# THE CHOW RING OF $K3$ SURFACES AND HK VARIETIES

KIERAN G. O'GRADY

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## 0. Introduction

The Chow rings of smooth projective varieties with trivial canonical bundle have special properties. We will discuss $K3$ surfaces and more generally Hyperkähler varieties. For results on Abelian varieties see [2, 5], for results on Calabi-Yau varieties see [30, 12].

0.1. **Conventions.** The base field is $\mathbb{C}$. Unless we specify differently a point of a scheme $X$ is a *closed* point. A *variety* is an integral and separated scheme of finite type over $\mathbb{C}$. A curve is a variety of dimension 1, a surface a variety of dimension 2, etc. A *rational curve* is a projective curve whose normalization is isomorphic to $\mathbb{P}^1$. Let $X$ be a smooth projective variety and $F$ a coherent sheaf on $X$: we denote by $c_i(F)$ the Chern classes of $F$ in the *Chow ring* of $X$ and by $c_i^{\rm hom}(F)$ the Chern classes of $F$ in the *singular cohomology ring* of $X$.

## 1. The Beauville-Voisin ring and splittings of the Bloch-Beilinson filtration

We will present the result of Beauville and Voisin [7] on intersection of divisors on $K3$ surfaces and the conjectural generalization to Hyperkähler varieties of arbitrary dimension.

1.1. **Rational curves on $K3$ surfaces.** The key point is that every complete linear system on a $K3$ surface contains divisors whose support is the union of rational curves. The following is a result attributed to Bogomolov and Mumford.

**Theorem 1.1** (Bogomolov and Mumford [23]). *Let $X$ be a $K3$ surface and $L$ an ample line-bundle on $X$. There exists $D \in |L|$ whose support is the union of rational curves.*

There are also results motivated by the question: are there rational curves $D \in |L|$ ? First we introduce some notation. Let $X$ be a $K3$ surface and $L$ a line-bundle on $X$. We let

$$g(L) := \chi(L) - 1 = \frac{1}{2}\deg(L \cdot L) + 1. \tag{1.1}$$

If $D$ is a divisor on $X$ we let $g(D) := g(\mathcal{O}_X(D))$. Let $C \subset X$ be an integral curve: then $g(C)$ is the arithmetic genus of $C$. Now assume that $L$ is ample. Let $0 \le \delta \le g(L)$. We let $V_\delta(X, L) \subset |L|$ be the (Severi) variety parametrizing integral curves whose geometric genus is $(g(L) - \delta)$ - thus $V_\delta(X, L)$ is locally closed. If $V_\delta(X, L)$ is non-empty then it has pure dimension

$$\dim V_\delta(X, L) = \dim |L| - \delta = g(L) - \delta.$$

By results of X. Chen and Bogomolov - Hassett - Tschinkel we know that $V_\delta(X, L)$ is non-empty if $(X, L)$ is generic. Let us be more precise. Let

$$\pi \colon \mathcal{X} \longrightarrow T_g \tag{1.2}$$

be a complete family of $K3$ surfaces with a polarization of degree $(2g - 2)$ i.e. the following hold:

(1) $\pi$ is a projective and smooth map, we let $\mathcal{M}$ be "the"relatively ample line-bundle.

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(2) Let $t \in T_g$: then $X_t = \pi^{-1}(t)$ is a $K3$ surface.
(3) Let $t \in T_g$ and $M_t := \mathcal{M}|_{X_t}$: then $c_1^{\rm hom}(M_t)$ is indivisible and $g(M_t) = g$.
(4) if $X$ is a $K3$ surface equipped with an indivisible ample line-bundle $M$ with $g(M) = g$ there exist $t \in T_g$ and an isomorphism $f\colon X \xrightarrow{\sim} X_t$ such that $f^* M_t \cong M$.

Such a family exists, moreover we may assume that $T_g$ is irreducible by the Global Torelli Theorem for $K3$ surfaces. Below is the result that we mentioned (see Ch. 11 of [16] for a detailed treatment of the proof by Bogomolov - Hassett - Tschinkel).

**Theorem 1.2** (Chen [9], Bogomolov - Hassett - Tschinkel [8])**.** *Keep notation as above. Let $n > 0$ be an integer. There exists an open dense $U_g(n) \subset T_g$ such that the following holds. Let $0 \le \delta \le g(M_t^{\otimes n})$ and $t \in U_g(n)$: then $V_\delta(X_t, M_t^{\otimes n})$ is non-empty.*

## 1.2. Intersection of divisors on $K3$ surfaces.

**Proposition 1.3.** *Let $X$ be a projective $K3$ surface. Let $C_1, C_2 \subset X$ be rational curves. Let $p_i \in C_i$ for $i = 1, 2$. Then $p_1 \equiv p_2$.*

*Proof.* Since $X$ is projective there exists an ample line-bundle $L$ on $X$. By **Theorem 1.1** there exists $D \in |L|$ whose support is the union of rational curves. Since $D$ is ample there exists $q_i \in D \cap C_i$, and $p_i \equiv q_i$ because $C_i$ is rational. On the other hand $D$ is connected because it is ample, and since every component of $D$ is rational $q_1 \equiv q_2$. Thus $p_1 \equiv p_2$. □

By **Proposition 1.3** the following definiton makes sense.

**Definition 1.4.** Let $X$ be a projective $K3$ surface. The *Beauville-Voisin class* in $\mathrm{CH}_0(X)$ is the class $c_X$ represented by a point on an arbitrary rational curve in $X$.

