# Seminar 13. An Introduction to Operads

Domenico Fiorenza

June 1, 1999

## Introduction

This short note contains a quick introduction to the theory of operads. It finds its collocation in this book dedicated to Maxim Kontsevich's theorem in deformation quantization of Poisson structures [5] as a consequence of his new paper on the same subject [6] in which he points out the role played by operads in his proof.

The note is organized as follows. In the first part basic definitions and examples from ribbon category theory are recalled, following mostly Bojko Bakalov and Aleksander Kirillov's [1]. In the second part Vladimir Turaev's graphical calculus for morphisms is introduced. Basic references for this part are [1] and Tuaraev's book [9]. In the third part, the concept of operad is finally introduced, as a category with "additional structure" on morphisms. This differs a bit from usual definitons of operads one can find in literature, and actually it is a slight generalization of those (see Example 4.3), close relative both of closed and enriched categories of Samuel Eilenberg and Gregory Maxwell Kelly ( [2] and [4] ) and of Saunders MacLane's props ( in a recent discussion on the subject I had with Ezra Getzler, he told me that operads can be considered as "the most important example of props"). The concepts of base change for an operad and of free operads are then introduced and a few basic examples are provided. The last part of the note is dedicated to algebras on operads, and the case of Gerstenhaber algebras (in a rather general sense) is discussed in detail.

### Acknowledgments

There are a lot of people whose contribution to this note has been essential. I wish to thank them all, and especially Enrico Arbarello, who organized the seminars, all the authors of the notes contained in this book, Aleksander Aleksandrovic Kirillov, Bojko Bakalov, Ezra Getzler, Marcello Paris and Paolo Caressa, for the many interesting discussions I had with them. Finally, a special aknowledgment goes to Federico De Vita who has realized all the graphical part of this note.

# 1 Ribbon categories

A ribbon category is basically a category with an associative tensor product, a symmetry corresponding to the permutation of factors in the tensor product, a concept of duality and natural isomorphisms of each object with his bidual (the basic example one has to have in mind is the category of finite dimensional vector spaces on a fixed base field). We are going to introduce the definition of ribbon category by steps, through the concepts of monoidal category and of braided and symmetric tensor categories. A further step could be made defining modular categories. We won't do it here.

**Definition 1.** *A* monoidal category $\mathcal{C} = (\mathcal{C}_0, \otimes, I, a, r, l)$ *consists of the following set of data:*

*i) A category* $\mathcal{C}_0$,

*ii) A bifunctor* $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$ *called* tensor product,

*iii) A* unit object $I \in Ob(\mathcal{C}_0)$,

*iv) A functorial isomorphism* $a_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ *called* associativity isomorphism,

*v) Functorial isomorphisms* $l_V : I \otimes V \to V$ *and* $r_V : V \otimes I \to V$,

*subject to the conditions given by the commutativity of the following diagrams:*

$$\begin{array}{ccccc} & & ((U \otimes V) \otimes W) \otimes Z & & \\ & \swarrow^{a} & & \searrow^{a} & \\ (U \otimes (V \otimes W)) \otimes Z & & & & (U \otimes V) \otimes (W \otimes Z) \\ \downarrow a & & & & \downarrow a \\ U \otimes ((V \otimes W) \otimes Z) & & \xrightarrow{Id \otimes a} & & U \otimes (V \otimes (W \otimes Z)) \end{array}$$

$$\begin{array}{ccc} (V \otimes I) \otimes W & \xrightarrow{a} & V \otimes (I \otimes W) \\ & {}_{r \otimes Id}\searrow \quad \swarrow_{l \otimes Id} & \\ & V \otimes W & \end{array}$$

$$I \otimes I \underset{l}{\overset{r}{\rightrightarrows}} I$$

**Example 1.1.** *If* $\mathcal{C}_0$ *is a category with finite products,* $\mathcal{C}_0$ *has a natural structure of monoidal category, taking as tensor product the product of* $\mathcal{C}_0$*, as unit element the final object of* $\mathcal{C}_0$ *and as* $a, l, r$ *the canonical isomorphisms induced by the universal property of the product and by the defining property of the final object. Such monoidal categories are called* cartesian monoidal categories.

*Examples of these are the category* **Sets** *of sets, the category* **Top** *of topological spaces, the category* $\mathbf{Cov}(X)$ *of coverings of a topological space* $X$ *(with product given by the fibre product over* $X$*), and the category* $\mathbf{Vect}(X)$ *of vector bundles over a topological space* $X$ *(again with the fibre product as product). Examples of non-cartesian monoidal categories are the category* **Ab** *of abelian groups, the category* $R$**-mod** *of modules on a commutative ring with identity* $R$*, the category* **dg-**$R$**-mod** *of differential graded* $R$*-modules, the categories* $\mathbf{Rep}(G)$*,* $\mathbf{Rep}_f(G)$*,* $\mathbf{Rep}(\mathfrak{g})$*,* $\mathbf{Rep}_f(\mathfrak{g})$*, of representations and finite dimensional representations of a group* $G$ *and of a Lie algebra* $\mathfrak{g}$*, all with the usual tensor products and unit objects. Another important example is the following. Given any unitary semigroup* $G$*, the discrete groupoid over* $G$ *with tensor product given by multiplication in* $G$ *is a monoidal category. Note that for any set* $S$*, the set of all finite words in the alphabet* $S$*,* $W_f(S) := \cup_{n=0}^{\infty} S^n$ *has a natural structure of unitary semigroup given by* $(a_1, a_2, \ldots, a_m) \cdot (b_1, b_2, \ldots, b_n) = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ *(the unit is the void word).*

Now that we have introduced the class of monoidal categories, we have to describe morphismps between them. Since our objects are a paricular class of categories, our morphisms will be a particular class of functors.

