DG (co)algebras, DG Lie algebras and L_∞ algebras

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1 DG Algebras, DG Coalgebras and DG Lie Algebras

Definition 1.1 *A* graded complex over the field k *is a graded k-vector space $C_* = \bigoplus_{i \in \mathbf{Z}} C_i$, together with a differential d_C of degree $+1$ (i.e. $d_C(C_i) \subset C_{i+1}$). A morphism from the graded complex (C_*, d_C) to the graded complex (D_*, d_D) (over k) is a homogeneous k-linear map $\phi : C_* \to D_*$, such that $d_D\phi = \phi d_C$. The category of graded complexes over k is indicated with $\mathcal{C}(k)$.*

We first recall some operations on graded vector spaces and graded complexes. The base field k is assumed to be fixed unless otherwise stated.

Definition 1.2 *1) Given two graded complexes of vector spaces $V = (V_*, d_V)$ and $W = (W_*, d_W)$, their tensor product (over the base field) is defined as follows:*

$$(V \otimes W)_r = \bigoplus_{p+q = r} V_p \otimes W_q$$

and the differential is expressed as the sum of its graded components as:

$$For\ x \in V_p, y \in W_q,\ d_{V\otimes W}(x \otimes y) = d_V(x) \otimes y + (-1)^p x \otimes d_W(y)$$

2) Given a graded complex of vector spaces $V = (V_*, d_V)$, the Twisting map

$$\mathbf{T} : V \otimes V \to V \otimes V$$

is defined as the linear extension of the map defined on homogeneous vectors by

$$\mathbf{T}(x \otimes y) = (-1)^{deg(x)deg(y)} y \otimes x$$

Remark 1.3 *As a rule of thumb to "get the signs right" is formulas like the ones above, which appear frequently when dealing with graded objects, one could use the following: "whenever an object of degree r passes on the other side of an object of degree s, a sign $(-1)^{rs}$ must be inserted".*

The proof of the following proposition is elementary, and omitted.

Proposition 1.4 *1) Given three graded complexes V, W, Z there is a canonical natural isomorphism*

$$(V \otimes W) \otimes Z \;\cong\; V \otimes (W \otimes Z)$$

2) For an integer i, let $k[i]$ be the graded differential vector space having zero differential, and with

$$(k[i])_p \;=\; (0) \; if \; p \neq \; -i, \quad (k[i])_{-i} \;=\; k$$

We identify as customary k with $k[0]$. For any object V of $\mathcal{C}(k)$ we then have a canonical isomorphisms

$$V \otimes k \;\cong\; k \otimes V \;\cong\; V$$

For any integer i, and any object V, we write $V[i] \;=\; k[i] \otimes V$.
3) There is a canonical isomorphism for any two objects V, W and any integers i, j

$$V[i] \otimes W[j] \;\cong\; (V \otimes W)\,[i+j]$$

obtained by sending $x \otimes y$, with $x \in V_m$, $y \in W_m$ (and hence $x \otimes y \in (V[i] \otimes W[j])_{n+m-i-j}$) to $(-1)^{jm} x \otimes y$

From now on we will work in the category $\mathcal{C}(k)$ of differential graded k-vector spaces, with differential of degree $+1$, and with maps defined accordingly (in particular they are homogeneous of degree 0). Given an object V, we denote with $\mathbf{Id}_V$ its identity map.

Definition 1.5 *A* differential graded algebra *(briefly DG Algebra) over the field k is an object $A \;=\; (A, \mathbf{d}_A)$ of $\mathcal{C}(k)$, together with a morphism of $\mathcal{C}(k)$*

$$\mu \;:\; A \otimes_k A \;\to\; A$$

such that, using the canonical identification $(A \otimes A) \otimes A \;\cong\; A \otimes (A \otimes A)$,

$$\mu \otimes (\mathbf{Id}_A \otimes \mu) \;=\; \mu \otimes (\mu \otimes \mathbf{Id}_A)$$

A morphism of DG algebras over k from (A, μ_A) to (B, μ_B) is a morphism $\phi \in \mathcal{C}(k)$ from A to B, such that

$$\mu_B \phi \otimes \phi \;=\; \phi \mu_A$$

We indicate with $\mathbf{DGA}(k)$ the category of DG algebras over k

Definition 1.6 $A \in \mathbf{DGA}(k)$
1) is said to be with unit *if it is given a morphism in $\mathcal{C}(k)$*

$$\mathbf{1}_A \;:\; k \;\to\; A$$

such that, using the canonical identifications $A \;\cong\; k \otimes A \;\cong\; A \otimes k$,

$$\mu_A \left(\mathbf{1}_A \otimes \mathbf{Id}_A\right) \;=\; \mu_A \left(\mathbf{Id}_A \otimes \mathbf{1}_A\right) \;=\; \mathbf{Id}_A$$

2) is said to be commutative *if*

$$\mu_A \mathbf{T} \;=\; \mu_A$$

where $\mathbf{T}$ *is the twisting map of the category.*

Example 1.7 *1) Given a manifold M, the algebra of differential forms on it, with the natural grading and the exterior differential, is a DG Algebra.*
2) The cohomology of a manifold M, together with the product induced by the wedge product on forms, the natural grading and zero differential, is a DG Algebra.

Note that the category of vector spaces (without grading or differential) is a full sub category of the category of graded differential vector spaces, and that if we consider the DG algebras (resp. with unit, commutative) which lie in the image of the immersion, we simply recover the notion of an Algebra (resp with unit, commutative).
We now come to the concept of a DG coalgebra, which can be viewed as the dual to the notion of a DG Algebra, in a sense which will be made precise later.

Definition 1.8 *A* differential graded coalgebra *(briefly DG coalgebra) over the field k is an object $C \;=\; (C_*, \mathbf{d}_C)$ of $\mathcal{C}(k)$, together with a morphism of $\mathcal{C}(k)$*

$$\Delta_C \;:\; C \;\rightarrow\; C \otimes C$$

such that, using the canonical identification $(C \otimes C) \otimes C \;\cong\; C \otimes (C \otimes C)$,

$$(\Delta_C \otimes \mathbf{Id}_C) \otimes \Delta_C \;=\; (\mathbf{Id}_C \otimes \Delta_C) \otimes \Delta_C$$

A morphism of DG coalgebras from (C, Δ_C) to (D, Δ_D) is a morphism $\phi \;:\; C \rightarrow D$ of $\mathcal{C}(k)$ such that

$$\phi \otimes \phi \Delta_C \;=\; \Delta_D \phi$$

We indicate with $\mathbf{DGcoA}(k)$ *the category of DG coalgebras over k*

Definition 1.9 $(C, \Delta_C) \in \mathbf{DGcoA}(k)$
1) is said to be with counit *if it is given a morphism in $\mathcal{C}(k)$*

$$\epsilon_c \;:\; C \;\rightarrow\; k$$

such that, using the canonical identifications $C \;\cong\; k \otimes C \;\cong\; C \otimes k$,

$$(\epsilon_c \otimes \mathbf{Id}_C)\, \Delta_C \;=\; (\mathbf{Id}_C \otimes \epsilon_c)\, \Delta_C \;=\; \mathbf{Id}_C$$

2) is said to be cocommutative *if*

$$\mathbf{T}\Delta_C \;=\; \Delta_C$$

where $\mathbf{T}$ *is the twisting map of the category.*

Example 1.10 *1) The algebra of polynomial functions on a vector space V, $k[V^*]$, with zero differential, and Δ given by*

$$\Delta(v) \;=\; v \otimes 1 \;+\; 1 \otimes v, \quad v \in V^*$$

extended to all of $k[V^*]$ by requiring it to be multiplicative with respect to the natural algebra structures on $k[V^*]$ and $k[V^*] \otimes k[V^*]$. Here the space V^* is considered in degree 0, and therefore the whole coalgebra lives in degree zero. We will elaborate this example more extensively in a proposition later.
2) If $A \in \mathbf{DGA}(k)$, with $dim_k(A) < \infty$,then the space $(\bigoplus_q A_q^*[q], d_A^*)$, with Δ induced by the adjoint to the multiplication, gives a DG coalgebra structure (with zero differential).