**Theorem 1.5.** *Let $X$ be a projective $K3$ surface and $D_1, D_2 \in \mathrm{CH}_1(X)$: then*

$$D_1 \cdot D_2 = (\deg D_1 \cdot D_2) c_X. \tag{1.3}$$

*Proof.* Since every divisor is linearly equivalent to the difference of ample divisors we may assume that $D_1, D_2$ are ample. By **Theorem 1.1** we may further assume that each of $D_1, D_2$ is a sum of rational curves (with suitable positive coefficients), and then the statement is obvious. □

*Remark* 1.6. One may ask for which projective surfaces the image of

$$\begin{array}{ccc} \mathrm{CH}_1(X) \times \mathrm{CH}_1(X) & \longrightarrow & \mathrm{CH}_0(X) \\ (D_1, D_2) & \mapsto & D_1 \cdot D_2 \end{array} \tag{1.4}$$

is a subgroup of rank 1. By considering the blow-up of a surface with $p_g > 0$ we see that the image of (1.4) does *not* always have rank 1. See 1.4 of [12] for examples of smooth surfaces in $\mathbb{P}^3$ for which the image of (1.4) does not have rank 1.

1.3. **The Chow ring of HK varieties.** A compact Käbler manifold $X$ is *hyperkähler* (HK) if it is simply connected and $H^{2,0}(X)$ is spanned by the class of a holomorphic *symplectic* form. Notice that a HK manifold has trivial canonical bundle and is of even dimension. A HK manifold of dimension 2 is a $K3$ surface. Higher-dimensional HK manifolds behave like $K3$ surfaces in many respects, see [3, 14, 15, 25, 21]. An example (of Beauville [3]) of HK manifold of dimension $2n$ is the Douady space $S^{[n]}$ parametrizing length-$n$ analytic subsets of a $K3$ surface $S$. Let $n > 1$: the generic deformation of $S^{[n]}$ has no non-zero divisor, since $S^{[n]}$ contains the non-zero divisor $\Delta_n$ parametrizing non-reduced subsets it follows that the generic deformation of $S^{[n]}$ is *not* isomorphic to $(K3)^{[n]}$. Suppose that $S$ is a projective $K3$ surface and hence $S^{[n]}$ is the (projective) Hilbert scheme parametrizing length-$n$ subschemes of $S$. Then $h^{1,1}_{\mathbb{Z}}(S^{[n]}) \ge 2$ because the classes of $\Delta_n$ and an ample divisor $H$ are linearly independent. On the other hand deformation theory gives that a very generic *projective* deformation of $S^{[n]}$ keeping $H$ of type $(1,1)$ will have $h^{1,1}_{\mathbb{Z}} = 1$: thus it is not isomorphic to $(K3)^{[n]}$. Beauville and Donagi [4] have shown that one gets a locally complete family of projective HK 4-folds by considering the variety of lines of a smooth cubic 4-folds. Other explicit locally complete families of projective deformations of $K3^{[2]}$ are constructed in [19, 20, 24, 10].

Let $X$ be a smooth projective variety. We let $\mathcal{D}(X) \subset \mathrm{CH}^\bullet(X)$ be the subring generated by divisor classes and $\mathcal{D}(X)_{\mathbb{Q}} \subset \mathrm{CH}^\bullet(X)_{\mathbb{Q}}$ its tensor-product with $\mathbb{Q}$. Beauville [6] proposed the following conjecture.

**Conjecture 1.7** (Beauville)**.** *Let $X$ be a HK variety. The restriction of the cycle-class map*

$$\begin{array}{ccc} \mathcal{D}(X)_{\mathbb{Q}} & \longrightarrow & H^{\bullet}(X;\mathbb{Q}) \\ Z & \mapsto & cl(Z) \end{array} \tag{1.5}$$

*is injective.*

*Remark* 1.8.
(1) Let $X$ be a HK variety of dimension $2n$ and assume that (1.5) is injective. Then there exists a unique $c_X \in \mathrm{CH}_0(X)$ of degree 1 such that
$$D_1 \cdot D_2 \cdot \ldots \cdot D_{2n} = \deg(D_1 \cdot D_2 \cdot \ldots \cdot D_{2n}) c_X \tag{1.6}$$
for arbitrary $D_1 \cdot D_2 \cdot \ldots \cdot D_{2n} \in \mathrm{CH}^1(X)$. In fact let $Z$ be the left-hand side of (1.6). If $\deg Z = 0$ then $Z = 0$ by injectivity of (1.5). Next suppose that $\deg Z \neq 0$. Then $Z$ is rationally equivalent to $(\deg Z)Z_1$ where $Z_1 \in \mathrm{CH}_0(X)$ has degree 1 (represent $Z$ by a 0-cycle supported on a smooth curve), and such a $Z_1$ is unique by a celebrated Theorem of Roitman (if $V$ is a complex smooth projective variety the restriction of the Albanese map to the torsion of $\mathrm{CH}_0(V)$ is injective). Injectivity of (1.5) gives that $Z_1$ is independent of $Z$.
(2) If $X$ is a $K3$ surface the statement of **Conjecture 1.7** reduces to the statement of **Theorem 1.5**.
(3) Injectivity of (1.5) has been tested on various families of HK varieties [29, 11].
(4) Notice that if $X$ is a HK variety of dimension $2n > 2$ there exists no divisor $D \subset X$ with rational desingularization $\phi\colon \widetilde{D} \to D$. In fact let $\sigma$ be a regular symplectic form on $X$: then $\pi^*\sigma$ is a non-zero regular 2-form on $\widetilde{D}$ because the maximum dimension of a lagrangian subspace of a $2n$-dimensional symplectic vector-space is equal to $n < \dim D$. Thus the proof of **Theorem 1.5** does not lend itself to an immediate generalization.
(5) Let $X$ be the variety of lines on a smooth cubic 4-fold $Y \subset \mathbb{P}^5$: Voisin [29] has proved that (1.5) is injective. The class $c_X$ (see Item (1) above) is described as follows. Let $H \subset \mathbb{P}^5$ be a hyperplane transversal to $Y$: the variety $S := F(Y \cap H)$ of lines in $Y \cap H$ is a lagrangian surface in $X$. By degenerating $H$ one gets a degeneration $S_0$ of $F(Y \cap H)$ which has rational desingularization: then $c_X$ is represented by any point of $S_0$. A similar picture holds for the locally complete family of projective HK 4-folds given by double EPW-sextics [24], this was proved by Ferretti [11]. In that case the analogue of the surface $S_0$ is an Enriques surface $S_0$. These examples suggest the following questions:
   (a) Let $X$ be a HK 4-dimensional variety and $S \subset X$ a Lagrangian surface: among deformations of $S$ does there exists a surface $S_0$ whose desingularization(s) has vanishing geometric genus ?
   (b) Let $S_0$ be as above and assume the validity of Bloch's conjecture for surfaces. Then any two points of $S_0$ are rationally equivalent: one is tempted to conjecture that any such point represents the class $c_X$ whose existence is predicted by **Conjecture 1.7**.