**Definition 2.** *Let* $\mathcal{C}, \mathcal{C}'$ *be monoidal categories. A* monoidal functor $\Phi : \mathcal{C} \to \mathcal{C}'$ *is given by a triple* $\Phi = (\phi, \tilde{\phi}, \phi^0)$*, where*

*i)* $\phi : \mathcal{C}_0 \to \mathcal{C}'_0$ *is a functor*

*ii)* $\tilde{\phi}_{UV} : \phi(U) \otimes \phi(V) \to \phi(U \otimes V)$ *is a natural trasformation*

*iii)* $\phi^0 : I' \to \phi(I)$ *is a morphism in* $\mathcal{C}'_0$

*and* $\phi, \tilde{\phi}, \phi^0$ *are such that the following diagrams commute:*

$$\begin{array}{ccc}
(\phi(A) \otimes \phi(B)) \otimes \phi(C) & \xrightarrow{a'} & \phi(A) \otimes (\phi(B) \otimes \phi(C)) \\
\downarrow{\scriptstyle \tilde{\phi} \otimes Id} & & \downarrow{\scriptstyle Id \otimes \tilde{\phi}} \\
\phi(A \otimes B) \otimes \phi(C) & & \phi(A) \otimes \phi(B \otimes C) \\
\downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \tilde{\phi}} \\
\phi((A \otimes B) \otimes C) & \xrightarrow{\phi(a)} & \phi(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccc}
I' \otimes \phi(A) & \xrightarrow{l'} & \phi(A) \\
\downarrow{\scriptstyle \phi_0 \otimes Id} & & \uparrow{\scriptstyle \phi(l)} \\
\phi(I) \otimes \phi(A) & \xrightarrow{\tilde{\phi}} & \phi(I \otimes A)
\end{array}$$

$$\begin{array}{ccc}
\phi(A) \otimes I' & \xrightarrow{r'} & \phi(A) \\
\downarrow{\scriptstyle Id \otimes \phi_0} & & \uparrow{\scriptstyle \phi(r)} \\
\phi(A) \otimes \phi(I) & \xrightarrow{\tilde{\phi}} & \phi(A \otimes I)
\end{array}$$

**Example 1.2.** *Two basic examples of monoidal functors are the* singular chains *(with coefficients in the ring R) functor,*

$$C_*(-,R) : \mathbf{Top} \to \mathbf{dg\text{-}}R\mathbf{\text{-}modules}$$

*and the* homology *functor*

$$H_* : \mathbf{dg\text{-}}R\mathbf{\text{-}modules} \to \mathbf{dg\text{-}}R\mathbf{\text{-}modules\ with\ zero\ differential}$$

*Their composition is the singular homology functor from the category of topological spaces to that of graded R-modules.*

In most of the examples of monoidal categories presented so far the objects $U \otimes V$ and $V \otimes U$ were isomorphic in a natural way. Formalizing this concept lead us to the following

**Definition 3.** *A* braided tensor category *is* $\mathcal{C} = (\mathcal{C}_0, \otimes, I, a, l, r, c)$ *where*

*i)* $(\mathcal{C}_0, \otimes, I, a, l, r)$ *is a monoidal category*

*ii)* $c_{UV} : U \otimes V \to V \otimes U$ *is a natural isomorphism (*commutativity isomorphism*) such that the following two diagrams commute:*

$$\begin{array}{ccccccc} & & U \otimes (V \otimes W) & \xrightarrow{c} & (V \otimes W) \otimes U & & \\ & \overset{a}{\nearrow} & & & & \overset{a}{\searrow} & \\ (U \otimes V) \otimes W & & & & & & V \otimes (W \otimes U) \\ & \underset{c \otimes Id}{\searrow} & & & & \underset{Id \otimes c}{\nearrow} & \\ & & (V \otimes U) \otimes W & \xrightarrow{a} & V \otimes (U \otimes W) & & \end{array}$$

$$\begin{array}{ccccccc} & & U \otimes (V \otimes W) & \xrightarrow{c^{-1}} & (V \otimes W) \otimes U & & \\ & \overset{a}{\nearrow} & & & & \overset{a}{\searrow} & \\ (U \otimes V) \otimes W & & & & & & V \otimes (W \otimes U) \\ & \underset{c^{-1} \otimes Id}{\searrow} & & & & \underset{Id \otimes c^{-1}}{\nearrow} & \\ & & (V \otimes U) \otimes W & \xrightarrow{a} & V \otimes (U \otimes W) & & \end{array}$$

*A braided tensor category* $\mathcal{C}$ *is called* symmetric *if moreover* $c_{VU}c_{UV} = Id_{U \otimes V}$.

**Example 1.3.** *All monoidal cartesian categories are symmetric, taking as c the isomorphism induced by the universal property of the product. The category of R-modules is symmetric with* $c_{UV}(x \otimes y) = y \otimes x$ *for any* $x \in U$ *and* $y \in V$. *The categories of linear representations of groups and Lie algebras are symmetric with the same c. The category of (differential) graded R-modules has two natural commutativity isomorphisms. One is the same as for R-modules, the other is* *the* graded *commutativity isomorphism given by* $c(x \otimes y) = (-1)^{xy}(y \otimes x)$. *An example of a braided tensor category that is not symmetric is the category* $\mathcal{C}(\mathfrak{g})$ *of the finite dimensional representations of the quantum group* $U_q(\mathfrak{g})$ *(see [1] for details). The discrete groupoid of a unitary semigroup* $G$ *is not a braided tensor category unless* $G$ *is commutative. In this case it is a symmetric tensor category.*

Now we come to the definition of morphisms between braided tensor categories.

**Definition 4.** *Let* $\mathcal{C}$, $\mathcal{C}'$ *be braided tensor categories. Then a* tensor functor $\Phi : \mathcal{C} \to \mathcal{C}'$ *is a triple* $\Phi = (\phi, \tilde{\phi}, \phi^0)$ *such that*

*i)* $\Phi$ *is a monoidal functor*

*ii) The following diagram commutes:*

$$\begin{array}{ccc} \phi(A) \otimes \phi(B) & \xrightarrow{c'} & \phi(B) \otimes \phi(A) \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \tilde{\phi}} \\ \phi(A \otimes B) & \xrightarrow{\phi(c)} & \phi(B \otimes A) \end{array}$$

**Example 1.4.** *The singular chains, homology and singular homology functors are examples of tensor functors.*

The next ingredient we are going to introduce to define ribbon categories is the concept of duality.