One is tempted to say that a DG coalgebra (resp. with counit, cocommutative) is simply the dual of a DG algebra (resp. with unit, commutative). There however a fundamental obstruction to making this statement rigorous. Namely, unless the algebra A is finite dimensional over k, $(A \otimes A)^* \;\ncong\; A^* \otimes A^*$, and hence there is no natural way to induce a comultiplication from the multiplication. This problem can be overcome in some interesting situations in the way that is best explained by the second example above. We will also see that in some situations arising from geometry one can replace the DG Algebra at hand with one (called its "minimal model") satisfying the a condition regarding dimensions that is enough to guarantee that $(A \otimes A)^* \;\cong\; A^* \otimes A^*$ (taking * to be the graded dual).

Definition 1.11 *A* Differential graded Lie algebra *(briefly DGLA) over the field k is an object $\mathbf{g} \;=\; (\mathbf{g}, \mathbf{d}_{\mathbf{g}})$ of $\mathcal{C}(k)$, together with a morphism of $\mathcal{C}(k)$*

$$[\;,\;]_{\mathbf{g}} \;:\; \mathbf{g} \otimes \mathbf{g} \;\rightarrow\; \mathbf{g}$$

(called the bracket*) such that,*
1)

$$[\;,\;]_{\mathbf{g}} \mathbf{T} \;=\; -[\;,\;]_{\mathbf{g}}$$

2) Using the canonical identification $(\mathbf{g} \otimes \mathbf{g}) \otimes \mathbf{g} \;\cong\; \mathbf{g} \otimes (\mathbf{g} \otimes \mathbf{g})$,

$$[\;,\;]_{\mathbf{g}}(\mathbf{Id}_{\mathbf{g}} \otimes [\;,\;]_{\mathbf{g}}) \;+\; [\;,\;]_{\mathbf{g}}(\mathbf{Id}_{\mathbf{g}} \otimes [\;,\;]_{\mathbf{g}})(\mathbf{Id}_{\mathbf{g}} \otimes \mathbf{T})(\mathbf{T} \otimes \mathbf{Id}_{\mathbf{g}}) \;+$$
$$+\; [\;,\;]_{\mathbf{g}}(\mathbf{Id}_{\mathbf{g}} \otimes [\;,\;]_{\mathbf{g}})(\mathbf{T} \otimes \mathbf{Id}_{\mathbf{g}})(\mathbf{Id}_{\mathbf{g}} \otimes \mathbf{T}) \;=\; 0 \quad (\textit{the zero map})$$

A morphism of DG Lie algebras from $(\mathbf{g}, [\;,\;]_{\mathbf{g}})$ to $(\mathbf{h}, [\;,\;]_{\mathbf{h}})$ is a morphis $\phi \;:\; \mathbf{g} \rightarrow \mathbf{h}$ of $\mathcal{C}(k)$, such that

$$[\;,\;]_{\mathbf{h}} \phi \otimes \phi \;=\; \phi[\;,\;]_{\mathbf{g}}$$

We indicate with $\mathbf{DGLA}(k)$ the category of DG Lie algebras over k.

Remark 1.12 *If one writes what condition* 2) *above means when applied to a, b, c of degrees α, β, γ respectively, one finds*

$$[a, [b, c]] \;+\; (-1)^{(\alpha\beta + \alpha\gamma)}[b, [c, a]] \;+\; (-1)^{(\beta\gamma + \alpha\gamma)}[c, [b, a]] \;=\; 0$$

which is the usual Jacobi identity when $\alpha = \beta = \gamma = 0$. Therefore, the degree zero part of a DG Lie Algebra is a Lie Algebra in the usual sense.

Example 1.13 *1) If M is a manifold, $\underline{\mathbf{g}}$ is a smooth bundle of DG Lie algebras on it, then the space $\Omega^*(M, \underline{\mathbf{g}})$ is a DG Lie algebra, with the bracket defined on homogeneous elemesnt as*

$$[\alpha \otimes g, \beta \otimes h]_{\Omega^*(M,\underline{\mathbf{g}})} \;=\; (-1)^{|\beta||g|}(\alpha \wedge \beta) \otimes [g, h]_{\underline{\mathbf{g}}}$$

Note that it is essential here to consider graded *differential forms, i.e. finite sums of differential forms with values in homogeneous components of $\underline{\mathbf{g}}$. We will come back to graded vector bundles in the lecture on Q-manifolds.*
2) (See [K], page 9) If A is an associative algebra, the complex of Hoschild cochains with coefficients in A (and shifted in degree by one) is a DG Lie algebra, with the Hoschild differential and the Gerstenhaber bracket. See the reference for details.

The category of graded vector spaces (without a differential) is in a natural way a full sub category of $\mathcal{C}(k)$, with the map defined by taking the differential on a graded object to be zero. In this way, restricting the definitions above to this sub category, we get the notions of Graded Algebras, Graded Coalgebras, and Graded Lie Algebras. There is also a forgetful functor from $\mathbf{DGA}(k)$ (resp. $\mathbf{DGcoA}(k)$, $\mathbf{DGLA}(k)$) to their graded counterparts. There is also a forgetful functor from $\mathbf{DGA}(k)$ (resp. $\mathbf{DGcoA}(k)$, $\mathbf{DGLA}(k)$) to $\mathcal{C}(k)$. In the next section we will build adjoints to some of these forgetful functors.

Remark 1.14 *Instead of $\mathbf{Z}$-graded vector spaces we could consider, more generally, vector spaces graded over an abelian group Γ, $C = \bigoplus_{g\in\Gamma} C_g$, together with an assigned homomorphism $|\ | : \Gamma \to \mathbf{Z}$ and an endomorphism of degree $+1$ with respect to this group homomorphism, $d_C(C_g) \subset \bigoplus_{h:|h|=|g|+1} C_h$, $d_C^2 = 0$. In this setting we may build different twisting maps T, and therefore we can have different notions of (graded) commutativity.*

2 Free Graded DG Algebras and DG Coalgebras

Remark 2.1 *Let V be an object of $\mathcal{C}(k)$, and $\mathbf{T}$ the twisting map introduced before. Then, if we indicate $V^{\otimes n} = V \otimes (V \otimes (\cdots \otimes V)\cdots)$ (n times), given any positive integer n, there is a natural action of the symmetric group*

$$\mathcal{S}_n \;\to\; Aut\left(V^{\otimes n}\right)$$

Note that we need to also use the canonical isomorphism

$$V \otimes (V \otimes V) \;\cong\; (V \otimes V) \otimes V$$

to give this action.