Beauville [6] derived injectivity of (1.5) from a conjectural splitting of the (conjectural) Bloch-Beilinson filtration of the Chow ring of $X$ (and called it the *weak splitting property*). Motivation for this line of thought comes from what is known to hold for abelian varieties: if there exists a filtration on the Chow ring of abelian varieties which satisfies the conjecture of Bloch-Beilinson then it is the filtration associated to a *ring graduation* of the Chow ring. For some evidence in favour of Beauville's splitting conjecture see [27].

## 2. Decompositions of small diagonals

In [7] Beauville and Voisin proved the following result.

**Theorem 2.1.** *Let $X$ be a projective $K3$ surface. Then*
$$c_2(X) = 24 c_X. \tag{2.1}$$

It is amusing to prove (2.1) for particular classes of $K3$ surfaces. Let $S \subset \mathbb{P}^3$ be a smooth quartic. We have an exact sequence of locally-free sheaves
$$0 \longrightarrow T_S \longrightarrow T_{\mathbb{P}^3}|_S \longrightarrow \mathcal{O}_S(4) \longrightarrow 0. \tag{2.2}$$
Let $h := c_1(\mathcal{O}_S(1))$. Since $c(T_{\mathbb{P}^3}) = (1+h)^4$ it follows from Whitney's formula and (1.3) that (2.1) holds for $S$. A similar argument works for a $K3$ surface which is a double covering of $\mathbb{P}^2$ or a complete

intersection. Another class of $K3$'s for which (2.1) is clearly true is that of elliptic $K3$'s: considering an elliptic fibration $f\colon S\to\mathbb{P}^1$ we see that $c_2(S)$ is represented by a 0-cycle supported on the singular fibers of $f$ (because $K_S$ is trivial away from singular fibers) and since the singular fibers are unions of rational curves we get (2.1). Equation (2.1) follows from an interesting relation in the Chow ring of $X\times X\times X$ involving the small diagonal. First we will introduce the relevant cycles, after that we will sketch the proof of **Theorem 2.1**.

2.1. **Modified diagonals.** Let $X$ be a smooth projective variety of dimension $n$. Fix an integer $m>1$ and $c\in X$. Let $I\subset\{1,2,\ldots,m\}$ be a *non-empty* subset: we let

$$\Delta_I(c):=\{(x_1,\ldots,x_m)\in X^m\mid x_i=x_j \text{ if } i,j\in I,\ x_h=c \text{ if } h\notin I\}. \tag{2.3}$$

Thus $\Delta_{1,2,\ldots,m}(c)$ is the small diagonal $\Delta_{1,2,\ldots,m}\subset X^m$ (independent of $c$), and each $\Delta_I(c)$ is a closed subset of $X^m$ isomorphic to $X$. We will consider the $n$-cycle on $X^m$

$$\Gamma_m(c):=\sum_{\emptyset\neq I\subset\{1,2,\ldots,m\}}(-1)^{m-|I|}\Delta_I(c)\in Z_n(X^m). \tag{2.4}$$

For $m=2$ and $m=3$ we get

$$\Gamma_2(c)=\Delta_{12}-\Delta_1(c)-\Delta_2(c),\qquad \Gamma_3(c)=\Delta_{123}-\Delta_{12}(c)-\Delta_{13}(c)-\Delta_{23}(c)+\Delta_1(c)+\Delta_2(c)+\Delta_3(c). \tag{2.5}$$

**Proposition 2.2.** *Keep notation as above. Let $\alpha_i\in H^{d_i}(X;\mathbb{Q})$ for $i=1,\ldots,m$ and suppose that $\sum_{i=1}^m d_i=2n$ and $d_1\cdot d_2\cdot\ldots\cdot d_m=0$. Let $\pi_i\colon X^m\to X$ be the projection to the $i$-th factor; then*

$$cl(\Gamma_m(c))\cup\pi_1^*\alpha_1\cup\ldots\cup\pi_m^*\alpha_m=0. \tag{2.6}$$

*Proof.* By the symmetry of $\Gamma_m(c)$ we may assume that $0=d_1=\ldots=d_s$ and $d_i>0$ for $0<s<i\le m$. Let $\rho_I\colon X\xrightarrow{\sim}\Delta_I$ be the obvious isomorphism. We must prove that

$$\sum_{\emptyset\neq I\subset\{1,2,\ldots,m\}}(-1)^{m-|I|}\int_{[X]}\rho_I^*(\pi_1^*\alpha_1\cup\ldots\cup\pi_m^*\alpha_m)=0. \tag{2.7}$$