**Definition 5.** *Let* $\mathcal{C}$ *be a monoidal category. A* right duality *in* $\mathcal{C}$ *is a contravariant functor* $V \mapsto V^*$ *together with natural trasformations*

$$e_V : V^* \otimes V \to I$$

$$i_V : I \to V \otimes V^*$$

*such that the following diagrams commute:*

$$\begin{array}{ccc} V & \xrightarrow{i_V \otimes Id} & V \otimes V^* \otimes V \\ & {\scriptstyle Id} \searrow & \downarrow{\scriptstyle Id \otimes e_V} \\ & & V \end{array}$$

$$\begin{array}{ccc} V^* & \xrightarrow{Id \otimes i_V} & V^* \otimes V \otimes V^* \\ & {\scriptstyle Id} \searrow & \downarrow{\scriptstyle e_V \otimes Id} \\ & & V^* \end{array}$$

*Analoguosly, a* left duality *is a contravariant functor* $V \mapsto {}^*V$ *toghether with natural trasformations*

$$e'_V : V \otimes {}^*V \to I$$

$$i'_V : I \to {}^*V \otimes V$$

*such that the following diagrams commute*

$$\begin{array}{ccc} V & \xrightarrow{Id\otimes i'_V} & V \otimes {}^*V \otimes V \\ & \searrow^{Id} & \downarrow{\scriptstyle e'_V \otimes Id} \\ & & V \end{array}$$

$$\begin{array}{ccc} {}^*V & \xrightarrow{i'_V\otimes Id} & {}^*V \otimes V \otimes {}^*V \\ & \searrow^{Id} & \downarrow{\scriptstyle Id \otimes e'_V} \\ & & {}^*V \end{array}$$

**Remark 1.5.** *Note that a monoidal category $\mathcal{C}$ can have at most one right duality and one left duality, up to a canonical isomorphism. In fact, if $V \mapsto V^{*_1}$ and $V \mapsto V^{*_2}$ are two right dualities, we have the map $\varphi_V : V^{*_1} \to V^{*_2}$ given by the composition*

$$V^{*,1} \xrightarrow{Id\otimes i_2} V^{*,1} \otimes V \otimes V^{*,2} \xrightarrow{e_1\otimes Id} V^{*,2}$$

*that is a natural isomorphism between the two right dualities. The same argument works for left dualities.*

**Definition 6.** *A monoidal category $\mathcal{C}$ is said to be* rigid *if it has both left and right duals.*

**Example 1.6.** *The category vector spaces, vector bundles on a topological space $X$ and of linear representations of groups and of Lie algebras are rigid (one has to be careful in defining the tensor product in the infinite dimensional case: the standard tensor product won't work and one has to take a completion. Note, however that the usual tensor product and its completion coincide on finite dimensional spaces). The categories of sets, topological spaces and coverings of a topological space $X$ are not.*

**Remark 1.7.** *Duality implies the existence of natural isomorphism*

$$Hom(U \otimes V, W) \longrightarrow Hom(U, W \otimes V^*)$$

*and*

$$Hom(U, V \otimes W) \longrightarrow Hom(V^* \otimes U, W)$$

*given by sending $\psi \in Hom(U \otimes V, W)$ to the composition*

$$U \xrightarrow{Id\otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi\otimes Id} W \otimes V^*$$

*and $\psi \in Hom(U, V \otimes W)$ to the composition*

$$V^* \otimes U \xrightarrow{Id\otimes\psi} V^* \otimes V \otimes W \xrightarrow{e_V\otimes Id} W$$

*In physicists folklore this amount to saying that an incoming particle is the same thing as an outgoing antiparticle. Note that the isomorphism between Hom-spaces defined above reduced the knowledge of all the spaces $Hom(U,V)$ to that of the spaces $Hom(I,W)$, for $W$ varying in $\mathcal{C}_0$. Again, in physicists folklore, this corresponds to saying that each particle interaction can be considered as transition from the vacuum state.*

**Remark 1.8.** *Assume now that our category is braided. From the above observation follows that there are natural isomorphisms*

$$Hom(V,V) \simeq Hom(V, V \otimes I) \xrightarrow{\sim} Hom(V^* \otimes V, I)$$

*and*

$$Hom(V \otimes V^*, I) \xrightarrow{\sim} Hom(V, I \otimes V^{**}) \simeq Hom(V, V^{**}).$$

*Since $c_{VV^*}$ induces a pullback isomorphism $c^*_{VV^*} : Hom(V^* \otimes V, I) \tilde{\to} Hom(V \otimes V^*, I)$, the composition of the two isomorphisms above gives us an isomorphism $Hom(V,V) \tilde{\to} Hom(V, V^{**})$. Thus, to the identity in $V$ corresponds a morphism $\psi_V : V \to V^{**}$. In general $\psi_V$ is not an isomorphism (think to non reflexive Banach spaces), but there are many interesting cases in which it is, such as finite dimensional spaces or Hilbert spaces. We give to this property a name.*

**Definition 7.** *A rigid braided tensor category is said to be* reflexive *if the morphism $\psi_V$ is an isomorphism for each $V$.*

**Example 1.9.** *Finite dimensional vector spaces are reflexive, infinite dimensional vector spaces are not. A less trivial example of a reflexive category are Hilbert spaces.*

**Remark 1.10.** *The morphisms $\psi$ don't behave too well respect to tensor products. In fact, due to the presence of $c$ in their definition, one has*

$$\psi_{V \otimes W} = c_{VW} c_{WV} (\psi_V \otimes \psi_W)$$

*To control the action of the twists, we can multiplicate $\psi_V$ with some* balancing twist *$\theta_V$ such that*

$$\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c^{-1}_{WV} c^{-1}_{VW}$$

The existence of such a $\theta$ doesn't follow from the axioms of reflexive braided ribbon category, so we arrive to the definition of ribbon category:

**Definition 8.** *A* ribbon category *is a reflexive braided tensor category endoved with a natural isomorphism $\theta_V : V \to V$ such that*

*i)* $\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c^{-1}_{WV} c^{-1}_{VW}$

*ii)* $\theta_I = id_I$

*iii)* $\theta_{V^*} = (\theta_V)^*$

**Example 1.11.** *Finite dimensional vector spaces, finite dimensional vector bundles on a space $X$, finite dimensional linear representations of groups or Lie algebras, Hilbert spaces, unitary representations of compact groups are ribbon categories with the trivial twist $\theta_V = id_V$. On the category of (differential) graded vector spaces there are two natural twists: the trivial one and the one defined by $\theta(x) = (-1)^x x$ for homogeneus $x$.*

## 2 Turaev's graphical calculus

Graphical calculus is a powerful way of representing morphisms and their compositions in any ribbon category; it will be the basic tool for the costruction of free operads in the third section of this note.

