is an equivalence of categories with inverse given by $C^* \mapsto H^0(C^*)^\sim$.*

Proof. Since the functor $\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \longrightarrow D^-(\mathrm{Mod}_{S,\mathrm{gr}})$ has a left inverse $H^0(-)^\sim$, it induces an equivalence between $\mathfrak{B}$ and the full subcategory $A \subset D^-(\mathrm{Mod}_{S,\mathrm{gr}})$ of complexes quasi-isomorphic to complexes in the image of $\tau_{<N}\mathbf{R}\Gamma_*$. But by Lemma 1.5, the full subcategory $B \subset K^-(\mathrm{Mod}_{S,\mathrm{gr}})$ has the properties that $\mathrm{ob}(B) \subset \mathrm{ob}(A)$ and that the minimal projective resolution of every object of A belongs to B. Hence the natural functor $B \longrightarrow A$ is also an equivalence of categories by Lemma 1.1. Since P^* is exactly the composition of the equivalence $\tau_{<N}\mathbf{R}\Gamma_* : \mathfrak{B} \longrightarrow A$ with the inverse of the equivalence $B \longrightarrow A$, it is an equivalence. The inverse of P^* remains the same as that of $\tau_{<N}\mathbf{R}\Gamma_*$, namely $H^0(-)^\sim$. □

Now the graded module associated to a vector bundle $\mathcal{E}$ on $\mathbb{P}^N$ has a minimal projective resolution:

$$0 \to Q^{-(N-1)} \to \cdots \to Q^{-1} \to Q^0 \to \Gamma_*(\mathcal{E})$$

For any $\mathcal{E}$ we now define the following complexes in addition to the $P^*(\mathcal{E})$ defined above. First we set:

$$Q^*(\mathcal{E}) : \qquad \cdots \to 0 \to Q^{-(N-1)} \to \cdots \to Q^{-1} \to Q^0 \to 0 \to \cdots .$$

We then let $R^*(\mathcal{E})$ be the natural concatenation of $Q^*(\mathcal{E})$ with $P^*(\mathcal{E})$ induced by the composition $Q^0 \twoheadrightarrow \Gamma_*(\mathcal{E}) \hookrightarrow P^0$:

$$R^*(\mathcal{E}) : \qquad \cdots \to 0 \to Q^{-(N-1)} \to \cdots \to Q^0 \to P^0 \to \cdots \to P^{N-1} \to 0 \to \cdots$$

Thus $R^i(\mathcal{E}) = P^i(\mathcal{E})$ for $i \geq 0$, and $R^i(\mathcal{E}) = Q^{i+1}(\mathcal{E})$ for $i < 0$. Note that although the projective complexes $P^*(\mathcal{E})$ and $Q^*(\mathcal{E})$ are minimal, $R^*(\mathcal{E})$ may not be minimal, because there may be a direct factor of $Q^0(\mathcal{E})$ which is mapped isomorphically onto a direct factor of $P^0(\mathcal{E})$. However, one may write $R^*(\mathcal{E})$ as the direct sum of a minimal complex of projectives $R^*_{\min}(\mathcal{E})$

$$R^*_{\min}(\mathcal{E}) : \qquad \cdots \to Q^{-2} \to Q^{-1} \to Q^0_{\min} \to P^0_{\min} \to P^1 \to P^2 \to \cdots$$

and of an exact complex of projectives

$$\cdots \to 0 \to L \xrightarrow{\mathrm{Id}} L \to 0 \to \cdots . \qquad (4)$$

The complexes $Q^*(\mathcal{E})$, $R^*(\mathcal{E})$, and $R^*_{\min}(\mathcal{E})$ are all functorial (in the homotopy category) in $\mathcal{E}$. Moreover, we may use the identification between the categories $\mathfrak{B}$ and B to define complexes $Q^*(P^*)$, $R^*(P^*)$, and $R^*_{\min}(P^*)$ for P^* in B. Namely, $Q^*(P^*)$ is the minimal projective resolution of $H^0(P^*)$, $R^*(P^*)$ is the concatenation of $Q^*(P^*)$ with P^*, etc.