Clearly we have

$$\int_{[X]}\rho_I^*(\pi_1^*\alpha_1\cup\ldots\cup\pi_m^*\alpha_m)=\begin{cases}\int_{[X]}\alpha_{s+1}\cup\ldots\cup\alpha_m & \text{if } I\supset\{s+1,\ldots,m\},\\ 0 & \text{if } I\not\supset\{s+1,\ldots,m\}.\end{cases} \tag{2.8}$$

It follows that the left-hand side of (2.7) is equal to

$$\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}\cdot\int_{[X]}\alpha_{s+1}\cup\ldots\cup\alpha_m=(1-1)^s\int_{[X]}\alpha_{s+1}\cup\ldots\cup\alpha_m=0. \tag{2.9}$$

□

**Corollary 2.3.** *Let $X$ be a smooth projective variety of dimension $n$. If $m>2n$ then the homology class of $\Gamma_m(c)$ is torsion. If $H^1(X;\mathbb{Q})=0$ then the homology class of $\Gamma_m(c)$ is torsion as soon as $m>n$.*

*Proof.* By the Künneth decomposition it suffices to prove that (2.6) holds for any choice of $\alpha_i\in H^{d_i}(X;\mathbb{Q})$, where $1\le i\le m$. Our hypotheses ensure that at least one of the $d_i$'s vanishes and hence (2.6) holds by **Proposition 2.2**. □

B. Gross and C. Schoen have studied $\Gamma_3(c)$ for $X$ a curve, see [13]. The key result that we will need is the following.

**Proposition 2.4** (Gross-Schoen [13])**.** *Let $E$ be a curve of genus 1 and $c\in E$. Then $6\Gamma_3(c)=0$.*

*Proof.* The symmetric group on 3 elements $\mathcal{S}_3$ acts on $E^3$ with quotient the symmetric product $E^{(3)}$. We will identify $E^{(3)}$ with the variety parametrizing effective divisors of degree 3 on $E$. Let $\pi\colon E^3\to E^{(3)}$ be the quotient map. The cycle $\Gamma_3(c)$ is invariant under $\mathcal{S}_3$; since $\deg\pi=6$ it follows that

$$\pi^*(\pi_*\Gamma_3(c))=6\Gamma_3(c). \tag{2.10}$$

Thus it suffices to prove that

$$0=\pi_*\Gamma_3(c)=\{3x\mid x\in E\}-3\{c+2x\mid x\in E\}+3\{2c+x\mid x\in E\}. \tag{2.11}$$

The tautological map

$$\rho\colon E^{(3)}\to\mathrm{Pic}^3(E)\cong E \tag{2.12}$$

is the projectivization of a rank-3 vector-bundle. In order to simplify notation we choose an isomorphism $\mathrm{Pic}^3(E) \xrightarrow{\sim} E$. Let $h \in \mathrm{Pic}(E^{(3)})$ be a divisor which restricts to $\mathcal{O}_{\mathbb{P}^2}(1)$ on the fibers of $\rho$: then

$$\mathrm{CH}_1(E^{(3)}) = \{ah^2 + h \cdot \rho^*\eta \mid a \in \mathbb{Z}, \quad \eta \in \mathrm{Pic}(E)\}. \tag{2.13}$$

By **Corollary 2.3** the class of $\pi_*\Gamma_3(c)$ in $H^4(E^{(3)};\mathbb{Q})$ is zero: it follows that

$$\pi_*\Gamma_3(c) = h \cdot \rho^*\eta, \quad \deg\eta = 0. \tag{2.14}$$

Thus it suffices to prove that

$$0 = \eta = \rho_*(h \cdot \pi_*\Gamma_3(c)). \tag{2.15}$$

Let $p \in E$; as $h$ we choose the class represented by the divisor

$$D := \{A \in E^{(3)} \mid A - p \geq 0\}. \tag{2.16}$$

We have

$$\rho_*(D \cdot \pi_*\Gamma_3(c)) = 3[3p] - 6[c + 2p] + 3[2c + p]. \tag{2.17}$$

Here $[3p], [c+2p], [2c+p]$ are the *points* of $\mathrm{Pic}^3(E)$ represented by $3p$, $c+2p$ and $2c+p$ respectively - the coefficients are given by a straightforward multiplicity computation. Notice that the degree of the right-hand side of (2.17) is zero, as expected. The map

$$\begin{array}{ccc} \mathrm{Pic}^0(\mathrm{Pic}^3(E)) & \xrightarrow{\sigma} & \mathrm{Pic}^0(E) \\ \sum_i r_i[A_i] & \mapsto & \sum_i r_i A_i \end{array} \tag{2.18}$$

is an isomorphism. Applying $\sigma$ to the right-hand side of (2.17) we get 0: it follows that (2.15) holds. □

2.2. **Decomposition of the small diagonal for $K3$ surfaces.**

**Theorem 2.5** (Beauville-Voisin [7]). *Let $X$ be a projective $K3$ surface and $c_X \in \mathrm{CH}_0(X)$ the Beauville-Voisin class. Then $\Gamma_3(c_X)$ is a torsion class.*