**Definition 9.** *Let $\mathcal{C}$ be a ribbon category. A* Turaev $\mathcal{C}$-ribbon graph *is defined by the following set of data:*

*i) The two lines $t_0 := \mathbf{R} \times \{0\}$ and $t_1 := \mathbf{R} \times \{1\}$ in the real plane $\mathbf{R}^2$ with the standard orientation. These lines are called line of the initial state and line of the final state.*

*ii) $m$ distinct points on $t_0$ and $n$ distinct points on $t_1$, up to orientation preserving diffeomorphisms of $t_0$ and $t_1$.*

*iii) A directed ribbon graph (up to isotopy) with exactly $n+m$ external vertices corresponding to the distinct points of $t_0$ and $t_1$ and a cyclic ordering of the half edges occurring in any vertex. The graph need not to be connected.*

*iv) An object of $\mathcal{C}_0$ for every edge of the graph. The object corresponding to the edge $e$ is called* label *of $e$ and we write $V_e$ to denote it. The $m$-ple of objects corresponding to the points in $t_0$ is called the* initial labelling *of the graph, and the $n$-ple corresponding to the points in $t_1$ is called the* final label. *Note that, for every internal vertex $v$, the tensor products*

$$In(v) := \bigotimes_{e \text{ incoming in } v} V_e$$

*and*

$$Out(v) := \bigotimes_{e \text{ outgoing from } v} V_e$$

*are well defined, due to the ordering on the half edges occurring at the vertex. $In(v)$ and $Out(v)$ are called* incoming state *and* outgoing state *of the vertex.*

*v) For every internal vertex $v$, a morphism $\varphi_v : In(v) \to Out(v)$.*

Given two finite sequences of objects of $\mathcal{C}_0$, $\underline{A} = (A_1, A_2, \ldots, A_m)$ and $\underline{B} = (B_1, B_2, \ldots, B_n)$, denote as $Graphs_{\underline{A}}^{\underline{B}}$ the set of all Turaev graphs with initial labelling equal to $\underline{A}$ and final labelling equal to $\underline{B}$. Then we have two basic operations: *composition*

$$\circ : Graphs_{\underline{B}}^{\underline{C}} \times Graphs_{\underline{A}}^{\underline{B}} \to Graphs_{\underline{A}}^{\underline{C}}$$

that consists of putting the first graph on top of the second one, identifying the final label of the second with the initial label of the first

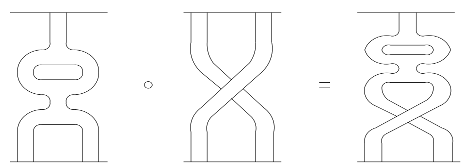


Figure 1:

and *tensor product*

$$\otimes : Graphs_{\underline{A}}^{\underline{B}} \times Graphs_{\underline{C}}^{\underline{D}} \to Graphs_{\underline{A},\underline{C}}^{\underline{B},\underline{D}}$$

consisting in putting the second graph on the right of the first one.

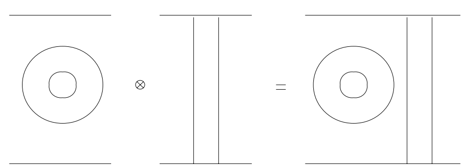


Figure 2:

Note that the labels of Turaev graphs are the objects of the tensor category of finite sequences of objects of $\mathcal{C}_0$. So *Graphs* can be thought as morphisms of an *enrichment* of this category, i.e., we can consider the category $Graphs(\mathcal{C})$ that has for objects the finite sequences of objects of $\mathcal{C}_0$ and for any two such sequences, $\underline{A}$ and $\underline{B}$,

$$Hom_{Graphs(\mathcal{C})}(\underline{A}, \underline{B}) := Graphs_{\underline{A}}^{\underline{B}}$$

# 3 The morphism associated to a Turaev graph

To each $\mathcal{C}$-Turaev graph $\Gamma$ corresponds a morphism $\varphi_\Gamma$ in $\mathcal{C}$ in such a way that

$$\varphi_{\Gamma_1 \circ \Gamma_2} = \varphi_{\Gamma_1} \circ \varphi_{\Gamma_2}$$

and

$$\varphi_{\Gamma_1 \otimes \Gamma_2} = \varphi_{\Gamma_1} \otimes \varphi_{\Gamma_2}$$

To define $\varphi_\Gamma$, assume, in a first moment, that all vertices, stationary points, twists and crossings (special points for short) of the graph $\Gamma$ lie at different vertical levels. Then choose $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n < \tau_{n+1} = 1$ such that in each strand $[\tau_i, \tau_{i+1}] \times \mathbf{R}$ lies at most one special point. Then change every downgoing arrow with the corrisponding upgoing one, changing the label of such strands with the dual label. Having done this, the graph $\Gamma$ has been expressed as the composition of the graphs $\Gamma_{i,i+1}$ and each of these is a tensor product of the form

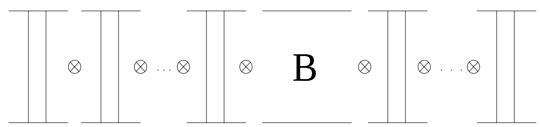



Figure 3:

where $B$ is one of these basic graphs:

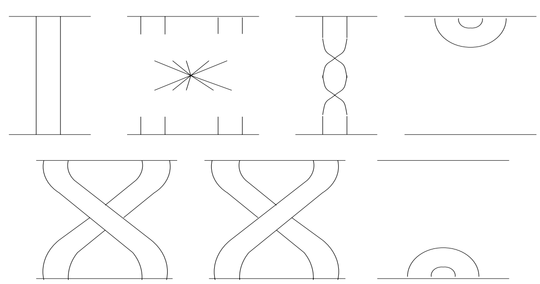


Figure 4:

Now we make the following associations: we let the first graph correspond to $Id_V$, the second one to the morphism $\varphi_V$ labelling the vertex, the third one to $\theta_V$, the fourth one to $i_V$, the fifth one to $c_{VW}$, the sixth one to $c^{-1}_{WV}$ and the seventh one to $e_V$. The morphisms $\varphi_{\Gamma_{i,i+1}}$ are then defined as tensor products of the $\varphi_B$ with copies of the identity and $\varphi_\Gamma$ is finally defined as the composition of the $\varphi_{\Gamma_{i,i+1}}$. It's important to point out that this construction of $\varphi_\Gamma$ is based on a particular choiche of a representative for the isotopy class of the graph $\Gamma$. Reshetikhin and Turaev's theorem tells us that $\varphi_\Gamma$ is actually independent of such a choice.

**Theorem 3.1.** *$\varphi_\Gamma$ depends only on the isotopy class of $\Gamma$, thus giving a modular functor*

$$\varphi : Graphs(\mathcal{C}) \to \mathcal{C}$$

*Proof.* We just sketch it here (one can find the complete proof in [7]). The basic observation needed to prove the theorem is that one can pass fron one representative for an isotopy class to another by a finite sequence of *simple moves* called *Reidmeister moves*. Then one has just to check that axioms for a ribbon category are such that $\varphi_\Gamma$ is unchanged when one applies a Reidmeister move to $\Gamma$. □

# 4 Operads

In our toy model of a ribbon category, finite dimensional vector spaces on a fixed base field, there is another property we still haven't considered: given two vector spaces $V$ and $W$, their morphisms $Hom(V, W)$ are again a vector space. This concept naturally generalizes to the naive idea of a category $\mathcal{C}_0$ whose hom-sets are objects of another category $\mathcal{C}'$ (when $\mathcal{C}_0 = \mathcal{C}'$ as in the case of vector spaces, one has the *closed categories* of Eilenberg and Kelly). Formalizing this idea we get the definition of $\mathcal{C}'$-category of [2].

**Definition 10.** *A $\mathcal{C}'$-category is $\mathcal{C} = (\mathcal{C}', \mathcal{C}_0, h, m, j)$ where*

*i) $\mathcal{C}_0$ is a category and $\mathcal{C}'$ is a monoidal one. $\mathcal{C}_0$ is called* category of the objects *and $\mathcal{C}'$ is called* category of the morphisms*.*

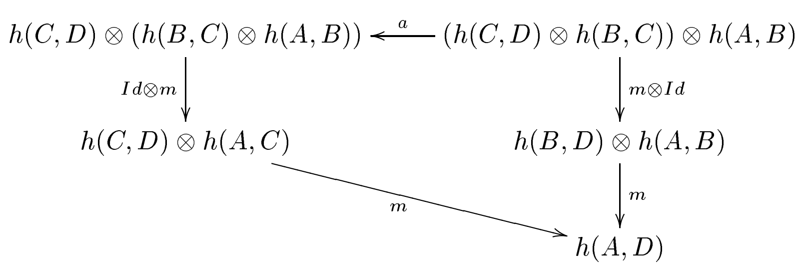
*ii) A* hom-space *functor $h : \mathcal{C}_0^{op} \times \mathcal{C}_0 \to \mathcal{C}'$.*

*iii) A natural trasformation $m_{ABC} : h(B, C) \otimes h(A, B) \to h(A, C)$ called* composition law*.*

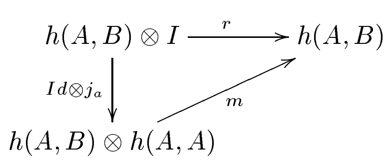
*iv) An* identity element *$j_A : I \to h(A, A)$ depending functorially on $A$.*

*such that the following diagrams commute:*

*i)*




*ii)*



$$\begin{array}{ccc} I \otimes h(A,B) & \xrightarrow{\;l\;} & h(A,B) \\ \downarrow{\scriptstyle j_b \otimes Id} & \nearrow{\scriptstyle m} & \\ h(B,B) \otimes h(A,B) & & \end{array}$$

**Remark 4.1.** *Every category can be seen as a* **Sets***-category by taking as $h, m, j$ the usual $Hom$, compositions and identity elements. Conversely, for each $\mathcal{C}'$ category there is an* underlying category $\pi(\mathcal{C})$. *To define it, one has to observe that we have a natural functor $\pi : \mathcal{C}' \to \mathbf{Sets}$ given by $\pi(A) := Hom(I, A)$. By abuse of language, the elements of $\pi(A)$ are called* elements of $A$ *(note that in many important examples, such as vector spaces or topological spaces, these are actually the elements of the space: vectors of a vector space correspond to the images of the element 1 in a morphism from the base field to the vector space, points of a topological space correspond to morphisms from a point to the space). Now it's clear that the underlyng category of $\mathcal{C}$ is defined as the category that has the same objects of $\mathcal{C}_0$ and as morphisms between them the elements of the hom-spaces $h(-,-)$. The compositions of two morphisms $f : I \to h(A, B)$ and $g : I \to h(B, C)$ is defined as the morphism $g \circ f$ given by the composition*

$$I \longrightarrow I \otimes I \xrightarrow{g \otimes f} h(B, C) \otimes h(A, B) \xrightarrow{m} h(A, C)$$

*With these definitions $m$ induces the composition of morphisms in $\pi(\mathcal{C})$ and $j$ induces the identities on the objects. One can notice that the category structure obtained this way, corresponds to the* **Sets***-category structure given by defining*

$$h' = \pi \circ h; \qquad m' = \pi \circ m; \qquad j' = \pi \circ j$$

*i.e. the underlying category is obtained by the* base change $\pi$ *(on this point we'll come back later).*