We now define a homotopy category of complexes of type $R^*_{\min}$. More formally, let $Z \subset K^b(\mathrm{Mod}_{S,\mathrm{gr}})$ be the full subcategory of minimal complexes of projective modules of finite rank of the form

$$\cdots \to 0 \to R^{-N} \to \cdots \to R^{-1} \to R^0 \to \cdots \to R^{N-1} \to 0 \to \cdots \qquad (5)$$

such that the cohomology modules $H^i(R^*)$ are of finite length for $0 < i < N$ and vanish for all other i.

We need one more lemma before proving Theorem 1.4.

Lemma 1.7. *The natural functor $Z \to \mathfrak{Z}$ is an equivalence of categories.*

Proof. Let R^* be the minimal projective resolution of an object C^* of $\mathfrak{Z}$. Since $H^i(R^*) = H^i(C^*) = 0$ for $i \geq N$, we have $R^i = 0$ for $i \geq N$ by Lemma 1.3. Moreover, all the $H^i(C^*)$ are of finite length, so $H^i_{\mathfrak{m}}(C^*) = H^i(C^*)$ for all i. In particular, $H^i_{\mathfrak{m}}(R^*) = H^i_{\mathfrak{m}}(C^*) = 0$ for $i \leq 0$. So $R^i = 0$ for $i \leq -N-1$ by Lemma 1.3. Thus the minimal projective resolution of any object of $\mathfrak{Z}$ is in Z. The lemma now follows from Lemma 1.1. □

Proof of Theorem 1.4. Lemma 1.6 permits us to identify a vector bundle $\mathcal{E}$ with the complex $P^*(\mathcal{E})$ of B. Since $P^*(\mathcal{E})$ is already quasi-isomorphic to $\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$, the complex $\zeta(\mathcal{E}) = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{E})$ is quasi-isomorphic to the complex

$$\cdots \to 0 \to \Gamma_*(\mathcal{E}) \to P^0 \to P^1 \to \cdots \to P^{N-1} \to 0 \to \cdots$$

and hence to the complexes $R^*(\mathcal{E})$ and $R^*_{\min}(\mathcal{E})$. Hence the object $\zeta(\mathcal{E})$ in $\mathfrak{Z}$ is quasi-isomorphic to the object $R^*_{\min}(\mathcal{E})$ of Z. Hence after identifying $\mathfrak{B}$ and $\mathfrak{Z}$ with B and Z by Lemmas 1.6 and 1.7, the functor ζ may be identified with the functor from B to Z which associates to any complex P^* in B the corresponding complex $R^*_{\min}$ as described earlier.

Similarly, given any object C^* of $\mathfrak{Z}$ with minimal projective resolution R^*, the definitions say that $P^*(\mathcal{H}(C^*)) = \sigma_{\geq 0}(R^*)$, the naive truncation. Thus the map $\mathcal{H} : \mathrm{ob}(\mathfrak{Z}) \to \mathrm{ob}(\mathfrak{B})$ may be identified with $\sigma_{\geq 0} : \mathrm{ob}(Z) \to \mathrm{ob}(B)$. Note that since all objects of Z and B are minimal complexes of projective modules, homotopy equivalence classes of objects of Z and B coincide with isomorphism classes. Hence the map $\sigma_{\geq 0} : \mathrm{ob}(Z) \to \mathrm{ob}(B)$ preserves homotopy equivalence.

Since Z and B are subcategories of the homotopy category, this means that $\sigma_{\geq 0}$ is well-defined on objects of Z. However, $\sigma_{\geq 0}$ and hence $\mathcal{H}$ are not well-defined on morphisms of Z.