Definition 2.2 *Let V be an object of $\mathcal{C}(k)$. Then we define the following graded algebras:*

1)

$$T(V) \;=\; \bigoplus_{n\geq 0} V^{\otimes n}, \;\; \bar{T}(V) \;=\; \bigoplus_{n>0} V^{\otimes n}$$

with the multiplication defined on $\bar{T}(V)$ *as*

$$\mu_T(v_1 \otimes \cdots \otimes v_m, w_1 \otimes \cdots \otimes w_n) \;=\; v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n$$

and defined on $T(V)$ *as the extension of that of* $\bar{T}(V)$ *in which the canonical generator of* $T^0(V) \;=\; k[0]$ *acts as unity.*
$T(V)$ *is called the* tensor algebra *over* V*, while* $\bar{T}(V)$ *is sometimes called the* reduced *tensor algebra over* V*.*
2) Let $I_S \;\subset\; \bar{T}(V) \;\subset\; T(V)$ *be the two-sided ideal generated by homogeneous elements of the form*

$$v \otimes w \;-\; \mathbf{T}(v \otimes w), \;\; v, w \;\in\; V \textit{ homogeneous}$$

We define the symmetric algebra *over* V *to be*

$$S(V) \;=\; T(V)/I_S, \;\; \bar{S}(V) \;=\; \bar{T}(V)/I_S$$

$\bar{S}(V)$ *is sometimes called the* reduced *symmetric algebra over* V*.*
π_S *is the natural surjection from* $T(V)$ *to* $S(V)$*. Given* $v_1, ..., v_n$ *in* $T(V)$*, we indicate with the same symbols* v_i *their images under the projection in* $S(V)$*, and write*

$$v_1 \cdots v_n \;=\; \pi_S(v_1 \otimes \cdots \otimes v_n)$$

The image under projection of $V^{\otimes n}$ *is indicated with* $S^n(V)$*.*
3) there is a natural injective map (in $\mathcal{C}(k)$ *, but not preserving the DG algebra structures)*

$$N \;:\; S(V) \;\rightarrow\; T(V)$$

defined for $v_1, ..., v_n$ *in* V *(and then extended linearly) as*

$$N(v_1 \cdots v_n) \;=\; \sum_{\sigma \in \mathcal{S}_N} \sigma(v_1 \otimes \cdots \otimes v_n)$$

4) Let $I_\Lambda \;\subset\; \bar{T}(V) \;\subset\; T(V)$ *be the two-sided ideal generated by homogeneous elements of the form*

$$v \otimes w \;+\; \mathbf{T}(v \otimes w), \;\; v, w \;\in\; V \textit{ homogeneous}$$

We define the exterior algebra *over* V *to be*

$$\Lambda(V) \;=\; T(V)/I_\Lambda, \;\; \bar{\Lambda}(V) \;=\; \bar{T}(V)/I_\Lambda$$

$\bar{\Lambda}(V)$ *is sometimes called the* reduced *exterior algebra over* V*.*
π_Λ *is the natural surjection from* $T(V)$ *to* $\Lambda(V)$*. Given* $v_1, ..., v_n$ *in* $T(V)$*, we indicate with the same symbols* v_i *their images under the projection in* $\Lambda(V)$*, and write*

$$v_1 \wedge \cdots \wedge v_n \;=\; \pi_\Lambda(v_1 \otimes \cdots \otimes v_n)$$

The image under projection of $V^{\otimes n}$ *is indicated with* $\Lambda^n(V)$*.*

Remark 2.3 *We omit the verification, necessary to make sense of the above definitions, that the map μ_T is a morphism in $\mathcal{C}(k)$, and that the ideals I_S and I_Λ are* differential *ideals, i.e. are closed with respect to the differentials.*

Definition 2.4 *Let V be an object of $\mathcal{C}(k)$. There is then a canonical isomorphism in $\mathcal{C}(k)$*

$$dec_n \ : \ S^n(V[1]) \ \rightarrow \ \Lambda^n(V)[n]$$

called décalage *isomorphism, and indicated with dec_n. An explicit formula for dec_n is, if $x_i \ \in \ V_{p_i}$,*

$$dec_n\left((1_{k[1]} \otimes x_1) \cdots (1_{k[1]} \otimes x_n)\right) \ = \ (-1)^{\sum_{i=1}^n (n-i)(p_i-1)} 1_{k[n]} \otimes (x_1 \wedge \cdots \wedge x_n)$$

In the following we will use the natural identifications

$$V^{\otimes 1} \ \cong \ \pi_S(V^{\otimes 1}) \ \cong \ V$$

in order to see V as a subspace of both $T(V)$ and $S(V)$. It is clear that V generates both these as unitary algebras (i.e. the smallest unitary subalgebra of them which contains V is the whole algebra itself).

Definition 2.5 *1) Let V be an object of $\mathcal{C}(k)$. Then we define the comultiplication Δ_T on $T(V)$ by giving its value on homogeneous elements:*

$$\Delta_T(v_1 \otimes \cdots \otimes v_n) \ = \ \sum_{r=0}^{n} (v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n)$$

The counit ϵ_T is defined as the canonical projection onto $V^{\otimes 0} \ \cong k[0] \ \subset \ T(V)$. The canonical (split) projection of $T(V)$ onto $\bar{T}(V)$ can be used to induce a comultiplication on $\bar{T}(V)$ which is, however, without counit.
2) We define a comultiplication Δ_S on $S(V)$ by defining it for $v \ \in \ V$, an extending it as an algebra map, as

$$\Delta_S(\bar{v}) \ = \ 1 \otimes v \ + \ v \otimes 1$$

Proposition 2.6 *Let V be an object of $\mathcal{C}(k)$. Then*

$$(T(V), \mu_T, 1) \ \ (S(V), \mu_S, 1)$$

and their reduced counterparts are graded unitary algebras.

$$(T(V), \Delta_T, \epsilon_T) \ \ (S(V), \Delta_S, \epsilon_S)$$

and their reduced counterparts are graded counitary coalgebras.
Moreover, the map $\pi_S \ : \ T(V) \ \rightarrow \ S(V)$ is a map of DG algebras, while $N \ : \ S(V) \ \rightarrow \ T(V)$ is a map of DG coalgebras. Both π_S and N restrict to the reduced spaces.

Definition 2.7 ([Q]) *Let C be a DG counitary coalgebra. Suppose that there exists an element 1_{C} of C such that $\Delta(1_{C}) = 1_{C} \otimes 1_{C}$, $\epsilon_{C}(1_{C}) = 1$. We define subspaces $F_rC, r \geq 0$ as follows:*

$$F_0C = k1_{C}; \quad F_rC = \{x \in C \mid \Delta_{C}(x) - x \otimes 1_{C} - 1_{C} \otimes x \in F_rC \otimes F_rC\}$$

C is said to be coconnected *if*

$$C = \bigcup_{r \geq 0} F_rC$$

Proposition 2.8 ([Q], pages 282-283) *1) A sub DG counitary coalgebra of a coconnected coalgebra is coconnected.*
2) Let V be an object of $\mathcal{C}(k)$. Then

$$(T(V), \Delta_T, \epsilon_T) \quad (S(V), \Delta_S, \epsilon_S)$$

are both coconnected coalgebras.