The functor $\pi$ forgets the additional structure we had on morphisms. Each time we can speak of a *free* object in $\mathcal{C}'$ generated by a set we have the adjoint of $\pi$, $free_{\mathcal{C}'}$ that changes a category in a $\mathcal{C}'$-category, simply by changing the hom-set $Hom(A, B)$ in the object of $\mathcal{C}'$ given by $Free(Hom(A, B))$. To be precise, we must define the functor $Free$. To do this we need $\mathcal{C}'$ to have coproduts. Then

$$Free(S) := \coprod_{s \in S} I$$

The concept of $\mathcal{C}'$ category is the right formalization of the intuitive idea of a category with structure on morphisms. The concept of *operad* corresponds to such a formalization for the concept of monoidal category with additional structure on morphisms. Then, to braided (symmetric) tensor categories correspond braided (symmetric) operads, and to ribbon categories correspond ribbon operads. To a particular kind of ribbon categories, *modular categories* should correspond objects to be called *modular operads.* Curiosly, objects called modular operads have been defined in [3] without any reference to modular categories.

At the moment I don't know whether the two definitions of a modular operad do coincide.

In defining a monoidal category with additional structure on morphisms the only thing to do is to define understand what should be a $\mathcal{C}'$-tensor product on $\mathcal{C}_0$. We need a preliminary definition.

**Definition 11.** *Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be $\mathcal{C}'$-categories. A $\mathcal{C}'$-functor $F : \mathcal{C} \to \tilde{\mathcal{C}}$ is a function $F_0 : \mathcal{C}_0 \to \tilde{\mathcal{C}}_0$ together with a natural trasformation $F_{AB} : h(AB) \to \tilde{h}(F_0(A), F_0(B))$ such that the following diagrams commute:*

$$\begin{array}{ccc} h(B,C)\otimes h(A,B) & \xrightarrow{\quad m \quad} & h(A,C) \\ \downarrow{\scriptstyle F\otimes F} & & \downarrow{\scriptstyle F} \\ \tilde{h}(F_0(B),F_0(C))\otimes \tilde{h}(F_0(A),F_0(B)) & \xrightarrow{\tilde{m}} & \tilde{h}(F_0(A),F_0(C)) \end{array}$$

$$\begin{array}{ccc} & & h(A,A) \\ & \nearrow^{j} & \\ I & & \downarrow{\scriptstyle F} \\ & \searrow_{\tilde{j}} & \\ & & \tilde{h}(F(A),F(A)) \end{array}$$

Now note that, if $\mathcal{C}$ is a $\mathcal{C}'$-category, $\mathcal{C}\times\mathcal{C}$ has a natural structure of $\mathcal{C}'$-category, setting

$$h\left((A,B),(C,D)\right) := h(A,C)\otimes h(B,D)$$

Then we come to the definition of $\mathcal{C}'$-tensor product.

**Definition 12.** *Let $\mathcal{C}$ be a $\mathcal{C}'$-category. A $\mathcal{C}'$-tensor product on $\mathcal{C}$ is a bi-$\mathcal{C}'$-functor $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$.*

We are now ready to define operads.

**Definition 13.** *A $\mathcal{C}$-*operad *is a $\mathcal{C}$-category $\mathcal{O}$, with a $\mathcal{C}$-tensor product such that the category underlying $\mathcal{O}$ is a monoidal one.* Braided tensor, symmetric, *and* ribbon *operads are defined in the natural way. By abuse of notation we often write $\mathcal{O}(A,B)$ for $h_{\mathcal{O}}(A,B)$.*

**Example 4.2.** *Each monoidal category can be seen as a* **Sets***-operad. Analoguosly, each ribbon category can be seen as a* **Sets***-ribbon operad. Again, the functor Free allows one to give more structure to morphisms of any monoidal or ribbon category, giving us examples of $\mathcal{C}$-(ribbon) operads.*

**Example 4.3.** *If $\mathcal{C}_0 = \{\mathbf{N}, +\}$, the data for a $\mathcal{C}$-operad reduce to*

*i) A monoidal category $\mathcal{C}$.*

*ii) A map $\mathcal{O} : \mathbf{N} \times \mathbf{N} \to Ob(\mathcal{C})$.*

*iii) For each $m, n, k \in \mathbf{N}$, composition maps*

$$\circ_{mnk} : \mathcal{O}(n,k) \otimes \mathcal{O}(m,n) \to \mathcal{O}(m,k).$$

*iv) For each $n \in \mathbf{N}$ a distinguished element (identity) $j_n \in \mathcal{O}(n,n)$.*

*v) For each $m, n, k, l \in \mathbf{N}$ a tensor product*

$$\otimes_{mnkl} : \mathcal{O}(m,n) \otimes \mathcal{O}(k,l) \to \mathcal{O}(m+k, n+l)$$

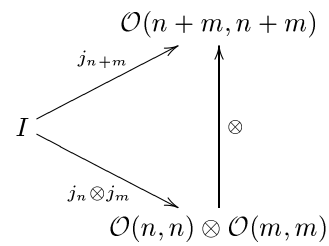
*such that the following diagrams commute:*

$$\begin{array}{ccc}
\mathcal{O}(c,d) \otimes (\mathcal{O}(h,c) \otimes \mathcal{O}(a,h)) & \xleftarrow{a} & (\mathcal{O}(c,d) \otimes \mathcal{O}(h,c)) \otimes \mathcal{O}(a,h) \\
\downarrow{\scriptstyle Id \otimes m} & & \downarrow{\scriptstyle m} \\
\mathcal{O}(c,d) \otimes \mathcal{O}(a,c) & & \mathcal{O}(h,d) \otimes \mathcal{O}(a,h) \\
& \searrow{\scriptstyle m} & \downarrow{\scriptstyle m} \\
& & \mathcal{O}(a,d)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{O}(m,n) \otimes I & \xrightarrow{r} & \mathcal{O}(m,n) \\
\downarrow{\scriptstyle Id \otimes j_m} & \nearrow{\scriptstyle m} & \\
\mathcal{O}(m,n) \otimes \mathcal{O}(m,m) & &
\end{array}$$