(a) The above identifications now say if $\mathcal{E} \in \mathrm{ob}(\mathfrak{B})$, then $\mathcal{H}\zeta(\mathcal{E})$ is the object of $\mathfrak{B}$ corresponding to the complex $\sigma_{\geq 0}(R^*_{\min}(\mathcal{E}))$:

$$\sigma_{\geq 0}(R^*_{\min}(\mathcal{E})): \qquad \cdots \to 0 \to P^0_{\min} \xrightarrow{\mu} P^1 \to \cdots \to P^{N-1} \to 0 \to \cdots .$$

By Lemma 1.6, the sheaf $\mathcal{H}\zeta(\mathcal{E})$ is $\ker(\mu)^\sim$. So $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E}) \oplus \widetilde{\mathcal{L}}$ where L is the projective module of (4). Since $\widetilde{L}$ is now a direct sum of line bundles, (a) follows.

(b) If C^* is an object of $\mathfrak{Z}$ with minimal projective resolution R^* in Z of the form (5), then the above computations identify $\mathcal{H}(C^*)$ in $\mathfrak{B}$ with $P^*(\mathcal{H}(C^*)) = \sigma_{\geq 0}(R^*)$ in B. Thus $\zeta\mathcal{H}(\mathcal{C}^*)$ becomes identified with $R^*_{\min}(\mathcal{H}(\mathcal{C}^*))$ which is just R^* again. Since R^* is quasi-isomorphic to C^*, we have $\zeta\mathcal{H}(\mathcal{C}^*) \simeq \mathcal{C}^*$ as desired.

(c) After identifying $\mathfrak{B}$ with B and $\mathfrak{Z}$ with Z, assertion (c) becomes the statement: For any pair of objects E^* and F^* in B, the natural map

$$\mathrm{Hom}_B(E^*, F^*) \to \mathrm{Hom}_Z(R^*_{\min}(E^*), R^*_{\min}(F^*)) \tag{6}$$

is surjective and its kernel is the subspace of morphisms which factor through an object of B of the form

$$\cdots \to 0 \to L \to 0 \to \cdots \tag{7}$$

with L a free graded S-module of finite rank appearing in degree 0.

We first prove surjectivity. Suppose $\phi \in \mathrm{Hom}_Z(R^*_{\min}(E^*), R^*_{\min}(F^*))$. Since Z is a homotopy category, ϕ is actually a homotopy equivalence class of maps in $C(\mathrm{Mod}_{S,\mathrm{gr}})$. So we may choose a chain map f in the class ϕ. Then f may be extended to a chain map $\overline{f} : R^*(E^*) \to R^*(F^*)$ by defining it to be 0 on the exact factor of the type (4). Then $\sigma_{\geq 0}\overline{f}$ maps E^* to F^*, and its homotopy class in B has image ϕ in Z. This proves surjectivity.

We now compute the kernel of (6). First if $\alpha \in \mathrm{Hom}_B(E^*, F^*)$ factors through a complex L^* of the form (7), then $R^*_{\min}(\alpha)$ factors through $R^*_{\min}(L^*) = 0$ and so vanishes. So the kernel of (6) contains all morphisms which factor through complexes of the form (7).

Conversely, suppose α is in the kernel of (6). Since α is a morphism in B, it is a homotopy class of chain maps from which we may choose a member β. We may complete β to a chain map $\rho : R^*(E^*) \to R^*(F^*)$.

$$\begin{array}{ccccccccccccccc}
R^*(E^*) & & \cdots \to 0 \to & \overline{E}^{-N} & \to \cdots \to & \overline{E}^{-1} & \to & E^0 & \to \cdots \to & E^{N-1} & \to & 0 & \to \cdots \\
\downarrow{\scriptstyle \rho} & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & \downarrow & \\
R^*(F^*) & & \cdots \to 0 \to & \overline{F}^{-N} & \to \cdots \to & \overline{F}^{-1} & \to & F^0 & \to \cdots \to & F^{N-1} & \to & 0 & \to \cdots
\end{array}$$