Proof
1) Let C' be a sub DG counitary coalgebra of C. We first show that $1_C \in C'$. Indeed, if not, there is a minimum r, call it r_0, such that $C' \cap F_rC \neq (0)$. Note that the composition of inclusion with the projection map

$$C \to C/_{F_{r_0-1}C}$$

is an injective map of coalgebras. If $x \in C' \cap F_{r_0}C$ is not zero, then

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \rho$$

with $\rho \in F_{r_0-1}C \otimes F_{r_0-1}C$. However, by composing with the projection mention above, we see that this implies $\Delta(x) = 0$. On a counitary coalgebra, this implies $x = 0$, which is absurd. We have therefore proved that $1_C \in C'$. It follows clearly that if we define

$$F_rC' = F_rC \bigcap C'$$

we have $C' = \bigcup_{r\geq 0} F_rC'$ and moreover

$$F_rC' = \left\{x \in C' \mid \Delta_{C'}(x) - x \otimes 1_{C'} - 1_{C'} \otimes x \in F_rC' \otimes F_rC'\right\}$$

2) For $T(V)$ it is enough to show that taking as 1_T the natural one (coming from the algebra structure),

$$F_rT(V) = \bigoplus_{n \leq r} V^{\otimes n}$$

By induction, assume the thesis true for $r \geq 0$, and let $x \in T(V)$ be such that

$$\Delta_T(x) - x \otimes 1 - 1 \otimes x \in \left(\bigoplus_n \leq rV^{\otimes n}\right) \otimes \left(\bigoplus_n \leq rV^{\otimes n}\right)$$

From the explicit definition of Δ_T, it follows immediately that this holds if and only if $x \in \bigoplus_{n\leq r} V^{\otimes n}$ as desired.
The proof for $S(V)$ follows now from the first point and the existence of the injective map of counitary coalgebras $N : S(V) \to T(V)$.

Proposition 2.9 *1) Let $\phi : V \to W$ be a morphism in $\mathcal{C}(k)$. We obtain from ϕ two maps:*

$$T(\phi) : T(V) \to T(W), \quad S(\phi) : S(V) \to S(W)$$

which preserve the structures of DG algebra and coalgebra introduced before on these graded complexes, and induce also maps on the reduced spaces.
2) The definitions of $T(\phi)$ for all ϕ fit together to provide a functor from the category $\mathcal{C}(k)$ to the category $\mathbf{DGA}(k)$ (resp. $\mathbf{DGcoA}(k)$). The definitions of $S(\phi)$ provide a functor from the category $\mathcal{C}(k)$ to the categories of commutative objects of $\mathbf{DGA}(k)$ (resp. of cocommutative objects of $\mathbf{DGcoA}(k)$).
3) The functor T (resp. S) gives a right adjoint to the forgetful functor from the category of coconnected counitary coalgebras (resp. cocommutative) to $\mathcal{C}(k)$ (with some conditions on the maps).

Proof
The proofs of 1) and 2) are straightforward, and are left to the reader.
3) **([Q], Proposition 4.1 page 285)**. Let V be an object of $\mathcal{C}(k)$, and j the natural projection from $T(V)$ to V (and from $S(V)$ to V). Let C be a coconnected DG coalgebra. Then we will show that the map $\theta \to j\theta$ provides a bijection $Hom_{DGC}(C, T(V)) \to Hom_{\mathcal{C}(k),0}(C, V)$, where $Hom_{\mathcal{C}(k),0}$ are the maps which send 1_C to zero. Note that due to this requirement on the image of 1_C the funftor T is not an adjoint to the forgetful functor. The assumption can be dropped when, for example, V does not have a part of degree zero.
Going to the proof, define $\Delta^{(n)}\ C \to C^{\otimes(n+1)}$ inductively as $\Delta^{(-1)} = \epsilon_T$, $\Delta^{(0)} = \mathbf{Id}_C$, $\Delta^1_C = \Delta_C$; $\Delta_C^{(n+1)} = (\mathbf{Id}_C \otimes \Delta_C^{(n)})\Delta_C$. It follows inductively that if $x \in F_r C$ then $\Delta_C^{(r)}(x)$ is a linear combination of expressions of the form $x_0 \otimes \cdots \otimes x_r$, where at least one x_i is 1_C. Therefore, if $u : C \to V$ is such that $u(1_C) = 0$, then $u^{\otimes n}\Delta_C^{(n-1)}(x) = 0\ for\ x \in F_{n-1}C$. It follows that the map $\theta : C \to T(V)$ given by $\theta(x) = \sum_{n\geq 0} u^{\otimes n}\Delta_C^{(n-1)}(x)\mathrm{ST}$ is well defined in $\mathcal{C}(k)$, because the entries of the infinite sum which defines it are almost all zero, for any fixed x. We now show that θ is a map of DG coalgebras:

$$(\theta\otimes\theta)\Delta = \left((\sum_{n\geq 0} u^{\otimes n}\Delta_C^{(n-1)}) \otimes (\sum_{m\geq 0} u^{\otimes m}\Delta_C^{(m-1)}) \right) \Delta =$$

$$= \left(\sum_{n,m} u^{\otimes n}\Delta_C^{(n-1)} \otimes u^{\otimes m}\Delta_C^{(m-1)} \right) \Delta =$$

$$= \sum_{n,m} (u^{\otimes n}\otimes u^{\otimes m})(\Delta_C^{(n-1)}\otimes\Delta_C^{(m-1)})\Delta = \sum_k \sum_{n+m=k} (u^{\otimes n}\otimes u^{\otimes m})\Delta_C^{(k-1)} = \Delta\theta$$

We have used the relation $(\Delta_C^{(n)} \otimes \Delta_C^{(m)})\Delta_C = \Delta_C^{(n+m+1)}$. To conclude the proof, we still have to show that a map θ is determined by its projection $j\theta$ (and therefore coincides with the image of the right inverse to j just built applied to $j\theta$). This is done showing by induction on the filtration F_rC that $j\theta_1 = j\theta_2$ for two maps θ_1, θ_2, then $\theta_1 = \theta_2$ on C. The statement is clear for $r = 0$, as $\theta(x) = 0$ for $x \in F_0C$. Assume the thesis true for $r - 1 \geq 0$, and let $x \in F_rC$. Then $\Delta\theta_1(x) = \theta_1 \otimes \theta_1\Delta(x) = \theta_1 \otimes \theta_1 (x \otimes 1 + 1 \otimes x + \rho) = \theta_1 \otimes \theta_1(\rho)$. We know from the definition of F_rC that $\rho \in F_{r-1} \otimes F_{r-1}$, therefore by induction the value of $\theta_1 \otimes \theta_1$ on it is the same as the value of $\theta_2 \otimes \theta_2$. We can therefore write $\Delta\theta_1(x) = \theta_1 \otimes \theta_1(\rho) = \theta_2 \otimes \theta_2(\rho) = \Delta\theta_2(x)$. The last equality is obtained by reversing the previous chain of equalities with θ_2 in place of θ_1. The induction step is concluded by the observation that Δ is injective on a cofree counitary coconnected coalgebra. The proof in the cocommutative case is exactly the same, with the only minor modification that the θ built starting from a u is $\theta(x) = \sum_{n\geq 0} \pi_n u^{\otimes n}\Delta_C^{(n-1)}(x)$ where, if π_S indicates the projection from $T(V)$ to $S(V)$, $\pi_n = \frac{1}{n!}\pi_{S_{|_{V^{\otimes n}}}}$. $\square$

Remark 2.10 *In view of the preceding propositions, the objects obtained applying the functors T (resp. S) are called free graded (co)algebras (resp. free (co)commutative graded (co)algebras).*

3 L_∞ Algebras

Definition 3.1 *1) A L_∞ Algebra is a pair $(\mathbf{g}, Q)$, where $\mathbf{g}$ is a graded k-vector space, and Q is a graded coalgebra differential of degree $+1$ on the graded coalgebra $\bar{S}(\mathbf{g}[1])$.*
2) Given two L_∞ algebras $(\mathbf{g}_1, Q_1)$ and $(\mathbf{g}_1, Q_1)$, a morphism of L_∞ algebras *between them is a morphism of DG coalgebras*

$$\left((\bar{S}(\mathbf{g}_1[1]), Q_1), \Delta_{\bar{S}(\mathbf{g}_1[1])}\right) \to \left((\bar{S}(\mathbf{g}_2[1]), Q_2), \Delta_{\bar{S}(\mathbf{g}_2[1])}\right)$$