$$\begin{array}{ccc}
I \otimes \mathcal{O}(m,n) & \xrightarrow{l} & \mathcal{O}(m,n) \\
\downarrow{\scriptstyle j_n \otimes Id} & \nearrow{\scriptstyle m} & \\
\mathcal{O}(n,n) \otimes \mathcal{O}(m,n) & &
\end{array}$$

$$\begin{array}{ccc}
(\mathcal{O}(a,b) \otimes \mathcal{O}(c,d)) \otimes \mathcal{O}(e,f) & \xrightarrow{a} & \mathcal{O}(a,b) \otimes (\mathcal{O}(c,d) \otimes \mathcal{O}(e,f)) \\
\downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes} \\
\mathcal{O}(a+c, b+d) \otimes \mathcal{O}(e,f) & & \mathcal{O}(a,b) \otimes \mathcal{O}(c+e, d+f) \\
& \searrow{\scriptstyle \otimes} & \downarrow{\scriptstyle \otimes} \\
& & \mathcal{O}(a+c+e, b+d+f)
\end{array}$$




*Thus we obtain the structure commonly called operad in literature.*

**Remark 4.4.** *Let $\mathcal{O}$ be a $\mathcal{C}$-operad, and let $\Phi : \mathcal{C} \to \tilde{\mathcal{C}}$ be a monoidal functor. Then, defining $\tilde{h}$, $\tilde{m}$, $\tilde{j}$ and $\tilde{\otimes}$ respectively as $\phi \circ h$, $\phi(m) \circ \tilde{\phi}$, $\phi(j) \circ \phi^0$ and $\phi(\otimes) \circ \tilde{\phi}$, we obtain a $\tilde{\mathcal{C}}$-operad denoted $\Phi_*(\mathcal{O})$ (one has to check that diagrams in the definition of operad commute. This is an easy but tedious calculation we leave to the suspicious reader). The operad $\Phi_*(\mathcal{O})$ is said to be obtained from $\mathcal{O}$ by the* base change $\Phi$. *Note that tensor functors and ribbon functors provide base changes for braided and ribbon operads.*

**Example 4.5.** *The "forget the structure" functor $\pi$ is a base change from $\mathcal{C}$-operads to* **Sets**-*operads. The singular chains functor is a base change from topological operads to differential grade ones; the homology functor is a base change from dg-operads to graded operads.*

The last topic in the abstract theory of operads we want to introduce here is the concept of generators and relations for an operad. This will allow us to produce operads once we are given a basic set of operations. As an enlightning example think to the set of all possible operations in an algebra that you can write composing in all the possible ways the basic operationd of the algebra. We need some definitions.

**Definition 14.** *A $\mathcal{C}$-collection is a triplet $Coll = (\mathcal{C}, \mathcal{C}_0, h_{Coll})$, where $\mathcal{C}$ is a monoidal category, $\mathcal{C}_0$ a category, and $h_{Coll}$ is a functor $h_{Coll} : {\mathcal{C}_0}^{op} \times \mathcal{C}_0 \to \mathcal{C}$. By abuse of notation we often write $Coll(A, B)$ for $h_{Coll}(A, B)$.*

**Remark 4.6.** *From the definition above it's clear that a collection is just an operad without composition rules and tensor products. This means that we have a natural forgetful functor $\pi_{Coll}$ from the category of operads to that of collections. To construct its adjoint, we mimic the costruction of the operad $Graphs(\mathcal{C})$. For a collection $C$, we define $Graphs(C)$ to be the set of all Turaev graphs with internal vertices $v$ marked by $\varphi_v$ with $\varphi_v$ element of $C(In(v), Out(v))$. It's immediate to see that, defining composition and tensor product of graphs as above, $Graphs(C)$ is a* **Sets**-*operad with objects the finite sequences of objects of the category $\mathcal{C}_0$ of objects of the collection $C$. Now apply the functor $Free_{\mathcal{C}}$ to the hom-sets of $Graphs(C)$. This way we have defined a $\mathcal{C}$ operad $Free_{Op}(C)$ having as category of object the monoidal category of finite sequences of objects of $\mathcal{C}_0$. For two such sequences $\underline{A}$ and $\underline{B}$, one has*

$$Free_{Op}(C)(\underline{A}, \underline{B}) := Free_{\mathcal{C}}\left(Graph(C)^{\underline{B}}_{\underline{A}}\right)$$

*The functor $Free_{Op}$ so defined is clearly the adjoint of $\pi_{Coll}$. In fact, given a $\mathcal{C}$-collection $C$ and a $\mathcal{C}$-operad $\mathcal{O}$, every morphism of collections $\varphi : C \to \pi_{Coll}(\mathcal{O})$ (i.e. just a collection of morphisms $\varphi_{AB} : C(A,B) \to \mathcal{O}(A,B)$ defined for every $A, B \in Ob(\mathcal{C}_0)$), lifts in a natural way to $\tilde{\varphi} : Graphs(C) \to Graphs(\mathcal{O})$ and thus to $\tilde{\varphi} : Free_{Op}(C) \to Free_{\mathcal{C}}(Graphs(\mathcal{O}))$. Composing $\tilde{\varphi}$ with the canonical projection $Free_{\mathcal{C}}(Graphs(\mathcal{O})) \to \mathcal{O}$, one gets the desired lift of $\varphi$.*

# 5 Algebras over an operad

An *algebra* is just an object $A$ of a ribbon operad $\mathcal{O}$ toghether with multplications $m_i \in \mathcal{O}(A \otimes A, A)$ satisfying certain properties. A more precise way of expressing this concept is to consider the free operad $\mathcal{F}$ generated over $\mathbf{N}$ by the multiplications $m_i$, the twist $\theta_A$ and the switches $c_{AA}$. Then we have a morphism $\varphi : \mathcal{F} \to \underline{End}(A)$, where $\underline{End}(A)$ is the suboperad generated by $A$ inside $\mathcal{O}$, i.e. $\underline{End}(A)(m,n) := \mathcal{O}(A^{\otimes m}, A^{\otimes m})$. Relations among multiplications of $A$, switches of factors and the $\theta$'s give a factorization of $\varphi$ through a quotient $\mathcal{A}$ of $\mathcal{F}$. The operad $\mathcal{A}$ is the operad describing the structure of the algebra $A$. Thus we come to the definition of $\mathcal{O}$-algebra.