The homotopy class of ρ is the image of α under R^* and so must vanish by hypothesis. (Note that R^* and $R^*_{\min}$ are homotopy equivalent.) Thus ρ is homotopic to 0. Thus if we write δ^i for the differentials of $R^*(E^*)$, and ϵ^i for the differentials of $R^*(F^*)$, then there is a chain homotopy $h = (h^i)$ such that $\rho^i = h^{i+1}\delta^i + \epsilon^{i-1}h^i$ for all i. Now restrict h to a chain homotopy $\widehat{h} = (\widehat{h}^i)$ with $\widehat{h}^i : E^i \to F^{i-1}$ defined by defined by $\widehat{h}^i = h^i$ for all $i \geq 1$, and $\widehat{h}^i = 0$ for all $i \leq 0$. Then β is homotopic to a morphism whose components are

$$\beta^i - (\widehat{h}^{i+1}\delta^i + \epsilon^{i-1}\widehat{h}^i) = \begin{cases} \rho^i - (h^{i+1}\delta^i + \epsilon^{i-1}h^i) = 0 & \text{if } i \geq 1, \\ \rho^0 - h^1\delta^0 = \epsilon^{-1}h^0 & \text{if } i = 0, \\ 0 & \text{if } i \leq -1. \end{cases}$$

Hence the homotopy class α of β factors through the complex

$$\cdots \to 0 \to \overline{F}^{-1} \to 0 \to \cdots$$

of type (7). So the kernel of (6) is as asserted. This completes the proof of the theorem. □

We will use three further results concerning the Horrocks correspondence. The first will permit us to use the Horrocks correspondence to constuct locally free resolutions of coherent sheaves.

Proposition 1.8. *Let $\mathcal{Q}$ be a quasi-coherent sheaf on $\mathbb{P}^N$, let $C^* \in \mathrm{ob}(\mathfrak{Z})$, and let $\beta : C^* \to \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\mathcal{Q})$ be a morphism in $D^b(\mathrm{Mod}_{S,\mathrm{gr}})$. Then there exists a morphism of quasi-coherent sheaves $\widetilde{\beta} : \mathcal{H}(C^*) \to \mathcal{Q}$ such that $\beta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$. In particular, the induced morphisms $H^i_*(\mathcal{H}(C^*)) \to H^i_*(\mathcal{Q})$ are the same as $H^i(\beta)$ for $1 \le i \le N-1$.*

Proof. Let R^* be a minimal projective resolution of C^*, and let $\mathcal{I}^*$ be an injective resolution of $\mathcal{Q}$. Then β may be identified with an actual chain map

$$\begin{array}{ccccccccccccc} \cdots \to R^{-2} \to & R^{-1} & \to & R^0 & \xrightarrow{\lambda} & R^1 & \to \cdots \to & R^{N-2} & \to & R^{N-1} & \to 0 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to 0 & \to \Gamma_*(\mathcal{Q}) & \to & \Gamma_*(\mathcal{I}^0) & \xrightarrow{\mu} & \Gamma_*(\mathcal{I}^1) & \to \cdots \to & \Gamma_*(\mathcal{I}^{N-2}) & \to & \ker(\delta^{N-1}) & \to 0 \end{array}$$

Thus β induces a morphism $\widetilde{\beta}$ from $\mathcal{H}(C^*) = \ker(\lambda)^\sim$ to $\mathcal{Q} = \ker(\mu)^\sim$.

We now need to calculate $\mathbf{R}\Gamma_*(\widetilde{\beta})$. Consider the complex

$$P^* : \qquad \cdots \to 0 \to R^0 \to R^1 \to \cdots \to R^{N-1} \to 0 \to \cdots .$$

The previous diagram induces a new commutative diagram

$$\begin{array}{ccc} \widetilde{P}^*: & & \cdots \to 0 \to \widetilde{R}^0 \to \widetilde{R}^1 \to \cdots \to \widetilde{R}^{N-2} \to \widetilde{R}^{N-1} \to 0 \to 0 \to \cdots \\ \downarrow \overline{\beta} & & \downarrow \quad \downarrow \quad \downarrow \qquad \downarrow \qquad \downarrow \quad \downarrow \quad \downarrow \\ \mathcal{I}^*: & & \cdots \to 0 \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^{N-2} \to \mathcal{I}^{N-1} \to \mathcal{I}^N \to 0 \to \cdots \end{array}$$