*Sketch of proof.* Let $H$ be an ample primitive divisor on $X$. If $(X, H)$ is generic there exists an irreducible 1-dimensional family, say $\overline{\mathcal{E}} \to \overline{B}$ of curves on $X$ whose generic member is a curve of geometric genus 1 i.e. its normalization is a smooth curve of genus 1. Let $R \subset X$ be a rational curve: by base change $B \to \overline{B}$ we get an elliptic surface $\rho\colon \mathcal{E} \to B$ with a section $\sigma\colon B \to \mathcal{E}$ and a regular surjective map $f\colon \mathcal{E} \to X$ such that $f(\sigma(B)) = R$. One obtains the result for $X$ by applying **Proposition 2.4** to the cycle on the triple fiber-product $\mathcal{E} \times_\rho \mathcal{E} \times_\rho \mathcal{E}$ which restricts to $\Gamma_3(\sigma(b))$ on $E_b \times E_b \times E_b$ for regular values $b$. (A toy-model is that of $X$ an elliptic fibration with a section.) The result for arbitrary $X$ follows from the result for a generic polarized $X$. □

Now let's show that **Theorem 2.1** follows from **Theorem 2.5**. Let $\tau\colon X^3 \to X$ be the projection to the *third* factor. Let

$$\Omega := [\{(x, x, y)\}] \in \mathrm{CH}^2(X \times X \times X). \tag{2.19}$$

In order to simplify notation we will denote $\Delta_I(c_X)$ (see (2.3)) by $\Delta_I$. **Theorem 2.5** gives that in the rational Chow ring $\mathrm{CH}(X)_{\mathbb{Q}}$ we have the following equality:

$$0=\tau_*(\Gamma_3(c_X)\cdot\Omega)=\tau_*(\Delta_{123}\cdot\Omega)-\tau_*(\Delta_{12}\cdot\Omega)-\tau_*(\Delta_{13}\cdot\Omega)-\tau_*(\Delta_{23}\cdot\Omega)+\tau_*(\Delta_1\cdot\Omega)+\tau_*(\Delta_2\cdot\Omega)+\tau_*(\Delta_3\cdot\Omega). \tag{2.20}$$

We claim that

$$\tau_*(\Delta_{123}\cdot\Omega)=c_2(X), \quad \tau_*(\Delta_{12}\cdot\Omega)=24c_X, \quad \tau_*(\Delta_{13}\cdot\Omega)=c_X, \quad \tau_*(\Delta_{23}\cdot\Omega)=c_X, \quad \tau_*(\Delta_1\cdot\Omega)=c_X, \quad \tau_*(\Delta_2\cdot\Omega)=c_X, \quad \tau_*(\Delta_3\cdot\Omega)=0. \tag{2.21}$$

In fact the first and second equalities are computed as follows. Since $\Delta_{123}, \Delta_{12} \subset \Omega$ the cycles $\Delta_{123} \cdot \Omega$, $\Delta_{12} \cdot \Omega$ are the push-forward of $c_2(N_{\Omega/X^3})|_{\Delta_{123}}$ and $c_2(N_{\Omega/X^3})|_{\Delta_{12}}$ respectively. On the other hand both $\Delta_{123}$ and $\Delta_{12}$ are naturally identified with $X$ and via this identification $N_{\Omega/X^3}|_{\Delta_{123}}$ and $N_{\Omega/X^3}|_{\Delta_{12}}$ are identified with the tangent bundle of $X$; the first and second equalities follow. The other intersections are transverse with the exception of the last. Since we may represent $\Delta_3$ as a sum of cycle of the form $\{(d_j, e_j, x) \mid x \in X\}$ with $d_j \neq e_j$ we get the last equality. Plugging the equalities of (2.21) into (2.20) we get that $(c_2(X) - 24c_X)$ is a torsion class in $\mathrm{CH}_0(X)$. On the other hand a celebrated Theorem of Roitman asserts that on a regular projective smooth complex variety $V$ the group $\mathrm{CH}_0(V)$ has no torsion: that proves **Theorem 2.1**.

2.3. **Voisin's conjecture on the Chow ring of HK's.** Let $X$ be a Hyperkähler variety. We let $\mathcal{E}(X) \subset \mathrm{CH}^\bullet(X)$ be the subring generated by divisor classes and the Chern classes of $X$ and $\mathcal{E}(X)_\mathbb{Q} \subset \mathrm{CH}^\bullet(X)_\mathbb{Q}$ its tensor-product with $\mathbb{Q}$. (Notice that the odd Chern classes of $X$ are 2-torsion and hence they do not contribute to $\mathcal{E}(X)$.) Voisin [29] formulated the following conjecture.

**Conjecture 2.6** (Voisin). *Let $X$ be a HK variety. The restriction of the cycle-class map*

$$\begin{array}{ccc} \mathcal{E}(X)_\mathbb{Q} & \longrightarrow & H^\bullet(X;\mathbb{Q}) \\ Z & \mapsto & cl(Z) \end{array} \tag{2.22}$$

*is injective.*

Evidence in favour of the above conjecture has been given by Voisin [29] and by Ferretti [11].

## 3. Moduli of sheaves on $K3$ surfaces and the Chow ring

Throughout the present section $X$ is a complex projective $K3$ surface. The main result asserts essentially that the set whose elements are $c_2(F)$ for $F$ varying among (semi)stable sheaves on $X$ parametrized by a moduli space of sheaves with fixed topological Chern classes depends only on the dimension of the moduli space. Before formulating the main result we will introduce a filtration of $CH_0(X)$ by subsets and we will recall notation and results valid for semistable sheaves on $X$.