Remark 3.2 *It is customary to write $C(\mathbf{g})$ instead of $\bar{S}(\mathbf{g}[1])$ for a graded vector space $\mathbf{g}$. Note that there is no* a priori *differential on $C(\mathbf{g})$. We adopt the notation from [K], in which $C(\mathbf{g})$ is a coalgebra* without *counit.*

The above definition provides us with a category (as L_∞ morphism can be clearly composed, and there is an identity morphism for any object). There is also a morphism from the category of L_∞ algebras to the category of DG coalgebras, which factors through the subcategory of cofree cocommutative (coconnected) coalgebras without counit: $(\mathbf{g}, Q) \to \left((\bar{S}(\mathbf{g}[1]), Q), \Delta_{\bar{S}(\mathbf{g}[1])}\right)$ (the functor on maps is defined in the obvious way).

Definition 3.3 *Given a L_∞ algebra $(\mathbf{g}, Q)$, we indicate with Q^i the composition of Q with the canonical projection $S^{\geq 1}(\mathbf{g}[1])[1] \to S^i(\mathbf{g}[1])[1]$. We also indicate with $Q^i_j : S^j(\mathbf{g}[1]) \to S^i(\mathbf{g}[1])[1]$ the restriction of Q^i to $S^j(\mathbf{g}[1])$.*

The Q^1_n can be thought of as multilinear symmetric operators on $\mathbf{g}[1]$. More generally, with the help of the décalage isomorphism we can define a map $(dec_m[-n+1])(Q^m_n[-n])(dec_n[-n])^{-1} \ : \ \Lambda^n(\mathbf{g}) \ \rightarrow \ \Lambda^m(\mathbf{g})[m-n+1] \ \ (n \ \geq \ 1)$ A commonly used notation is

$$[...]_n \ = \ (dec_1[-n+1])(Q^1_n[-n])(dec_n[-n])^{-1}\pi_\Lambda \ : \ \mathbf{g}^{\otimes n} \ \rightarrow \ \mathbf{g}[2-n]$$

Recall the definition of j as the canonical projection $S(V) \ \rightarrow \ V$. We use the same notation for the map $C(\mathbf{g}) \ \rightarrow \ \mathbf{g}[1]$.

Proposition 3.4 *Let $(\mathbf{g}, Q)$ be a L_∞ algebra. We then have the explicit formula* $Q^n \ = \ \frac{1}{(n-1)!}\pi_S \left(Q^1 \otimes j^{\otimes n-1}\right) \Delta^{(n-1)} \ : \ C(\mathbf{g}) \ \rightarrow \ S^n(\mathbf{g}[1])[1]$

Proof
Define $\tilde{Q} \ = \ \pi_S \sum_{n\geq 1} \frac{1}{(n-1)!} \left(Q^1 \otimes j^{\otimes n-1}\right) \Delta^{(n-1)}$. We must first verify that $\Delta\tilde{Q} \ = \ \left(\tilde{Q} \otimes I \ + \ sgn \cdot I \otimes \tilde{Q}\right) \Delta$, where sgn is the (integer valued) sign function on $\mathbf{g}[1]$.

$$\Delta\tilde{Q}(v_1 \cdots v_p) \ = \ \Delta\pi_S \sum_{n\geq 1} \frac{1}{(n-1)!} \left(Q^1 \otimes j^{\otimes n-1}\right) \Delta^{(n-1)}(v_1 \cdots v_p) \ =$$

$$= \ \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{k_1+k_2 \ = \ n} \pi_{k_1} \otimes \pi_{k_2} \sum_{\sigma\in\mathcal{S}_n} \sigma \left(Q^1 \otimes j^{\otimes n-1}\right) \Delta^{(n-1)}(v_1 \cdots v_p) \ =$$

$$\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{k_1+k_2 \ = \ n} \pi_{k_1} \otimes \pi_{k_2} \left(\sum_{\sigma\in\mathcal{S}_n, \sigma(1)\leq k_1} \sigma \left(Q^1 \pi_{p-n+1} \otimes I^{\otimes n-1}\right) + \right.$$

$$\left. \sum_{\sigma\in\mathcal{S}_n, \sigma(1)\geq k_1+1} \sigma \left(Q^1 \pi_{p-n+1} \otimes I^{\otimes n-1}\right) \right) \sum_{\rho\in\mathcal{S}_p} (\rho(v_1 \otimes \cdots \otimes v_p)) \ =$$

$$\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{k_1+k_2 \ = \ n} \pi_{k_1} \otimes \pi_{k_2} \left(k_1(k_1+k_2-1)! Q^1 \pi_{p-n+1} \otimes I^{\otimes k_1 -1} \otimes I^{\otimes k_2} + \right.$$

$$\left. k_2(k_1+k_2-1)! sgn(I^{\otimes k_1}) I^{\otimes k_1} \otimes Q^1 \pi_{p-n+1} \otimes I^{\otimes k_2 -1}\right) \sum_{\rho\in\mathcal{S}_p} (\rho(v_1 \otimes \cdots \otimes v_p)) \ =$$

$$\sum_{k_1+k_2 \ \geq \ 1} \left(k_1 \pi_{k_1} \left(Q^1 \pi_{p-k_1-k_2+1} \otimes I^{\otimes k_1-1}\right) \otimes \pi_{k_2} + \right.$$

$$\left. \pi_{k_1} sgn(I^{\otimes k_1}) \otimes k_2 \pi_{k_2} \left(Q^1 \pi_{p-n+1} \otimes I^{\otimes k_2 -1}\right)\right) \sum_{\rho\in\mathcal{S}_p} (\rho(v_1 \otimes \cdots \otimes v_p)) \ =$$

$$\sum_{k_1+k_2 \ \geq \ 1} \left(k_1 \pi_{k_1} \left(Q^1 \otimes j^{\otimes k_1 -1}\right) \Delta^{(k_1-1)} \pi_{p-k_2} \otimes \pi_{k_2} + \right.$$

$$\left. \pi_{k_1} sgn(I^{\otimes k_1}) \otimes k_2 \pi_{k_2} \left(Q^1 \otimes I^{\otimes k_2 -1}\right) \Delta^{(k_2-1)} \pi_{p-k_1}\right) \sum_{\rho\in\mathcal{S}_p} (\rho(v_1 \otimes \cdots \otimes v_p)) \ =$$

$$\sum_k \left(\tilde{Q}\pi_{p-k} \otimes \pi_k + \right.$$

$$\left. \pi_k sgn(I^{\otimes k_1}) \otimes \tilde{Q}\pi_{p-k} \right) \sum_{\rho \in \mathcal{S}_p} (\rho(v_1 \otimes \cdots \otimes v_p)) \ =$$

$$\left(\tilde{Q} \otimes I \ + \ sgn \cdot I \otimes \tilde{Q} \right) \Delta(\rho(v_1 \cdot \cdots \cdot v_p))$$

as desired. To conclude the proof, we must show that if two maps satisfy this property, and have the same projection (via j), then they coincide. This is left as an easy exercise to the reader. One needs to do an induction using the standard filtration on $C(\mathbf{g})$. □

Corollary 3.5 *1)* $Q^n_m \ = \ 0 \ for \ n \ > \ m$
2) $Q \ = \ \pi_S \sum_{n \geq 1} \frac{1}{(n-1)!} \left(Q^1 \otimes j^{\otimes n-1} \right) \Delta^{(n-1)}$

4 Quillen's construction and Homotopy Lie algebras

Quillen (see for example [Q]) introduced a way to pass from a DG Lie Algebra to a L_∞ algebra. In this section we will show that this is a special case of a more general correspondence between Homotopy Lie Algebras (which generalize DG Lie Algebras) and L_∞ algebras. However, because Quillen's construction is much easier to handle than the general one, it is worth to treat it separately.