between resolutions of $\mathcal{H}(C^*)$ and $\mathcal{Q}$ extending $\widetilde{\beta}$. Let $\gamma : P^* \to J^*$ be an injective resolution of P^*. Then $\overline{\beta}$ factors through $\widetilde{\gamma}$ as $\widetilde{P}^* \to \widetilde{J}^* \to \mathcal{I}^*$. Applying Γ_* now gives a factorization

$$P^* \to \Gamma_*(\widetilde{J}^*) \to \Gamma_*(\mathcal{I}^*). \tag{8}$$

Now $\overline{\beta}$ is a map between resolutions of $\mathcal{H}(\mathcal{C}^*)$ and $\mathcal{Q}$, respectively, which extends $\widetilde{\beta} : \mathcal{H}(\mathcal{C}^*) \to \mathcal{Q}$, while $\widetilde{\gamma}$ is a quasi-isomorphism. So the map $\widetilde{J}^* \to \mathcal{I}^*$ is a map between injective resolutions of $\mathcal{H}(\mathcal{C}^*)$ and $\mathcal{Q}$ extending $\widetilde{\beta}$. So by definition, the second arrow of (8) is $\mathbf{R}\Gamma_*(\widetilde{\beta}) : \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*)) \to \mathbf{R}\Gamma_*(\mathcal{Q})$. On the other hand, the proof of Lemma 1.5(a) shows that the first arrow of (8) can be identified with the truncation $\tau_{<N}(\mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))) \to \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))$ because it induces isomorphisms $H^i(P^*) \cong H^i(\Gamma_*(\widetilde{J}^*)) = H^i_*(\mathcal{H}(\mathcal{C}^*))$ for $i < N$. Hence $\Gamma_*(\overline{\beta}) : P^* \to \Gamma_*(\mathcal{I}^*)$ can be identified with the composition of the truncation $\tau_{<N}(\mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))) \to \mathbf{R}\Gamma_*(\mathcal{H}(\mathcal{C}^*))$ with $\mathbf{R}\Gamma_*(\widetilde{\beta})$. Thus $\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$ may be identified with the diagram

$$\begin{array}{ccccccccccc} \cdots \to 0 \to & R^0 & \to & R^1 & \to \cdots \to & R^{N-2} & \to & R^{N-1} & \to 0 \to \cdots \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \cdots \to 0 \to & \Gamma_*(\mathcal{I}^0) & \to & \Gamma_*(\mathcal{I}^1) & \to \cdots \to & \Gamma_*(\mathcal{I}^{N-2}) & \to & \ker(\delta^{N-1}) & \to 0 \to \cdots \end{array}$$

induced by β. Truncating on the left, we reach a diagram equivalent to the first diagram of the proof of the proposition. So $\beta = \tau_{>0}\tau_{<N}\mathbf{R}\Gamma_*(\widetilde{\beta})$. □

We now need two homological criteria for maps of vector bundles to be isomorphisms.

Lemma 1.9. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^N$ with neither containing a line bundle as a direct factor. If $\alpha : \mathcal{E} \to \mathcal{F}$ is a map such that $H^i_*(\alpha) : H^i_*(\mathcal{E}) \to H^i_*(\mathcal{F})$ is an isomorphism for $0 < i < N$, then α is an isomorphism.*

Proof. We use the notation of the proof of Theorem 1.4. Let $E^* = P^*(\mathcal{E})$ and $F^* = P^*(\mathcal{F})$, and let $\overline{\alpha} : E^* \to F^*$ be the map induced by α. The hypothesis $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E})$ implies that E^* is homotopy equivalent to $\sigma_{\ge 0}R^*_{\min}(E^*)$, or equivalently that $R^*(E^*)$ is a minimal complex of projectives. Similarly, $R^*(F^*)$ is a minimal complex of projectives. The hypothesis on α

implies that $\zeta(\alpha) : \zeta(\mathcal{E}) \to \zeta(\mathcal{F})$ is a quasi-isomorphism. This in turn translates into $R^*_{\min}(\overline{\alpha})$ being a homotopy equivalence. But because of the earlier hypotheses, this means that $R^*(\overline{\alpha}) : R^*(E^*) \to R^*(F^*)$ is a homotopy equivalence between the minimal complexes of projectives. Hence by Lemma 1.2 $R^*(\overline{\alpha})$ is actually an isomorphism of complexes. So its naive truncation $\sigma_{\geq 0} R^*(\overline{\alpha}) = \overline{\alpha}$ is also an isomorphism. Therefore α is an isomorphism. □

We will also need a slight generalization of the previous lemma.