### 3.1. A filtration on $\mathrm{CH}_0(K3)$.

**Definition 3.1.** Let $S_g(X) \subset CH_0(X)$ be the set of classes $[Z] + ac_X$ where $Z = p_1 + \ldots + p_g$ is an effective 0-cycle of degree $g$ and $a \in \mathbb{Z}$.

Notice that $S_0(X) = \mathbb{Z}c_X$.

**Claim 3.2.** *Let $C$ be an irreducible smooth projective curve of genus $g$ and $f\colon C \to X$ be a non-constant map. Then $f_*CH_0(C) \subset S_g(X)$.*

*Proof.* There exists $p \in C$ such that $f_*[p] = c_X$. In fact let $H$ be a primitive ample divisor on $X$, by [23] there exists $D \in |H|$ whose irreducible components are rational curves. Since $f$ is not constant and $D$ is ample $f(C) \cap D \neq \emptyset$: if $p \in f^{-1}(D)$ then $i_*[p] = c_X$. Now let $\mathfrak{z} \in CH_0(C)$. By Riemann-Roch there exists an effective cycle $p_1 + \ldots + p_g$ on $C$ such that $\mathfrak{z} = [p_1 + \ldots + p_g] + (\deg \mathfrak{z} - g)p$: thus $f_*\mathfrak{z} = ([f(p_1) + \ldots + f(p_g)] + (\deg \mathfrak{z} - g)c_X) \in S_g(X)$. □

Multiplication by $\mathbb{Z}$ maps $S_g(X)$ to itself and hence we may say that $S_g(X)$ is a cone. On the other hand $S_g(X)$ is a subgroup of $CH_0(X)$ only if $g = 0$. We have a filtration

$$S_0(X) \subset S_1(X) \subset \ldots \subset S_g(X) \subset S_{g+1}(X) \subset \ldots \subset CH_0(X). \tag{3.1}$$

In fact let $\mathfrak{z} = ([p_1 + \ldots + p_g] + ac_X) \in S_g(X)$. Let $p_{g+1} \in X$ be a point lying on a rational curve: then $[p_{g+1}] = c_X$ and hence $\mathfrak{z} = ([p_1 + \ldots + p_g + p_{g+1}] + (a-1)c_X) \in S_{g+1}(X)$. This proves (3.1). We also have that

$$\bigcup_{g=0}^{\infty} S_g(X) = CH_0(X). \tag{3.2}$$

In fact let $\mathfrak{z} \in CH_0(X)$. There exist a smooth curve $\iota\colon C_0 \hookrightarrow X$ of genus $g$ and a cycle $D_0 \in Z^1(C_0)$ such $\mathfrak{z} = [\iota_* D_0]$. By **Claim 3.2** we get that $\mathfrak{z} \in S_g(X)$; this proves (3.2).

### 3.2. Moduli of sheaves on a K3-surface.

The *Mukai pairing* on $H^\bullet(X;\mathbb{Z})$ is the symmetric bilinear form defined by

$$\langle \alpha, \beta \rangle := -\int_X \alpha^\vee \cup \beta, \qquad (\alpha_0 + \alpha_2 + \alpha_4)^\vee := \alpha_0 - \alpha_2 + \alpha_4, \quad \alpha_p \in H^p(X;\mathbb{Z}). \tag{3.3}$$

Let

$$v = (r, \ell, s) \in H^\bullet(X;\mathbb{Z}). \tag{3.4}$$

(We identify $H^4(X;\mathbb{Z})$ with $\mathbb{Z}$ via the orientation class.)

**Definition 3.3.** A *Mukai vector* (for $X$) is a $v$ as in (3.4) such that the following hold:

1. $r \geq 0$,
2. $\ell \in H^{1,1}_\mathbb{Z}(X)$,
3. if $r = 0$ then $\ell$ is effective.

Given a coherent sheaf $F$ on $X$ the *Mukai vector of $F$* is

$$v(F) := (\mathrm{ch}_0^{\mathrm{hom}}(F) + \mathrm{ch}_1^{\mathrm{hom}}(F) + \mathrm{ch}_2^{\mathrm{hom}}(F)) \cup \sqrt{\mathrm{Td}_X} \tag{3.5}$$

where $c_p^{\mathrm{hom}}(F) \in H^{2p}(V;\mathbb{Z})$ is the topological $p$-th Chern class of $F$. Suppose that $X$ is projective and $H$ is an ample divisor on $X$. Let $v \in H^\bullet(X;\mathbb{Z})$ be a Mukai vector and $\mathfrak{M}_v(X,H)$ be the moduli space of $S$-equivalence classes of pure $H$-semistable sheaves on $X$ with $v(F) = v$, see [18, 28]. Thus $\mathfrak{M}_v(X,H)$ is a projective complex scheme. Let $\mathfrak{M}_v(X,H)^{\mathrm{st}}$ be the open subscheme of $\mathfrak{M}_v(X,H)$ parametrizing isomorphism classes of pure $H$-stable sheaves. Suppose that $\mathfrak{M}_v(X,H)^{\mathrm{st}}$ is not empty: then it is smooth of pure dimension given by

$$\dim \mathfrak{M}_v(X,H)^{\mathrm{st}} = 2 + v^2 = 2d(v). \tag{3.6}$$

(We let $v^2 := \langle v, v\rangle$.) Notice that $d(v)$ is an integer because the Mukai pairing is even. We let $\overline{\mathfrak{M}}_v(X,H)^{\mathrm{st}}$ be the closure of $\mathfrak{M}_v(X,H)^{\mathrm{st}}$ in $\mathfrak{M}_v(X,H)$. Let

$$c_2(v) := r + \frac{\ell \cdot \ell}{2} - s. \tag{3.7}$$

Thus $c_2(v)$ is the degree of $c_2(F)$ where $F$ is a coherent sheaf such that $v(F) = v$.