Proposition 4.1 (Quillen's construction) *Let* $(\mathbf{g}, \mathbf{d}_\mathbf{g}, [\ ,\ ]_\mathbf{g})$ *be a DG Lie Algebra. Then there exists a* Q *giving on* $\mathbf{g}$ *the structure of a* L_∞ *algebra, and such that if* $[..]_i$ *are the products associated to* Q*,* $[..]_1 \ = \ \mathbf{d}_\mathbf{g}, \ \ [..]_2 \ = \ [\ ,\ ]_\mathbf{g}, \ \ [..]_i \ = \ 0 \ for \ i > 2$.

Proof
The plan is to use the products $[..]_i$ to give Q^1, to use the explicit formulas above to obtain a Q from this Q^1, and then to verify that $QQ \ = \ 0$. This will be enough, because we already verified that any Q defined using the explicit formulas above behaves correctly with respect to the coproduct. The proof is straightforward, but the signs introduced by the décalage isomorphism must be tracked carefully. If $\alpha \in \mathbf{g}$, we will indicate with $\overline{\alpha}$ its canonical image in $\mathbf{g}[1]$. Define $Q^1_1(\overline{\alpha}) \ = \ \overline{\mathbf{d}_\mathbf{g}(\alpha)}, \ \ Q^1_2(\overline{\alpha}\overline{\beta}) \ = \ (-1)^{|\alpha|-1}\overline{[\alpha, \beta]_\mathbf{g}}$. The actual computation of $Q^2 = 0$ is straightforward once we obtain the general formula

$$Q(\overline{\alpha_1} \cdot \cdots \cdot \overline{\alpha_p}) \ = \ \sum_{i=1}^{p} \overline{\alpha_1} \cdot \cdots \cdot \overline{\mathbf{d}_\mathbf{g}(\alpha_i)} \cdot \cdots \cdot \overline{\alpha_p} \ +$$

$$+ \sum_{1 \leq i,j \leq p} (-1)^{\sigma_{ij} + |\alpha_i| - 1} \overline{[\alpha_i, \alpha_j]_\mathbf{g}} \cdot \overline{\alpha_1} \cdot \cdots \cdot \hat{\overline{\alpha_i}} \cdot \cdots \cdot \hat{\overline{\alpha_j}} \cdot \cdots \cdot \overline{\alpha_p}$$

where σ_{ij} is the sign of the permutation of the product $\overline{\alpha_1} \cdot \cdots \cdot \overline{\alpha_p}$ which sends it to $\overline{\alpha_i \alpha_j \alpha_1} \cdots \hat{\overline{\alpha_i}} \cdots \hat{\overline{\alpha_j}} \cdots \overline{\alpha_p}$ □

We now recall the notion of homotopy category. See also [H, Page 25] for more details.

Definition 4.2 *Let k be a vector space. We indicate with $\mathbf{K}(k)$ the category, called* Homotopy category of complexes*, which has as objects the objects of $\mathcal{C}(k)$, and as morphisms the homotopy equivalence classes of morphisms from $\mathcal{C}(k)$. If $\phi \in Mor(\mathcal{C}(k))$, we use the same symbol ϕ to indicate its class as an element of $Mor(\mathbf{K}(k))$.*

Definition 4.3 *A* Homotopy Lie Algebra *is a Lie algebra in the category $\mathbf{K}(k)$, namely it is an object $(\mathbf{g}, \mathbf{d}_\mathbf{g})$ of $\mathbf{K}(k)$, together with a morphism $[\ ,\ ]_\mathbf{g} : \mathbf{g} \otimes \mathbf{g} \rightarrow \mathbf{g}$ in $\mathbf{K}(k)$ such that*

$$[\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g}) \ + \ [\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g})(\mathbf{Id}_\mathbf{g} \otimes \mathbf{T})(\mathbf{T} \otimes \mathbf{Id}_\mathbf{g}) \ +$$

$$+ \ [\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g})(\mathbf{T} \otimes \mathbf{Id}_\mathbf{g})(\mathbf{Id}_\mathbf{g} \otimes \mathbf{T}) \ = \ 0 \ \ (in\ \mathbf{K}(k))$$

A morphism of Homotopy Lie algebras from $(\mathbf{g}, [\ ,\ ]_\mathbf{g})$ to $(\mathbf{h}, [\ ,\ ]_\mathbf{h})$ is a morphis $\phi\ :\ \mathbf{g} \rightarrow \mathbf{h}$ of $\mathbf{K}(k)$, such that

$$[\ ,\ ]_\mathbf{h} \phi \otimes \phi \ = \ \phi[\ ,\ ]_\mathbf{g} \ \ (in\ \mathbf{K}(k))$$

We indicate with $\mathbf{HLA}(k)$ the category of homotopy Lie algebras over k.

Remark 4.4 *Another way to give the definition of the objects of $\mathbf{HLA}(k)$ is that there exists a morphism $\rho_\mathbf{g} : \mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g} \rightarrow \mathbf{g}[-1]$ of graded k-vector spaces such that*

$$[\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g}) \ + \ [\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g})(\mathbf{Id}_\mathbf{g} \otimes \mathbf{T})(\mathbf{T} \otimes \mathbf{Id}_\mathbf{g}) \ +$$

$$+ \ [\ ,\ ]_\mathbf{g}(\mathbf{Id}_\mathbf{g} \otimes [\ ,\ ]_\mathbf{g})(\mathbf{T} \otimes \mathbf{Id}_\mathbf{g})(\mathbf{Id}_\mathbf{g} \otimes \mathbf{T}) \ = \ \mathbf{d}_\mathbf{g} \rho_\mathbf{g} \ + \ \rho_\mathbf{g} \mathbf{d}_{\mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g}}$$

When $\rho_\mathbf{g} = 0$, we recover the notion of a DGLA. Note however that even when we have two HLA's coming from DGLA's, the morphisms between them in $\mathbf{HLA}(k)$ are different from the morphisms in $\mathbf{DGLA}(k)$.