Lemma 1.10. *Let $\mathcal{E} = \mathcal{H}\zeta(\mathcal{E}) \oplus \bigoplus \mathcal{O}_{\mathbb{P}^N}(n_i)$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^N$, and let $\mathcal{Q}$ be a coherent sheaf on $\mathbb{P}^N$. Suppose that there exist morphisms $\alpha : \mathcal{E} \to \mathcal{F}$ and $\beta : \mathcal{F} \to \mathcal{Q}$ such that*
(i) $H^i_*(\alpha) : H^i_*(\mathcal{E}) \to H^i_*(\mathcal{F})$ is an isomorphism for $0 < i < N$,
(ii) $\beta\alpha$ takes the generators of the factors $S(n_i)$ of $\Gamma_*(\mathcal{E})$ onto a minimal set of generators of the module $\overline{Q} := \Gamma_*(\mathcal{Q})/\beta\alpha(\Gamma_*(\mathcal{H}\zeta(\mathcal{E})))$,
(iii) $\mathcal{E}$ and $\mathcal{F}$ have the same rank.
Then α is an isomorphism.

Proof. Write $\mathcal{F} = \mathcal{H}\zeta(\mathcal{F}) \oplus \bigoplus \mathcal{O}_{\mathbb{P}^N}(m_j)$. The splittings of $\mathcal{E}$ and of $\mathcal{F}$ into direct factors are not canonical. But choosing such splittings gives an injection $\mathcal{H}\zeta(\mathcal{E}) \hookrightarrow \mathcal{E}$ and a projection $\mathcal{F} \twoheadrightarrow \mathcal{H}\zeta(\mathcal{F})$. Then the composition

$$\overline{\alpha} : \quad \mathcal{H}\zeta(\mathcal{E}) \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \to \mathcal{H}\zeta(\mathcal{F})$$

is, like α, an isomorphism on H^i_* for $0 < i < N$. So $\overline{\alpha}$ is an isomorphism by Lemma 1.9. Hence by identifying $\mathcal{H}\zeta(\mathcal{F})$ with $\alpha(\mathcal{H}\zeta(\mathcal{E})) \subset \mathcal{F}$, we see that α induces a morphism of diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}\zeta(\mathcal{E}) & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus \mathcal{O}_{\mathbb{P}^N}(n_i) & \longrightarrow & 0 \\ & & \| & & \downarrow \alpha & & \downarrow \alpha_1 & & \\ 0 & \longrightarrow & \mathcal{H}\zeta(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \bigoplus \mathcal{O}_{\mathbb{P}^N}(m_j) & \longrightarrow & 0 \end{array} \tag{9}$$

The morphisms α and β therefore induce maps

$$\bigoplus S(n_i) \xrightarrow{\Gamma_*(\alpha_1)} \bigoplus S(m_j) \xrightarrow{\overline{\beta}} \overline{Q} = \Gamma_*(\mathcal{Q})/\beta\alpha(\Gamma_*(\mathcal{H}\zeta(\mathcal{E}))).$$

The composition is a surjection corresponding to a minimal set of generators of $\overline{Q}$ by hypothesis (ii). Hence the righthand map $\overline{\beta}$ must be a surjection corresponding to a set of generators of $\overline{Q}$. However, the two free modules have the same rank by hypothesis (iii). Hence $\overline{\beta}$ also corresponds to a minimal set of generators, and $\Gamma_*(\alpha_1)$ must be an isomorphism. So returning to diagram (9), α_1 and hence α are isomorphisms. □

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