*Remark* 3.4. Let $[F] \in \mathfrak{M}_v(X,H)$ with $F$ not $H$-stable i.e. properly $H$-semistable. The same point of $\mathfrak{M}_v(X,H)$ is represented by any $H$-semistable pure sheaf $G$ which is $S$-equivalent to $F$ i.e. such that $\mathrm{gr}^{JH}(F) \cong \mathrm{gr}^{JH}(G)$ where $\mathrm{gr}^{JH}(F)$, $\mathrm{gr}^{JH}(G)$ are the the direct-sums of the successive quotients of Jordan-Holder filtrations of $F$ and $G$. It follows that although $F$, $G$ may not be isomorphic the Chern classes $c_2(F)$ and $c_2(G)$ are equal. Thus we may associate to $[F] \in \mathfrak{M}_v(X,H)$ a well-defined class $c_2(F) \in \mathrm{CH}_0(X)$.

### 3.3. **The main result.**

**Theorem 3.5** (Huybrechts, O'Grady, Voisin)**.** *Let $X$ be a complex projective $K3$ surface and $H$ an ample divisor on $X$. Let $v = (r, \ell, s)$ be a Mukai vector. Suppose that $\mathfrak{M}_v(X,H)^{\mathrm{st}}$ is not empty. Then*

$$\{c_2(F) \mid [F] \in \overline{\mathfrak{M}}_v(X,H)^{\mathrm{st}}\} = \{\mathfrak{z} \in S_{d(v)}(X) \mid \deg \mathfrak{z} = c_2(v)\}. \tag{3.8}$$

*(Here* $\deg\colon CH_0(X) \to \mathbb{Z}$ *is the degree homomorphism.)*

Huybrechts [17] proved the result for $v^2 = -2$ under additional hypotheses on $v$ or $X$; essentially either that $\ell$ is indivisible or the Picard number of $X$ is at least 2. O'Grady [26] introduced Filtration (3.1) and proved the result under additional hypotheses on $v$ or $X$: the conditions are similar to those of Huybrechts or one may assume that $r \le 2$ (plus an epsilon). The proof of O'Grady is an improved version of the proof of Huybrechts. Voisin proved the result without additional assumptions - the key step in her proof is a very interesting characterization of Filtration (3.1), more precisely of the subset of $S_g(X)$ consisting of 0-cycles of degree greater than $g$. Actually Voisin showed that simplicity of the sheaves (as opposed to stability) is the key property (as conjectured in [26]) - see **Subsection 3.5**.

3.4. **Voisin's characterization of the filtration.** Let $Z$ be a 0-cycle on $X$ such that $[Z] \in S_g(X)$ and suppose that

$$d := \deg Z > g. \tag{3.9}$$

Then $Z \equiv (p_1 + \ldots + p_g + (d-g)c_X)$. Now let $R \subset X$ be a rational curve: since any point in $R$ represents $c_X$ we get that every 0-cycle $p_1 + \ldots + p_g + x_1 + \ldots + x_{d-g}$ with $x_1, \ldots, x_{d-g} \in R$ is *effective* and rationally equivalent to $Z$. Thus

$$\dim\{Y \in X^{[d]} \mid \gamma(Y) \equiv Z\} \ge (d-g)\}. \tag{3.10}$$

(We recall that $\gamma\colon X^{[d]} \to X^{(d)}$ is the Hilbert-Cow morphism.)

**Theorem 3.6** (Voisin, Thm. 2.1 of [31])**.** *Let $d > g \ge 0$ and $V \subset X^{[d]}$ be a closed irreducible subset such that the following hold:*

1. $\dim V \ge (d-g)$.
2. *If $Z_1, Z_2 \in V$ then $\gamma(Z_1) \equiv \gamma(Z_2)$.*

*Then $[\gamma(Z)] \in S_g(X)$ for every $Z \in V$.*

*Remark* 3.7. If $g = 0$ and $d = 1$ then **Theorem 3.6** follows from (1.3).

*Remark* 3.8. We give the initial idea in the proof of **Theorem 3.6**. For simplicity we assume that $\dim \gamma(V) \geq (d-g)$. Let $R \subset X$ be a rational ample curve and

$$\Sigma_R := \{Z \in X^{(d)} \mid Z \cap R \neq \emptyset\}. \tag{3.11}$$

Since $\dim V > 0$ and $V$ is closed the intersection $V \cap \Sigma_R$ is not empty. (The divisor $\Sigma_R$ is ample on $X^{(d)}$.) Thus $V \cap \Sigma_R$ is closed of dimension at least $(\dim V - 1)$. By a dimension count we expect that

$$\{Z \in V \mid \deg(Z \cap R) \geq 2\} \tag{3.12}$$

has dimension at least $(d-2)$, and so on. This argument by itself is *not* sufficient: think of the closed

$$V := \{p_0 + p \mid p \in X\} \tag{3.13}$$

and suppose that $R$ does not contain $p_0$. Then there is *no* $Z \in V$ which is supported on $R$, contrary to what is suggested by a dimension count. Thus one must use the hypothesis that $\gamma(Z_1) \equiv \gamma(Z_2)$ for arbitrary $Z_1, Z_2 \in V$: that is what Voisin does in [31].