**Definition 15.** *Given an operad $\mathcal{O}$, an $\mathcal{O}$-algebra $A$ is simply an object $A$ of an operad together with a morphism $\varphi : \mathcal{O} \to \underline{End}(A)$.*

**Example 5.1.** *Algebras with one multiplication. The free operad governing them is generated by trivalent graphs, with vertices marked by the multiplication $\bullet$. Quotienting respect to the relation*

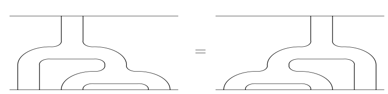


Figure 5:

*we get $\underline{Assoc}$ that governs associative algebras; quotienting respect to the relation*

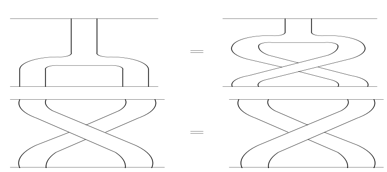


Figure 6:

*we get the operad $\underline{Comm}$ that governs commutative algebras.*

*Algebras with two multiplications. This time the free operad governing them is generated by trivalent graphs with two possible markings,* $\bullet$ *and* $\circ$ *on the vertices, corresponding to the two operations. Our basic example of operads with two operations is the operad* $\underline{Gerst}$*, governing Gerstenhaber algebras. The relations defining* $\underline{Gerst}$ *are the following:*

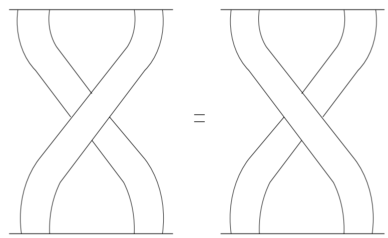


Figure 7:

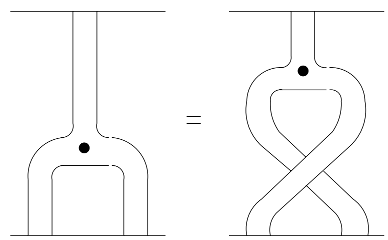


Figure 8: Commutativity of $\bullet$

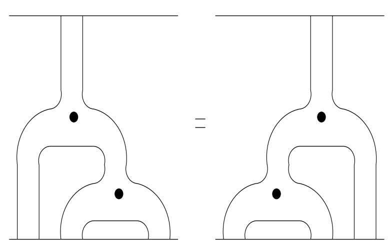


Figure 9: Associativity of $\bullet$

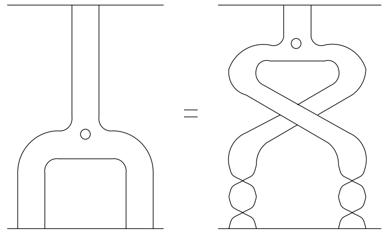


Figure 10: Odd anticommutativity of $\circ$

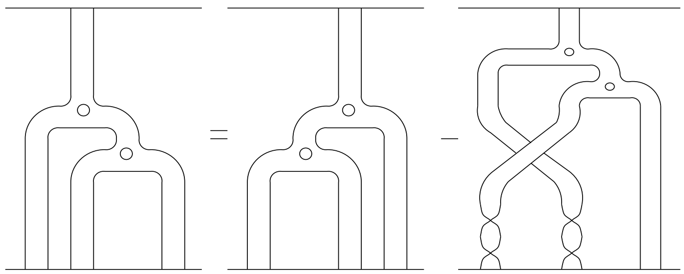


Figure 11: Odd Jacobi identity

*and the odd Poisson identity*

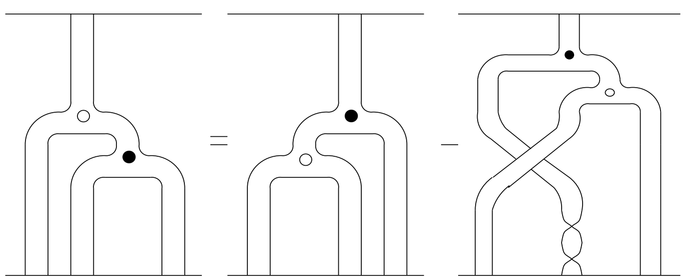


Figure 12: Odd Poisson identity

*Note that if one takes $\sigma(x \otimes y) = (-1)^{xy}(y \otimes x)$ and $\theta(x) = (-1)^x x$ in the category of supervector spaces and writes $\cdot$ for the $\bullet$ and $[,]$ for the $\circ$, one gets the classical definition of Gerstenhaber algebra (cfr [8]). If one adds the relation which equals*

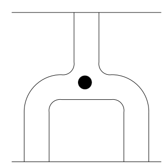


Figure 13:

*to 0, one obtains the operad odd Lie governing odd Lie superalgebras as a quotient of Gerst. This corresponds to the trivial statement that each odd Lie superalgebra can be seen as a Gerstenhaber algebra with vanishing $\cdot$ product. As a final observation, note that each vector space $A$ can be made "super" just putting all is elements in odd degree (and thus giving to the elements of the tensor product $A^{\otimes n}$ the same parity of $n$). With this assumption, the odd Lie superalgebra structure on $A$ is simply an usual Lie algebra structure.*

# References


[1] B. Bakalov, A. A. Kirillov, *Lectures on tensor categories*, unpublished.

[2] S. Eilenberg, G. M. Kelly, *Closed categories*, in *Proceedings of the Conference on Categorical Algebra*, La Jolla 1965, S.Eilenberg editor.

[3] E. Getzler and M.M. Kapranov, *Modular operads*, dg-ga/9408003

[4] G. M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society LNS 64, Cambridge University Press

[5] M. Kontsevich, *Deformation quantization of Poisson manifolds*, q-alg/9709040

[6] M. Kontsevich, *Operads in deformation quantization*, q-alg/9904055

[7] N.Y. Reshetikhin, V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inv.Mat. 103 (1991), 547-597

[8] R. Silvotti, *Topological vertex algebras*, Seminar 10 in this volume

[9] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, W. de Gruyter, Berlin, 1994