Proposition 4.5 *Let $(\mathbf{g}, Q)$ be a L_∞ algebra. The maps $\mathbf{d}_\mathbf{g} = [\ ]_1$, $[\ ,\ ]_\mathbf{g} = [\ ,\ ]_2$ give to $\mathbf{g}$ the structure of a homotopy Lie algebra.*

Proof
Given a, b, c homogeneous elements of $\mathbf{g}$, let us write explicitely $QQ(\alpha\beta\gamma) = 0$, where α, β, γ are the objects in $\mathbf{g}[1]$ corresponding to a, b, c. First, observe that, if we indicate with j lower script the projection on the "tensors" of degree j, the condition $Q^2 = 0$ implies the equations

$$0 \ = \ (QQ)_1(\alpha) \ = \ Q^1 Q^1(\alpha),$$

$$0 \ = \ (QQ)_1(\alpha\beta) \ = \ Q^1 Q^1(\alpha\beta) \ + \ Q^1 Q^2(\alpha\beta),$$

$$0 \ = \ (QQ)_1(\alpha\beta\gamma) \ = \ Q^1 Q^1(\alpha\beta\gamma) \ + \ Q^1 Q^2(\alpha\beta\gamma) \ + \ Q^1 Q^3(\alpha\beta\gamma),$$

$$0 = (QQ)_2(\alpha\beta) = Q^2Q^1(\alpha\beta\gamma) + Q^2Q^3(\alpha\beta\gamma),$$
$$0 = (QQ)_2(\alpha\beta\gamma) = Q^2Q^1(\alpha\beta\gamma) + Q^2Q^2(\alpha\beta\gamma) + Q^2Q^3(\alpha\beta\gamma),$$
$$0 = (QQ)_3(\alpha\beta\gamma) = Q^3Q^3(\alpha\beta\gamma).$$

Once translated in terms of the $[\ ]_i,\ i \geq 1$, the relations above are exactly what one needs to see that $[\ ,\ ,\ ]_3$ induces the homotopy of a homotopy Lie algebra with respect to $\mathbf{d}_{\mathbf{g}} = [\ ]_1,\ [\ ,\ ]_{\mathbf{g}} = [\ ,\ ]_2$. □

There is also a partial converse to the previous proposition. We omit the proof, which is contained in [HS].

Proposition 4.6 *Let $(\mathbf{g}, [\ ,\ ]_{\mathbf{g}})$ be a Homotopy Lie Algebra. Then there exists a Q giving the structure of L_∞ algebra on $\mathbf{g}$, and such that if $[..]_1, [..]_2$ are the first two products associated to Q, $\mathbf{d}_{\mathbf{g}} = [\ ]_1$ and $[\ ,\ ]_2$ represents the homotopy class of $[\ ,\ ]_{\mathbf{g}}$*

Note that in going from a L_∞ algebra to a homotopy Lie algebra the process is uniquely determined. However, in going from a homotopy Lie algebra to a L_∞ algebra we have some freedom in choosing the successive products. We cannot therefore say that a L_∞ algebra *is* a homotopy Lie algebra. Perhaps this is the reason why some authors call L_∞ algebras *Strong* homotopy Lie algebras: the structure of a L_∞ algebra contains more information, namely the choice of a "model".

5 From connected DG Algebras to Minimal models

In this section we prove a theorem which is an analogue for DG algebras graded over $\mathbf{Z}$ of the usual minimal model construction for $\mathbf{N}$-graded algebras (see [S] for this). It is useful for example in the study of the homotopy type of Q-manifolds (see [G] for Q-manifolds), where the classical minimal model construction does not apply.

Definition 5.1 *We say that a unitary commutative DG Algebra A is* connected *if there is a map of DG algebras $\epsilon : A \rightarrow k[0]$ such that:*
1) $\epsilon(1_A) = 1$
2) Let $A^+ = Ker(\epsilon)$. Then we have that the natural map of unitary DG algebras

$$A \rightarrow Lim_k \left(A/(A^+)^k\right)$$

is injective.
We sometimes write (A, ϵ_A) if we want to single out one such ϵ, and we call it a connecting augmentation.

Remark 5.2 *1) Let M be a smooth manifold. Then $A = (\Omega^*(M), d_{DR})$ is a connected DGA over $\mathbf{R}$ if and only if M is a connected manifold.*

2) Condition 2) *of the definition of connectedness is clearly equivalent to*

$$\bigcap_k (A^+)^k \ = \ (0)$$

3) The map ϵ *in the definition of connectedness is not necessarily unique. Take for example the polynomial ring* $k[x]$ *in one variable in degree* 0*. Then any element* λ *of* k *identifies a map* ϵ_λ *given by the quotient map*

$$k[x] \ \to \ k[x]/(x-\lambda) \ \cong \ k$$

Because they all have different kernels, all these maps are different.

Definition 5.3 *1) A unitary commutative connected non-negatively graded DG algebra* (A, ϵ_A) *with* $A^1 = 0$ *is said to be* minimal *(with respect to* ϵ_A*) if it is free as a graded unitary connected commutative algebra, and its differential* d_A *has the property that*

$$d_A(A^+) \subset \ A^+ A^+$$

for $A^+ \ = \ Ker(\epsilon_A)$*.*
2) A unitary commutative connected DG algebra $(S(V), d)$ *(with* V *a graded vector space) is said to be* contractible *if its differential* d *has the property that*

$$d(S^1(V)) \ \subset \ S^1(V)$$

and moreover the induced differential on $S^1(V) \ \cong \ V$ *is acyclic.*
3) A unitary commutative connected DG algebra (A, ϵ_A) *is said to be* almost minimal *(with respect to* ϵ_A*) if there exist a graded vector space* V*, a differential* $d_{\mathcal{F}}$ *on* $S(V)$ *and an ascending filtration* $\mathcal{F}_i$*,* $i \in \mathbf{Z}$*,* $(i < 0 \Rightarrow \mathcal{F}_i = (0))$ *of* V *(in the category of graded vector spaces) such that:*
a) $((S(V), d_{\mathcal{F}}), \epsilon_S)$ *is a connected unitary DG algebra, isomorphic to* (A, ϵ_A)*;*
b) $d_{\mathcal{F}}(\mathcal{F}_{i+1}) \subset S^{\geq 1}(\mathcal{F}_i)$ *for all* i*.*

Theorem 5.4 *Let* (A, ϵ_A) *be a connected unitary commutative DG algebra. Then there exist an almost minimal DG algebra* $\mathcal{F} \ = \ (F, d_{\mathcal{F}})$ *and a morphism of unitary connected DG algebras* $\phi \ : \ (F, d_{\mathcal{F}}) \ \to \ (A, \epsilon_A)$ *inducing an isomorphism in cohomology.*

Definition 5.5 *Any* $\mathcal{F}$ *satisfying the conditions of the theorem is called an* almost minimal model *of* (A, ϵ_A)*.*

Proof
In the following argument whenever we encounter a pair of spaces V and $\tilde{V}$, the space $\tilde{V}$ is an isomorphic copy of V. We will write $I_V \ : \ \tilde{V} \ \to \ V$ for the map which identifies V with $\tilde{V}$.
Let W_0 be a complementary subspace of $Im(d)$ inside $Ker(d_{|_{A^+}})$, and let $\mathcal{F}_0 \ = \ V_0$, with ,$V_0 \ = \ \tilde{W}_0$, $d_0 \ = \ 0$. The lower index numbers a sequence of DG algebras which we will build. Put $d_0 = 0$. We clearly have a map of