3.5. **Proof of the main result.** The key result is the following.

**Theorem 3.9** (Voisin)**.** *Let $F$ be a simple locally-free sheaf on $X$. Let $v := v(F)$ and $g := d(v)$. Then $c_2(F) \in S_g(X)$.*

*Proof.* Tensorizing $F$ by a sufficiently ample divisor we may assume (because of **Theorem 1.5**) that $F$ is globally generated and has no higher cohomology. Let $v(F) = (r, c_1^{\mathrm{hom}}(\mathcal{O}_X(H)), s)$. Let

$$g = d(v) = 1 + \frac{v^2}{2} = 1 + \frac{H \cdot H}{2} - rs. \tag{3.14}$$

Let

$$d := \deg c_2(F) = c_2(v) = r + \frac{H \cdot H}{2} - s. \tag{3.15}$$

Clearly $d > g$ (we may assume that $r > 1$ because if $r = 1$ then $c_2(F) = 0$). Choosing a generic $U \in \mathrm{Gr}(r-1, H^0(F))$ we get an exact sequence

$$0 \longrightarrow U \otimes \mathcal{O}_X \stackrel{\mathrm{ev}_U}{\longrightarrow} F \longrightarrow \mathcal{I}_{Z_U}(D) \longrightarrow 0 \tag{3.16}$$

where $Z_U \subset X$ is a 0-dimensional subscheme of $X$. By Whitney's formula $c_2(F)$ is represented by the cycle $\gamma(Z_U)$ associated to $Z_U$. Let $\mathrm{Gr}(r-1, H^0(F))^0 \subset \mathrm{Gr}(r-1, H^0(F))$ be the open dense subset such that $\mathrm{ev}_U \colon U \otimes \mathcal{O}_X \to F$ has cokernel which is torsion-free of rank 1. We have a regular map

$$\begin{array}{ccc} \mathrm{Gr}(r-1, H^0(F))^0 & \stackrel{\rho}{\longrightarrow} & X^{[d]} \\ U & \mapsto & Z_U \end{array} \tag{3.17}$$

We claim that $\rho$ is injective. In fact this amounts to proving that for $Z \in \mathrm{Im}\,\rho$ we have

$$\dim \mathrm{Hom}(F, \mathcal{I}_Z(H)) = 1. \tag{3.18}$$

The above equation follows from simplicity of $F$ (apply the functor $\mathrm{Hom}(F, \cdot)$ to Exact Sequence (3.16).) Let $V := \overline{\mathrm{Im}\,\rho}$. Then $V$ is an irreducible closed subset of $X^{[d]}$ and if $Z_1, Z_2 \in V$ then $\gamma(Z_1) \equiv \gamma(Z_2)$. By injectivity of $\rho$

$$\dim V = \dim \mathrm{Gr}(r-1, H^0(F)) = (r-1)(h^0(F) - r + 1) = (r-1)(r+s-r+1) = (r-1)(s+1). \tag{3.19}$$

On the other hand (3.14) and (3.15) give that

$$d - g = r - s - 1 + rs = (r-1)(s+1). \tag{3.20}$$

By **Theorem 3.6** we get that $\gamma(Z) \in S_g(X)$. □

**Theorem 3.5** follows from **Theorem 3.9** and the existence of a regular (holomorphic) symplectic form on $\mathfrak{M}_v(X,H)^{\mathrm{st}}$ - see Proposition 1.3 of [26]. Actually Voisin proves also a result similar to **Theorem 3.5** in which stable sheaves are replaced by simple locally-free sheaves, see Corollary 1.11 of [31].

3.6. **Generalized Franchetta conjecture.** Let $g \geq 3$. Let $\mathfrak{F}_g$ be the moduli space of $K3$ surfaces with a polarization of degree $(2g-2)$. Let $\mathfrak{F}_g^0 \subset \mathfrak{F}_g$ be the open dense subset parametrizing polarized $K3$ surfaces with trivial automorphism group (of the polarized $K3$). There is a tautological family of $K3$ surfaces $\rho\colon \mathcal{X}_g \to \mathfrak{F}_g^0$. Given $t \in \mathfrak{F}$ we let $X_t := \rho^{-1}(t)$ and $H_t \in \mathrm{Pic}(X_t)$ be the class of the polarization. The following question is quite natural:

*Question* 3.10. Let $\mathcal{Z} \in CH^2(\mathcal{X}_g)$. Let $t \in \mathfrak{F}_g^0$ and $Z_t = \mathcal{Z}|_{X_t}$. Is it true that there exist $a, b \in \mathbb{Z}$ such that $Z_t = ac_2(X_t) + b$ ?

The statement of the above question is similar to Franchetta's conjecture on rationally defined line-bundles on the tautological family of curves on $\mathfrak{M}_g$ - now a Theorem, see [1, 22] (there is also a version for families of curves embedded by linear systems other than the canonical one). Franchetta's conjecture may be proved for very low values of $g$ by a simple direct argument. The proof may be adapted in order to give an affirmative answer to **Question 3.10** for those values of $g$ such that the generic $K3$ surface of genus $g$ is a complete intersection in projective space i.e. $g = 3, 4, 5$ (I thank Daniel Huybrechts for bringing that to my attention). Notice that **Question 3.10** implies **Theorem 3.5** for $v^2 = -2$.

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