DG algebras $\phi_0 : (S(\mathcal{F}_0), d_0) \to A$ inducing a surjective map in cohomology. Moreover, the kernel of the map $S(\mathcal{F}_0) \to H(A)$ is included in the kernel of the natural map $S(\mathcal{F}_0) \to k$, because this last one can be obtained as a composition $S(\mathcal{F}_0) \to H(A) \to k$, where $H(A) \to k$ is induced by ϵ_A. Therefore, there is a set of homogeneous representatives in $S^{\geq 1}(V_0)$ for the kernel. Assume inductively that we have built $\mathcal{F}_i = \bigoplus_{j \leq i} V_i$, d_i on $S(\mathcal{F}_i)$, as an extension (as DG algebras) of $(S(\mathcal{F}_{i-1}), d_{i-1})$, such that $d_i \left(S^{\geq 1}(\mathcal{F}_i)\right) \subset S^{\geq 1}(\mathcal{F}_i)$ together with a map of DG algebras $\phi_i : ((S(\mathcal{F}_i), d_i), \epsilon_S) \to (A, \epsilon_A)$ inducing a surjective map in cohomology, commuting with the inclusion $((S(\mathcal{F}_{i-1}), d_{i-1}), \epsilon_S) \subset ((S(\mathcal{F}_i), d_i), \epsilon_S)$ (when $i > 0$), and such that $d_i(\mathcal{F}_i) \subset S(\mathcal{F}_{i-1})$ and we have that the space $d_i(\mathcal{F}_i) / \left(d_{i-1}(S(\mathcal{F}_{i-1})) \cap d_i(\mathcal{F}_i)\right) \subset H(S(\mathcal{F}_{i-1}))$ generates the kernel of the map in cohomology $H(\mathcal{F}_{i-1}) \to H(A)$. This last condition is assumed to be vacuously true when $i = 0$. We want to build a $\mathcal{F}_{i+1}$. To do that, let W_{i+1} be a homogeneous complementary subspace inside $Ker(d_i) \bigcap \left(S^{\geq 1}(\mathcal{F}_i)\right) \subset S(\mathcal{F}_i)$ of the space $\{x \in Ker(d_i) \mid \phi(x) \in Im(d_A)\}$ Write $W_i = K_i \oplus \Lambda_i$ with $K_i = Ker(\phi_{i|_{W_i}})$. Let $H_i \subset A$ be a subspace such that there is a vector space isomorphism $d_A : H_i \xrightarrow{\cong} \phi_i(\Lambda_i)$ We define $V_{i+1} = \tilde{K}_i[1] \oplus \tilde{H}_i$. The map ϕ_{i+1} is defined to be 0 on $\tilde{K}_i$, I_{H_i} on $\tilde{H}_i$, and is extended multiplicatively to the rest of $S(\mathcal{F}_{i+1})$, where $\mathcal{F}_{i+1} = \mathcal{F}_i \oplus V_{i+1}$. The differential d_{i+1} is defined to be I_{K_i} on $\tilde{K}_i[1]$, and $\phi_i^{-1} d_A I_{H_i}$ on $\tilde{H}_i$, and is extended to all of $S(\mathcal{F}_{i+1})$ in the unique way compatible with the Leibnitz rule and the fact that it extends d_i on $S(\mathcal{F}_i)$. We now verify that the map and differential satisfy the inductive hypotheses. First, it is clear from the construction that $d_{i+1}(V_{i+1}) \subset S^{\geq 1}(\mathcal{F}_i) \subset S^{\geq 1}(\mathcal{F}_{i+1})$ and hence also $d_{i+1}\left(S^{\geq 1}(\mathcal{F}_{i+1})\right) \subset S^{\geq 1}(\mathcal{F}_{i+1})$. Moreover, still by construction, $((S(\mathcal{F}_{i+1}), d_{i+1}), \epsilon_S)$ is an extension of $((S(\mathcal{F}_i), d_i), \epsilon_S)$ (as connected DG algebras). To verify that ϕ_{i+1} is a map of DG algebras, we show it for "monomials", and proceed by induction on the number k of terms coming from V_{i+1}. If $k = 0$, the statement is true by induction on i and the fact that ϕ_{i+1} extends ϕ_i. For $k = 1$, write $v = (x_1 + x_2)y$, with $x_1 \in \tilde{K}_i$ and $x_2 \in \tilde{H}_i$ homogeneous elements of the same degree (with respect to the grading) and $v \in S(\mathcal{F}_i)$. We have then

$$\phi_{i+1}\left(d_{i+1}\left((x_1 + x_2)y\right)\right) =$$

$$\phi_{i+1}\Big(d_{i+1}(x_1)y + d_{i+1}(x_2)y + (-1)^{deg(x_1)}(x_1 + x_2)d_{i+1}(y)\Big) =$$

$$\phi_{i+1}\left(I_{K_i}(x_1) + \phi_i^{-1} d_A I_{H_i}(x_2)\right)\phi_{i+1}(y) +$$

$$+ (-1)^{deg(x_1)}\phi_{i+1}(x_1 + x_2)\,\phi_{i+1}(d_{i+1}(y)) =$$

$$0 + d_A I_{H_i}(x_2)\phi_{i+1}(y) +$$

$$+ (-1)^{deg(x_1)}\phi_{i+1}(x_1 + x_2)\, d_A(\phi_i(y)) =$$

$$d_A(\phi_{i+1}(x_1) + \phi_{i+1}(x_2))\,\phi_{i+1}(y) +$$

$$+ (-1)^{deg(x_1)}\phi_{i+1}(x_1 + x_2)\, d_A(\phi_{i+1}(y)) =$$

$$d_A\left((\phi_{i+1}(x_1) + \phi_{i+1}(x_2))\,\phi_{i+1}(y)\right) = d_A\left(\phi_{i+1}((x_1 + x_2)y)\right)$$

For $k > 1$ the argument is easier: write $x = yz$ with both y and z with lower k. Then the equality $\phi_{i+1}(d_{i+1}(x)) = d_A(\phi_{i+1}(x))$ follows immediately from the inductive hypotheses, and the fact that ϕ_{i+1} is a map of graded algebras, and d_{i+1} is a differential (of a graded algebra). The last properties, namely that ϕ_{i+1} commutes with the inclusion $S(\mathcal{F}_i) \subset S(\mathcal{F}_{i+1})$ and the map ϕ_i, the fact that ϕ_{i+1} induces a surjective map in cohomology, and finally the fact that $d_{i+1}(V_{i+1}) \subset S(\mathcal{F}_i)$ and $d_{i+1}(V_{i+1})/d_i(\mathcal{F}_i) \cap d_{i+1}(V_{i+1}) \subset H(\mathcal{F}_i)$ generates the kernel of the map in cohomology $H(\mathcal{F}_i) \rightarrow H(A)$ are all clear from the construction. We have therefore complete the inductive step in the construction of the $(S(\mathcal{F}_{i+1}), d_{i+1})$ and of ϕ_{i+1}. Take $V = \bigcup_i \mathcal{F}_i$, $d_{\mathcal{F}} = Lim_i d_i$. We have a map of DG algebras $\phi = Lim_i \phi_i : S(V) \rightarrow A$ which is surjective in cohomology because all the ϕ_i are. Assume $x \in S(V)$, $d_{\mathcal{F}}(x) = 0$, and $\phi(x) = d_A(y)$ for some $y \in A$. It must be that $x \in S(\mathcal{F}_i)$ for some i, from the definition of V. Therefore we know that there is an element $z \in \mathcal{F}_{i+1}$ for which $d_{i+1}(z) - x \in Im(d_i)$ and hence $x \in Im(d_{\mathcal{F}})$ as desired. This shows that, ϕ is injective in cohomology, and therefore induces an isomorphism in cohomology. Moreover, $d_{\mathcal{F}}\left(S^{j\geq 1}(V)\right) \subset S^{j\geq 1}(V)$ because this is true at each step of the inductive construction of $\mathcal{F}$. By construction, $d_{\mathcal{F}}(\mathcal{F}_{i+1}) \subset S(\mathcal{F}_i)$, which was the last thing to check to show almost minimality. $\square$

As a final remark, note that the proof of this theorem can be extended to provide almost minimal models for connected DG algebras graded over more general groups (see Remark 1.14).
There is an analogue of the notion of minimal model also for coalgebras, at least when they are free of algebraic relations. This subject is treated more extensively in [C]. See also the Lemma of section 4.5.1. in [K], page 13.

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